1972

Observability in the state estimation of power systems

Erwin Enrique Fetzer
Iowa State University

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Observability in the state estimation of power systems

by

Erwin Enrique Fetzer

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I. INTRODUCTION

The purpose of an electric power system is the conversion and delivery of energy to users when, where and for as long as required and as economically and dependably as possible. The objective of the energy conversion system is to operate each plant at optimum efficiency at the power generation level assigned to it. The objective of interconnecting the plants by a transmission network is to assign and maintain plant generation levels and transmission line power flows which will give the best overall system economy. In order to determine whether a system is performing properly, and to improve the system performance, it is important to know what the system is doing at any instant of time, i.e., we must know the state of the system. The state of the system is determined by its state variables which are defined as an independent set of \( n \) variables that completely characterize the operation of the system.

A. Power System Operation

The operation of a power system can be considered static or dynamic. By static operation, we mean the system is operating under steady state or slowly varying conditions. This is the type of operation that has been used in recent papers [13], [26], [39], and is the basis for system monitoring and control functions. The dynamic type of operation considers
major changes in the system, e.g., a fault in the network, a failure in a piece of equipment, or the sudden application of a major load or a loss of a generating unit [2]. The number of state variables to represent the dynamic operation of the system is much larger than for the static operation since control systems are operating. So we define our system to be a network, operating in steady state conditions, with \( N \) nodes (not including the neutral or reference node). In the nodal frame of reference with \( N \) nodes, there are \( N \) independent complex voltages. Thus we can choose the voltage magnitudes and voltage phase angles as the state variables which completely determine the network operating condition. Furthermore, if one of the voltage phase angles is used as a reference, we will have only \( 2N-1 \) states as we will see later in Chapter V.

The primary function of the power system is to provide the real and reactive powers demanded by the loads. At the same time, the frequency and bus voltages must be kept within specified tolerance. At constant frequency, by manipulation of the prime mover torque (which is accomplished by the turbine governor) we can maintain an exact balance between the generated real power and the demanded real power plus real power losses. At constant bus voltage magnitude, by manipulation of the field current and thus the generated emf, we keep an exact balance between the generated reactive power
and demanded reactive power plus reactive losses. Thus, in the static operation where the real and reactive power demand is determined by the consumers, the boundary conditions or limitations for buses \( i = 1, N \) are:

(a) The voltage magnitudes \( |E_i| \) must be within certain limits, (5% in some cases), i.e.,

\[
|E_i|_{\text{min}} < |E_i| < |E_i|_{\text{max}} \quad i = 1, \ldots, N \quad (1.1)
\]

(b) The phase angle \( \delta_i \) must satisfy the inequality

\[
(\delta_i - \delta_j) < (\delta_i - \delta_j)_{\text{max}} \quad i, j = 1, \ldots, N \quad (1.2)
\]

which specifies the maximum power transfer angle for a transmission line between buses \( i \) and \( j \).

(c) There are also practical limitations of the real and reactive power generation.

\[
P_{gi,\text{min}} < P_{gi} < P_{gi,\text{max}}
\]

\[
Q_{gi,\text{min}} < Q_{gi} < Q_{gi,\text{max}} \quad i = 1, \ldots, N \quad (1.3)
\]

or if some buses do not have power generation

\( P_{gi} = 0 \) and/or \( Q_{gi} = 0 \).

B. Power System Mathematical Description in the Steady State

The power system can be described by either nodal or branch equations. In the nodal frame of reference, which we
will use for this thesis, the system is described by bus voltages and related to the injection currents, i.e.,

\[ I = Y E_{bus} \]  \hspace{1cm} (1.4)

where

\[ I = (N \times 1) \text{ vector of injection currents} \]
\[ Y = (N \times N) \text{ admittance matrix} \]
\[ E_{bus} = (N \times 1) \text{ vector of phase-to-neutral voltages measured with respect to} \]
\[ \text{the reference node} \]

Thus, if we have \( N \) nodes, we have \( N \) phasor equations or \( 2N \) algebraic equations.

On the other hand, we can use branch currents and solve for the branch voltages, i.e.,

\[ E_{BR} = Z_{BR} I_{BR} \]  \hspace{1cm} (1.5)

where

\[ E_{BR} = (b \times 1) \text{ vector of voltages across the branches} \]
\[ Z_{BR} = (b \times b) \text{ branch impedance matrix} \]
\[ I_{BR} = (b \times 1) \text{ vector of currents through the branches} \]

If \( b \) is the number of branches, we will have \( b \) equations and this set of equations usually overspecifies the network, i.e., the equations are not independent if \( b>N \) (as is usually the
The constraints on the voltage magnitudes described by equation (1.1) are very important in the operation of the power system. On the other hand, the branch currents may vary over a wide range from 0 to some maximum.

The metered quantities that are normally available for power system monitoring are bus voltage magnitudes referred to a node reference (ground), injection power, power flow in certain branches, and sometimes current magnitudes. It is very difficult to measure phase angle.

C. The State of the System

Power systems are subject to random disturbances, so the static state of the power system is also random. In order to determine the state of a power system, we take measurements or observations on the system. These measurements are usually contaminated with noise from several sources in the measurement devices and data communication channels. The problem of determining the state of a system from noisy measurements is called estimation or filtering. This is of central importance in power systems where the static state estimate is required for centralized monitoring and control of the system.

The process of estimation is the process of making a decision or judgment concerning the approximate value of the state of the system when the decision is weighted or influenced by all available information. The state estimator is
the heart of the system monitor because of its ability to simultaneously process redundant data from different sources, automatically taking into account the relative value of each piece of data. Redundancy implies reliability. However, the amount of information available for a power system estimation is very large considering all the types of measurements and a priori knowledge of some variables (called pseudomeasurements by Schweppe, [36]), that can be used in the state estimator. Since most power systems are physically large, the problem gets out of hand very rapidly.

The estimator not only generates a state estimate, but also the covariance or error associated with it. This quantity provides a measure of accuracy of the estimate. Since we cannot use all the information available from a power system, we must make a selection of the measurements and pseudomeasurements to be used in the state estimator. Hopefully, we can then determine the best set of measurements and also the best meter placements. As pointed out by Schweppe [36],

"Performance depended heavily on meter placement, accuracy, and type.... Meter placement and type as well as meter accuracy effected the accuracy of the final estimate. The effect of adding or removing one meter was sometimes dramatic and sometimes unobservable; it all depended on the situation.... A wide range of meter placement and types was explored in the hopes of developing some rules of thumb for choosing meter placement and types which are economical and yet provide good estimates. Unfortunately, no such insight that could be extrapolated to larger systems was obtained."

The meter placements have great importance in accuracy of the
estimate. In addition, if we do not make proper selection of
the measurements, the problem gets out of hand in size and
also in the number of computations to be made in order to
arrive at a good estimate. This problem is discussed in the
literature by Larson et al. [27]. If \( m \) is the number of meas-
urements and \( n \) the variables to be found, using least-squares
state estimation,

"These computations require basic arithmetic oper-
ations on the order of \( mn^2 \). Using one of the faster
present-day computers, the computations for \( n = 100, \)
m\( > n \), would require only a few seconds; for \( n = 1000, \)
m\( > n \), the computations would require several minutes,
which is too long for most practical on-line applica-
tions. The high-speed storage requirements are on
the order of \( mn \) locations. If \( n = 100, m > n \), the
number of locations is on the order of \( 10^5 \), which
is large but manageable. However, if \( n = 1000, \) the
number is \( 10^6 \), which is clearly infeasible for any
readily available system."

Thus, for a given network, the accuracy or variance of an
estimate is function of meter types and placement, measure-
ment accuracy and the number of measurements considered. The
number of measurements is bounded by the number of arithmet-
cal operations and memory size for the given on-line computer
application.

The present work is directed at finding a quantitative
method of comparing different sets of measurements. In other
words, we seek a set of measurements which will give the best
accuracy or smallest variance. We also attempt to show which
measurement offers the least improvement, or conversely, which
will be the 'best' measurement to be added to a given
collection of measurements.

D. A Mathematical Definition of State and State Estimation

The static state estimation is a method to obtain an estimate of the n component state vector $\mathbf{x}$ of the static system using the m component measurement vector, $\mathbf{z}$, containing random error, $\mathbf{v}$, which is independent of the state $\mathbf{x}$. For any given system, a state variable is defined as an independent set of n variables that completely characterizes the operation of the system. In the nodal frame of reference, the n state variables will number 2N-1 and will correspond to N independent nodal phasor equations. Our state variables will be all the phase voltage angles and voltage magnitudes at each node, i.e.,

$$\mathbf{x} = [\delta_1, \delta_2, \ldots, \delta_N, |E_1|, |E_2|, \ldots, |E_N|]^T$$

As shown in Chapter V, we are only interested in the difference in phase angles $\delta_i - \delta_j$. Consequently, we can choose one phase angle as a reference which, in our case, will be $\delta_1$, i.e., $\delta_1 = 0$. Therefore, our state vector will be

$$\mathbf{x} = [\delta_2, \delta_3, \ldots, \delta_N, |E_1|, |E_2|, \ldots, |E_N|]^T \quad (1.6)$$

All our measurements are nothing more than approximations to the truth, i.e.,
where
\[ z = z^* + v \] (1.7)

\[ z = \text{raw noisy measurements, an } (m \times 1) \text{ vector} \]
\[ z^* = \text{noiseless measurements, an } (m \times 1) \text{ vector} \]
\[ v = \text{an } (m \times 1) \text{ vector that represents the uncertainty of our measurement, } z, \text{ specified by the statistics mean and covariance} \]

The uncertainty or error of the measurements comes from many sources such as meter inaccuracies. We also know that the noiseless measurement \( z^* \) is a function of the state, i.e.,
\[ z^* = f(x) \] (1.8)

Substituting equation (1.8) into (1.7)
\[ z = f(x) + v \] (1.9)

So from the measurement \( z \), we want to find the state of the system.

The measurements could consist of any or all the following variables: power injection (real and reactive), line power flows (real and reactive), voltage magnitudes and perhaps currents and voltage phase angles. We can also include in our measurement vector any \textit{a priori} knowledge of the variables mentioned above as pseudomeasurements [26], [39], namely, known voltage and other regulator settings, known generation and interchange schedules, or short-term load
forecasts by nodes.

We could make the same measurements of unknown variables and solve the system of equations (1.8) as is done in computer power flow studies, but in this way we have not made use of all the information or measurements and pseudomeasurements available, since we have more measurements than unknown variables.

In a power system, \( f(x) \) in equation (1.9) is nonlinear. To equation (1.9), we can apply the least-square method which will give us an estimate of the state \( x \) of the system that minimizes the scalar

\[
J(x) = (z - f(x))^T D (z - f(x))
\]  
(1.10)

where \( D \) is a weighting matrix that can be chosen. The minimization of \( J(x) \) of equation (1.10) with respect to \( x \) cannot be done in closed form because \( f(x) \) is a nonlinear function of \( x \). However, we can linearize \( f(x) \) around an operating point \( x_0 \) where \( z_0 \) is defined as

\[
z_0 = f(x_0)
\]  
(1.11)

Expanding \( z \) and \( f(x) \) to first order, we get

\[
f(x) = f(x_0) + H\Delta x
\]  
(1.12)

\[
z = z_0 + \Delta z
\]  
(1.13)

where \( H \) is the Jacobian matrix.
Substituting equations (1.12) and (1.13) into (1.11), we get

\[ J(x) = \begin{bmatrix} z_0 + \Delta z - f(x_0) - H\Delta x \end{bmatrix}^T \ \begin{bmatrix} z_0 + \Delta x \\ \Delta x \end{bmatrix} \]

\[ = (\Delta z - H\Delta x)^T D (\Delta z - H\Delta x) \]  \hspace{1cm} (1.14)

This is the same scalar \( J \) as for the problem defined by

\[ \Delta z = H\Delta x + v \]

or

\[ z = Hx + v \]  \hspace{1cm} (1.15)

Now, if we make use of a linear model of a power system, we can use the Kalman filter.

The main problem in both the Kalman and least-square methods, is in deciding how to choose the appropriate \( f(x) \) of equation (1.8) that represents a set of equations.

E. Summary

A brief review of literature for this research is given in Chapter II. Chapter III contains a simple but general model of the power system in steady state. Based on this model, we develop the observability criteria in Chapter IV in answer to the problem related to meter placement in state estimation. In Chapter V we develop the linearized measurement equations for a power system considering the following
measurements: power injection (real and reactive), line power flows (real and reactive) and voltage magnitudes. In order to show the advantages of meter placement in a clearer form, we develop the Kalman filter equations in Chapter VI and also show its relation to the least-squares method. A simple example of a power system is shown in Chapter VII, which will serve in testing the theory developed in Chapter IV and Chapter VI. Chapter VIII consists of a discussion and gives results of the theory applied to the example. Chapter IX contains the conclusions and some suggestions for future work. A list of references is included in Chapter X. The appendices contain the computer programs and format used to obtain the results in Chapter VIII.
II. REVIEW OF LITERATURE

In recent decades, considerable work has been done in the aerospace and defense industries which has served to lay the groundwork for research and application of state estimation in power systems. We find applications of the theory of linear filtering and estimation in satellite orbit determination, submarine and aircraft navigation, and space flights where the principal questions are the vehicle location, how to keep track of it, how to get where it has to go and how to orient the vehicle so as to execute the proper maneuvers. The sciences related to these questions are orbit determination, navigation, guidance and control. These same basic questions apply to power systems. Although the questions are the same and the methods of solution are similar, the systems are different and the constraints in a power system are much different. Much research and development is needed to adapt and extend these methods to power systems.

In the state estimation of power systems, various articles have appeared in recent years. The first to develop the idea of state estimation in power systems as a necessary step to real-time monitoring and control, and to point out some of its problems was Schweppe and Wilkes [39] who used static state estimation based on the weighted least-squares theory (equation 1.10), where the weighting matrix $D$ is $E(vv^T)$. He followed the standard method of expanding the nonlinear
function $f(x)$ around an initial or operating point $x_0$ and used only the two first terms of the expansion and got the following estimate $\hat{x}$ as

$$\hat{x} = x_0 + \left[H^T D H\right]^{-1} H^T D \left[z - f(x_0)\right]$$

(2.1)

where $H$ is the Jacobian matrix of $f(x)$ evaluated at $x_0$. He iterated equation (2.1) until $J(x)$ of equation (1.10) approached a minimum. He noted "that it is possible for $J(x)$ to have a local minimum and flat spots and thus $x$ may converge but not to $\hat{x}$. It is also possible for $x$ to oscillate and never converge to anything". He pointed out the dynamic state estimation as a second step after a dynamic model for the power system has been developed, and also discussed the significance of meter placement in the estimator (which is quoted in the Introduction).

An important contribution was made by Larson and Peschon [25] and Larson et al. [26], who also linearized the measurement function $f(x)$ around a nominal point, $x_0$. For the case where there are more measurements than states, they used the following formulas:

$$\Delta \hat{x} = \left[H^T D H\right]^{-1} H^T D \left[z - x_0\right]$$

$$C_x = \left[H^T D H\right]^{-1}$$

$$\Delta \hat{z} = H \hat{x}$$

$$C_z = H C_x H^T$$

(2.2)
Figure 2.1. Flow chart for state estimation
where $C_X$ and $C_Z$ are the covariance matrices. They used the flow chart in Figure 2.1 for the estimator, and also pointed out some computational procedure improvements, such as: (a) partitioning of the measurements into a basic set plus redundant ones; (b) sequential processing of the redundant measurements; and (c) sequential processing using only the diagonal elements of the state covariance matrix. In the practical implementation of the state estimator, they also carried out some experimental work with 400-node network and pointed out the computational problem, quoted in the Introduction, that existed in the order of upper bounds for the number of computations and for memory size. They also pointed out that small errors in the parameters of the system will not have a significant effect upon the quality of the estimate if the system is large.

How the errors in the model of the system will affect the accuracy of the estimate is pursued by Stuart [45], who followed a sensitivity approach to find which parameters are more critical, and used the least-squares method for the estimation.

Load flow and least-squares method were compared by Stagg et al. [43] who found difficulties with the least-squares estimation and stated "Regardless of the approach used, an important factor to consider is the appropriate selection of redundant information". However, Dopazo et al. [13, 14] coupled the
use of load-flow procedures with the weighted least-squares to obtain a method for on-line calculation of the state of the system, and also analyzed the relative weight of the measurement in terms of the accuracy of the measurement.

A dynamic estimator was used by Debs and Larson [10]. He used a simple model of a power system and a measurement model similar to all the preceding articles. The estimation method is based on the Kalman-Bucy filtering technique.

A report from North American Rockwell [32], describes how the Kalman filter can accurately and dynamically track the state of the power system. The algorithm used by them is slow, but they point out that the efficient employment of diakoptics or optimal ordering techniques will effectively reduce computation time.

The problem of estimating the state of a dynamical system dates back almost two centuries to the work of Gauss, who developed the technique known today as least-squares. This technique is found in Bryson and Ho [7], Deutsch [12], Jazwinski [20], among others. Recent developments in the area of estimation theory have been introduced by Kalman [21], who made use of the state variable description of the system and developed the filtering at discrete instants of time, after which he developed the continuous filter [23]. Many good references have appeared since, such as Brown [4], Jazwinski [20], Meditch [29], and Sorenson [42], all of whom make use
of a mathematical description of systems, which can be found in Brown [4, 5], DeRusso et al. [11], and Jazwinski [20].

The time-discrete Kalman filter is composed of a group of matrix recursive relations. From these recursive relations we obtain an estimate, $\hat{x}$, and its covariance which is a measure of the estimation error. The solution of the covariance equation is an integral part of the implementation of the Kalman filter and the stability of this equation is very important. Therefore, the following questions arise:

(a) Does a unique solution exist for all $t > t_0$, and if so, under what conditions is it stable?
(b) Does the initial value of the covariance matrix have any effect on the solution after the system has been operating for a long period of time?
(c) Can small errors in the computation of the solution be permitted?

These problems in the Kalman filter solution have been treated by Duven [16] and Potter [33].

The observability concept, as the name implies, is concerned with observations or measurements, $z$. A system is said to be observable if, by making noiseless measurements for an interval of time 0 to $T$, we get enough information to determine the initial state $x_0$. In Kalman [22], Kreindler and Sarachik [24], and Sorenson [41], we find the basic theory of observability and tests that can be used in order to find
out if a system is observable. But Ablin [1] and Brown [5] found a criterion to measure how observable the system is or the degree of observability. The criterion of how observable the system is requires us to find the eigenvalues and eigenvectors of a matrix for which we have the basic theory and methods used in Ralston [35] and also in Wilkinson [47]. Chuang [8] developed two types of algorithms to find the eigenvalues and eigenvectors of a matrix based on norm reduction and we use these algorithms to test the results of the computer program developed by Argonne National Laboratory Applied Mathematics Division. The subroutines used are: TRED2, to reduce a real symmetric matrix to a symmetric tridiagonal matrix accumulating the orthogonal transformations, and TQL2 to determine the eigenvalues and eigenvectors of a symmetric tridiagonal matrix.

In all the papers on state estimation of power systems, the problem of meter-placement or where the measurements should be made is not solved. There are no references which show how to choose this basic set of measurements. On the other hand, if we want to add more measurements, the question of which should be added in order to get a closer estimate is also of interest. These topics will be the subject of this research.
III. MATHEMATICAL MODELS

The field of modern control and estimation theory is heavily dependent on the concept of state variables. As pointed out in the Introduction, the Kalman filter technique is based on a linear model of the power system and a linear measurement model. In the case of using the least-squares method, we only make use of the linearized measurement model. From equation (1.6), our state vector will be

\[ \mathbf{x} = \begin{bmatrix} \delta_2, \delta_3, \ldots, \delta_N, |E_1|, |E_2|, \ldots, |E_N| \end{bmatrix}^T \quad (3.1) \]

A. System Model

The state-variable formulation of a linear dynamic system is a set of n first order differential equations arranged in the form

\[ \dot{\mathbf{x}} = A \mathbf{x} + \mathbf{f} \quad (3.2) \]

The solution of equation (3.2) is usually written in the form

\[ \mathbf{x} = \Phi \mathbf{x}_0 + \mathbf{q} \quad (3.3) \]

where

\( \mathbf{x} \) = is the n-dimensional column vector of the state variables, at time t.

\( \dot{\mathbf{x}} \) = is the n-dimensional column vector of the time derivative of the state variables.
\( A = n \times n \) matrix giving the relation between \( \dot{x} \) and \( x \).

\( f \) = driving function.

\( \Phi \) = transition matrix \((n \times n)\). It is the solution of equation (3.2) when \( f \) is zero and relates the state of the system at time \( t \) to a previous state at time \( t_0 \).

\( x_0 \) = is the \( n \)-dimensional column vector at a previous or initial state at time \( t_0 \).

\( q \) = column vector of state responses due to driving functions that occur in the interim between \( t \) and \( t_0 \).

A power system delivers power from generating stations to loads via transmission lines. These systems rarely achieve a true steady state operating point since their power demands are always changing, minute by minute, hourly, daily, weekly, seasonally and yearly. Whereas the individual loads may be entirely random in character, a certain average pattern is recognizable even at the distribution feeder level. At the subtransmission level, this averaging effect is still more pronounced. Finally, at the transmission level, we reach an almost predictable situation.

It is not only the loads that are varying. The transmission line parameters, i.e., line inductance, line shunt capacitance and line resistance are changing slightly due
to many factors, such as weather, temperature, current density, atmospheric humidity, etc. Since the loads and transmission line parameters are varying, the automatic controls are also varying. These automatic controls, like load tap changing transformers and switched capacitor banks, are slow acting, 10-120 sec., and act in discrete steps. The generator voltage regulator, load frequency controls and boiler controls are much faster in the order of a few seconds or less. But for the steady state period of observation, these controls are nearly constant. If we want to take into account these automatic controls, we have to take into consideration their dynamic behaviour and we have to introduce more states to specify the system [3].

There are many other items to be considered before the power system is completely modeled. However, a basic but general model of the power system can be assumed which makes several approximations and simplifications of the real system so that the analysis can be accomplished. These approximations are the following:

1. System operation is in quasi steady state with a frequency of 60 Hz. By quasi steady state, we mean that the time scale of interest is relatively long and that the network operating condition (state) at any time t can be estimated for our purpose from a knowledge of the state of
the network at \((t - \Delta t)\) by a linear extrapolation.

2. The network parameters are completely known and constant.

3. The three-phase lines are balanced. Hence, single-phase representation can be used.

In the steady state, we can select the \(N\) voltage magnitudes and \(N\) phase angles at each of the nodes as the state variables. These variables completely determined the system. However, as shown in Chapter V, the network equations are functions only of the difference in phase angles between nodes, so that the total number of independent states is \(n = 2N - 1\). For convenience, we usually fix the phase angle reference at one arbitrarily chosen node to be zero.

A real power system, as mentioned previously, is never in steady state; however, for a short interval of a few minutes, the system is nearly in steady state. So the state of the system will be the same for a small interval of time \(t\), where \(x_1 = x_0\). But since there is a small variation from state \(x_0\) to the state \(x_1\), some uncertainty is introduced into the state \(x\) that can be taken into account by the vector \(\mathbf{g}\) which represents a random variable. Later, a restriction will be placed on the vector \(\mathbf{g}\). These restrictions are not needed in the development of the following observability criteria. In mathematical terms, our power system model is
\[ x_1 = x_0 + q \]  \hspace{1cm} (3.4)

If we compare it with equation (3.3), we see that the transition matrix, \( \Phi \), is the unit matrix. The transition matrix is the solution of equation (3.2) when the driving function \( \mathbf{f} \) is equal to zero or

\[ \dot{x} = A \cdot x \]

The solution to this equation for a period of time \( \Delta t \) is

\[ x = e^{A \Delta t} x_0 = \Phi \cdot x_0 = x_0 \]  \hspace{1cm} (3.5)

In order that equation (3.5) is satisfied, the matrix \( A \) is equal to the zero matrix since \( t \) is nonzero. The result \( A = 0 \) will simplify greatly the applicability of the observability criteria developed later.

B. Measurement Model

For our measurement equations, we use equation (1.9) in linearized form. Since \( f(x) \) is nonlinear for a power system and the dimension of the measurement vector, \( z \), (the total number of measurements and pseudomeasurements) is large, it is important to simplify equation (1.9) to conserve computing cost. Therefore, equation (1.9), in linearized form, is written as

\[ z = H \cdot x + v \]  \hspace{1cm} (3.6)
and this linearization for a power system is shown in Chapter V, where

\[ \mathbf{z} = \text{is the m-dimensional column vector or measurement vector, including noise.} \]

\[ \mathbf{H} = (m \times n) \text{ measurement matrix which relates linearly the measurement vector } (\mathbf{z}) \text{ with state vector } (\mathbf{x}). \text{ (Under noise-free conditions of observation } \mathbf{z} = \mathbf{H} \mathbf{x}. \)

\[ \mathbf{v} = \text{is the m-dimensional column vector that represents the uncertainty of the measurements and is specified by the mean and covariance statistics.} \]

We will use the nodal frame of reference in obtaining the relation between power injected and voltages for our measurement matrix. The nodal solution is Kirchhoff's current law which states that the sum of currents entering a node is equal to the sum of currents leaving the node. The currents injected into the nodes are labeled \( I_1, \ldots, I_N \), the bus voltages are called \( E_1, \ldots, E_N \), as shown in Figure 3.1. The \( y_{i0} \) are the shunt admittance at node \( i \), \( y_{ij} = y_{ji} \) are the line admittance between nodes \( i \) and \( j \). Thus,

\[
\begin{align*}
I_1 &= y_{10}E_1 + (E_1 - E_2)y_{12} + (E_1 - E_3)y_{13} + \ldots + (E_1 - E_N)y_{1N} \\
I_2 &= (E_2 - E_1)y_{12} + y_{20}E_2 + (E_2 - E_3)y_{23} + \ldots + (E_2 - E_N)y_{2N} \\
&\vdots \\
I_N &= (E_N - E_1)y_{N1} + (E_N - E_2)y_{N2} + \ldots + (E_N - E_{N-1})y_{N(N-1)} + y_{N0}E_N
\end{align*}
\]

(3.7)
Figure 3.1. N node system
or in matrix form

\[
\begin{bmatrix}
I_1 \\
I_2 \\
\vdots \\
I_N
\end{bmatrix} =
\begin{bmatrix}
Y_{11} & \cdots & Y_{1N} \\
\vdots & \ddots & \vdots \\
Y_{N1} & \cdots & Y_{NN}
\end{bmatrix}
\begin{bmatrix}
E_1 \\
\vdots \\
E_N
\end{bmatrix}
\]

(3.8)

where the diagonal terms are

\[
Y_{11} = Y_{10} + Y_{12} + \cdots + Y_{1N}
\]

\[
Y_{22} = Y_{21} + Y_{20} + Y_{23} + \cdots + Y_{2N}
\]

\[
\vdots
\]

\[
Y_{NN} = Y_{N1} + \cdots + Y_{N(N-1)} + Y_{N0}
\]

(3.9)

and the off-diagonal terms are

\[
Y_{ij} = -Y_{ji}
\]

(3.10)

Equation (3.8) in compact form is

\[
I = Y E
\]

(3.11)

Thus, the network behaviour is described by currents entering at nodes 1 through N and voltages measured as voltage drops to our external reference as shown in Figure 3.1. So we have N complex equations or 2N real equations. The complex power injected at bus p is given by

\[
P_p + jQ_p = E_p I_p^*
\]

(3.12)
where $I_p^*$ is the complex conjugate of the phasor current $I_p$.
The power injected at all nodes is the algebraic addition of
the power generated and the load power. Equation (3.12) gives
us the relation between the injected power and the bus volt-
age ($I_p^*$ can be substituted in terms of $E_p$).

Also, we make use of the power flow between buses $p$ and
$q$ as the power leaving bus $p$, i.e.,

$$P_{pq} + jQ_{pq} = E_p I_p^*$$  \hspace{1cm} (3.13)

where

- $P_{pq}$ is real power flow from bus $p$ to bus $q$.
- $Q_{pq}$ is the reactive power flow from bus $p$ to
  bus $q$.
- $I_{pq}$ is the current flow from bus $p$ to bus $q$.
- $E_p$ is the voltage at bus $p$.

Note that the first subscript in $P_{pq}$ and $Q_{pq}$ also indicates
the bus where the measurement is made.

The operating point in a power system is always changing.
Thus, the linearization of the equation is only valid for
limited periods of time, depending upon the load variation.
We will choose different operating points for our measurement
model.
IV. OBSERVABILITY CRITERIA

We will first elaborate the criterion of observability for the general model of a linear dynamic system which is described by equations (3.2) and (3.6), where the matrices $A$ and $H$ are fixed constants and do not vary with time.

**Definition 4.1:** A system is said to be observable if, and only if, in some finite time after $t_0$, with the knowledge of the state variable description of the system and with $f$ and $g$ equal to zero, the initial state at time, $t_0$, can be determined by observing the measurement vector $z$.

**Theorem 4.1:** The system is observable if, and only if, the rank of the matrix $Q$ is $n$ and where $Q$ is the $(n \times mn)$ matrix

$$Q = \begin{bmatrix} H^T, A^T H^T, (A^T)^2 H^T, \ldots, (A^T)^{N-1} H^T \end{bmatrix}$$

From Definition 4.1, equations (3.2) and (3.6) are reduced to

$$\dot{X} = AX$$

$$z = HX$$

We can sample our measurement vector $z$ (m-dimensional) at intervals of time $\Delta t$ and we can form the higher derivatives of the variable $z$, i.e., $\dot{z}, \ddot{z}, \ldots$, etc., by finite differences, i.e.,

$$\dot{z}(t) = \frac{z(t + \Delta t) - z(0)}{\Delta t}$$
In the same manner, we can find $\mathbf{y}$, $\mathbf{z}$, and so on. On the other hand, we have equation (4.3) and its higher derivatives

\[ \mathbf{z} = \mathbf{H} \mathbf{x} \]

and substituting (4.2), we compute

\[ \mathbf{z} = \mathbf{H} \mathbf{A} \mathbf{x} \]

In the same manner, we obtain

\[ \mathbf{y} = \mathbf{H}(\mathbf{A})^2 \mathbf{x} \]

\[ \vdots \]

\[ \mathbf{z}^{n-1} = \mathbf{H}(\mathbf{A})^{n-1} \mathbf{x} \]

Where $n$ is the order of the matrix $\mathbf{A}$ and further differentiation is superfluous [Ablin, 1]. Written in matrix form, the above equations would be

\[
\begin{bmatrix}
\mathbf{z} \\
\mathbf{z} \\
\mathbf{y} \\
\vdots \\
\mathbf{z}^{n-1}
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{H} \\
\mathbf{H} \mathbf{A} \\
\mathbf{H}(\mathbf{A})^2 \\
\vdots \\
\mathbf{H}(\mathbf{A})^{n-1}
\end{bmatrix} \begin{bmatrix}
\mathbf{x}
\end{bmatrix} = \mathbf{Q}^T \mathbf{x}
\]

(4.5)

This vector is of order $mn$ and the $\mathbf{Q}^T$ matrix is $(mn \times n)$. We need $n$ independent equation or rows of equation (4.5) to solve for our $n$ unknowns. In order to know if there are $n$
independent equations in (4.5), we resort to finding the rank of the matrix $Q^T$ or $Q$. The rank of $Q^T$ is equal to the rank of $Q$. If the rank of $Q$ is $n$, we have $n$ independent equations and the system is observable. Q.E.D.

We have determined that, for a static state estimator, $A$ is equal to zero; consequently,

$$Q = H^T$$

(4.6)

Being $A = 0$ means that there is no coupling between states, i.e., $x_i \neq x_j$. It also means that the derivatives are all zero, i.e., steady state.

A geometric interpretation of the columns of $Q$ is given by a simple example. Given $\dot{x} = 0$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ \alpha_1 & -\alpha_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(4.7)

we have

$$Q = \begin{bmatrix} 1 & \alpha_1 \\ -1 & -\alpha_2 \end{bmatrix} = [v_1 \ , \ v_2]$$

(4.8)

the rank of $Q$ will be two if $\alpha_1 \neq \alpha_2$. If we graph the two vectors $v_1$ and $v_2$ of equation (4.8), the result is shown in Figure 4.1.

Even if the two vectors, $v_1$ and $v_2$, are nearly coincident, the system is observable, but the solution or intersection of these two equations will be very difficult to
Figure 4.1. A plot of the vectors $v_1$ and $v_2$ from equation (4.8)

Figure 4.2. A plot of vectors $v_1$ and $v_2$ including noise
obtain. From equation (4.7) we can get

\[
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \frac{1}{\alpha_1 - \alpha_2} \begin{bmatrix}
  -\alpha_2 & 1 \\
  -\alpha_1 & 1
\end{bmatrix} \begin{bmatrix}
  z_1 \\
  z_2
\end{bmatrix}
\] (4.9)

and we can observe that when \( \alpha_1 \) is nearly equal to \( \alpha_2 \), their difference will be small and the inverse will be large. So the error or uncertainty of the measurements, \( z \), will be magnified. In other words, small measurement errors will reflect as large errors in the unknowns, \( x \). Including the error in the measurements, Figure 4.2, the solution or intersection of these two equations will be in the shaded area, assuming \( z \) equals zero. If these two equations are orthogonal, the maximum values of \( x_1 \) and \( x_2 \) will be minimum. So if we want to find or add another equation to our original set of equations, which will be the best equation or measurement to add? In our two-dimensional example, the best equation or measurement to be added would be the one which is the most orthogonal to \( v_1 \) and \( v_2 \), such as \( v_3 \) of Figure 4.1. Therefore, we want to find the "most orthogonal" equation and we can work with the rows of the matrix \( H \) or the columns of \( Q \). We will use the columns of \( Q \) in this research.

A geometric interpretation becomes cumbersome for higher order systems, but the idea of the "most orthogonal" equation is still valid. The system is strongly observable [Brown, 5] if the column vectors of \( Q \) fill the \( n \) space, or we can not
find a vector which is approximately orthogonal to all of the
column vectors of \( \mathbf{Q} \). The "most orthogonal" vector contains
considerable information and we proceed to find it next. We
know that the inner product of two orthogonal vectors is zero.

**Definition 4.2:** The most orthogonal vector, \( \mathbf{u} \), is defined as
the vector that minimizes the sum of the squares of the inner
products between it and each of the columns of the matrix \( \mathbf{Q}^\top \),
and \( \mathbf{u} \) is of unit length, i.e., \( \mathbf{u}\mathbf{u}^\top = 1 \).

We normalize the column vector \( \mathbf{N} \) of the \( \mathbf{Q} \) matrix. This
normalization will not affect our results since we are inter­
ested in the angles between these column vectors. Therefore,
by normalization, we get the \( (n \times mn) \) normalized matrix \( \mathbf{Q}_n \)
where

\[
\mathbf{Q}_n = \begin{bmatrix}
\mathbf{w}_1^\top & \mathbf{w}_2^\top & \cdots & \mathbf{w}_m^\top
\end{bmatrix}
\tag{4.10}
\]

**Theorem 4.2:** For any system, the vector \( \mathbf{u} \) that corresponds
to the most orthogonal vector to the columns of the matrix
\( \mathbf{Q}_n \), is the eigenvector associated with the smallest eigenvalue
of the matrix \( \mathbf{Q}_n\mathbf{Q}_n^\top \).

We define a loss function \( L \) of the form

\[
L = (\mathbf{w}_1^\top \mathbf{u})^2 + (\mathbf{w}_2^\top \mathbf{u})^2 + \ldots + (\mathbf{w}_m^\top \mathbf{u})^2
\tag{4.11}
\]
and \( u \) is chosen such as to minimize \( L \). From equation (4.11),

\[
L = \left[ (w_1^T u)^T (w_1^T u) + (w_2^T u)^T (w_2^T u) + \ldots + (w_m^T u)^T (w_m^T u) \right]
\]

\[
= u^T w_1 w_1^T u + u^T w_2 w_2^T u + \ldots + u^T w_m w_m^T u
\]

\[
= u^T (w_1 w_1^T + w_2 w_2^T + \ldots + w_m w_m^T) u
\]

(4.12)

Substituting equation (4.10) into (4.12)

\[
L = u^T Q_n Q_n^T u
\]

(4.13)

where \( Q_n \) is the \((n \times mn)\) matrix defined by (4.10). In order to minimize \( L \), we can apply Lagrange multipliers, forming the function \( v \),

\[
v = u^T Q_n Q_n^T u - \lambda (u^T u - 1)
\]

(4.14)

to find the minimum

\[
\frac{\partial v}{\partial u} = 0
\]

(4.15)

\[
\frac{\partial}{\partial u} (u^T Q_n Q_n^T u - \lambda (u^T u - 1)) = 0
\]

(4.16)

By elementary calculus, we know that \( \frac{\partial}{\partial x} x^T B x = 2 B x \)

where \( B \) is symmetric. So we have

\[
2 Q_n Q_n^T u - \lambda 2 u u = 0
\]

(4.17)

\[
(Q_n Q_n^T - \lambda U) u = 0
\]

(4.18)

where capital \( U \) is the unit matrix.
The theory of simultaneous linear algebraic equations shows there is a nontrivial solution if, and only if, the matrix \( Q_n Q_n^T - \lambda I \) is singular. That is, \( \det(Q_n Q_n^T - \lambda I) = 0 \). The \( n \) roots are called the eigenvalues of the matrix \( Q_n Q_n^T \) and corresponding to any eigenvalue, \( \lambda_i \), the set of equations (4.18) has at least one nontrivial solution \( u \). Such a solution is called an eigenvector.

We must find the eigenvectors of the matrix \( Q_n Q_n^T \). If \( \xi_i \) is the eigenvector or solution of (4.18) corresponding to eigenvalue \( \lambda_i \), we can substitute this vector into equation (4.13) and our loss function will be

\[
L_\zeta = \xi_i^T Q_n Q_n^T \xi_i
\]  
(4.19)

and by equation (4.15)

\[
L_\zeta = \xi_i^T (Q_n Q_n^T) \xi_i = \xi_i^T \lambda_i \xi_i = \lambda_i \xi_i \xi_i^T = \lambda_i \xi_i
\]  
(4.20)

So the minimum loss function will correspond to the minimum, \( \lambda_i \), eigenvalue and the vector \( u \) (most orthogonal vector) will correspond to the associated eigenvector. Q.E.D.

The measurement desirable to add to our initial set of measurements should be as close as possible to the direction of the most orthogonal vector, or at least we can match one or more of their highest components.

**Theorem 4.3:** The best case of observability will be when all the eigenvalues of \( Q_n Q_n^T \) are equal.
We know by DeRusso et al. [11], that the sum of the diagonal elements (trace) of a square matrix, is equal to the sum of its eigenvalues. We see that the trace of $Q_nQ_n^T$ is equal to the number of nonzero columns of $Q$. So the trace will be $n$. The best case of observability is when all the columns of $Q$ are orthogonal and no "most orthogonal" vector can be found. This case will correspond to best observability and all the eigenvalues will be the same. Since the sum of eigenvalues is $n$ and there are $m$ nonzero columns in $Q_n$, all the eigenvalues will be equal to $m/n$. Q.E.D.

If we compare two sets of measurements, the one closest to the best case (Theorem 4.3) will be the better set of measurements. A zero eigenvalue means no observability or lack of observability in the direction of the associated eigenvector.

It is important to note that when the two smallest eigenvectors $\lambda_1$ and $\lambda_2$ are equal, any linear combination, $y$, of the associated eigenvectors is also an eigenvector.

$$\lambda_1 = \lambda_2 \quad \text{and} \quad y = c_1y_1 + c_2y_2$$

Thus

$$Q_1y_1 = \lambda_1y_1 \quad ; \quad Q_2y_2 = \lambda_2y_2$$

$$c_1Q_1y_1 = c_1\lambda_1y_1 \quad ; \quad c_2Q_2y_2 = c_2\lambda_2y_2$$

adding these last two equations and we obtain
The same procedure can be applied to any two or more equal eigenvalues.

So when the situation arises (equal eigenvalues), we can make use of the above relation to find the most convenient measurement to add. The direction of the "most orthogonal" vector is a linear combination of the eigenvectors corresponding to equal smallest eigenvalues.

It is important to note that Theorems 4.1 through 4.3 are for the general matrix $Q$ and not only for the static state estimator ($A = 0$). If we have $A \neq 0$, we still can apply the theorems and find the most orthogonal vector (Definition 4.1). This method will be of great value for the dynamic estimator.

In summary, we wish to find the eigenvalues and eigenvectors of the matrix $Q_n Q_n^T$, where the matrix $Q$ is equal to $H^T$, the measurement matrix. In the next chapter, we show how to find the measurement matrix for a power system.
V. MEASUREMENT MATRIX

The measurement matrix, $H$, gives an approximate linear relation between the measurement vector $z$ and the state vector $x$. For a power system, this relation is nonlinear; consequently, the system is linearized around a quiescent or operating point.

The states will be the voltage phase angles and voltage magnitudes at each of the $N$ nodes, measured with respect to one node voltage angle as a reference. The measurement vector, $z$, will usually consist of any or all of the following variables:

(a) Real power injection, $P_p$, at any of the $N$ nodes, i.e., $p = 1, \ldots, N$.

(b) Reactive power injection, $Q_p$, at any of the $N$ nodes, i.e., $p = 1, \ldots, N$.

(c) Real power flow $P_{pq}$ from node $p$ to node $q$ in any of the $b$ branches of the system where $b$ is the branch number. We can obtain a maximum of $2b$ measurements of this type.

(c) Reactive power flow $Q_{pq}$ from node $p$ to node $q$, at any of the $b$ branches of the system. We can obtain a maximum of $2b$ measurements of this type.

(e) Voltage magnitude, $|E_p|$, at any of the $N$ nodes.
The current measurement magnitude has not been taken into consideration since the power system operators are often more interested in power flows. The current magnitude is of interest in the design of the power system and is commonly metered in operation. We could eventually include this measurement into our estimator but, for the present, we will confine our analysis to measurements (a) through (e) above.

The general linearized form of the measurement equation in matrix form is given by equations (5.1) - (5.3). Thus

\[ z = H x \] (5.1)

which may be written in expanded form as

\[
\begin{bmatrix}
\Delta P_1 \\
\vdots \\
\Delta P_N \\
\Delta Q_1 \\
\vdots \\
\Delta Q_N \\
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial P_1}{\partial \delta_2} & \cdots & \frac{\partial P_1}{\partial \delta_N} & \frac{\partial P_1}{\partial |E_1|} & \cdots & \frac{\partial P_1}{\partial |E_N|} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial P_N}{\partial \delta_2} & \cdots & \frac{\partial P_N}{\partial \delta_N} & \frac{\partial P_N}{\partial |E_1|} & \cdots & \frac{\partial P_N}{\partial |E_N|} \\
\frac{\partial Q_1}{\partial \delta_2} & \cdots & \frac{\partial Q_1}{\partial \delta_N} & \frac{\partial Q_1}{\partial |E_1|} & \cdots & \frac{\partial Q_1}{\partial |E_N|} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial Q_N}{\partial \delta_2} & \cdots & \frac{\partial Q_N}{\partial \delta_N} & \frac{\partial Q_N}{\partial |E_1|} & \cdots & \frac{\partial Q_N}{\partial |E_N|} \\
\end{bmatrix}
\begin{bmatrix}
\Delta \delta_2 \\
\vdots \\
\Delta \delta_N \\
\Delta |E_1| \\
\vdots \\
\Delta |E_N| \\
\end{bmatrix}
\]
or in partitioned form as

\[
\begin{bmatrix}
\Delta P_p \\
\Delta Q_p \\
\Delta P_{pq} \\
\Delta Q_{pq} \\
\Delta |E_p|
\end{bmatrix} =
\begin{bmatrix}
J_1 & J_2 \\
J_3 & J_4 \\
J_5 & J_6 \\
J_7 & J_8 \\
0 & J_g
\end{bmatrix}
\begin{bmatrix}
\Delta \delta_p \\
\Delta \delta_N \\
\Delta |E_1| \\
\Delta |E_N|
\end{bmatrix}
\]

where \( z \) is \((m \times 1)\) vector, \( H \) is \((m \times n)\) matrix and \( x \) is an \( n \) or \( 2N-1 \) vector.

In accordance with industry-wide standards, a positive reactive power flow from \( p \) to \( q \) refers to a situation where
the current $I_{pq}$ lags the voltage.

From Chapter II, in the nodal frame of reference, the complex power injected at bus $p$ is given by

$$p_p + j0_p = E_p I_p^*$$  \hspace{1cm} (5.4)

where the injected current entering node $p$ is

$$I_p = \sum_{q=1}^{N} Y_{pq} E_q.$$  \hspace{1cm} (5.5)

$E_p$ and $E_q$ are the phasor voltages at buses $p$ and $q$, $N$ is the number of network buses and $I_p^*$ is the complex conjugate of the phasor current $I_p$. Note that $Y_{pq}$ is the element $p,q$ of the admittance matrix, where the diagonal elements, $Y_{pp}$, are obtained as algebraic sum of all admittances incident to bus $p$. The off diagonal elements $Y_{pq} = Y_{qp}$ are obtained as negative of the actual total admittance directly connecting buses $p$ and $q$. Substituting the value of $I_p$ in equation (5.5) into (5.4), we obtain

$$p_p + j0_p = E_p (\sum_{q=1}^{N} Y_{pq} E_q)^*$$  \hspace{1cm} (5.6)

This equation can be simplified by expressing the bus voltages and admittances in polar form. Thus we define the angles $\delta_p$ and $\theta_{pq}$ to write

$$E_p = |E_p| e^{j\delta_p} \quad \text{and} \quad Y_{pq} = |Y_{pq}| e^{j\theta_{pq}}$$  \hspace{1cm} (5.7)
Then

\[ P_p + jQ_p = |E_p| e^{j\delta_p} (\sum_{q=1}^{N} |Y_{pq}| e^{j\theta_{pq}}|E_q| e^{j\phi}) \]

\[ = \sum_{q=1}^{N} |E_p Y_{pq} E_q| e^{j(\delta_p - \delta_q - \theta_{pq})} \quad (5.8) \]

where we use a simplified notation \(|E_p Y_{pq} E_q|\) to mean \(|E_p||Y_{pq}||E_q|\). We also define \(\delta_{pq} = \delta_p - \delta_q\). Then we may separate the real and imaginary components as

\[ P_p = \sum_{q=1}^{N} |E_p Y_{pq} E_q| \cos (\delta_{pq} - \theta_{pq}) \quad (5.9) \]

\[ Q_p = \sum_{q=1}^{N} |E_p Y_{pq} E_q| \sin (\delta_{pq} - \theta_{pq}) \quad (5.10) \]

For the entries of matrix \(J_1\) of (5.3), we compute

\[ \frac{\partial P}{\partial \delta_q} = |E_p Y_{pq} E_q| \sin (\delta_{pq} - \theta_{pq}) \]

for \(q = 1, \ldots, N; \quad q \neq p \quad (5.11) \)

\[ \frac{\partial P}{\partial \delta_p} = - \sum_{q=1}^{N} |E_p Y_{pq} E_q| \sin (\delta_{pq} - \theta_{pq}) \quad (5.12) \]

For the entries of matrix \(J_2\) of (5.3), we compute
\[ \frac{\partial P_p}{\partial |E_q|} = |E_p Y_{pq}| \cos (\delta_{pq} - \Theta_{pq}) \]

for \( q = 1, \ldots, N; \ q \neq p \) (5.13)

\[ \frac{\partial P_p}{\partial |P_p|} = 2 |E_p Y_{pp}| \cos (-\Theta_{pp}) \]

\[ + \sum_{q=1}^{N} |Y_{pq} E_q| \cos (\delta_{pq} - \Theta_{pq}) \] (5.14)

For the entries of matrix \( J_3 \) of (5.3), we compute

\[ \frac{\partial Q_p}{\partial \delta_q} = -|E_p Y_{pq} E_q| \cos (\delta_{pq} - \Theta_{pq}) \]

for \( q = 1, \ldots, N; \ q \neq p \) (5.15)

\[ \frac{\partial Q_p}{\partial \delta_p} = \sum_{q=1}^{N} \sum_{q \neq p} |E_p Y_{pq} E_q| \cos (\delta_{pq} - \Theta_{pq}) \] (5.16)

For the entries of matrix \( J_4 \) of (5.3), we compute

\[ \frac{\partial Q_p}{\partial |E_q|} = |E_p Y_{pq}| \sin (\delta_{pq} - \Theta_{pq}) \]

for \( q = 1, \ldots, N; \ q \neq p \) (5.17)

\[ \frac{\partial Q_p}{\partial |P_p|} = 2 |E_p Y_{pp}| \sin (-\Theta_{pq}) \]

\[ + \sum_{q=1}^{N} \sum_{q \neq p} |Y_{pq} E_q| \sin (\delta_{pq} - \Theta_{pq}) \] (5.18)
To obtain the active and reactive power flows in the lines, we find the current leaving bus \( p \) in the line connecting bus \( p \) to bus \( q \) as shown in Figure 5.1.

\[
\begin{align*}
I_{pq}^p & = Y_{pq} (E_p - E_q) + Y_{pq0} E_p \\
I_{qp}^q & = Y_{qp} (E_q - E_p) + Y_{qp0} E_q
\end{align*}
\]

Figure 5.1. The \( \pi \) equivalent of a line

We note that the first subscript of \( I \) indicates where the measurement will be made and, similarly for \( P_{pq} \) and \( Q_{pq} \), the first subscript indicates where the measurement is made.

Then

\[
I_{pq} = -Y_{pq} (E_p - E_q) + Y_{pq0} E_p \tag{5.19}
\]

where

- \( Y_{pq} \) is the element \( p,q \) of bus admittance matrix equal to negative of the admittance \( y_{pq} \) on the line \( pq \).
- \( E_p \) and \( E_q \) are the voltages at buses \( p \) and \( q \).
\( y_{pq0} \) is the shunt admittance of the line \( pq \) at bus \( p \).

We usually consider the case where \( y_{pq0} \) and \( y_{qp0} \) are purely capacitive, i.e.,

\[
y_{pq0} = g_{pq0} + jb_{pq0} = 0 + jb_{pq0} = b_{pq0} \square 90
\]

Thus

\[
\varphi_{pq0} = +90
\]

The complex power flowing from \( p \) to \( q \) in line \( pq \) measured at bus \( p \) is

\[
P_{pq} + jQ_{pq} = E_p I_p^q
\]

\[
= E_p \left( -y_{pq} (E_p - E_q) + y_{pq0} E_p \right)^* \\
= E_p y_{pq}^* E_p^* - y_{pq} |E_p|^* + y_{pq0} |E_p|^2 \quad (5.20)
\]

or in polar coordinates

\[
P_{pq} + jQ_{pq} = |E_p| y_{pq} E_q^2 e^{j(\delta_{pq0} - \theta_{pq})} - |y_{pq0}| E_p^2 e^{-j\theta_{pq0}} \\
+ |y_{pq0}| E_p^2 e^{-j\theta_{pq0}} \quad (5.21)
\]

Separating the real and imaginary parts

\[
P_{pq} = |E_p| y_{pq} E_q^2 \cos (\delta_{pq0} - \theta_{pq}) - |y_{pq0}| E_p^2 \cos (\theta_{pq}) \quad (5.22)
\]
\[ Q_{pq} = |E_p Y_{pq} E_q| \sin (\delta_{pq} - \theta_{pq}) + |Y_{pq} E_p^2| \sin (\theta_{pq}) \]

\[ + |E_p^2 Y_{pq}^0| \]

(5.23)

For the entries of matrix \( J_5 \) of (5.3), we have

\[ \frac{\partial P_{pq}}{\partial \delta_p} = -|E_p Y_{pq} E_q| \sin (\delta_{pq} - \theta_{pq}) \]

(5.24)

\[ \frac{\partial P_{pq}}{\partial \delta_q} = |E_p Y_{pq} E_q| \sin (\delta_{pq} - \theta_{pq}) \]

(5.25)

For the entries of matrix \( J_6 \) of (5.3), we have

\[ \frac{\partial P_{pq}}{\partial |E_p|} = |Y_{pq} E_q| \cos (\delta_{pq} - \theta_{pq}) \]

\[ - 2|Y_{pq} E_p| \cos \theta_{pq} \]

(5.26)

\[ \frac{\partial P_{pq}}{\partial |E_q|} = |E_p Y_{pq}^0| \cos (\delta_{pq} - \theta_{pq}) \]

(5.27)

For the entries of matrix \( J_7 \) of (5.3), we have

\[ \frac{\partial Q_{pq}}{\partial \delta_p} = |E_p Y_{pq} E_q| \cos (\delta_{pq} - \theta_{pq}) \]

(5.28)

\[ \frac{\partial Q_{pq}}{\partial \delta_q} = -|E_p T_{pq} E_q| \cos (\delta_{pq} - \theta_{pq}) \]

(5.29)

For the entries of matrix \( J_8 \) of (5.3), we have
For the entries of matrix of (5.3), we have

\[ \frac{\partial Q_{pq}}{\partial |E_p|} = |Y_{pq} E_q| \sin (\delta_{pq} - \theta_{pq}) - 2 |E_p Y_{pq}| \sin \theta_{pq} \]

\[ + 2 |E_p b_{pq0}| \]  \hspace{1cm} (5.30)

\[ \frac{\partial Q_{pq}}{\partial |E_q|} = |E_p Y_{pq}| \sin (\delta_{pq} - \theta_{pq}) \]  \hspace{1cm} (5.31)

For the entries of matrix \( J_q \) of (5.3), we have

\[ \Delta |E_p| = U \], the unity matrix \hspace{1cm} (5.32)

Equations (5.11) through (5.18) and (5.29) through (5.32) are used in a computer program given in the Appendix in order to find the measurement matrix \( H \).

Note that in the equations of injected power, (5.9) and (5.10), and power flow, (5.22) and (5.23), the phase angles \( \delta_p \) and \( \delta_q \) appear in the difference form \( \delta_p - \delta_q \) which we have called \( \delta_{pq} \). Thus we are only interested in the difference of angles and the reference is arbitrary. Consequently, we can choose one phase angle as a reference which, in our case, will be the phase angle at bus one, i.e., \( \delta_1 = 0 \). Therefore, our state vector will be dimensioned 2N-1.
VI. KALMAN FILTER APPLICATION

Our mathematical model of a power system in steady state in Chapter III fits the Kalman filter with certain restrictions [4]. One restriction is in the way we define the column vector \( q \) of equation (3.3) which is the state response due to all of the independent white noise driving functions that occur in the interim between \( t \) and \( t_0 \).

From equation (3.2), \( \dot{x} = A \dot{x} + f \), we have the solution

\[
x(t) = \phi(t - t_0) x(t_0) + \int_{t_0}^{t} \phi(t - \tau) f(\tau) d\tau
\]

(6.1)

or in more compact form, \( x = \phi x_0 + q \) which is equation (3.3).

For the discrete case, we let

\[
t_0 = t_k
\]
\[
t = t_{k+1}
\]

(6.2)

Substituting these values into equation (6.1), we have

\[
x(t_{k+1}) = \phi(t_{k+1} - t_k) x(t_k) + \int_{t_k}^{t_{k+1}} \phi(t_{k+1} - \tau) f(\tau) d\tau
\]

(6.3)

or

\[
x_{k+1} = \phi_k x_k + q_k
\]

(6.4)
where

\[ Z_k = \int_{t_k}^{t_{k+1}} \phi(t_{k+1}-\tau)f(\tau)d\tau \]

Furthermore, from equation (3.6), \( z = H x + v \), we compute in discrete form

\[ z_k = H_k x_k + v_k \quad (6.5) \]

We will assume that all measurement errors are uncorrelated i.e., we may compute the expected value

\[ E(v_i, v_j^T) = \begin{cases} W_i & i = j \\ 0 & i \neq j \end{cases} \quad (6.6) \]

where i and j are time indices.

For the Kalman filter, we have the following definitions [4]. Let

- \( \hat{x}_k \) = best estimate of \( x_k \) based on all past measurements up to \( z_k \), also called the a posteriori estimate of \( x_k \).
- \( \hat{x}_{k}' \) = best estimate of \( x_k \) based on all past measurements up to \( z_{k-1} \), also called the a priori estimate of \( x_k \).
- \( K_k \) = weighting or gain matrix.
- \( z_k \) = measurement vector at time \( t_k \).
- \( z_k' \) = measurement vector corresponding to \( x_k' \).
\( e_k \) is a **posteriori** estimation error \((\hat{x}_k - x_k)\)

\( e_k' \) is a **priori** estimation error \((\hat{x}'_k - x_k)\)

and the error covariance matrices are

\[
C_k = E(e_k e_k^T)
\]

\[
C_k = E(e_k' e_k'^T)
\]

\[
W_k = E(v_k v_k^T)
\]

\[
G_k = E(g_k g_k^T)
\]

where \( E \) means the expected value.

The linear relation for the Kalman filter is usually given as [4]

\[
\hat{x}_k = \hat{x}'_k + K_k (z_k - \hat{z}'_k)
\] (6.7)

and we wish to choose \( K_k \) to minimize the loss function \( L \), where

\[
L = E((\hat{x}_k - x_k)^T (\hat{x}_k - x_k)) = E(e_k e_k^T) = E(T_x (e_k e_k^T)) = T_x (C_k)
\] (6.8)

where \( T_x \) is the trace of a matrix.

Because the driving functions are white, the best thing to do is to ignore its contribution to the **priori** estimate \( \hat{x}'_k \). Thus

\[
\hat{x}_k = \hat{x}'_{k-1} + \hat{x}'_k
\] (6.9)
and its measurement vector is

\[ \hat{\xi}_k = H_k \hat{x}_k \]  

(6.10)

We have defined

\[ e_k = \hat{\xi}_k - \xi_k \]  

(6.11)

and substituting equation (6.7) into (6.11), we get

\[ e_k = \hat{\xi}_k + K_k (z_k - \hat{\xi}_k) - \xi_k \]  

(6.12)

Then substituting equation (6.5) and (6.10) into equation (6.12), we compute the estimation error

\[ e_k = \hat{\xi}_k - x_k + K_k (H_k x_k + v_k - H_k \hat{\xi}_k) \]

\[ = (\hat{\xi}_k - x_k) - K_k H_k (\hat{\xi}_k - x_k) + K_k v_k \]

\[ = (U - K_k H_k) e_k + K_k v_k \]  

(6.13)

Substituting the value of \( e_k \) of equation (6.13) into (6.8), we get the loss function, i.e.,

\[
L = E \left\{ T_x \left( (U - K_k H_k) e'_k + K_k v_k \right) \left( (U - K_k H_k) e'_k + K_k v_k \right)^T \right\} \\
= E \left\{ T_x \left( (U - K_k H_k) e'_k + K_k v_k \right) (e'_k^T (U - K_k H_k)^T + v_k^T K_k^T) \right\} \\
= E \left\{ T_x \left( (U - K_k H_k) e'_k e'_k^T (U - K_k H_k)^T \right) \\
+ T_x 2 (U - K_k H_k) e'_k v_k^T K_k + T_x (K_k v_k v_k^T K_k^T) \right\} 
\]

(6.14)
Now let \( E(e^T_k v^T_k) = 0 \), i.e., \( v_k \) must be uncorrelated with \( x_k \), so from equation (6.14) we have

\[
L = T_x ((U-K_k H_k) C^*_k (U-K_k H_k)^T + K_k W_k K_k)^T \quad (6.15)
\]

To find the minimum we compute \( \frac{\partial L}{\partial K_k} = 0 \) and solve for \( K_k \) to get

\[
K_k = C^*_k H_k (H_k C^*_k H_k + W_k)^{-1} \quad (6.16)
\]

which is the optimum weighting matrix.

The inverse of \( (H_k C^*_k H_k + W_k) \) exists if \( W_k \) is positive definite and \( C^*_k \) is non-negative definite [16]. These conditions are satisfied in most applications of the Kalman filter. Furthermore, even if the matrix \( (H_k C^*_k H_k + W_k) \) is singular, the solution can be obtained by the use of the generalized inverse [16]. For the covariance matrix \( C_k \), we get

\[
C_k = E(e^T_k e^T_k) = (U-K_k H_k) C^*_k (U-K_k H_k)^T + K_k W_k K_k \quad (6.17)
\]

If we choose the optimum weighting matrix on the basis of (6.16), we get a special case for \( C_k \).

From equation (6.16), we have

\[
K_k (H_k C^*_k H_k + W_k) = C^*_k H_k
\]

\[
K_k W_k = C^*_k H_k - K_k H_k C^*_k H_k
\]

\[
K_k W_k = (U-K_k H_k) C^*_k H_k \quad (6.18)
\]
substituting equation (6.18) into (6.17)

\[ C_k = (U-k) C_k^* (U-k) H_k^T + (U-k) C_k H_k H_k^T (U-k) \]

\[ = (U-k) C_k^* ((U-k) H_k^T + H_k H_k^T) \]

(6.19)

or

\[ C_k = (U-k) C_k^* \]

(6.20)

Thus the error covariance matrix is

\[ C_k = C_k^* - K_k (H_k C_k H_k^T + W_k) K_k^T \]

(6.21)

We may also compute

\[ e_{k+1}^i = \hat{x}_{k+1}^i - x_{k+1} = \phi_k e_k - (\phi_k x_k + q_k) \]

\[ = \phi_k e_k - q_k \]

(6.22)

so the covariance matrix \( C_{k+1}^* \) is

\[ C_{k+1}^* = E(e_{k+1}^i e_{k+1}^T) = \phi_k C_k \phi_k^T + G_k \]

(6.23)

where \( G_k = E(q_k q_k^T) \)

and where we let

\[ \hat{x}_{k+1}^i = \phi_k \hat{x}_k \]

(6.24)

For the Kalman filter we can apply iteratively equations (6.16), (6.7), (6.21), (6.24) and (6.23). We define \( C_\infty \) as the equilibrium solution for the covariance equation (6.21) when \( K \to \infty \).
Duven [16] has shown that the covariance equation (6.23) has a stable solution if the random process has no random walk or no unobservable unstable mode, and that the stable equilibrium solution is the only non-negative equilibrium solution, $C_e$, if the random process is regular. Thus, any solution whose initial value is non-negative definite approaches $C_e$ as $K \to \infty$. Also, since every solution whose initial value is non-negative definite converges to the same equilibrium value, small errors in the numerical solution of the covariance equation have a relatively minor effect on the operation of the filter. A random process is regular [33] if no eigenvector of $A$ whose corresponding eigenvalue has a non-negative real part is a null vector of $H W^{-1} H$ and no eigenvector of $A$ whose corresponding eigenvalue has a non-negative real part is a null vector of $G$. Potter [33] also showed that if a system is regular, then a single, stable, non-negative definite equilibrium solution exists and any solution whose initial value is non-negative definite approaches this equilibrium solution exponentially fast.

For our analysis of variance, we will use equations (6.16), (6.21) and (6.23) iteratively, and this is shown in Figure 6.1.

We can appreciate which set of measurements, $H$, give the higher accuracy or the smaller variance after the first iteration and also which set converges to $C_e$ faster or to a
Figure 6.1. Flow chart for covariance Kalman filter

\[
K_k = \frac{C_k^*}{H_k} H_k^T \left( (H_k C_k^* H_k^T) + W_k \right)^{-1}
\]

\[
C_{k+1} = C_k^* - \frac{K_k}{H_k C_k^* H_k^T + W_k} \left( H_k C_k^* H_k^T + W_k \right)^T
\]

\[
C_{k+1}^* = \frac{\Phi_k}{C_k} \Phi_k^T + G_k
\]

No

Is \( k > 20 \)

Yes

END
given covariance threshold.

We now point out the interesting relation that exists between the estimate obtained by the least-squares method and by the Kalman filter (equation (6.16)), [42].

In the classical least-squares estimation, an approximate relation between the measurements and the state is assumed, i.e.,

$$z = Hx$$  \hspace{1cm} (6.25)

In the least-squares sense, the best estimate of $x$ or $\hat{x}$, is chosen so that

$$(z - Hx)^TD(z-Hx)$$  \hspace{1cm} (6.26)

is minimized. The matrix $D$ is an arbitrary positive definite weighting matrix. By elementary techniques, we get

$$\hat{x} = (H^TD^{-1}H)^{-1}H^TD^{-1}z$$  \hspace{1cm} (6.27)

Implicit in the estimate is the assumption that $x$ is constant, i.e., $\dot{x} = 0$. In order to compare the estimate of equation (6.27) with the estimate of equation (6.16), we let $k = 1$, and associate the measurements $z_1$ with $z$, $x_1$ with $x$ and $H_1$ with $H$. So we can apply equation (6.7), (6.10), (6.20) and (6.23) where $G_k$ is equal to zero. By equation (6.23) we have $C_1^* = C_0$.

Now we make use of the matrix inversion lemma. Let $A$
be a n x n matrix related to B, C and D by
\[ A = B + D B D^T (D B D^T + C)^{-1} D B, \]
where B and C are n x n positive definite matrices. Then \( A^{-1} \) is given by \( A^{-1} = B^{-1} + D^T C^{-1} D. \)

We can substitute equation (5.16) into (6.20) and get,

\[
C_k = C_k^* - K_k H_k C_k^* \\
= C_k^* - C_k^* H_k^T (H_k C_k^* H_k^T + W_k)^{-1} H_k C_k^*
\]

Then by the matrix inversion lemma, we have

\[
C_k^{-1} = (C_k^*)^{-1} + H_k^T W_k^{-1} H_k
\]  

(6.28)

assuming that \( C_k^* \) and \( W_k \) are positive definite. Also, from equation (6.16), we write

\[
K_k = C_k^* H_k^T (H_k C_k^* H_k^T + W_k)^{-1} \\
= C_k^* H_k^T W_k^{-1} (H_k C_k^* H_k^T + W_k)^{-1}
\]  

(6.29)

Substituting equation (6.28) into (6.29), we compute

\[
K_k = C_k \left[ U + H_k^T W_k^{-1} H_k C_k^* H_k^T W_k^{-1} \right]^{-1} \left[ H_k C_k^* H_k^T W_k^{-1} + U \right]^{-1}
\]

or

\[
K_k = C_k H_k^T W_k^{-1}
\]  

(6.30)

Now, using this alternative form for \( C_k \) and \( K_k \), we get

\[
C_1^{-1} = C_0^{-1} + H_1^T W_1^{-1} H_1
\]  

(6.31)

\[
K_1 = C_1 H_1^T W_1^{-1}
\]  

(6.32)
If no information is available on the initial state, we can assume \( C_0 = \sigma U \) and \( C_0^{-1} \) will vanish. Then equation (6.31) will become

\[
C_1^{-1} = H_1^T W_1^{-1} H_1
\]

or

\[
C_1 = (H_1^T W_1^{-1} H_1)^{-1}
\]  

(6.33)

By equation (6.7), we have

\[
\hat{x}_1 = \hat{x}_1' + K_1(z_1 - \hat{z}_1') = \hat{x}_0 + K_1(z_1 - H_1 \hat{x}_0)
\]

(6.34)

Substituting equations (6.32) and (6.33) into (6.34)

\[
\hat{x}_1 = \hat{x}_0 + (H_1^T W_1^{-1} H_1)^{-1} H_1^T W_1^{-1} (z_1 - H_1 \hat{x}_0)
\]

\[
\hat{x}_1 = (H_1^T W_1^{-1} H_1)^{-1} H_1^T W_1^{-1} z_1
\]  

(6.35)

If the weighting matrix, \( D \), is taken to be equal to the covariance of the measurement noise, \( W \), the two equations (6.27) and (6.35) are identical. When the weighting matrix \( D \) or \( W \) is set equal to a scalar matrix, in the deterministic sense, this means that each measurement has the same influence, which in the probabilistic sense means that the noise in each measurement is independent and equally distributed. So we have shown that the Kalman filter technique with certain restrictions is similar to the least-squares method. The Kalman filter technique is more general and takes into account
more information about the system and consequently their estimates will be much better. For this reason, we have chosen the Kalman filter as a method to demonstrate the quality of our observability tests.

In summary, in the derivation of the Kalman filter, it was assumed that all driving functions, \( f \), in our state variable model of the system were white noise functions, but if they were another spectral function, \( f_1 \), we can resort to a shaping filter or another artifice, i.e.,

\[
\begin{array}{ccc}
  f & \rightarrow & \text{Shaping filter} & \rightarrow & f_1 \\
  \text{White noise} & \rightarrow & \text{Desired function} \\
\end{array}
\]

Figure 6.2. Shaping filter

by which we can fit the Kalman filter format. Also, it was assumed that the measurement error was uncorrelated from step to step timewise, which might be questionable for a power system, and that the measurements and states must also be uncorrelated \( (E(e'_k v'_k^T) = 0) \).

Note that our measurements are assumed to be at discrete samples. Now, if we have a continuous measurement and want
to use all the information contained in this measurement, at
discrete time, we have to resort to prefiltering, or some
other method, of this measurement in order to update all
the information to the sample point, which is difficult to
do.

The prediction problem is that of estimating the state
of the system at some point in time ahead of the present
time. If we project our present estimate $\hat{x}_k$ through the
transition matrix $\Phi_k$, we obtain our predicted value, and the
formulas to use are (6.24) and (6.23). Note that the trans­
ition matrix $\Phi_k$ will be for the interval desired and not only
for one step as the filter problem.
VII. SAMPLE SYSTEM

The example of the system that we will use is the one used by Ward and Hale [46] and is shown in Figure 7.1.

The transmission line and transformer data is shown in Table 7.1. The voltampere base is 100 MVA.

Table 7.1. Transmission line and transformer data.

<table>
<thead>
<tr>
<th>Line P Q</th>
<th>R(%)</th>
<th>X(%)</th>
<th>KVAC</th>
<th>Tap Ratio</th>
</tr>
</thead>
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<tr>
<td>1 4</td>
<td>16.00</td>
<td>74.00</td>
<td>1406.80</td>
<td>-</td>
</tr>
<tr>
<td>1 6</td>
<td>24.60</td>
<td>103.60</td>
<td>1983.00</td>
<td>-</td>
</tr>
<tr>
<td>2 3</td>
<td>144.60</td>
<td>210.00</td>
<td>0.00</td>
<td>-</td>
</tr>
<tr>
<td>2 5</td>
<td>56.40</td>
<td>128.00</td>
<td>0.00</td>
<td>-</td>
</tr>
<tr>
<td>4 3</td>
<td>0.00</td>
<td>26.60</td>
<td>0.00</td>
<td>.909</td>
</tr>
<tr>
<td>4 6</td>
<td>19.40</td>
<td>81.40</td>
<td>1525.80</td>
<td>-</td>
</tr>
<tr>
<td>6 5</td>
<td>0.00</td>
<td>60.00</td>
<td>0.00</td>
<td>.976</td>
</tr>
</tbody>
</table>

Applying the following formulas for π equivalent of a transmission line

\[
Y_{pq0} = Y_{qp0} = \frac{KVAC}{2 \times 10^5} = \frac{BC}{2}(pu) \quad (7.1)
\]

\[
Y_{pq} = g + jb = \frac{1}{R + jX} \quad (7.2)
\]

Note that the lower case \( y \) is for actual admittance whereas upper case \( Y \) is for the \( Y \) matrix quantities.
Figure 7.1. One line diagram of the sample system
The admittance matrix is obtained as follows. The diagonal elements $Y_{pp}$ are obtained as the algebraic sum of all admittances incident to node $p$. The off-diagonal elements $Y_{pq} = Y_{qp}$ are obtained as the negative of the admittance connecting nodes $p$ and $q$. In order to include the tap effects in the admittance matrix, the off-nominal turns ratio is represented at bus $p$ for a transformer connecting nodes $p$ and $q$.

We make use of the following formulas in Stagg and El-Abiad [44] where we include the effect of the shunt capacitance of each line as the admittance $y_{pq0}$. And $\sum_{q=1}^{n} y_{pq0}$ represents the total admittance due to the shunt capacitance of all $n$ lines connected to node $p$.

Then, if node $p$ is connected to $n$ other nodes, including a transformer $pq$, with tap ratio $a$, we compute

$$Y_{pp} = y_{p1} + \cdots + \frac{y_{pq}}{a^2} + \cdots + y_{pn} + \sum_{q=1}^{n} y_{pq0} \quad (7.3)$$

$$Y_{pq} = \frac{y_{pq}}{a} \quad (7.4)$$

$$Y_{qp} = \frac{y_{qp}}{a} \quad (7.5)$$

Similarly, for node $q$ we have

$$Y_{qq} = y_{q1} + \cdots + y_{qp} + \cdots + y_{qn} + \sum_{p=1}^{n} y_{pq0} \quad (7.6)$$

Equations (7.3) through (7.6) can be used for the general
case by appropriate interpretation of subscripts. Using equations (7.1) and (7.2), we obtain Table 7.2.

Table 7.2. The \( y_{pq} \) elements.

<table>
<thead>
<tr>
<th>Line ( \text{P} )</th>
<th>( \text{Q} )</th>
<th>( y_{pq} = y_{qp} )</th>
<th>( g ) (pu)</th>
<th>( b ) (pu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 4</td>
<td></td>
<td>.0070</td>
<td>.2791</td>
<td>-1.2910</td>
</tr>
<tr>
<td>1 6</td>
<td></td>
<td>.0099</td>
<td>.2170</td>
<td>-.9137</td>
</tr>
<tr>
<td>2 3</td>
<td></td>
<td>.0000</td>
<td>.2224</td>
<td>-.3230</td>
</tr>
<tr>
<td>2 5</td>
<td></td>
<td>.0000</td>
<td>.2883</td>
<td>-.6542</td>
</tr>
<tr>
<td>4 3</td>
<td></td>
<td>.0000</td>
<td>.0000</td>
<td>-3.7594</td>
</tr>
<tr>
<td>4 6</td>
<td></td>
<td>.0076</td>
<td>.2771</td>
<td>-1.1625</td>
</tr>
<tr>
<td>6 5</td>
<td></td>
<td>.0000</td>
<td>.0000</td>
<td>-1.6667</td>
</tr>
</tbody>
</table>

The resulting admittance matrix, \( Y_{pq} \), is symmetric and the results of the upper nonzero diagonal elements are given in Table 7.3 using equations (7.3) through (7.6) where \( a \) will be unity if there is no tap effect.

In order to observe different operating points of the sample system, we have varied the generation of real power at bus 2, and have also varied the reactive load in buses 3 through 6. All these different conditions are shown in Cases A through F in Table 7.4. From these different conditions, we have used a load flow program to obtain the operating point of the system, given in Table 7.5, where the
operating points (states) are given as the voltage magnitudes and voltage phase angles at each of the buses.

We will use these operating points for our observability test and we linearize the measurement matrix, $H$, around these operating points.
Table 7.3. The nonzero upper diagonal elements of the admittance matrix $Y_{pq}$.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>Rectangular form</th>
<th>Polar form $Y_{pq}$</th>
<th>Polar form $\theta_{pq}$</th>
<th>BC/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.4961 - j2.1878</td>
<td>2.2433</td>
<td>-77.2238</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>-0.2791 + j1.2910</td>
<td>1.3208</td>
<td>102.1920</td>
<td>0.0070</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>-0.2170 + j0.9137</td>
<td>0.9385</td>
<td>103.3685</td>
<td>0.0099</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.5117 - j0.9772</td>
<td>1.1031</td>
<td>-27.6383</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>-0.2224 + j0.3230</td>
<td>0.3922</td>
<td>124.5492</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>-0.2883 + j0.6542</td>
<td>0.7149</td>
<td>113.7827</td>
<td>0.0000</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.2224 - j4.0824</td>
<td>4.0885</td>
<td>-3.1183</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>+ j4.1357</td>
<td>4.1357</td>
<td>90.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.5562 - j6.9891</td>
<td>7.0112</td>
<td>-85.4500</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>-0.2771 + j1.1625</td>
<td>1.1951</td>
<td>103.4072</td>
<td>0.0076</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0.2883 - j2.3209</td>
<td>2.3387</td>
<td>-7.0810</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>+ j1.7077</td>
<td>1.7077</td>
<td>90.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>0.4941 - j3.8088</td>
<td>3.84071</td>
<td>-82.6085</td>
<td></td>
</tr>
</tbody>
</table>
Table 7.4. Bus data for cases A through F.

| Case | Swing bus $|E|$ | MW | Generation 1 | Generation 2 | Load 3 | Load 4 | Load 5 | Load 6 |
|------|-----------|-----|----------|-------------|--------|--------|--------|--------|--------|
| A    | 1.05      | 25.00 | 27.5 | 6.5 | - | - | 15.0 | 9.0 | 25.0 | 2.5 |
| B    | 1.05      | 12.5 | 27.5 | 6.5 | - | - | 15.0 | 9.0 | 25.0 | 2.5 |
| C    | 1.05      | 50.0 | 27.5 | 6.5 | - | - | 15.0 | 9.0 | 25.0 | 2.5 |
| D    | 1.05      | 25.0 | 27.5 | 6.5 | - | - | 15.0 | 9.0 | 25.0 | 25.0 |
| E    | 1.05      | 25.0 | 27.5 | 9.2 | - | - | 15.0 | 5.0 | 25.0 | 8.1 |
| F    | 1.05      | 25.0 | 27.5 | 13.7 | - | - | 15.0 | 7.5 | 25.0 | 12.5 |
Table 7.5. Solution of load flow program for cases A through F.

<p>| Case | (|E|) | (\delta) | MW  | MVAR | (|E|) | (\delta) | MVAR |
|------|-------|-----|-----|------|-------|-----|------|
| A    | 1.05  | 0   | 47.6| 21.8 | 1.1   | -3.4| 9.3  |
| B    | 1.05  | 0   | 60.1| 30.5 | 0.998 | -12.9| 6.0  |
| C    | 1.05  | 0   | 28.9| 30.4 | 1.1   | 17.6| 11.0 |
| D    | 1.05  | 0   | 56.1| 75.8 | 0.881 | -1.4| 12.5 |
| E    | 1.05  | 0   | 47.9| 26.4 | 1.1   | -3.7| 10.1 |
| F    | 1.05  | 0   | 49.6| 43.1 | 1.037 | -3.6| 12.5 |</p>
<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th></th>
<th>4</th>
<th></th>
<th>5</th>
<th></th>
<th>6</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>δ</td>
<td>E</td>
<td>δ</td>
<td>E</td>
<td>δ</td>
<td>E</td>
<td>δ</td>
<td></td>
</tr>
<tr>
<td>1.001</td>
<td>-12.8</td>
<td>.93</td>
<td>-9.8</td>
<td>.919</td>
<td>-12.3</td>
<td>.919</td>
<td>-12.2</td>
<td></td>
</tr>
<tr>
<td>.965</td>
<td>-12.6</td>
<td>.897</td>
<td>-12.5</td>
<td>.867</td>
<td>-18.7</td>
<td>.876</td>
<td>-16.0</td>
<td></td>
</tr>
<tr>
<td>.946</td>
<td>-7.2</td>
<td>.898</td>
<td>-5.1</td>
<td>.854</td>
<td>-2.0</td>
<td>.891</td>
<td>-6.5</td>
<td></td>
</tr>
<tr>
<td>.797</td>
<td>-16.6</td>
<td>.755</td>
<td>-11.9</td>
<td>.595</td>
<td>-16.7</td>
<td>.616</td>
<td>-15.3</td>
<td></td>
</tr>
<tr>
<td>.975</td>
<td>-12.9</td>
<td>.911</td>
<td>-9.8</td>
<td>.923</td>
<td>-12.5</td>
<td>.898</td>
<td>-12.4</td>
<td></td>
</tr>
<tr>
<td>.889</td>
<td>-13.7</td>
<td>.847</td>
<td>-10.0</td>
<td>.830</td>
<td>-13.4</td>
<td>.819</td>
<td>-12.9</td>
<td></td>
</tr>
</tbody>
</table>
VIII. DISCUSSION AND RESULTS

A. Discussion

The state estimation of a power system using the Kalman filtering technique is based on the mathematical model of a power system (3.4) and the measurement model (3.6). The least-squares technique is based only on the measurement model.

In the measurement model, we can select our measurement matrix $H$ and the error or uncertainty, $v$, of the measurement, $z$. Thus, the estimate (6.7) is a function of the measurement matrix, $H$, and error $v$. The effect of the error $v$ on the estimate can be studied by the Kalman filter. It has been assumed that all measurements have the same 3% error. The effect of a specific measurement error on the estimate can be studied through Kalman filter techniques, but has not been pursued by this research. We have concentrated on the effect of choosing the measurements by which the matrix $H$ will vary. Each measurement contributes in one row of $H$ and, by choosing the $m$ measurements, we are choosing the $m$ rows of $H$. By choice of these rows, we change the accuracy of the estimate in the first iteration and its rate of convergence. In Chapter IV, we developed the observability criteria based on the measurement matrix $H$ and assumed no noise, i.e., $v = 0$, (4.3). By Theorem 4.3, we found that the best case of observability is when all the rows of $H$ are
orthogonal. Since the system is nth order, this means that \( H \) must be \( n \times n \) \((n = 2N-1)\). Suppose that we have \(2N-1\) measurements for our \(2N-1\) states. The best case of observability possible is when \( H = U \), i.e., where we measure directly all phase angles, \( \delta \), and voltage magnitudes \(|E|\). But if the one measurement fails, we will have an undetermined system of equations. Thus, we want to introduce redundant equations to ensure a solution, i.e., to improve reliability and also to improve our estimate and force a smaller variance of the estimate.

Now, if we want to improve our estimate by adding one more measurement, which measurement will it be? According to Theorem 4.2, we should add that measurement in the direction of poorest observability which is the direction given by the most orthogonal eigenvector.

In a power system, the measurement matrix \( H \) is nonlinear and the theory developed corresponds to a linear system. Thus, we have to linearize the matrix \( H \) around an operating point. The operating point in a power system varies with the load conditions. In order to observe the effect of the operating point on the most orthogonal vector for the system for a given set of measurements, we have used different loading conditions (Table 7.4, cases A through F). Then, using a load flow program, the corresponding operating points shown in Table 7.5, cases A through F are determined. In order to
observe the effect of the operating point on the most orthogonal vector, the set of measurements will be taken as the set of all the power injection measurements. The most orthogonal vector corresponds to the eigenvector associated to the smallest eigenvalue of $Q_nQ_n^T$ where $Q = H^T$.

The linearization of $H$ and the matrix multiplication $Q_nQ_n^T$ were made by computer (Appendix A) and the eigenvalues and eigenvectors were also found numerically (Appendix B). Therefore, we obtain the computer outputs labelled 1 through 6 corresponding to cases A through F in Table 7.5, which are tabulated in part B of this chapter as Tables 8.2 to 8.7.

It should be observed that in computer outputs 1 through 6, at least for this example, changing the operating point does not greatly change the major components of the most orthogonal vector and the largest component is always $\delta_5$, followed by $\delta_2$, in all cases A through F.

Observe from Tables 8.2 to 8.7, that the largest components of the most orthogonal eigenvector are always associated with the phase angles. The ranking in size of these components are summarized in Table 8.1 where it is clear that, in all cases except B and D, all five angle components are larger than the voltage components. Furthermore, if any voltage measurement is to be added, it should definitely be $E_5$ as this is always ranked highest among the E's. Table 8.1 also shows that $\delta_3$ and $\delta_6$ are always ranked
Table 8.1. Summary of eigenvector component rankings.

<table>
<thead>
<tr>
<th>Ranking R</th>
<th>Component with ranking R Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>1</td>
<td>δ₅</td>
</tr>
<tr>
<td>2</td>
<td>δ₂</td>
</tr>
<tr>
<td>3</td>
<td>δ₃</td>
</tr>
<tr>
<td>4</td>
<td>δ₆</td>
</tr>
<tr>
<td>5</td>
<td>δ₄</td>
</tr>
<tr>
<td>6</td>
<td>E₅</td>
</tr>
<tr>
<td>7</td>
<td>E₃</td>
</tr>
<tr>
<td>8</td>
<td>E₆</td>
</tr>
<tr>
<td>9</td>
<td>E₁</td>
</tr>
<tr>
<td>10</td>
<td>E₂</td>
</tr>
<tr>
<td>11</td>
<td>E₄</td>
</tr>
</tbody>
</table>

3 and 4, behind δ₅ and δ₂, and they have magnitudes of about 0.3 which is about double the magnitude of E₅. This tells us something significant about the most desirable measurements, viz, that angle measurements on almost any node are probably the best measurements to add. One way to obtain angle-related information is to add measurements of the real power flow on the branches. But, if an economical method of measuring angle differences directly becomes available, such a method should certainly be considered.
We know that the best way to change the magnitude of the transmitted power between two nodes is to change the phase angle $\delta_{ij}$, where $\delta_{ij}$ represents the angle between $E_i$ and $E_j$. Conversely, if we measure real power flow in the line, we can get information related to this angle. This should be valuable in our case since we want to improve the estimation of these angles. Therefore, we should add more measurements pertaining to real power flow.

If we want to improve the estimation on the voltages, we can measure voltages or we could measure the reactive power, since the reactive power flow is strongly influenced by the voltages of the system. However, in the system under study, additional voltage measurements would not appear to be necessary.

Now we will use only one operating point for the forthcoming discussion, which will be case A. Table 7.5. The accuracy of all the measurements is assumed to be 3% and the a priori covariance of each estimate (the diagonal terms of $C_0^*$) are assumed to be 10%. These data are required in the Kalman covariance analysis.

If we use the set of power injection measurements (which correspond to the eigenvalues and eigenvectors in Table 8.2), the measurement matrix $H$ is given in Table 8.8, and Table 8.9 is the output of the covariance of the Kalman filter (the diagonal terms of $C_k$) for 20 iterations. Observe that in
Table 8.9, the highest variance is in \( \delta_5 \) state, followed by the variance of \( \delta_2 \) state which agrees with our observability criteria. Also note that the covariances of each state are converging to a steady solution, \( C_e \). Now, if we want to improve our estimate at the first iteration, (smaller variance), or to reduce the number of iterations to achieve a given threshold, we should add more measurements. Which one?

We have the answer in the most orthogonal eigenvector, and should add a measurement where all the entries corresponding to this measurement in the matrix \( H \) match the components of the most orthogonal vector or at least affect the largest components. We observe in Table 8.2, that the largest component is in \( \delta_5 \). Thus, the best single measurement to add is the one whose components are proportional to this most orthogonal vector. This is not easy to accomplish with a single measurement and would compensate only for the highest component. The best single measurement to add for this case will be the phase angle \( \delta_5 \) which is measured with respect to the phase angle at bus one. In general, phase angle measurements are not available, although some meters for doing this are beginning to appear. By examining the circuit diagram or admittance matrix, we observe that the measurements that can be added to the power injection measurements to improve our observability are the measurements that include the phase angle \( \delta_5 \), which are the real and reactive power flow between
bus 5 and bus 2, and between bus 5 and bus 6. Note that we can add more than one measurement of any kind, but we will try to introduce a new measurement, i.e., one not used previously. Adding these four power measurements, and the voltage at bus 5, we get the matrix $H$ of these measurements in Table 8.10. Now observe that the component of greatest magnitude is $\delta_5$ in the row corresponding to $P_5$. Note, however, that the $\delta_5$ in the row corresponding to $P_{5-6}$ is also large. This result agrees with the preceding discussion by which we know that we should make a real power flow measurement. Therefore, the best measurement to add to our injection power measurements, is $P_{5-6}$.

We find the most orthogonal vector for the power injection plus real power flow $P_{5-6}$ in Table 8.11, and observe that the highest component now is in $\delta_2$ direction. As a comparison, Table 8.12 gives the most orthogonal vectors for the power injection plus real power flow $P_{5-2}$ where we observe that $\delta_5$ is again the highest component. If we compare the smallest eigenvalue of these two tables, we observe that the one corresponding to $P_{5-6}$ (0.053) is bigger than that for adding $P_{5-2}$ (0.017), which tells us that the observability is better for $P_{5-6}$. Table 8.13 describes the covariance analysis, $C_k$, considering all power injections plus the real power flow between buses 5 and 6, $P_{5-6}$. Table 8.14 describes the covariance analysis, $C_k$, considering all power injections
plus real power flow between buses 6 and 5, \( P_{6-5} \). As we know, the flow can be measured at any end of the line; consequently, we can measure the real power flow, \( P_{5-6} \), or \( P_{6-5} \). But if we choose \( P_{5-6} \), a small improvement is observed comparing Tables 8.13 and 8.14 at the first iteration, i.e., the variance at \( \delta_5 \) is smaller in Table 8.13 for \( P_{5-6} \) added. The row in matrix \( H \) corresponding to the measurement \( P_{6-5} \) is 
\[
[0.0, 0.0, 0.0, -1.44225, 1.44225, 0.0, 0.0, 0.0, 0.0, -0.00274, 0.00274]
\]
and comparing the row of matrix \( H \) in Table 8.10 corresponding to measurement \( P_{5-6} \) we observe that the sign in the \( \delta \) directions are interchanged, and \( E_6 \) has changed signs.

Table 8.15 describes the covariance analysis \( C_k \), considering all power injections plus real power flow between buses 5 and 2, \( P_{5-2} \). Table 8.16 describes the covariance analysis, \( C_k \), considering all power injections plus voltage at \( E_4 \).

In order to appreciate the improvement in the estimate from 12 measurements to 13 measurements, we can compare Tables 8.9, 8.13, 8.15 and 8.16 and observe the number of iterations that we have to make in order to achieve the same thresholds of covariance for each state. We could have used the same threshold for all the states, but in order to make it more noticeable, we have chosen thresholds corresponding to each of the states. With these computer outputs, we can prepare Table 8.17 where the number of iterations to achieve
a given threshold of variance is given for all the states. Observe that the addition of real power flow between buses 5 and 6 measured at 5 (Table 8.13) to the power injection measurements (Table 8.9) is the one that achieves the given threshold in the least number of iterations. Adding a measurement improves our estimate and the number of iterations to be made, but this improvement depends on how close we are to the steady solution $C_e$.

We can repeat the above procedure to find the most orthogonal eigenvector of this new set of measurements, (all power injections plus real power flow between buses 5 and 6) and we obtain Table 8.11. Observe that the highest component of the most orthogonal vector is now $\delta_2$. Suppose we add another measurement to this set; the new measurement could be real power flow between buses 2 and 5 measured at 2, and with this new measurement, we obtain Table 8.18 and observe a great improvement in $\delta_2$ direction compared to that of Table 8.13.

This technique of adding a power flow on a line connected to the node of the highest component of the eigenvector corresponding to the smallest eigenvalue, can be repeated again and again until the desired improvement is obtained or until all possible measurements have been added.
B. Results

The results computed for the 6 bus examples are listed in this section as Tables 8.2 through 8.18. These results are described in the narrative under part A of this chapter.
Table 8.2. Computer output no. 1. Case A.

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th>EIGENVECTORS</th>
<th>EIGENVALUES</th>
<th>EIGENVECTORS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00620547</td>
<td>0.55088907</td>
<td>1.32920207</td>
<td>1.54430940</td>
</tr>
<tr>
<td>0.30595652</td>
<td>0.40791952</td>
<td>1.82194260</td>
<td>2.11331670</td>
</tr>
<tr>
<td>0.35496546</td>
<td>0.73970507</td>
<td>2.34864235</td>
<td>2.34864235</td>
</tr>
<tr>
<td>1.00871152</td>
<td>0.10780572</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| δ₂ | 0.42645753 | 0.55088907 | -0.32948806 | 0.46038192 |
| δ₃ | -0.32948806 | 0.64212223 | 0.46038192 | -0.12291796 |
| δ₄ | -0.17155357 | 0.32076570 | 0.19112790 | 0.18061928 |
| δ₅ | 0.73970507  | 0.30651214 | 0.41109192 | 0.00175649 |
| δ₆ | 0.30485566  | 0.03911910 | 0.12995227 | 0.32560326 |
| E₁ | 0.04968039  | 0.25612824 | -0.1788766 | 0.67165719 |
| E₂ | -0.03783890 | -0.20766973 | 0.63913017 | 0.11556079 |
| E₃ | 0.08236781  | -0.15650027 | 0.02133988 | 0.05141680 |
| E₄ | 0.02673004  | 0.12632165 | -0.12414237 | 0.30279817 |
| E₅ | -0.16014257 | 0.13231593 | 0.03091475 | -0.03940202 |
| E₆ | 0.05332675  | 0.17151903 | -0.24601064 | 0.34050032 |

| δ₂ | -0.17155357 | -0.32648086 | 0.01211782 | 0.08230781 |
| δ₃ | 0.32076570  | 0.8255827   | 0.0124606 | -0.15650027 |
| δ₄ | -0.17535239 | 0.21650785  | -0.32056909 | 0.02133988 |
| δ₅ | -0.53888613 | 0.03545727  | 0.08182730 | 0.05141680 |
| δ₆ | 0.00283531  | -0.19080290 | 0.40791952 | 0.23025958 |
| E₁ | 0.55186936  | -0.29775388 | -0.1555984 | -0.08183143 |
| E₂ | 0.35665921  | -0.18147276 | 0.0503709 | 0.18061928 |
| E₃ | 0.18061928  | -0.12251796 | -0.41224533 | 0.35118193 |
| E₄ | -0.17535239 | 0.46038192  | -0.25672147 | -0.25231685 |
| E₅ | 0.21450785  | 0.08255827  | 0.55088907 | -0.35118193 |
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Table 8.8. Computer output no. 7. H matrix for all power injection. Case A.

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|   | Δ|E₁  | Δ|E₂  | Δ|E₃  | Δ|E₄  | Δ|E₅  | Δ|E₆  |
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| P₁ | 0.97294 | 0.00000 | 0.00000 | -0.5788 | 0.00000 | -0.2010 |
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| P₃ | 0.00000 | -0.27247 | 7.67235 | -0.21666 | 0.00000 | 0.00000 |
| P₄ | -0.45999 | 0.00000 | 0.20130 | 0.52215 | 0.00000 | -0.21221 |
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| Q₁ | 2.50472 | 0.00000 | 0.00000 | -1.38563 | 0.00000 | -0.98522 |
| Q₂ | 0.00000 | 0.13544 | -0.39052 | 0.00000 | -0.76001 | 0.00000 |
| Q₃ | 0.00000 | -0.28265 | -3.70628 | -4.13416 | 0.00000 | 0.00000 |
| Q₄ | -1.13896 | 0.00000 | -3.84093 | 6.50155 | 0.00000 | -1.90299 |
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| Q₆ | -0.77802 | 0.00000 | 0.00000 | -1.05676 | -1.56937 | 3.47284 |
Table 8.9. Computer output no. 8. Covariance analysis, $C$, considering all power injections. Case A.

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Table 8.10. Computer output no. 9. H matrix for all power injection plus real and reactive flows and voltage at bus 5. Case A.

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| \(\Delta|E_1|\) | \(\Delta|E_2|\) | \(\Delta|E_3|\) | \(\Delta|E_4|\) | \(\Delta|E_5|\) | \(\Delta|E_6|\) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| P1 | 0.972941 | 0.000000 | 0.000000 | -0.057882 | 0.000000 | -0.02010 |
| P2 | 0.000000 | 1.81432 | -0.18334 | 0.000000 | -0.20198 | 0.000000 |
| P3 | 0.000000 | -0.27247 | 7.67235 | -0.21666 | 0.000000 | 0.000000 |
| P4 | -0.45999 | 0.000000 | 0.20130 | 0.52215 | 0.000000 | 3.83836 |
| P5 | 0.000000 | -0.35477 | 0.000000 | 0.000000 | -0.29918 | 0.00274 |
| P6 | -0.37224 | 0.000000 | 0.000000 | -1.38563 | 0.000000 | -0.98522 |
| Q1 | 2.50472 | 0.000000 | 0.13544 | -0.39052 | 0.000000 | 0.76001 |
| Q2 | 0.000000 | -0.28265 | -3.70628 | -4.13416 | 0.000000 | 0.000000 |
| Q3 | -1.13896 | 0.000000 | -3.84093 | 6.50155 | 0.000000 | -1.09899 |
| Q4 | 0.000000 | -0.55297 | 0.000000 | 0.000000 | -1.70137 | -1.56937 |
| Q5 | -0.77802 | 0.000000 | 0.000000 | -1.05676 | -1.56937 | 3.47284 |
| Q6 | 0.000000 | -0.35477 | 0.000000 | 0.000000 | 0.10525 | 0.000000 |
| P5-2 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | -0.00274 | 0.000000 |
| P5-6 | 0.000000 | -0.55297 | 0.000000 | 0.000000 | -1.86429 | 0.000000 |
| Q5-2 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | -4.70812 | -1.56937 |
| Q5-6 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 1.00000 | 0.000000 |
Adding $P_{5-6}$ measurement.
Table 8.12. Computer output no. 11. Case A. Adding $P_{5-2}$ measurement.

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Table 8.13. Computer output no. 12. Covariance analysis, $C_{k'}$, considering all power injections plus $P_{5-6}$. Case A.

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| 2      | 0.00012 | 0.00038 | 0.00002 | 0.00002 | 0.00007 | 0.00007 |
| 3      | 0.00008 | 0.00026 | 0.00002 | 0.00001 | 0.00005 | 0.00005 |
| 4      | 0.00006 | 0.00020 | 0.00001 | 0.00001 | 0.00004 | 0.00004 |
| 5      | 0.00005 | 0.00016 | 0.00001 | 0.00001 | 0.00003 | 0.00003 |
| 6      | 0.00004 | 0.00014 | 0.00001 | 0.00002 | 0.00003 | 0.00003 |
| 7      | 0.00004 | 0.00012 | 0.00001 | 0.00002 | 0.00002 | 0.00002 |
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Table 8.14. Computer output no. 13. Covariance analysis, $C_{s}$, considering all power injections plus $P_{6-5}$. Case A.

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Table 8.15. Computer output no. 14. Covariance analysis, C, considering all power injections plus P_{5-2}. Case A.

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Table 8.16. Computer output no. 15. Covariance analysis, \( C_k \), considering all power injections plus \( E_4 \). Case A.

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| \( \Delta | E_1 | \) | \( \Delta | E_2 | \) | \( \Delta | E_3 | \) | \( \Delta | E_4 | \) | \( \Delta | E_5 | \) | \( \Delta | E_6 | \) |
|-----------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1         | 0.00022        | 0.00069        | 0.00007        | 0.00004        | 0.00029        | 0.00013        |
| 2         | 0.00012        | 0.00038        | 0.00004        | 0.00002        | 0.00018        | 0.00007        |
| 3         | 0.00008        | 0.00026        | 0.00003        | 0.00001        | 0.00013        | 0.00005        |
| 4         | 0.00006        | 0.00020        | 0.00002        | 0.00001        | 0.00010        | 0.00004        |
| 5         | 0.00005        | 0.00016        | 0.00002        | 0.00001        | 0.00008        | 0.00003        |
| 6         | 0.00004        | 0.00014        | 0.00002        | 0.00001        | 0.00007        | 0.00003        |
| 7         | 0.00004        | 0.00012        | 0.00001        | 0.00001        | 0.00006        | 0.00002        |
| 8         | 0.00003        | 0.00010        | 0.00001        | 0.00001        | 0.00005        | 0.00002        |
| 9         | 0.00003        | 0.00009        | 0.00001        | 0.00001        | 0.00005        | 0.00002        |
| 10        | 0.00002        | 0.00008        | 0.00001        | 0.00001        | 0.00004        | 0.00002        |
| 11        | 0.00002        | 0.00008        | 0.00001        | 0.00001        | 0.00004        | 0.00002        |
| 12        | 0.00002        | 0.00007        | 0.00001        | 0.00001        | 0.00004        | 0.00002        |
| 13        | 0.00002        | 0.00007        | 0.00001        | 0.00001        | 0.00004        | 0.00002        |
| 14        | 0.00002        | 0.00006        | 0.00001        | 0.00001        | 0.00003        | 0.00001        |
| 15        | 0.00002        | 0.00006        | 0.00001        | 0.00001        | 0.00003        | 0.00001        |
| 16        | 0.00002        | 0.00005        | 0.00001        | 0.00001        | 0.00003        | 0.00001        |
| 17        | 0.00001        | 0.00005        | 0.00001        | 0.00001        | 0.00003        | 0.00001        |
| 18        | 0.00001        | 0.00005        | 0.00001        | 0.00001        | 0.00003        | 0.00001        |
| 19        | 0.00001        | 0.00004        | 0.00001        | 0.00001        | 0.00002        | 0.00001        |
| 20        | 0.00001        | 0.00004        | 0.00001        | 0.00001        | 0.00002        | 0.00001        |
Table 8.17. Number of iterations to reach a given threshold.

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<td>14</td>
<td>14</td>
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Table 8.18. Computer output no. 17. Covariance analysis, $C_{k'}$, considering all power injections plus $P_{5-6}$ & $P_{2-5}$.

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IX. CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

A. Conclusions

The results of the study described in this thesis show the importance of the measurement location and type on the power system state estimator. We have shown that the addition of one particular measurement may improve the accuracy of the results more than any other chosen measurements and have provided a method to find this best measurement.

By adding the best possible measurement, we should be able to obtain the maximum improvement in the information content of a given number of measurements. The method developed does not make use of the different measurement accuracies. The measurement accuracy can also be studied by Kalman filter theory. We assumed a 3% accuracy for all our measurements.

All the necessary information to find better measurements is contained in the distribution of the eigenvalues of the $\mathbf{Q}_n\mathbf{Q}_n^T$ matrix and the components of the eigenvector associated with the smallest eigenvalue or most orthogonal vector.

Thus we must find the measurement matrix $\mathbf{H}$ for the network and meter placement desired, from which we find the eigenvalues and eigenvectors of $\mathbf{Q}_n\mathbf{Q}_n^T$, where $\mathbf{Q} = \mathbf{H}^T$. The computer program shown in Appendix B to find the eigenvalues and eigenvectors, can handle larger systems. Therefore, it is feasible to use the eigenvalue method up to about a 50-bus
system, which would correspond to at least a 99th order measurement matrix.

B. Suggestions for Future Work

The method proposed in this study was applied to a 6-bus system. It would be of obvious interest to apply it to a real and much larger power system. Hopefully, for a larger system, the difference in magnitude of the components of the most orthogonal vector would be greater. Consequently, the addition of a measurement would be even more effective.

The operating point in a power system varies due to changing loads. It would be possible to change the set of measurements along with the loads, therefore developing an adaptive selector of measurements.

The effect of the different measurement accuracies in this research was not considered. These effects can be studied through Kalman filter techniques, but a simple method applied to a power system would be of interest.

We found that the best case of observability was when all the eigenvalues of $Q_nQ_n^T$ were equal. On the other hand, when they are not equal, we observe the distribution of these eigenvalues. By this observation on different systems, we can say which system will be more observable. Consequently, it would be of interest to find a representative figure of
merit for each system in order to compare it automatically. This figure of merit could perhaps be weighted variance.

The method developed in this study was applied to obtain one new measurement at a time, so we could start with a small set of measurements and find one best measurement to add each time, and then repeat the method until satisfied with the result. This constitutes a measurement optimization technique. We could also add to this optimization, a cost function $K$ for each of the measurements, i.e.,

$$C = K_1 M_1 + K_2 M_2 + \ldots + K_M M_m$$

where $K_i$ is the cost of each measurement, and $M_i$ is the number of the measurement. The best economic solution would then be the minimization of the total cost $C$.

The effect of losing one measurement on the estimator will degenerate our estimate. How much? This suggests that the most vulnerable measurement may be valuable to know. If this measurement is unreliable, this may suggest steps of obvious improvement.

We are limited in the size matrix for which eigenvalues may be found. Systems of a single company can easily run to several hundred nodes. Even if equivalent circuits are used to reduce to just generation plus load buses, we have the problem that resulting admittances (of the equivalent) are not physical lines. Thus, we need either a method to estimate
only the smallest eigenvalue and its eigenvector of a large system or need to prove that the addition of line flows will always be the answer (but which line flows?).
X. BIBLIOGRAPHY


XI. ACKNOWLEDGMENTS

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The author also expresses his gratitude to all the institutions that contributed in making this dissertation possible.
XII. APPENDIX A - FORTRAN PROGRAM FOR CALCULATING THE MEASUREMENT MATRIX

A. Computer Program
DIMENSION F(6),C(6),RES(20,12),QQT(12,12)
DIMENSION IP(20),IQ(20),IV(20),IA(20)
COMMON PQY(13,5)
INTEGER YP
READ(5,3)NN,NL
3 FORMAT(215)
NT=NN+NL
NND=2*NN
READ(5,4)((PQY(I,J),J=1,5),I=1,NT)
4 FORMAT(F4.1,8X,F4.1,8X,F10.4,10X,F10.4,10X,F7.4)
WRITE(6,5)
5 FCMAT(' NODES P-Q   Y  Y-ANG')
   (PQY(I,J),J=1,5),I=1,NT)
DO 13 J=1,NT
13 PQY(J,4)=PQY(J,4)*0.174533
1 READ(5,6)(END=99)(C(J),J=1,NN)
READ(5,6)(D(J),J=1,NN)
6 FCMAT(0F8.3)
WRITE(6,7)
7 FCMAT('V  AT NODES 1.....')
WRITE(6,8)(F(J),J=1,NN)
WRITE(6,10)
8 FCMAT(' I  AT NODES 1.....')
WRITE(6,9)(O(J),J=1,NN)
WRITE(6,12)
9 CONTINUE
READ(5,15)(NPQ,NMP,NMQ,NV)
10 FCMAT(4I5)
READ(5,16)(IA(I),I=1,NPQ)
11 FCMAT(2I3)
WRITE(6,12)
12 FCMAT(' POWER INJECTION AT NODES')
1 FOR REAL 1...N, FOR reactive N...2N)
WRITE(6,21)(IA(I),I=1,NPQ)
21 FCMAT(0I4)
NS=NPQ+NMP+NMQ+NV
DO 17 I=1,NS
17 RES(I,J)=0.0
DO 75 JX=1,NPQ
I=IA(JX)
IF(I.GT.NN) GO TO 24
DO 18 J=1,NN
18
IF(IE.EQ.0) GO TO 40
PP=-PQY(IE,4)+D(I)-D(J)
RES(JX,J+NN)=E(I)*PQY(IE,3)*SIN(PP)
40 CONTINUE
EF=0
DO 43 IX=1,NN
IF(I.EQ.IX) GO TO 43
IE=YP(I,IX)
IF(IE.EQ.0) GO TO 43
PP=-PQY(IE,4)+D(I)-D(IX)
EF =EF+E(IX)*PQY(IE,3)*SIN(PP)
43 CONTINUE
IE=YP(I,1)
RES(JX,I+NN)=-2*E(I)*PQY(IE,3)*SIN(PQY(IF,4))+EF
75 CONTINUE
NX=2*NMP
IF(NMP.EQ.0) GC TO 57
WRITE(6,44) I,J
44 FORMAT(' REAL POWER FLOW IN LINE P-Q')
READ(5,55)(IP(I),I=1,NX)
55 FORMAT(20I3)
II=1
JJ=2
DO 76 IM=1,NMP
I=IP(II)
J=IP(JJ)
WRITE(6,45) I,J
75 CONTINUE
WRITE(6,44) I,J
45 FORMAT(5X,216)
IE=YP(I,J)
IF(IE.GT.0) GO TO 56
IE=YP(J,I)
56 A=E(I)*E(J)*PQY(IE,3)
IZ=NPQ+IM
RES(IZ,I)=-A*SIN(-PQY(IE,4)-D(J)+D(I))
RES(IZ,J)=-RES(IZ,I)
A=PQY(IE,3)*E(J)*COS(-PQY(IE,4)+D(I)-D(J))
RES(IZ,I+NN)=A-2.0*E(I)*PQY(IE,3)*COS(PQY(IF,4))
RES(IZ,J+NN)=E(I)*PQY(IE,3)*COS(-PQY(IF,4)+D(I)-D(J))
II=II+2
JJ=JJ+2
76 CONTINUE
57 NX=2*NMP
IF(NMP.EQ.0) GC TO 58
WRITE(6,91)
81 FORMAT(' REACTIVE POWER FLOW IN LINE P-Q')
REAC(5,55)(IQ(I),I=1,NX)
II=1
JJ=2
CC 77 I=1,NMQ
I=IQ(I)J
J=IQ(J)
WRITE(6,45) I,J
IE=YP(I,J)
IF(IE.GT.0) GO TO 59
IE=YP(J,I)
59 A=E(I)*E(J)*PQY(IE,3)
IZ=NPG+NP+IM
RES(IZ,I)=A*COS(-PQY(IE,4)-D(J)+D(I))
RES(IZ,J)=RES(IZ,I)
A=SIN(-PQY(IE,4)+D(I)-D(J))
B=2.0*E(I)*PQY(IE,3)*SIN(PQY(IE,4))
RES(IZ,I+NN)=PQY(IE,3)*E(J)*A-B+E(I)*PQY(IE,5)*2.
RES(IZ,J+NN)=E(I)*PQY(IE,3)*A
II=II+2
JJ=JJ+2
77 CONTINUE
58 IF(NV.EQ.0) GO TO 80
WRITE(6,82)
82 FORMAT(' VOLTAGES AT NODES ') Read(5,55)(IV(I),I=1,NV)
WRITE(6,55)(IV(I),I=1,NV)
IZ=NPG+NP+NMQ
CC 78 I=1,NV
IZ=IZ+1
I=IV(I)
RES(IZ,I+NN)=1.0
78 CONTINUE
80 CONTINUE
WRITE(6,53)
53 FORMAT(' PARTIAL MATRIX H AT V AND ANG ') WRITE(6,54)((RES(I,J),J=2,NN0),I=1,NS)
54 FORMAT(11F10.5)
DO 62 I=1,NS
XNCR=0.
CC 60 J=2,NN0
60 XNOR = XNCR*RES(I,J)**2
   XNCR = SCRT(XNOR)
   IF(XNOR.EQ.0) GO TO 62
   CC 61 J=2,NND
61 RES(I,J) = RES(I,J)/XNCR
62 CCNTINUE
   CC 65 I=2,NND
   CC 66 J=2,NND
   B=0
   DO 64 II=1,NNS
64 B = B + RES(II,I)*RES(II,J)
65 CCNTINUE
   CC 65 I=2,NND
   CC 66 J=2,NND
   CCNTINUE
95 CCNTINUE
96 FCRMAT(11IF10.7)
   GO TO 1
99 STOP
END

FLNC1ICA YP(IIX,IYY)
COMMON PCY(13,5)
INTEGER YP
   DC 100 II=1,13
   IP1=PCY(II,1)
   IP2=PCY(II,2)
   IF((IP1.EQ.IIX).AND.(IP2.EQ.IYY)) GO TO 101
   IF((IP1.EQ.IYY).AND.(IP2.EQ.IIX)) GO TO 101
100 CCNTINUE
   YP=0
   GO TO 1C2
101 YP=II
1C2 RETURN
END
B. Data Format

The first card specifies NN, NL in 215 Format where

NN is the number of nodes
NL is the number of lines

The subsequent NT cards (NT = NN + NL) specify the nonzero elements of the admittance matrix in polar form as shown in Table 7.3, one card per element in the format shown below.

<table>
<thead>
<tr>
<th>Description</th>
<th>Card columns</th>
<th>Format</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>1-4</td>
<td>F4.1</td>
</tr>
<tr>
<td>Q</td>
<td>13-16</td>
<td>F4.1</td>
</tr>
<tr>
<td></td>
<td>Y</td>
<td></td>
</tr>
<tr>
<td>Ang.</td>
<td>45-54</td>
<td>F10.4</td>
</tr>
<tr>
<td>EC</td>
<td>65-71</td>
<td>F7.4</td>
</tr>
</tbody>
</table>

The NT + 2 card specifies the voltages (E(j), j = 1, NN) at each node in the format 6F8.3.

The NT + 3 card specifies the angle (D(j), j = 1, NN) at each node in the format 6F8.3 where D(1) will be zero.

The NT + 4 card specifies NPQ, NMP, NMQ, NV in 4I5 Format where
NPQ is the number of real and reactive power injection measurements.

NMP is the number of real power flow measurements.

NMQ is the number of reactive power flow measurements.

NV is the number of voltage measurements.

The NT + 5 card specifies \((IA(j), j = 1, NPQ)\) at which nodes the power injection measurements have been made in Format (2014). For real power injection measurements, we use 1 through NN corresponding to the nodes in mention. For reactive power injection measurements, we use NN + 1 through 2NN corresponding to nodes 1 through NN.

The NT + 6 card specifies \((IP(j), j = 1, NMP)\) the measurements of real power flow between P, where the measurement is made, and node q in Format (2013).

The NT + 7 card specifies \((IQ(j), j = 1, NMQ)\) the measurements of reactive power flow between node p, where the measurement is made, and node q.

The NT + 8 card specifies \((IV(j), j = 1, NV)\) at which nodes (1,…,NN) the voltages measurements are made in 2013 Format.

As output, we will have all the input plus the \(H\) measurement matrix and the \(Q_n Q_n^T\) normalized matrix.
XIII. APPENDIX B - FORTRAN PROGRAM FOR CALCULATING THE
OBSERVABILITY CRITERIA

A. Computer Program
IMPLICIT REAL*8 (A-H,O-Z)
REAL*8 MACHEP
INTEGER ERROR
CIMENSION A(11,11),D(11),E(11),Z(11,11),F(11)
MACHEP=2.0D0*(-52)
N=11
TOL=1.0D0*(-60)
NND=11
75 CC 15 I=1,NND
READ(8,101,END=80)(A(I,J),J=1,I)
101 FORMAT(11F10.7)
15 CONTINUE
WRITE(6,600)
600 FORMAT(*')
16 CONTINUE
CALL TRED2(NM,N,TOL,A,D,E,Z)
CALL TCL2(KM,N,MACHEP,D,E,Z,ERROR)
IF(ERROR.EQ.0) GO TO 20
WRITE(6,102) ERROR
102 FORMAT('COMPUTATION FOR EIGENVALUE ',I3,' EXCEEDS 30 ITERATIONS'/ 'PRINTED EIGENVALUES ARE NOT ORDERED')
A=ERROR-1
20 WRITE(6,600)
N=1
A=6
30 CONTINUE
WRITE(6,103)
103 FORMAT(* EIGENVALUES')
WRITE(6,105)(D(I),I=NN,N)
105 FORMAT(6F14.8)
WRITE(6,104)
104 FORMAT(* EIGENVECTORS')
30 CONTINUE
N=NN+6
N=N+5
31 CONTINUE
GO TO 75
80 STOP
END
SUBROUTINE TRED2 (NM, N, TOL, A, D, E, Z)
IMPLICIT REAL*8 (A-H, O-Z)
REAL*8 A(NM, N), D(N), E(N), Z(NM, N)
DO 100 I=1, N
DO 100 J=1, I
Z(I,J)=A(I,J)
100 CONTINUE
IF(N.EQ.1) GO TO 320
DO 300 II=2, N
I=N+2-II
L=I-2
F=Z(I, I-1)
G=0.0D0
IF(L.LT.1) GO TO 140
DO 120 K=1, L
120 G=G+Z(I,K)*Z(I,K)
140 H=G+F*F
IF(G.GT.TOL) GO TO 160
E(I)=F
H=0.0D0
GO TO 280
160 L=L+1
G=-DSIGN(DSQR(T(H), F)
E(I)=G
H=H-F*G
DO 240 J=1, L
Z(J, I)=Z(I, J)/H
F=0.0D0
DO 180 K=1, J
100 G=G+Z(J,K)*Z(J,K)
JP1=J+1
IF(L.LT.JP1) GO TO 220
DO 200 K=JP1, L
200 G=G+Z(K, J)*Z(I, K)
220 E(J)=G/H
F=F+G*Z(J, I)
240 CONTINUE
HH=F/(H+H)
DO 260 J=1, L
F=Z(I, J)
G=E(J)-HH*F
E(J)=G
DO 260 K=1, J
Z(J, K)=Z(J, K)-F*E(K)-G*Z(I, K)
260 CONTINUE
280 D(I)=H
300 CONTINUE
320 D(I)=0.0D0
     E(I)=0.0D0
     DO 500 I=1,N
     L=I-1
     IF(D(I).EQ.0.0D0) GO TO 380
     DO 360 J=1,L
     G=0.0D0
     DO 340 K=1,L
     G=G+Z(I,K)*Z(K,J)
     DO 360 K=1,L
     Z(K,J)=Z(K,J)-G*Z(K,I)
     360 CONTINUE
80 D(I)=Z(I,I)
    Z(I,I)=1.0D0
    IF(L.LT.1) GO TO 500
    DO 400 J=1,L
    Z(I,J)=0.0D0
    Z(I,J)=0.0D0
700 CONTINUE
500 CONTINUE
RETURN
END
SUBROUTINE TQL2(NM,N,MACHEP,D,E,Z,ERROR)
IMPLICIT REAL*8 (A-H,O-Z)
REAL*8 MACHEP,D(N),E(N),Z(NM,N)
INTEGER ERROR
ERROR=0
IF(N.EQ.1) GO TO 1001
DO 100 I= 2,N
100 E(I-1)=E(I)
F=0.0D0
B=0.0D0
E(N)=0.0D0
DO 240 L=1,N
J=0
H=MACHEP*(DABS(D(L))+DABS(E(L)))
IF(B.LT.H) B=H
DO 110 M=L,N
IF(DABS(E(M)).LE.B) GO TO 120
110 CONTINUE
120 IF(M.EQ.L) GO TO 220
130 IF(J.EQ.30) GO TO 1000
J=J+1
P=(D(L+1)-D(L))/(2.0D0*E(L))
R=DSQRT(P*P+1.0D0)
H=D(L)-E(L)/(P+DSIGN(R,P))
DO 140 I=L,N
140 D(I)=D(I)-H
F=F+H
P=D(M)
C=1.0D0
S=0.0D0
MML=M-L
DO 200 II=1,MML
I=M-II
G=C*E(I)
H=C*P
IF(DABS(P).LT.DABS(E(I))) GO TO 150
C=E(I)/P
R=DSQRT(C*C+1.0D0)
E(I+1)=S*C*R
S=C/R
C=1.0D0/R
GO TO 160
150 C=P/E(I)
R=DSQRT(C*C+1.0D0)
E(I+1)=S*C*E(I)*R
S=1.0D0/R
C=C/R
160 P=C*D(I)-S*G
D(I+1)=H+S*(C*G+S*D(I))
DO 180 K=1,N
H=Z(K,I+1)
Z(K,I+1)=S*Z(K,I)+C*H
Z(K,I)=C*Z(K,I)-S*H
180 CONTINUE
200 CONTINUE
E(L)=S*P
D(L)=C*P
IF(DABS(E(L)).GT.B) GO TO 130
220 D(L)=D(L)+F
240 CONTINUE
NM1=N-1
DO 300 I=1,NM1
K=I
P=D(I)
IP1=I+1
DO 260 J=IP1,N
IF(D(J).GE.P) GO TO 260
K=J
P=D(J)
260 CONTINUE
IF(K.EQ.I) GO TO 300
D(K)=D(I)
D(I)=P
DO 280 J=1,N
P=Z(I,J)
Z(J,I)=Z(J,K)
Z(J,K)=P
280 CONTINUE
300 CONTINUE
GO TO 1001
1000 ERROR=L
1001 RETURN
END
B. Data Format

The first card specifies NN, which is the number of nodes in Format I3.

The subsequent cards (use output of Appendix A) specify the lower triangle of the normalized matrix $Q_n Q_n^T$ in Format 12F10.7. As output, we will obtain the eigenvalues and associate eigenvectors in ordered form.
XIV. APPENDIX C - FORTRAN PROGRAM FOR CALCULATING THE KALMAN COVARIANCE ANALYSIS

A. Computer Program
DIMENSION AM(20,12), AP(12,12), AV(20), AM(12,20), BB(20,20)
DIMENSION BB(20,20), L(20), M(20), BN(12,20), AL(20), APP(12,12)
DOUBLE PRECISION BB, D, BIGA, HOLD
READ(5,5) NL
9 FORMAT(13)
  NLL=20
  READ(8,10)((AM(I,J),J=1,11),I=1,NL)
10 FORMAT(11F10.7)
  WRITE(6,11)((AM(I,J),J=1,11),I=1,NL)
11 FORMAT(11F10.5)
  DO 12 I=1,11
    DO 12 J=1,11
      AP(I,J)=0.0
    12 AA(I,J)=0.0
    READ(5,14)(AP(I,I),I=1,11)
14 FORMAT(8F10.6)
    READ(5,14)(AV(I),I=1,NL)
    WRITE(6,11)(AP(I,I),I=1,NL)
    WRITE(6,11)(AV(I),I=1,NL)
    WRITE(6,600)
600 FORMAT(11)
    DO 60 JJJJ=1,20
      DO 60 I=1,11
        EF=0.0
        DO 20 JJ=1,11
          EF=EF+AP(I,JJ)*AM(J,JJ)
        20 AA(I,J)=EF
        DO 26 JJ=1,11
          EF=EF+AP(I,JJ)*AM(J,JJ)
        26 BB(I,J)=EF
        DO 28 I=1,11
          BB(I,J)=BB(I,J)+AV(I)
        28 BB(I,I)=BB(I,I)
        CALL DUPNV(BB, NL, NLL, D, L, M)
        DO 34 I=1,11
          DO 34 J=1,11
            EF=0.0
            DO 32 JJ=1,11
              EF=EF+AA(I,JJ)*BB(I,JJ)
            32 BN(I,J)=EF
            DO 34 I=1,11
              DO 34 J=1,11
EF=0.0
DO 42 JJ=1,NL
42 EF=EF+BN(I,JJ)*BB(JJ,J)
44 AL(J)=EF
DO 48 J=1,11
46 EF=EF+AL(JJ)*BN(J,JJ)
48 APP(I,J)=-EF+AP(I,J)
WRITE(6,11)(APP(I,I),I=1,11)
DO 54 I=1,11
DO 54 J=1,11
54 AP(I,J)=APP(I,J)
60 CONTINUE
STOP
END
SUBROUTINE DUMNV(A,N,NN,D,L,M)
DIMENSION A(I),L(I),M(I)
DOUBLE PRECISION A,D,BIGA,HOLD
D=1.DO
NK=-NN
DO 80 K=1,N
NK=NK+NN
L(K)=K
M(K)=K
KK=NK+K
BIGA=A(KK)
DO 20 J=K,N
IZ=NN*(J-1)
DO 20 I=K,N
IJ=IZ+I
10 IF(DABS(BIGA)-DABS(A(IJ))) 15,20,20
15 BIGA=A(IJ)
L(K)=I
M(K)=J
20 CONTINUE
J=L(K)
IF(J-K) 35,35,25
25 KI=K-NN
DO 30 I=1,N
KI=KI+NN
HOLD=-A(KI)
JI=KI-K+J
A(KI)=A(JI)
30 A(JI) =HOLD
35 I=M(K)
IF(I-K) 45,45,38
38 JP=NN*(I-1)
DO 40 J=1,N
JK=NK+J
JI=JP+J
HOLD=-A(JK)
A(JK)=A(JI)
40 A(JI) =HOLD
45 IF(BIGA) 48,46,48
46 D=0.DO
RETURN
48 DO 55 I=1,N
IF(I-K) 50,55,50
50 IK=NK+I
A(IK)=A(IK)/(-BIGA)
55 CONTINUE
DO 65 I=1,N
IK=IK+I
HOLD=A(IK)
IJ=I-NN
DO 65 J=1,N
IJ=IJ+NN
IF(J-K) 60,65,60
60 IF(J-K) 62,65,62
62 KJ=IJ-I+K
A(IJ)=HOLD*A(KJ)+A(IJ)
65 CONTINUE

KJ=K-NN
DO 75 J=1,N
KJ=KJ+NN
IF(J-K) 70,75,70
70 A(KJ)=A(KJ)/BIGA
75 CONTINUE

D=D*BIGA
A(KK)=1.0/BIGA
80 CONTINUE

K=N
100 K=(K-1)
IF(K) 150,150,105
105 I=L(K)
IF(I-K) 120,120,108
108 JQ=NN*(K-1)
JR=NN*(I-1)
DO 110 J=1,N
JK=JQ+J
HOLD=A(JK)
JI=JR+J
A(JK)=-A(JI)
110 A(JI) =HOLD
120 J=M(K)
IF(J-K) 100,100,125
125 KI=K-NN
DO 130 I=1,N
KI=KI+NN
HOLD=A(KI)
JI=KI-K+J
A(KI)=-A(JI)
130 A(JI) =HOLD
GO TO 100
150 RETURN
END
B. Data Format

The first card specifies NL in Format I3. NL is the total number of measurements.

The following cards (use output of Appendix A) specify the measurement matrix $\mathbf{H}$ in 12F10.7 Format.

The next card specifies $(\mathbf{AP}(I), I = 1, 12)$ which corresponds to the diagonal terms of the $\mathbf{C}_0^*$ covariance matrix in Format 8F10.6.

The next card specifies $(\mathbf{AV}(I), I = 1, NL)$ which corresponds to the diagonal terms of the $\mathbf{W}_0$ covariance matrix in Format 8F10.6.

The output gives a copy of the input and the covariance for each state variable per iteration.