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Solution of some theoretical soil drainage problems by generalized orthonormal functions

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by

William LeRoy Powers

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INTRODUCTION

This problem was initiated to provide an analytical mathematical method for solving certain types of drainage problems of non-rectangular flow regions. Generalized orthonormal functions are used.

Special types of orthonormal functions have been used for many years to solve physical and mathematical problems. In our work generalized orthonormal functions will be used. We have found no examples in published literature of the use, numerical calculations included, of orthonormal functions in applied mathematics work. One reason for the non-use of these functions may be the tremendous numerical work involved. Here we develop a program for the 360 digital computer for solving some problems; and we give numerical results.

Present methods for solving Laplace's equation by direct analytical solutions require that the flow region be rectangular. Solutions of Laplace's equation which are applicable to flow problems usually contain a series consisting of terms which are the product of a hyperbolic function and a trigonometric function. For example, such a solution could be of the form

\[ \phi(x,y) = \sum_{n=1}^{\infty} A_n (\cosh n\pi y/s) (\sin n\pi x/s) \]  

(1)

where \( A_n \) are arbitrary constants, \( n \) is an integer, \( x \) and \( y \) are coordinates and \( s \) is the spacing between the drains. If
the flow region is rectangular it may be that \( \phi \) satisfies three of the four boundary conditions exactly and reduces to

\[
\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{s}
\]

on the fourth boundary.

If \( \phi(x) \) behaves properly between \( x = 0 \) and \( x = s \), the arbitrary constants \( A_n \) are the Fourier constants and may be found.

If the flow region is not rectangular, solutions in the form of Equation 1 do not reduce to a Fourier series form on the boundary and the theory of Fourier series cannot be used to find the arbitrary constants \( A_n \).

We have two particular objectives in this thesis. One is to show a technique of using the properties of orthogonal functions to find solutions to Laplace's equation in a non-rectangular flow region. The second objective is to use this mathematical technique to solve two simple steady state drainage problems and to program the required computations for the IBM 360 digital computer.
LITERATURE REVIEW

There are many research publications on drainage. Over 60 references were listed by van Schilfgaarde, Kirkham, and Frevert (1956). Luthin (1957) edited a book on agricultural drainage which has a reference list of over 600 publications. Kirkham (1966) listed over 40 publications in a paper on steady state theories for drainage.

We can divide drainage theory into steady state theory and nonsteady state theory. A steady state system is one in which the flow rates and boundaries do not change with time. A nonsteady state system is time dependent. Because we have chosen to work steady state problems we will review mostly literature on steady state drainage theories.

We can classify steady state drainage theories as those obtained with the aid of laboratory models, as electrical analogues, and those obtained by direct mathematical analysis.

We will review first the literature on some theories associated with laboratory models.

Some investigators have used laboratory drainage models to determine drainage theories. Soil, sand, or glass beads have been used as the porous material in the models. Gross (1925) used a sand tank model to derive an hyperbolic expression for the shape of the water table in a tile drained soil. Kirkham (1941) used a sand tank model and dye to show the streamline patterns for the drainage of water into
tile over an impermeable barrier. The streamline patterns agreed with mathematical theory. Harding and Wood (1942) used the same device for tracing the streamlines into tiles when artesian pressure was present, when the sand was stratified, and when flow moved from a source tile to a sink tile. Donnan (1947) compared tile spacing determined from sand tank data with that determined from the "ellipse equation". We will discuss the "ellipse equation" when we consider mathematical theories of drainage. Bouwer (1955) and Keller and Robinson (1949) used laboratory models to determine water table shapes in sloping land with interceptor drains. Grover and Kirkham (1964) used a glassbead-glycerol model to measure drawdown rates for water tables. Childs and Youngs (1958) used a three dimensional drainage model to show that except for small gaps, the size of the gaps between the tiles has little effect on the water table height midway between the drains.

Electrical analogues can also be used to develop drainage theories. Laplace's equation in two dimensions is valid for both saturated water flow through soil and for the flow of electricity through a conducting medium. We can write the two dimensional form of Laplace's equation as

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

(3)

The symbols x and y denote the coordinate system and \( \Phi \) denotes
the hydraulic potential in the case of water flow; \( \phi \) denotes the electric potential in the case of electric current flow. Murphy, Shippy, and Luo (1963) devote a whole chapter to electric analogues of fluid flow.

Childs (1943, 1945a, 1945b, 1946) was one of the first to use electrical analogues to develop theories of soil drainage. Edwards (1956) used an electric analogue to draw theoretical flow nets for soils with anisotropic permeability.

Because we are most interested in mathematical theory, we will now review some mathematical theories.

Mathematical theories have been developed for steady state drainage by using (a) Dupuit-Forchheimer theory, (b) the hodograph and image methods, (c) limiting cases of already solved problems, (d) numerical analysis and by (e) deriving general analytical solutions to Laplace's equation.

De Wiest (1965) gives the historical development of the Dupuit-Forchheimer theory. The two basic assumptions of the Dupuit-Forchheimer theory as given by Muskat (1946) on page 359 are:

(a) for small inclinations of the water table of a gravity flow system, the streamlines can be taken as horizontal;

(b) the velocities along the streamlines are proportional to the slope of the water table and are independent of depth.

Aronovici and Donnan (1946) used the two assumptions to
develop an ellipse equation for the shape of the water table. Hooghoudt (1937) derived an expression for the quantity of flow into drainage ditches using the Dupuit-Forchheimer assumptions. Dagan (1964, 1965) used the Dupuit-Forchheimer assumptions in combination with a linearized differential equation to derive approximate tile spacing formulas for steady rainfall seepage into tile drains. Schmid and Luthin (1964) used the second Dupuit-Forchheimer assumption to derive an equation for the water table height during steady rainfall onto a sloping soil overlying an impermeable barrier. The soil was tile drained and its surface had the same slope as the impermeable barrier.

The validity of the Dupuit-Forchheimer theory has been discussed by many authors. Glover (1965), Bouwer (1965), and van Schilfgaarde (1965) have written a series of articles on the validity of the Dupuit-Forchheimer theory and the accuracy of calculations of quantity of flow, water table heights, and flow nets based on the theory. Although the theory is not correct, it does predict discharge from drains exactly and water table heights approximately, the latter especially for situations when the vertical extent of the flow region is small compared to the horizontal extent.

The hodograph has been used by some authors to find mathematical solutions to drainage problems. Gustafsson (1946) and Engelund (1957) used the hodograph to find
mathematical solutions for steady rainfall seepage through soil into drain tiles overlying an impermeable barrier. Van Deemter (1949) used the hodograph and relaxation method (numerical analysis) to determine the water table for steady rainfall seepage to great depths. By using a hodograph, Childs (1959) was able to determine mathematically the influence of the capillary fringe on steady state drainage into drains in a semi-infinite soil having an artesian flux. Childs (1960), later adapted his solution to include the case for an impermeable barrier at finite depth.

The method of images used to solve electricity problems has also been used to solve steady state water flow problems. By placing an image drain above the ponded water, Kirkham (1949) solved the problem of the flow of ponded water into tile drains overlying an impervious barrier. Kirkham (1951, 1954) later adapted this method to obtain solutions to the problem for stratified soils with impervious barriers and soils with artesian seepage in addition to ponded water seepage. Swartzendruber (1962) later modified this solution to obtain approximate simple formulas for the flow rate of water into tile drains. List (1964) used the image method to solve the problem of seepage of steady rainfall into drains above an impervious barrier. He obtained an approximate solution.

Kirkham (1950) used a limiting case for seepage into auger holes, solved earlier by Kirkham and van Bavel (1949),
to solve the problem for the seepage of water from a plane water surface into equally spaced ditches. The computed seepage rate does not account for the drawdown of the water table around the ditches, as his problem is for seepage of ponded surface water.

Because of the development of digital computers, numerical analysis has come into wide use as a method of solving seepage flow problems. Although in recent years, numerical analysis is used mostly for unsaturated water flow problems, it has been used to solve saturated water flow problems. Harr (1962) gives a detailed explanation of how to use the numerical method to solve Laplace's equation in two dimensions. Numerical analysis is sometimes called the relaxation method or the iteration method. By treating a falling water table as a series of steady state conditions, Kirkham and Gaskell (1951) used numerical analysis to solve some cases for a falling water table in soil draining into ditches.

The last mathematical technique used to derive theories we will review is one in which direct analytical solutions to Laplace's equation are obtained.

Kirkham (1960a) used a Fourier series to find a solution to the problem of ponded water seeping through soil over an impermeable barrier into drainage ditches of equal water level heights. Kirkham (1965) used this same technique to solve the same problem for seepage into drainage ditches
of unequal water level heights.

Kirkham (1958) again used a Fourier Series to solve the problem of steady rainfall seepage into drains in a soil where there was an impermeable barrier at finite depth. He used a physical approximation that the loss in hydraulic head in the region between the water table and a horizontal plane through the drain centers is small compared to the rest of the flow medium. Kirkham (1960b, 1964) later justified this physical approximation mathematically and derived a correction factor for the assumption. The theory of Kirkham (1958) agreed with the field experiments of Kirkham and De Zeeuw (1952) and Talsma and Haskew (1959). Wesseling (1964a) found good agreement between the theory of Kirkham (1958) and Hooghought (1940). Hinesly (1961) extended the work of Kirkham (1958) to solve the problem of steady state seepage reinforced by artesian water. Wesseling (1964b) used the solution given by Kirkham (1958) to show the effect of using continually submerged drains on tile spacing. Warrick (1964) used a Fourier transform to extend the work of Kirkham (1958) to include completely submerged drains and an impermeable barrier at infinite depth.

Kirkham and Powers (1966) used a fictitious porous media above the saturated flow region to find the shape of the water table for the seepage of steady rainfall into ditch drained land. Using this same technique, Kirkham (1966) solved the
problem for tile drained land.

Some of the physical and mathematical theories we have discussed here have been reviewed in Luthin (1957), van Schilfgaarde, Kirkham and Frevert (1956) and Kirkham (1966).
SOME PROPERTIES AND USES OF ORTHONORMAL FUNCTIONS

One of our objectives has been stated as "using properties of orthonormal functions to obtain solutions of Laplace's equation."

A set of functions $\lambda_n(x)$ ($n = 0, 1, 2, 3, \ldots$) which is orthonormal on the interval from $x = 0$ to $x = s$ is defined as one with the following property:

$$\int_0^s [\lambda_m(x) \lambda_n(x)] dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases} \quad (m, n = 0, 1, 2, 3, \ldots). \quad (4)$$

An example of a function which satisfies Equation 4 on the interval $x = 0$ to $x = s$ is

$$\lambda_n(x) = \left(\frac{2}{s}\right)^{1/2} \sin \frac{n\pi x}{s} \quad (n = 1, 2, 3, \ldots). \quad (5)$$

To show how we intend to use the property given in Equation 4, let us assume we want to approximate some function $f(x)$ between $x = 0$ and $x = s$ using the set of functions given in Equation 5. We assume $f(x)$ is approximated by

$$f(x) = \sum_{n=1}^{N} A_n \left(\frac{2}{s}\right)^{1/2} \sin \frac{n\pi x}{s} \quad (n = 1, 2, 3, \ldots N) \quad (6)$$

where $N$ is the number of terms in the series needed to get a close approximation of $f(x)$ and $A_n$ are arbitrary constants.

To find $A_n$ we multiply both sides of the last equation by $\left(\frac{2}{s}\right)^{1/2} \sin \frac{m\pi x}{s}$, ($m = 1, 2, 3, \ldots N$) which yields

$$f(x) \left(\frac{2}{s}\right)^{1/2} \sin \frac{m\pi x}{s} = \sum_{n=1}^{N} A_n \left(\frac{2}{s}\right)^{1/2} \sin \frac{n\pi x}{s} \left(\frac{2}{s}\right)^{1/2} \sin \frac{m\pi x}{s}.$$
Integrating both sides between \( x = 0 \) and \( x = s \) we have
\[
\int_{0}^{s} [f(x) (2/s)^{1/2} \sin m\pi x / s] \, dx =
\]
\[
\sum_{n=1}^{N} A_n \int_{0}^{s} [(2/s)^{1/2} \sin n\pi x / s (2/s)^{1/2} \sin m\pi x / s] \, dx,
\]
but from the property given in Equation 4 (which applies to the sine function), the integral in the right hand side is zero when \( m \) is not equal to \( n \) and one when \( m \) is equal to \( n \). Therefore the arbitrary coefficients \( A_n \) in Equation 6 are given by
\[
A_n = \int_{0}^{s} [f(x) (2/s)^{1/2} \sin n\pi x / s] \, dx. \quad (7)
\]
Equations 5 to 7 give a special case. In general, instead of Equation 6 we will use
\[
f(x) = B_1\lambda_1 + B_2\lambda_2 + \ldots + B_N\lambda_N \quad (8)
\]
where instead of Equation 7 we will have
\[
B_n = \int_{0}^{s} f(x) \lambda_n(x) \, dx. \quad (9)
\]
In Equations 6 and 8 we choose \( N \) large enough to approximate \( f(x) \) to a desired degree of accuracy. The function \( f(x) \) must have certain properties before it can be represented by the above technique. Churchill (1963) describes some of the necessary mathematical properties of \( f(x) \). Physically, we can say that our function \( f(x) \) has suitable properties if we plot \( f(x) \) as given by the right side of Equation 6 or 8 versus \( x \) on the same graph and find that the two graphs superpose
to a desired degree of accuracy.

For the drainage problems solved in this thesis we will generate an orthonormal set of functions from solutions of Laplace's equation. The members of this set will then be used to form a finite series which approximates a function of $x$ on a boundary to a desired degree of accuracy.
THEORY OF SURFACE WATER MOVING INTO AND THROUGH WATER-SATURATED SOIL BEDDING

Objectives and General Theoretical Procedure

Beauchamp (1952) reports that there are approximately 50,000,000 acres of tight soil in the midwest on which adequate surface drainage is needed. In Southern Iowa and Missouri there are soils which have from 9 to 20 inches of permeable soil over very slowly permeable or impervious claypans.

One surface drainage design for these soils is called a bedding system. Fig. 1 is a three dimensional diagram of a soil bedding system used for surface drainage of excess water. The bedding is constructed so that excess water runs off the surface of the soil from the center of the bedding to the drainage furrow which removes the water from the field. Water also moves through the soil in the bedding to the drainage furrows. The paths and the amount of water flowing through the soil are important factors governing fertilizer, herbicide, and insecticide movement in the soil.

To determine the paths of the saturated water flow in the permeable soil above the impermeable barrier we will solve Laplace's two dimensional equation and use this solution to draw flow nets. Using the solution to this equation we will also determine the amounts of water flowing through the soil.
Fig. 1. Soil bedding system used in surface drainage designs; adapted from Figure 14-6 of Luthin (1966).
DRAINAGE FURROW  

BED CENTER  

PLANT ROWS  

IMPERMEABLE BARRIER
We will use the following steps to solve the problem:

1. Select the flow region and determine the boundary conditions.

2. Obtain solutions to Laplace's equation which satisfy three out of four boundary conditions. The fourth boundary condition will be the one for the soil surface.

3. From the solutions in step 2 generate a set of orthonormal functions along the fourth boundary of the flow medium.

4. Form a series from the generated set of orthonormal functions and use the properties of orthonormal functions to find the arbitrary constant coefficients of each term in the series necessary to satisfy the fourth boundary condition.

5. Choose the number of terms in the series formed in step 4 to approximate the fourth boundary condition to a desired degree of accuracy (error less than one or two parts in a hundred) on a boundary.

6. Use the series from step 5 in the solution to Laplace's equation found in step 2 to find the potential function $\phi$.

7. Use the Cauchy-Riemann relations and the solution from step 6 to find the stream function $\psi$.

8. Convert the stream function $\psi$ and the potential function $\phi$ to a fractional basis and draw the
flow nets for different dimensions of the flow medium.

(9) Determine the quantity of water moving through the soil in relation to the rainfall rates.

We will show our computations for the first few terms of the developed series solution for an example problem. The orthonormalization process and formulas for $\phi$ and $\psi$ were programmed for the IBM 360 computer. This computer program is given in Appendix IV. Only the computations for the example problem will be shown. The computations for the various soil bedding cases investigated will not be given, but just the results presented.

Flow Medium

A two dimensional sketch of Fig. 1 is shown in Fig. 2 in which a rainfall rate $R$ has been added.

In Fig. 2 we have a soil which has been plowed so that the height from the impermeable barrier to the bottom of the drainage furrow is $a$ and the height from the impermeable barrier to the center of the bedding is $b$. The rainfall rate $R$ on the soil surface is just sufficient to keep the soil saturated and still keep the height of the water running in the drainage furrows negligible.

Because the flow region ABCDEO is symmetric about the line BE we can solve the problem by using just one half of the flow region and then draw in the other half of the flow
Fig. 2. A two dimensional drawing of steady rain falling on soil bedding overlying an impermeable barrier; The letter a denotes the distance from the barrier to the drainage furrow, the letter b the distance from the barrier to the top of the middle of the bedding, and the letter s the semiwidth of the bedding.
net by symmetry. We will solve just the left half $ABEO$ of the flow region of Fig. 2. This area is shown in Fig. 3.

We place the origin for our coordinate system directly below the drainage furrow at the top edge of the impermeable barrier. The semiwidth of the bedding is denoted by $s$. The boundaries are indicated by the four encircled numerals. We shall take the reference level for the hydraulic head as the impermeable barrier.

**Derivation of the Potential Function**

We will now turn to the mathematical solution of the problem. We let $\phi$ be the hydraulic head (potential). We assume that (a) both Darcy's Law and Laplace's equation (Equation 3) are valid in the flow region that (b) a sharp line exists between the permeable and impermeable soil and (c) that the soil has isotropic conductivity $K$ down to the impermeable barrier. Then for the boundaries indicated by the encircled numerals of Fig. 3 we have the following boundary conditions:

1. Along $AO$; $K(\partial \phi / \partial x) = 0$ or $\partial \phi / \partial x = 0$
2. Along $OE$; $K(\partial \phi / \partial y) = 0$ or $\partial \phi / \partial y = 0$
3. Along $BE$; $K(\partial \phi / \partial x) = 0$ or $\partial \phi / \partial x = 0$
4. Along $AB$; $\phi = a + cx$ where $c = (b-a)/s$

A set of solutions to Laplace's equation which satisfies boundary conditions 1, 2, and 3 is
Fig. 3. The left half of soil bedding over an impermeable barrier; the letter a denotes the distance from the impermeable barrier to drainage furrow, the letter b, the distance from the impermeable barrier to the top of the center of the bedding and the letter s, the semiwidth of the bedding; the encircled numerals denote the boundaries of the flow region.
\( \phi(x,y) = \frac{\cosh mny/s}{\cosh mnb/s} \cos mnx/s \quad (m = 0,1,2, \ldots). \) (10)

We use the factor \( \cosh mnb/s \) in the denominator to make later computations simpler.

If \( c \) is given by \( c = (b-a)/s \) we have for the fourth boundary condition \( \phi = a + cx \) along \( AB \) where \( y \) is also given by \( a + cx \).

To satisfy boundary condition 4 we must derive an expression for \( \phi \) from the set of functions in Equation 10 which yields \( a + cx \) along \( AB \). Because \( y \) is \( a + cx \) along \( AB \), the set of functions, the right hand side of Equation 10, becomes

\[
\frac{\cosh mna+cx/s}{\cosh mnb/s} \cos mnx/s \quad (m = 0,1,2, \ldots)
\]

along the fourth boundary.

The set of functions given in Expression 11 are not orthonormal on the interval \((0,s)\) so we won't form a series from this set to approximate \( \phi = a + cx \) because we will have no convenient way we know of finding the coefficients of the terms in the series. Instead we will generate a set of orthonormal functions \( \lambda_m(x) \) from the set in Expression 11 and use this set \( \lambda_m \) in a finite series to approximate \( \phi = a + cx \) along \( AB \). We will see later that the generated set \( \lambda_m \) satisfies Laplace's equation and boundary conditions 1, 2, 3, and 4.

Letting \( B_m \) \((m = 0,1,2, \ldots)\) be arbitrary constants we
approximate, by use of Equation 8, \( \phi = a + cx \) by

\[
a + cx = \sum_{m=0}^{N} B_m \lambda_m(x) ; \quad (m = 0,1,2, \ldots N)
\]

(12)

where \( N \) is the number of terms necessary to closely approximate \( \phi = a + cx \).

To get the \( B_m \) we put \( f(x) = a + cx \) in Equation 9 and find

\[
B_m = \int_0^s (a + cx) \lambda_m(x) \, dx ; \quad (m = 0,1,2, \ldots N).
\]

(13)

If we denote the set of functions in Expression 11 by \( u_m(x) \):

\[
u_m(x) = \frac{\cosh m\pi(x + cx)/s}{\cosh m\pi b/s} \cos m\pi x/s ; \quad (m = 0,1,2, \ldots)
\]

we can use the general formula for orthonormalization derived (Gram-Schmidt process) in Appendix I, Equation 158, to find \( \lambda_m(x) \). We will use the short notation of \( u_m \) for \( u_m(x) \) and \( \lambda_m \) for \( \lambda_m(x) \).

The appendix formula for \( \lambda_m \) is

\[
\lambda_m = \frac{u_m - \sum_{n=0}^{m-1} \lambda_n(u_m, \lambda_n)}{[(u_m, u_m) - \sum_{n=0}^{m-1} (u_m, \lambda_n)^2]^{1/2}}
\]

(14)

where \( (u_m, u_m) \) and \( (u_m, \lambda_n) \) are given by

\[
(u_m, u_m) = \int_0^s [u_m]^2 \, dx
\]

and

\[
(u_m, \lambda_n) = \int_0^s [u_m \lambda_n] \, dx.
\]

We shall also use the notation
\[(u_m, u_n) = \int_0^s [u_m u_n] dx.\]

We now use Equation 14 to develop the set \(u_m\) into the orthonormal set \(\lambda_m\). We shall see that in order to find \(\lambda_m\) as given by Equation 14 all we must know are the values of \((u_m, u_n)\) and \((u_m, u_m)\) for any \(a, b,\) and \(s\). Integration formulas for \((u_m, u_n)\) for any \(a, b,\) and \(s\) are derived in Appendix II. From Equations 182 and 183 we see that \((u_m, u_n)\) is given for \(m + n\) odd by

\[
(u_m, u_n) = \frac{sc}{4\pi(c^2 + 1) \cosh \frac{mb}{s} \cosh \frac{nb}{s}}

\left\{ \frac{2(m+n)^2c^2 + (m+n)^2 + (m-n)^2}{(m+n)[(m+n)^2c^2 + (m-n)^2]} \left[ \sinh (m+n)\pi a/s + (c) + \sinh (m+n)\pi a/s \right] \right. + \frac{2(m-n)^2c^2 + (m+n)^2 + (m-n)^2}{(m-n)[(m-n)^2c^2 + (m+n)^2]} \sinh (m+n)\pi a/s \left. \right\}.
\]

and for \((m+n)\) even it is given by

\[
(u_m, u_n) = \frac{sc}{4\pi(c^2 + 1) \cosh \frac{mb}{s} \cosh \frac{mb}{s}}

\left\{ \frac{2(m+n)^2c^2 + (m+n)^2 + (m-n)^2}{(m+n)[(m+n)^2c^2 + (m-n)^2]} \left[ \sinh (m+n)\pi (a/s + c) - \sinh (m+n)\pi a/s \right] \right. + \frac{2(m-n)^2c^2 + (m+n)^2 + (m-n)^2}{(m-n)[(m-n)^2c^2 + (m+n)^2]} \sinh (m+n)\pi a/s \left. \right\}.
\]
From Equation 199 in Appendix II we have \((u_m', u_m)\) given by

\[
(u_m', u_m) = \frac{s}{8m\pi (cosh m\pi b/s)^2} \left[ \frac{2c^2 + 1}{c(c^2 + 1)} [\sinh 2m\pi (a/s + c) - \sinh 2mA/s] + 2m\pi \right]
\]

(17)

for \(m\) not equal to zero. For \(m\) equal to zero we see from Equation 200 that \((u_m', u_m)\) is just \(s\).

If we were to use the general Equations 15, 16, and 17 to derive each \(\lambda_m\) of Equation 14 our expressions for each \(\lambda_m\) would become too lengthy to use without confusion. We therefore choose an example set of values for the parameters \(a, b,\) and \(s\). However, we have developed the IBM computer program to permit us to assign any values of \(a, b,\) and \(s\). As an example to follow throughout this problem we choose the values of 2, 4, and 10 for the parameters \(a, b,\) and \(s\). Table 1 gives \((u_m', u_n)\) and \((u_m', u_m)\) for these values of \(a, b,\) and \(s\). These values were calculated with a desk calculator for \(m\) and \(n\) equal to or less than five and served to "debug" the IBM 360 computer program.
Table 1. Values of the definite integrals \((u_m, u_n)\) and \((u_m, u_m)\) for \(a = 2\), \(b = 4\), and \(s = 10\)

<table>
<thead>
<tr>
<th>m</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10.0000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-0.7365</td>
<td>3.2810</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.2225</td>
<td>-0.8813</td>
<td>1.9397</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>-0.2341</td>
<td>0.3089</td>
<td>-0.8904</td>
<td>1.3494</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.1407</td>
<td>-0.2247</td>
<td>0.3911</td>
<td>-0.8060</td>
<td>1.0265</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>-0.1277</td>
<td>0.1527</td>
<td>-0.2493</td>
<td>0.4235</td>
<td>-0.7121</td>
<td>0.8249</td>
</tr>
</tbody>
</table>

Using Equation 14 we see that the zeroth order orthonormal function \(\lambda_0\) of Equation 12 is given by

\[
\lambda_0 = \frac{u_0}{(u_0, u_0)^{1/2}}. \tag{18}
\]

Using the value of 10 for \((u_0, u_0)\) given in Table 1 we can write Equation 18 as

\[
\lambda_0 = \frac{u_0}{10^{1/2}}. \tag{19}
\]

We again use Equation 14 and see that the first order orthonormal function \(\lambda_1\) is

\[
\lambda_1 = \frac{u_1 - \lambda_0 (u_1, \lambda_0)}{[(u_1, u_1) - (u_1, \lambda_0)^2]^{1/2}}. \tag{20}
\]

Substituting the value of \(\lambda_0\) from Equation 19 into the right hand side of 20 we have
\[ \lambda_1 = \frac{u_1 - \frac{(u_1, u_0)}{10} u_0}{[\left(u_1, u_1\right) - \frac{(u_1, u_0)^2}{10}]^{1/2}} \]  

(21)

Using the numerical values for \( (u_1, u_0) \) and \( (u_1, u_1) \) from Table 1 we can write \( \lambda_1 \) as

\[ \lambda_1 = \frac{u_1 - \frac{(-0.7365)}{10} u_0}{[3.2810 - \frac{(-0.7365)^2}{10}]^{1/2}} \]

or

\[ \lambda_1 = \frac{u_1 + 0.07365 u_0}{(3.2268)^{1/2}} \]  

(22)

where the square root in the denominator is not worked out as it will be multiplied by an equal term later.

For \( \lambda_2 \), the second order orthonormal function of Equation 12, we have from Equation 14

\[ \lambda_2 = \frac{u_2 - \lambda_0(u_2, \lambda_0) - \lambda_1(u_2, \lambda_1)}{[(u_2, u_2) - (u_2, \lambda_0)^2 - (u_2, \lambda_1)^2]^{1/2}}. \]  

(23)

Placing the expressions for \( \lambda_0 \) and \( \lambda_1 \) from Equations 18 and 22 into the right hand side of Equation 23 we have

\[ \lambda_2 = \frac{u_2 - \frac{(u_2, u_0)}{10} u_0 - (u_1 + 0.07365 u_0) \frac{(u_2, u_1) + 0.07365(u_2, u_0)}{3.2268}}{[(u_2, u_2) - \frac{(u_2, u_0)^2}{10} - \frac{[(u_2, u_1) + 0.07365(u_2, u_0)]^2}{3.2268}]^{1/2}} \]  

(24)

in which if we use the numerical values for \( (u_2, u_0) \), \( (u_2, u_1) \),
and \((u_2, u_2)\) from Table 1 and simplify becomes

\[
\lambda_2 = \frac{u_2 - 0.0025u_0 + 0.2680u_1}{(1.7029)^{1/2}}. \tag{25}
\]

The right hand sides of Equations 18, 21, and 24 illustrate that only the definite integrals \((u_m, u_n)\) and \((u_m, u_m)\) defined below Equation 14 are needed to get the orthonormal functions \(\lambda_m\) of Equations 12 and 14.

In a manner similar to that used to form \(\lambda_0, \lambda_1,\) and \(\lambda_2\) we have \(\lambda_3\) and \(\lambda_4\) given by

\[
\lambda_3 = \frac{u_3 + 0.0156u_0 + 0.0366u_1 + 0.4739u_2}{(0.9352)^{1/2}} \tag{26}
\]

and

\[
\lambda_4 = \frac{u_4 + 0.0017u_0 + 0.0389u_1 + 0.1235u_2 + 0.6702u_3}{(0.5261)^{1/2}}. \tag{27}
\]

We can generate any \(\lambda_m\) from Equation 14. However, for each succeeding \(\lambda_m\), the generation of \(\lambda_m\) if done without a computer would become more tedious.

If we let \(D_m^{1/2}\) be the denominator of each \(\lambda_m\) and let \(J_{mn}\) be the constant coefficient of each function \(u_n\) in a \(\lambda_m\) then we can write a general form for \(\lambda_m\) given by

\[
\lambda_m = \frac{u_m - \sum_{n=0}^{m-1} J_{mn} u_n}{D_m^{1/2}} \tag{28}
\]

where the \(u_m\) and \(u_n\) are defined below Equation 13. For example for \(\lambda_2\) we have from Equation 25 the following relations:
\[ u_m = u_2 \; ; \; J_{20} = 0.0025 \; ; \; J_{21} = 0.2680 \; ; \; D_m = 1.7029 \]

and \( \lambda_2 \) is given by (using the form of Equation 28)

\[ \lambda_2 = \frac{u_2 - 0.0025u_0 + 0.2680u_1}{(1.7029)^{1/2}} \]

which is the same as Equation 25.

Using the expression for \( \lambda_m \) from Equation 28 in the right hand side of Equation 13 we find the \( B_m \) as

\[ B_m = \int_0^s (a+cx) \left[ u_m - \sum_{n=0}^{m-1} J_{mn} u_n \right] \frac{D_m^{1/2}}{u_m} \, dx \]  

(29)

Moving the constant \( 1/D_m^{1/2} \) outside of the integral we have for \( B_m \) the expression

\[ B_m = \frac{1}{D_m^{1/2}} \int_0^s (a+cx)(u_m - \sum_{n=0}^{m-1} J_{mn} u_n) \, dx \]  

(30)

To find the constants \( B_m \) we need to evaluate the integral in the right hand side of Equation 30. Upon expanding the integral in Equation 30 we have

\[ \int_0^s (a+cx)(u_m - \sum_{n=0}^{m-1} J_{mn} u_n) \, dx = a \int_0^s (u_m - \sum_{n=0}^{m-1} J_{mn} u_n) \, dx + c \int_0^s x(u_m - \sum_{n=0}^{m-1} J_{mn} u_n) \, dx \]

or

\[ \int_0^s (a+cx)(u_m - \sum_{n=0}^{m-1} J_{mn} u_n) \, dx = \]

\[ a\int_0^s u_m \, dx - J_{m0}\int_0^s u_0 \, dx - J_{m1}\int_0^s u_1 \, dx - \ldots - J_{m(m-1)}\int_0^s u_{m-1} \, dx \]

\[ (31) \]
From Expressions 31 and 32 we see that we need to evaluate the two integrals
\[ \int_0^s u_m dx ; \quad (m = 0, 1, 2, \ldots) \]
and
\[ \int_0^s xu_m dx ; \quad (m = 0, 1, 2, \ldots) \]

The formulas for these integrals are derived in Appendix III. From Equations 231, 214, and 215 of Appendix III we have the first integral given by
\[ \int_0^s u_m dx = s \quad (33) \]
for \( m \) equal to zero; and
\[ \int_0^s u_m dx = \frac{sc}{m\pi(c^2+1) \cosh mnb/s} [\sinh m\pi(a/s+c) \]
\[ + \sinh m\pi a/s] \quad (34) \]
for \( m \) odd; and
\[ \int_0^s u_m dx = \frac{sc}{m\pi(c^2+1) \cosh mnb/s} [\sinh m\pi(a/s+c) - \sinh m\pi a/s] \quad (35) \]
for \( m \) even.

From Equations 233, 229, and 230 of Appendix III we have the integral \( \int_0^s xu_m dx \) given by
\[ \int_0^s xu_m dx = s^2/2 \]
for \( m \) equal to zero; and
\[
\int_0^s x u_m dx = -\frac{s^2}{m\pi(c^2+1)\cosh m\pi b/s} \left[ c \sinh m\pi(a/s+c) + \frac{1 - c^2}{m\pi(c^2+1)} \left( \cosh m\pi(a/s+c) + \cosh m\pi a/s \right) \right] \quad (36)
\]
for \( m \) odd; and
\[
\int_0^s xu_m dx = \frac{s^2}{m\pi(c^2+1)\cosh m\pi b/s} \left[ c \sinh m\pi(a/s+c) + \frac{1 - c^2}{m\pi(c^2+1)} \left( \cosh m\pi(a/s+c) - \cosh m\pi a/s \right) \right] \quad (37)
\]
for \( m \) even.

Table 2 shows the numerical values of \( \int_0^s u_m dx \) and \( \int_0^s xu_m dx \) for \( a = 2, b = 4, \) and \( s = 10. \)

Table 2. Numerical values of \( \int_0^s u_m dx \) and \( \int_0^s xu_m dx \) for \( a = 2, b = 4, \) and \( s = 10. \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int_0^s u_m dx )</td>
<td>10.0000</td>
<td>-0.7365</td>
<td>0.2225</td>
<td>-0.2341</td>
<td>0.1407</td>
<td>-0.1277</td>
</tr>
<tr>
<td>( \int_0^s xu_m dx )</td>
<td>50.0000</td>
<td>-19.8982</td>
<td>4.5818</td>
<td>-3.1926</td>
<td>2.0464</td>
<td>-1.5996</td>
</tr>
</tbody>
</table>

Upon substituting the Expressions 31 and 32 for the integral in the right hand side of Equation 30 we have \( B_m \) as
\[
B_m = \frac{1}{D_m^{1/2}} \left[ a \int_0^S u_m dx - J_{m0} \int_0^S u_0 dx - J_{m1} \int_0^S u_1 dx - \ldots \right]
\]
Using Equation 38 we see that $B_0$ is

$$B_0 = \frac{1}{D_0^{1/2}} \left\{ a \int_0^s u_0 dx + c \int_0^s xu_0 dx \right\}. \tag{38}$$

From Table 2 we obtain the values of the integrals above when $a$, $b$, and $s$ are 2, 4, and 10 respectively. We then have $B_0$ as

$$B_0 = \frac{1}{D_0^{1/2}} \left\{ 2[10] + 0.2[50] \right\}$$

or upon using the value of 10 for $D_0$ as seen in Equation 19, we have $B_0$ as

$$B_0 = \frac{30}{10^{1/2}}. \tag{39}$$

As earlier, we leave the denominator in the square root form because later the denominator will be multiplied by $D_0^{1/2}$ making the extraction of the square root unnecessary.

Using Equation 38 a second time we see that $B_1$ is

$$B_1 = \frac{1}{D_1^{1/2}} \left\{ a \int_0^s u_1 dx - J_10 \int_0^s u_0 dx + c \int_0^s \left( xu_1 dx - J_10 \int_0^s xu_0 dx \right) \right\}$$

or using Table 2 to find the integrals, looking at $\lambda_1$ to find $J_10 = -0.07365$, and remembering the value of $a$ and $c$ to be 2 and 0.2, we have
\[ B_1 = \frac{1}{D_1^{1/2}} \{ 2[-0.7365 - (-0.07365)(10)] + 0.2[-19.8982 - (-0.07365)(50)] \} \]

which for \( D_1 \) equal to 3.2268 as seen from Equation 22 yields

\[ B_1 = -\frac{3.2437}{(3.2268)^{1/2}}. \]  \hspace{1cm} (40)

Similarly using Equation 38 we find \( B_2, B_3, B_4, \) and \( B_5 \) to be given by

\[ B_2 = -\frac{0.1756}{(1.7029)^{1/2}}, \]  \hspace{1cm} (41)

\[ B_3 = -\frac{0.1931}{(0.9352)^{1/2}}, \]  \hspace{1cm} (42)

\[ B_4 = -\frac{0.0440}{(0.5262)^{1/2}}, \]  \hspace{1cm} (43)

and

\[ B_5 = -\frac{0.1422}{(0.2987)^{1/2}}. \]  \hspace{1cm} (44)

Placing the expressions for \( B_m \) from Equations 39, 40, 41, and 42 and the expressions for \( \lambda_m \) from Equations 19, 22, 25, and 26 into the right hand side of Equation 12 we have for four terms \( (N = 3) \) that \((a + cx)\) is approximated by

\[ (a+cx) = \frac{30}{10^{1/2}} \left[ \frac{u_0}{10^{1/2}} \right] + \frac{-3.2437}{(3.2268)^{1/2}} \left[ \frac{u_1 + 0.07365u_0}{(3.2268)^{1/2}} \right] + \\
\frac{-0.1756}{(1.7029)^{1/2}} \left[ \frac{u_2 - (0.0025)u_0 - (-0.2680)u_1}{(1.7029)^{1/2}} \right] + \]
\[
\frac{(-0.1931)}{(0.9352)^{1/2}} \left[ \frac{u_3 - (-0.0156)u_0 - (-0.0366)u_1 - (-0.4739)u_2}{(0.9352)^{1/2}} \right]
\]

or

\[
(a + cx) = \frac{30}{10} u_0 - \frac{3.2437}{3.2268} [u_1 + 0.0737u_0] - \frac{0.1756}{1.7029} [u_2 - 0.0025u_0 + 0.2681u_1] - \frac{0.1931}{0.9352} [u_3 + 0.0156u_0 + 0.0366u_1 + 0.4739u_2]
\]

where we remember that \( u_m \) is given by

\[
u_m = \frac{\cosh m\pi(a+cx)/s}{\cosh m\pi b/s} \cos m\pi x/s \quad (m = 0, 1, 2, \ldots)\] (46)

In the right hand side of Equation 45, the coefficients of each linear combination of \( u_m \) is just \( \xi^m \); \( D_m \) is the denominator of \( \lambda_m \). Using our previous definition for \( J_{mn} \) as used in Equation 28 and letting \( E_m \) be given by

\[
E_m = B_m / D_m^{1/2}
\]

we can write a general form for Equation 45 as

\[
(a + cx) = \sum_{m=0}^{N} E_m \left( u_m - \sum_{n=0}^{m-1} J_{mn} u_n \right).
\]

Using Equation 46 in the right hand side of Equation 48 we have

\[
(a + cx) = \sum_{m=0}^{N} E_m \left[ \frac{\cosh m\pi(a+cx)/s}{\cosh m\pi b/s} \cos m\pi x/s - \sum_{n=0}^{m-1} J_{mn} \frac{\cosh n\pi(a+cx)/s}{\cosh n\pi b/s} \cos n\pi x/s \right].
\]

(49)
If we replace $E_m$ of Equation 49 by the right hand side of Equation 47 and compare the results with Equation 12 we can see that $\lambda_m$ is given by

$$\lambda_m = \frac{1}{D_m^{1/2}} \left[ \frac{\cosh m \pi (a + cx)/s}{\cosh m \pi b/s} \cos m \pi x/s - \sum_{n=0}^{m-1} J_{mn} \frac{\cosh n \pi (a + cx)/s}{\cosh n \pi b/s} \cos n \pi x/s \right]$$

where $D_m^{1/2}$ and $J_{mn}$ are constants determined by the orthonormalization process. For example for the parameter values of $a = 2$, $b = 4$, and $s = 10$ we have from Equation 25 that for $\lambda_2$ the value of $D_m$ is 1.7029 and $J_{20}$ and $J_{21}$ are 0.0025 and -0.2680.

To show how well Equation 49 satisfies boundary condition 4 we have plotted in Fig. 4 and Fig. 5 the right hand side of Equation 49 versus $x$ and the left hand side of Equation 49 versus $x$ on the same graph. Fig. 4A shows these graphs for $N = 0$ and $N = 1$. Fig. 4B and Fig. 4C show the left and right hand side of Equation 49 for $N = 2$ and $N = 5$, i.e., for when 3 and 6 members of the orthonormal set are used. Fig. 4A, Fig. 5B and Fig. 5C show the left and right hand side of Equation 49 for $N = 10$, $N = 15$, and $N = 20$.

We can see from Fig. 4 and Fig. 5 that as more and more terms are added to our series, the approximation of the curve $a + cx$ becomes closer and closer to the true curve. When $N$ is taken as 20 in Fig. 5C the approximation differs perceptibly
Fig. 4. Approximations of the boundary condition $\phi = a + cx$ by Equation 49; the smooth curve passing through the circled points is a plot of the right hand side of Equation 49; the straight line form $\phi = 2$ to $\phi = 4$ is a plot of $a + cx$. 
Fig. 5. Approximations of the boundary condition $\phi = a + cx$ by Equation 49; the smooth curve passing through the circled points is a plot of the right hand side of Equation 49; the straight line from $\phi = 2$ to $\phi = 4$ is a plot of $a + cx$. 
from \( a + cx \) only at \( x = 0 \).

In our example (\( a = 2 \), \( b = 4 \) and \( s = 10 \)) we will use \( N = 20 \) to draw our flow nets; however, for the flow nets of cases shown in the results section we will see that the approximation using \( N = 10 \) is sufficient.

Because \( a + cx \) is just \( \phi(x,y) \) along the soil surface boundary of the flow medium and this boundary is given by \( y = a + cx \) we can see from Equation 49 that \( \phi(x,y) \) is given by a single and a double summation of terms which may be expressed as

\[
\phi(x,y) = \sum_{m=0}^{N} E_m \left( \frac{\cosh \frac{m \pi y}{s}}{\cosh \frac{m \pi b}{s}} \cos \frac{m \pi x}{s} - \sum_{n=0}^{m-1} \frac{J_{mn}}{\cosh \frac{n \pi b}{s}} \cos \frac{n \pi x}{s} \right) .
\] (50)

The potential function \( \phi(x,y) \) given in Equation 50 has the dimensions of length. The coefficients \( J_{mn} \) are known for each set of parameters and are a result of the orthonormalization process (see Equations 14 through 28). The constants \( E_m \) are determined from the properties of orthonormal functions (see Equation 47 and then Equations 29 through 47).

The computer computes and stores (see the computer program of Appendix IV) the \( E_m \) and \( J_{mn} \) of Equation 50 and uses these values with \( \cosh \) and \( \cos \) functions of \( x \) and \( y \) to prepare a net work of values of \( \phi(x,y) \) for specified values of \( (x,y) \).
Derivation of the Stream Function

Using the Cauchy-Riemann relations and the potential function of Equation 50 we can now determine the stream function $\psi$.

If $K$ is the hydraulic conductivity of the soil with dimensions of length per unit time, we define the velocity potential $\varphi$ by

$$\varphi = K\phi$$

and use the Cauchy-Riemann relations, given by

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}$$

and

$$\frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

to find the stream function $\psi$.

Using our expression for $\phi$ from Equation 50 in the right hand side of Equation 51 we have $\varphi$ given by

$$\varphi = K \sum_{m=0}^{N} E_m \left[ \frac{\cosh mn/s}{\cosh m\beta/s} \cos mn/s - \sum_{n=0}^{m-1} \frac{\cosh mn/s}{\cosh m\beta/s} \cos n\pi x/s \right].$$

To find $\psi$ we will use Equation 53. Differentiating $\varphi$ given by Equation 54 with respect to $y$ and setting this equal to $-\frac{\partial \psi}{\partial x}$ we have

$$-\frac{\partial \psi}{\partial x} = K \sum_{m=0}^{N} E_m \left[ (m\pi/s) \frac{\sinh mn/s}{\cosh m\beta/s} \cos mn/s - \right.$$
or upon multiplying through by -1 and integrating with respect to x we have

\[ \psi = -K \sum_{m=0}^{N} E_m \left[ \frac{\sinh mn/s}{\cosh mn/s} \sin mn/s - \right. \]

\[ \left. \sum_{n=0}^{m-1} J_{mn} \frac{\sinh mn/s}{\cosh mn/s} \sin mn/s \right] + f(y). \]  \hspace{1cm} (55)

We now use the Cauchy-Riemann relation in Equation 52 to find \( f(y) \). From Equation 54 we have \( \partial \phi/\partial x \) given by

\[ \partial \phi/\partial x = -K \sum_{m=0}^{N} E_m \left[ \frac{\cosh mn/s}{\cosh mn/s} \sin mn/s - \right. \]

\[ \left. \sum_{n=0}^{m-1} J_{mn} \frac{\cosh mn/s}{\cosh mn/s} \sin mn/s \right]. \]  \hspace{1cm} (56)

From Equation 55 we have \( \partial \psi/\partial y \) given by

\[ \partial \psi/\partial y = -K \sum_{m=0}^{N} E_m \left[ \frac{\cosh mn/s}{\cosh mn/s} \sin mn/s - \right. \]

\[ \left. - \sum_{n=0}^{m-1} J_{mn} \frac{\cosh mn/s}{\cosh mn/s} \sin mn/s \right] + f'(y). \]  \hspace{1cm} (57)

Substituting the right hand side of Equation 56 for \( \partial \phi/\partial x \) and the right hand side of Equation 57 for \( \partial \psi/\partial y \) into Equation 52 we see that \( f'(y) \) is given by

\[ f'(y) = 0. \]

Therefore \( f(y) \) of Equation 55 is just some constant say \( d \) and Equation 55, our expression for \( \psi \), becomes
In applied problems we are more interested in the amount of water flowing in the soil than the absolute value of the stream function at a point. The quantity of water moving between two points is just the difference between the stream functions at those points. The value of d does not change the magnitude of this difference and can therefore be any value we choose. For this problem we will assign d the value of zero. The stream function $\psi$ is then given as

$$\psi = -K \sum_{m=0}^{N} E_m \left[ \frac{\sinh m\pi y/s}{\cosh m\pi b/s} \sin m\pi x/s - \sum_{n=0}^{m-1} J_{mn} \frac{\sinh n\pi y/s}{\cosh n\pi b/s} \sin n\pi x/s \right] + d. \quad (58)$$

The dimensions of $\psi$ as given in Equation 58 are length squared per unit time.

The value of the hydraulic conductivity $K$ varies from soil to soil so it is convenient to calculate values for $\psi/K$ rather than values of $\psi$. The computer program given in Appendix IV computes values of $\psi/K$ for the parameters $a$, $b$, and $s$ and variables $x$ and $y$.

**Plotting of Potential and Stream Functions: Flow Nets**

We will draw flow nets using equipotential lines and streamlines calculated as a fraction of the total hydraulic
head loss through the medium and a fraction of the total flow through the medium respectively. To plot the equipotential lines and the streamlines we will use a technique similar to that used by Hinesly (1961).

To determine the equipotential lines as a fraction of the total potential drop across the medium we use the parameter \( \phi' \) defined by

\[
\phi' = \frac{\phi(x,y) - \phi(0,a)}{\phi(b) - \phi(0,a)} .
\]

We choose this parameter because it is unity at the point \( x = s \) and \( y = b \) where the potential is a maximum and zero at \( x = 0 \) and \( y = a \) where the potential is a minimum.

Next we plot a set of auxiliary curves as shown in Fig. 6. In Fig. 6 we have plotted \( \phi' \) versus \( x \) or \( y \). For example, the curve \( y = 1.5 \) was determined by using \( y = 1.5 \) in Equation 59 and plotting \( \phi' \) for various values of \( x \). In this case the abscissa would be \( x \). To obtain the curve \( x = 8 \) we plot \( \phi' \) for various values of \( y \) when \( x \) is 8 in the right hand side of Equation 59. In this case the abscissa would be \( y \).

We use Fig. 6 to draw the \( \phi' = 0.2 \), \( \phi' = 0.4 \), \( \phi' = 0.6 \), and \( \phi' = 0.8 \) equipotential lines. For example if we want to draw the 0.8 equipotential line we see from Fig. 6 that the line \( \phi' = 0.8 \) crosses the curves \( x = 10 \) and \( x = 9 \) at \( y = 2.4 \) and \( y = 2.65 \) (points A and B in Fig. 6) and it crosses the curves \( y = 3.0 \) and \( y = 2.5 \) at \( x = 8.55 \) and \( x = 9.5 \) (points D
Fig. 6. Auxiliary curves for plotting equipotential lines for the parameters $a = 2$, $b = 4$, and $s = 10$; the points A, B, D, and E are points along the $\phi' = 0.8$ equipotential line; the points D and E correspond to the points $D'$ and $E'$ of Fig. 7.
and E of Fig. 6). In other words this Fig. 6 tells us that the equipotential line \( \phi' = 0.8 \) passes through the points with coordinates (10.0, 2.4), (9.00, 2.65), (8.55, 3.00) and (9.5, 2.5) in our flow medium shown in Fig. 7. For illustration the last two points are shown as points D' and E' in Fig. 7. The points D' and E' of Fig. 7 correspond to the points D and E of Fig. 6. To form the equipotential lines in Fig. 7 we draw a smooth line through the points determined from the auxiliary curves.

The vertical velocity graph in Fig. 7 will be fully explained in the next section.

In a manner similar to that used to prepare the auxiliary curves for the equipotential lines we also prepare a set of auxiliary curves for the streamlines. These curves are shown in Fig. 8. To find the streamline auxiliary curves we plot the parameter \( \psi' \) given by

\[
\psi' = \frac{\psi(x,y) - \psi_{min}}{\psi(0,a) - \psi_{min}}
\]

(60)
on the ordinate for values of \( x \) or \( y \) on the abscissa. The value \( \psi_{min} \) is found by plotting the values of \( \psi \) along the line \( y = a + cx \) and determining the minimum value of \( \psi \) from the graph. In our example \( \psi_{min} \) is -0.575 K. The maximum value of \( \psi \) is \( \psi(0,a) \) and is equal to zero. Placing values of \( \psi \) from Equation 58 into Equation 60 we see that the parameter \( \psi' \) is independent of \( K \). This means that our flow nets will also be independent of \( K \). The parameter \( \psi' \) allows us to
Fig. 7. Flow net and vertical velocity graph for \( a = 2 \), \( b = 4 \), and \( s = 10 \); the arrows on the streamlines indicate the direction of the flow of water; lines without arrows are equipotential lines; the vertical velocity graph will be explained in the next section; the points \( D' \) and \( E' \) correspond to the points \( D \) and \( E \) of Fig. 6 and the points \( F' \) and \( G' \) correspond to the points \( F \) and \( G \) of Fig. 8.
VERTICAL VELOCITY DIVIDED BY K

STAG. PT. $\psi^* = 0$

F', G', D', E', .2, .4, .6, .8

a = 2

b = 4

SEMWIDTH (s = 10)
Fig. 8. Auxiliary curves for plotting streamlines for the parameters $a = 2$, $b = 4$, and $s = 10$; the points $F$ and $G$ on the 0.4 streamline and correspond to the points $F'$ and $G'$ of Fig. 7.
think of the quantity of water flowing between two streamlines as the fraction of the total quantity of water flowing through the medium. For example 0.2 of the total quantity of water flowing through the medium flows between the 0.6 streamline and the 0.8 streamline.

To plot the streamlines for the flow net we use the same technique as we did for the equipotential lines. For example, the points $F = (3.3, 2.0)$ and $G = (6.8, 2.5)$ on the 0.4 streamline in Fig. 8 correspond to $F'$ and $G'$ on the 0.4 streamline of Fig. 7.

Fig. 7 represents just the left half of the soil bedding. The streamlines have arrows on them which indicate the direction of the water flow. We have chosen $\psi' = 0$ to be the stagnation point (the point where water neither flows into or out of the soil—the velocity is zero). The stagnation point in our example is also the point on the soil surface where $\psi$ reaches a minimum. The impermeable barrier is the streamline $\psi' = 1$.

To show how the flow net for the whole width of the soil bedding looks we have included Fig. 9. The flow net in the left half of Fig. 9 is that of Fig. 7. Because the flow net of Fig. 9 is symmetric about the center of the bedding, we drew the flow net in the right half of the figure as the mirror image of the left half.

We will draw only the left half of each flow net for the
Fig. 9. Flow net for the entire width of the soil bedding for the parameters $a = 2$, $b = 4$, and $s = 10$. 
cases of the different parameters a, b, and s considered in the results section.

Theory of Water Movement in Soil in Relation to Rainfall

In this section we will use the vertical velocity of the water passing through the soil surface at the center of the soil bedding (x = s) to determine the minimum rainfall rate necessary to keep the soil saturated. We will also determine the percent of the total rain falling on the soil bedding which actually passes through the soil.

The vertical velocity of the water through the soil surface can be determined by using Darcy's law and the Cauchy-Riemann relation given by Equation 53.

Darcy's Law for a vertical soil sample is given by

$$Q = \frac{K(\phi_2 - \phi_1)A}{L}$$

(61)

where $Q$ is the quantity of water moving through the soil of length $L$ and cross sectional area $A$, $K$ is the hydraulic conductivity of the sample, $\phi_2$ is the hydraulic potential at the top of the sample and $\phi_1$ is the hydraulic potential at the bottom of the soil sample.

Upon dividing Equation 61 through by $A$ and defining the velocity by $v = Q/A$ we can write Equation 61 as

$$v = \frac{K(\phi_2 - \phi_1)}{L}.$$

(62)

If $L$ is the vertical length of the soil sample it can be
written as

\[ L = y_2 - y_1 \]  

(63)

where \( y_2 \) and \( y_1 \) are distances along the \( y \) axis. Using Equation 63 we write Equation 62 as

\[ v = \frac{K(\phi_2 - \phi_1)}{y_2 - y_1} \]

or mathematically, if we let the sample become very short, we have

\[ v = K \frac{\partial \phi}{\partial y} \]  

(64)

as the velocity in the \( y \) direction. Because the flow is in the negative \( y \) direction, a conventional minus sign does not appear in Equation 64. From Equation 51 we also have \( v \) given by

\[ v = \frac{\partial \omega}{\partial y} . \]  

(65)

From the Cauchy-Riemann relation given in Equation 53 we also have \( v \) as

\[ v = - \frac{\partial \psi}{\partial x} . \]  

(66)

Along the soil surface of the soil bedding \( y \) is given by \( y = a + cx \). If we divide through by \( K \) and substitute \( a + cx \) for \( y \) in Equation 58 we have

\[ \psi/K = - \sum_{m=0}^{N} E_m \left[ \frac{\sinh m\pi(a+cx)/s}{\cosh m\pi b/s} \sin m\pi x/s \right. \]

\[ \left. - \sum_{m=1}^{m-1} \sum_{n=0}^{N} J_{mn} \frac{\sinh n\pi(a+cx)/s}{\cosh n\pi b/s} \sin n\pi x/s \right] . \]  

(67)
We are interested in the vertical velocity $v$ along the soil surface which is a function of $x$ only as seen by Equation 67. Therefore we can write Equation 66 in terms of total derivatives as

$$v = -\frac{d\psi}{dx}.$$  \hspace{1cm} (68)

The vertical velocity $v$ across the soil surface is obtained by multiplying the slope of the curve of $\psi/K$ versus $x$ by $K$. We prefer to leave $v$ in terms of $K$ because this technique allows us to calculate $v$ for a soil of any hydraulic conductivity.

To enable the reader to better picture the distribution of the vertical velocity through the soil surface we have plotted $v/K$ for values of $x$ between 0 and 10 in a graph over the flow net in Fig. 7. Negative values of $v/K$ signify that the water movement is downward while positive values signify that it is upward. To obtain the vertical flow velocity through the soil surface all we need to do is multiply the value given in the graph by $K$.

One thing that is of interest to us is the minimum rainfall rate $R_{\text{min}}$ necessary to keep the soil profile saturated from the impermeable barrier to the soil surface. The rainfall rate must be large enough to keep the soil saturated at the soil surface at the center of the bedding, i.e. at the point $(s,b)$. If this surface soil is kept saturated, the remainder of the soil bedding will also be
saturated. Therefore all we need to know is the rainfall rate necessary to keep the soil bedding saturated at the point \( (s,b) \).

Because the streamline at the point \( (s,b) \) is vertical downward we know that the vertical velocity at that point is also the infiltration rate of the saturated soil. Further we know that in order to keep the soil saturated we must have a rainfall rate equal to or greater than the infiltration rate, which at \( (s,b) \) is just the vertical velocity.

For our example problem the vertical velocity at \( x = s \) \( (x = 10) \) and \( y = b \) \( (y = 4) \) is, to three decimal places, 0.750K as seen from the vertical velocity graph in Fig. 7. Therefore the minimum rainfall rate must be 0.750K.

Next we want to know what percent of the total amount of water falling as rain that will move through the soil.

The total amount of water moving through the soil is just the difference between \( \psi \) at the stagnation point and \( \psi \) at \( (s,b) \). The value of \( \psi \) at the stagnation point is just \( \psi_{\text{min}} \) which for our example we found to be -0.575K. The value of \( \psi \) at \( (s,b) \) is zero. Therefore the total flow of water moving through the soil \( Q \) is given by

\[
Q = \psi(s,b) - \psi_{\text{min}}
\]

which for our example is

\[
Q = 0 - (-0.575K) = 0.575K.
\]

The total amount of water falling on the soil surface
is just Rs. This value must be at least vs, the minimum rainfall rate to keep the soil saturated multiplied by s. If we let $R_{\text{min}}$ be called the minimum rainfall rate, then the percent $W$ of the total amount of water falling on the soil surface which passes through the soil is given by

$$W = \frac{\psi(s, b) - \psi_{\text{min}}}{R_{\text{min}}s} \times 100$$

(70)

where $\psi_{\text{min}}$ is the minimum value of $\psi$ and is found from the plot of $\psi/K$ in Equation 67 versus $x$, $R_{\text{min}}$ is the minimum rainfall rate and is equal to the vertical velocity $v$ of the water across the soil surface at $x = s$ and $y = b$, $s$ is the semiwidth of the soil bedding and $b$ is the height of the center of the soil bedding above the impermeable barrier. The vertical velocity $v$ through the soil surface is the slope of the curve of $\psi/K$ of Equation 67 versus $x$ multiplied by $K$.

In our example we have $\psi(s, b) = 0$, $\psi_{\text{min}} = -0.575K$, $R_{\text{min}} = 0.750K$ and $s = 10$. Therefore $W$ as calculated from Equation 70 is given by

$$W = \frac{0 - (-0.575K)}{(0.75K)(10)} \times 100 = 7.67 \text{ percent}.$$  

Thus only about 7.67 percent of the total rainfall from the minimum rainfall rate necessary to keep the soil saturated will flow through the soil; 92.33 percent never enters the soil. For greater rainfall rates we can expect even a smaller percentage of the total rain falling on the soil surface to pass through the soil.
Results of the Theory for Field Drainage Cases

Using the methods discussed in the previous sections we can now draw a set of flow nets and vertical velocity graphs for different values of the parameters $a$, $b$, and $s$.

In our solutions for $\varphi$ and $\psi$ we will let $N$ be 10, i.e., 11 members of the orthonormal set will be used. This value for $N$ allows us to approximate the boundary condition on the fourth boundary to about one percent or better, as seen in the tables and flow nets. We will use tables to show numerically how well the boundary conditions are met rather than graphs because the differences between actual and approximated boundary conditions are too small to show graphically as in Fig. 4 and Fig. 5.

We will investigate seven drainage cases resulting from seven sets of values for the parameters $a$, $b$, and $s$. The seven different cases with their respective values of $a$, $b$, and $s$ are given in Table 3. The units of measurement for $a$, $b$, and $s$ are arbitrary. However, we will use feet to agree with the drainage designs specified on pages 173 and 174 of Schwab et al. (1957).

To show how well the soil surface boundary condition is approximated by letting $N$ be 10 we have tabulated the values of $\varphi = a + cx$ and the approximated values of $x$ in Table 4, Table 5, and Table 6 for the seven cases investigated.
Table 3. The values of the parameters $a$, $b$, and $s$ of Fig. 2 for each soil bedding case

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Table 4. Comparison of $\phi = a + cx$ and approximated values of $\phi$ along the soil surface boundary for cases I, II, III, and IV of Table 3 for $N = 10$

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<th>Case II</th>
<th>$a+cx$ Appr. Value</th>
<th>Case III</th>
<th>$a+cx$ Appr. Value</th>
<th>Case IV</th>
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Table 5. Comparison of $\phi = a + cx$ and approximated values of $\phi$ along the soil surface boundary for cases V and VI of Table 3 for $N = 10$

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Table 6. Comparison of $\phi = a + cx$ and approximated values of $\phi$ along the soil surface boundary for case VII for $N = 10$

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</table>

We have grouped the flow nets and the vertical velocity graphs of our seven cases into three figures. Fig. 10, part A, shows case I; part B, case II; and part C, case III. In Fig. 10 the grouping shows how the flow nets and the vertical velocity graphs change as the slope of the soil surface changes while the semiwidth of the bedding and the depth to the impermeable barrier at the drainage furrow remain constant. Fig. 11, part A, shows case IV; part B, case II; and part C, case V. Fig. 11 shows how the flow nets and the vertical velocity graphs change as the depth to the impermeable barrier increases for constant semiwidth of the bedding and slope of the soil surface. Fig. 12, part A, shows case I; part B, case VI; and part C, case VII. Fig. 12 shows how the vertical velocity graph and the flow net change as the semiwidth of the bedding increases while the slope of the soil surface and the depth to the impermeable barrier are held constant.

The minimum rainfall rate necessary to keep the soil saturated, the value of at the stagnation point $v_{min}$,
Fig. 10. Flow nets and vertical velocity graphs for soil surface slopes of 3.33, 6.67, and 10.00 percent.
Fig. 11. Flow nets and vertical velocity graphs for depths to the impermeable barrier of 1.00, 2.00, and 3.00.
VERTICAL VELOCITY
DIVIDED BY K

STAG. PT.

\( \phi = 0 \)

\[ \alpha = 1 \]

0 1 2 BARRIER 6 8 10 12 14 15

SEMIWIDTH (s=15)

\[ \alpha = 2 \]

\[ \alpha = 3 \]

\[ b = 2 \]

\[ b = 3 \]

\[ b = 4 \]
Fig. 12. Flow nets and vertical velocity graphs for semiwidths of bedding of 15.00, 25.00, and 35.00.
the total flow of water through the soil, and the percent of all the water falling on the soil bedding from the minimum rainfall \( R_{\text{min}} \) which passed through the soil are shown in Table 7.

Table 7. The minimum rainfall rate \( R_{\text{min}} \), the value of the stream function at the stagnation point \( \psi_{\text{min}} \), the quantity \( Q \) of water flowing through the soil, and the percent \( W \) of the total amount of water falling at the minimum rainfall rate which passes through the soil, and the figure containing the flow nets for Cases I, II, III, IV, V, VI, and VII of Table 3

<table>
<thead>
<tr>
<th>Case</th>
<th>( R_{\text{min}} )</th>
<th>( \psi_{\text{min}} )</th>
<th>( Q )</th>
<th>( W )</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0.0500K</td>
<td>-0.0757K</td>
<td>0.0757K</td>
<td>10.09</td>
<td>10A, 12A</td>
</tr>
<tr>
<td>II</td>
<td>0.1200K</td>
<td>-0.1685K</td>
<td>0.1685K</td>
<td>9.36</td>
<td>10B, 11B</td>
</tr>
<tr>
<td>III</td>
<td>0.2000K</td>
<td>-0.2750K</td>
<td>0.2750K</td>
<td>9.17</td>
<td>10C</td>
</tr>
<tr>
<td>IV</td>
<td>0.0909K</td>
<td>-0.1135K</td>
<td>0.1135K</td>
<td>8.33</td>
<td>11A</td>
</tr>
<tr>
<td>V</td>
<td>0.1165K</td>
<td>-0.2250K</td>
<td>0.2250K</td>
<td>12.88</td>
<td>11C</td>
</tr>
<tr>
<td>VI</td>
<td>0.0440K</td>
<td>-0.0870K</td>
<td>0.0870K</td>
<td>7.91</td>
<td>12B</td>
</tr>
<tr>
<td>VII</td>
<td>0.0400K</td>
<td>-0.0965K</td>
<td>0.0965K</td>
<td>6.89</td>
<td>12C</td>
</tr>
</tbody>
</table>

\(^a\)The absolute value of the vertical velocity \( v \) at \( x = s, y = b \).

\(^b\)The minimum value of \( \psi \) taken from graph of \( \psi/K \) from Equation 67 versus \( x \).

\(^c\)Calculated from Equation 69.

\(^d\)Calculated from Equation 70.

From Fig. 10 we see that increasing the slope of the soil surface from about 3 to 10 percent has little effect on the shape of the flow net. However, from the sequence of case I, case II, and case III from Table 7 we see that the increase
in slope increases the minimum rainfall rate necessary to keep the soil saturated from 0.05K to 0.2K. The amount of water flowing through the soil is increased from 0.0757K to 0.2750K, but the percentage of the total water falling on the bedding that moves through the soil is reduced from 10.09 to 9.17 percent.

The streamlines in the flow nets in Fig. 11 become further apart as the depth to the impermeable barrier increases. The velocity of the water moving through the soil should decrease because the hydraulic gradient is decreasing. This decrease in velocity along with the spreading streamlines indicates that the amount of water passing through the soil increases with increasing depth to the impermeable barrier. The sequence of case IV, case II and case V from Table 7 shows this to be true. As the depth to the impermeable barrier increases, the amount of water passing through the soil increases from 0.1135K to 0.2250K. The percentage of the water falling from the minimum rainfall on the soil bedding that passes through the soil is increased from 8.33 to 12.88 percent.

As the semiwidth of the soil bedding is increased, the flow nets in Fig. 12 become elongated and a larger portion of the flow through the soil is horizontal. The sequence of Case I, Case VI, and Case VII from Table 7 shows that the minimum rainfall rate necessary to keep the soil saturated
and the percent of the water falling at the minimum rainfall rate along the soil surface that passes through the soil decrease with increasing semiwidth of the soil bedding. However, the quantity of water flowing through the soil increases with increasing semiwidth.
THEORY OF STEADY RAINFALL SEEPING THROUGH SOIL INTO DRAINAGE DITCHES OF UNEQUAL WATER LEVELS

Objectives and General Theoretical Procedures

In the first problem we found a theoretical solution to the problem of surface drainage in a soil bedding design. Another surface drainage system is the field-ditch design. In this design, open ditches are dug in a regular parallel pattern across the field.

The field-ditch system is used for water table control. When the water table is too high, the ditches are used as drains to lower the water table. When the water table is too low, the ditches may be used to supply water to raise the water table. According to Luthin (1966) water table control is of special importance in the high organic soils such as peat and muck soils. If the water table is too high, the crops suffer due to the poor growing conditions, but if the water table is too low, the organic soils tend to dry out and shrink. If allowed to get too dry, the organic matter oxidizes and over a period of years will disappear. Water table control can also be important on sands and highly permeable mineral soils (see Beauchamp (1952)).

Because ditches are used for water table control, mathematical expressions for the water table height between the ditches are of great value to drainage engineers. Although the ditches can be used for both drainage and irrigation we
will solve only the case of drainage under a steady rainfall.

We will use steps similar to those outlined for the soil bedding design. We will use the following steps:

1. Select a flow region and determine the boundary conditions.
2. Obtain solutions to the stream function form of Laplace's equation for three of the four boundaries of the problem.
3. Use some of the solutions in step 2 to generate a set of orthonormal functions along the fourth boundary.
4. From the set of orthonormal functions generated in step 3 form a finite series to approximate the fourth boundary condition.
5. Use the series in step 4 in the solution to Laplace's equation to find an expression for the stream function $\psi$.
6. Use the Cauchy-Riemann relations and the expression for $\psi$ found in step 5 to find the potential function $\phi$.
7. Use $\phi$ to determine the water table shape.
8. Calculate values for $\phi$ and $\psi$ and draw flow nets.
9. From the flow nets and values of $\psi$ determine the relative amounts of water seeping into the two ditches.

As before, sample computations will be given for an example problem. We have programmed the orthonormalization
process and formulas for $\Theta$ and $\psi$ for the 360 IBM computer. The computer program is given in Appendix VII. Results for
the various drainage cases investigated will be shown, but
only the computations for the example problem will be given.

To enable the reader to better follow the steps in both
problems we will use, as much as possible, the same notation
and symbols in this problem as we did in the last problem.

Flow Medium

A series of drainage ditches at a distance $s$ apart are
cut into a homogeneous soil of constant hydraulic conductivity
$K$, overlying an impermeable barrier. The ditches just reach
or penetrate into the barrier and the water stands at differ­
ent heights $a$ and $b$ in the ditches. Fig. 13 shows the
geometry of the system.

Steady rainfall $R$ maintains a steady state water table
arch $ABCDPEA$. The origin for the $x$ and $y$ coordinates is
taken as the point where the right wall of the left hand
ditch intersects the impermeable barrier.

We will find the shape of the water table $ABCD$ and a
flow net for the region $AEPDPO$. To simplify the mathematics
of the problem, we assume as did Kirkham (1958) that the
water moves vertically downward from the soil surface to the
water table and vertically downward across the water table
arch $ABCFDEFA$. We also assume that there is no loss of
hydraulic head across this arch. These assumptions are
Fig. 13. Two-dimensional drawing of a field-ditch drainage design.
close to field conditions for small water table arches resulting from small rainfall rates and for small differences in the water level heights between the ditches.

Physically we can force the water to move vertically downward across the water table arch by introducing an infinite number of parallel strips of infinitesimally small thickness such as EB and FC of Fig. 13. The lower edge of the strips touch the line AD, but the upper edges need not terminate at the water table ABCD. The assumption that no loss in hydraulic head occurs across the water table arch implies that no soil exists between the strips. We will show later how this assumption can be adjusted to give a water table height when we have soil between the strips. This adjustment results in an upper limit for the water table height and hence will give a result pertinent to a safe design (see Kirkham (1960b, 1964)).

We will not consider a capillary fringe along the water table or a surface of seepage along the ditch walls. Childs (1945a) has shown that the neglect of the capillary fringe causes little error in estimating the position of the water table. Also Childs (1946) showed that for small slopes and "foreign", non-rainfall, water the surface of seepage is negligible.

With these physical assumptions we can simplify the flow medium OABCDPO of Fig. 13 to that of OAEPDPO as singled out in Fig. 14.
Fig. 14. The flow medium OAEFDPO of Fig. 13.
Derivation of the Stream and Potential Functions

We can now turn to the mathematical solution of the problem. Let \( \psi \) be the stream function. We will assume that the stream function form of Laplace's equation is valid for this two-dimensional problem. Laplace's equation for this problem is given by

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 .
\]  

(70)

The boundary conditions as indicated by the four encircled numerals of Fig. 14 may be given as:

1. Along OA ; \( \partial \psi / \partial x = 0 \)
2. Along OP ; \( \psi \) is some constant value say \( d \)
3. Along PD ; \( \partial \psi / \partial x = 0 \)
4. Along AD (\( y = a + cx \)) ; \( c = (b-a)/s \); \( \psi = -Rx \)

A solution to Equation 70 which satisfies boundary conditions 1, 2, and 3 is

\[
\psi = d + gy + \frac{\sinh m\pi y/s}{\sinh m\pi b/s} \cos m\pi x/s ; (m = 1, 2, ...) 
\]  

(71)

where \( g \) is an arbitrary constant and \( d \) is the value of \( \psi \) along OP. The value of \( d \) and \( g \) will be determined later.

To satisfy boundary condition 4 we note that if \( c \) is defined by

\[
c = (b-a)/s
\]  

(72)

we can write the boundary AD as

\[
y = a + cx .
\]  

(73)

If we replace \( y \) in Equation 71 with the right hand side
of Equation 73 we have
\[ \psi = d + g(a + cx) + \frac{\sinh m \pi (a + cx) / s}{\sinh m \pi b / s} \cos m \pi x / s ; \]
\[ (m = 1, 2, \ldots ) \] (74)

Equation 74 will not satisfy boundary condition 4
(\( \psi = -Rx \) along \( y = a + cx \)). To satisfy this boundary condi-
tion we generate a set of functions \( \lambda_m(x) \) orthonormal between
\( x = 0 \) and \( x = s \) from the set \( v_m(x) \) given by
\[ v_m(x) = \frac{\sinh m \pi (a + cx) / s}{\sinh m \pi b / s} \cos m \pi x / s . \] (75)

Using the members of the orthonormal set \( \lambda_m(x) \) in a
finite series we approximate boundary condition 4 by
\[ -Rx = d + g(a + cx) + \sum_{m=1}^{N} B_m \lambda_m(x) \] (76)
where \( B_m \) are arbitrary constants and \( N \) is the number of terms
necessary to closely approximate \( \psi = -Rx \) along \( y = a + cx \).

As in the first problem we will use shorter notations for
\( \lambda_m(x) \) and \( v_m(x) \), namely \( \lambda_m \) and \( v_m \).

Equation 76 can be written in the form of Equation 8 as
\[ -Rx = d + g(a + cx) - gcx = \sum_{m=1}^{N} B_m \lambda_m . \]

To get \( B_m \) we put \( f(x) = -[(R + gc)x + (d + ga)] \) in Equation 9
and find
\[ B_m = -\int_0^s [(R + gc)x + (d + ga)] \lambda_m \, dx . \] (77)

To generate the orthonormal set \( \lambda_m \) from the set \( v_m \) we
use the same formula we used in the first problem except we
substitute $v_m$ for $u_m$. The formula was derived in Appendix I and is Equation 158 of the appendix. For this problem we have from Equation 158 that $\lambda_m$ is given by

$$\lambda_m = \frac{v_m - \sum_{n=0}^{m-1} \lambda_n (v_m, \lambda_n)}{[(v_m, v_m) - \sum_{n=0}^{m-1} (v_m, \lambda_m)^2]^{1/2}} \quad (78)$$

where $(v_m, \lambda_n), (v_m, v_m)$ and $(v_m, \lambda_n)^2$ are given by

$$(v_m, \lambda_n) = \int_0^s v_m \lambda_n \, dx,$$

$$(v_m, v_m) = \int_0^s v_m^2 \, dx,$$

and

$$(v_m, \lambda_n)^2 = [\int_0^s v_m \lambda_n \, dx]^2.$$

Also we define $(v_m, v_n)$ by

$$(v_m, v_n) = \int_0^s v_m v_n \, dx \quad (80)$$

As in the first problem we shall see that we need only to know the definite integral $(v_m, v_n)$ and the definite integral $(v_m, v_m)$ to find $\lambda_m$ of Equation 78.

The formulas for the integrals in Equations 79 and 80 are derived for any $a, b, c,$ and $s$ in Appendix V. The formula for the integral of Equation 80 depends upon whether $(m+n)$ is odd or $(m+n)$ is even. From Equations 255 and 256 of Appendix V, the value of $(v_m, v_n)$ for $(m+n)$ odd is given by
\begin{align*}
(v_m, v_n) &= \frac{\text{sc}}{4\pi(c^2+1) \sinh m\pi b/s \sinh n\pi b/s} \\
&= \frac{2(m-n)^2c^2 + (m+n)^2 + (m-n)^2}{(m-n)[(m-n)^2c^2 + (m+n)^2]} \left[ \sinh (m-n) \pi (a/s+c) \right. \\
&\quad + \sinh (m-n) \pi a/s \right] - \frac{2(m-n)^2c^2 + (m+n)^2 + (m-n)^2}{(m+n)[(m+n)^2c^2 + (m-n)^2]} \\
&\quad \left[ \sinh (m+n) \pi (a/s+c) + \sinh (m+n) \pi a/s \right] \quad (81)
\end{align*}

and the value of \((v_m, v_n)\) for \((m+n)\) even is given by

\begin{align*}
(v_m, v_n) &= \frac{\text{sc}}{4\pi(c^2+1) \sinh m\pi b/s \sinh n\pi b/s} \\
&= \frac{2(m+n)^2c^2 + (m+n)^2 + (m-n)^2}{(m+n)[(m+n)^2c^2 + (m-n)^2]} \left[ \sinh (m+n) \pi (a/s+c) \right. \\
&\quad - \sinh (m+n) \pi a/s \right] - \frac{2(m-n)^2c^2 + (m+n)^2 + (m-n)^2}{(m-n)[(m-n)^2c^2 + (m+n)^2]} \\
&\quad \left[ \sinh (m-n) \pi (a/s+c) - \sinh (m-n) \pi a/s \right] \quad (82)
\end{align*}

From Equation 271 of Appendix V we have the integral in Equation 79 given by

\begin{align*}
(v_m, v_m) &= \frac{s}{8\pi^2 \sinh m\pi b/s^2} \left\{ \frac{2c^2+1}{c(c^2+1)} \left[ \sinh 2m\pi (a/s+c) \right. \right. \\
&\quad - \sinh 2m\pi a/s \right\} - 2mn \quad (83)
\end{align*}

Having derived formulas for \((v_m, v_n)\) and \((v_m, v_m)\) we can use Equation 78 to generate each one of our orthonormal functions \(\lambda_m\). The use of the general formulas in Equations 81,
82, and 83 to determine each \( \lambda_m \) by Equation 78 would be very difficult to follow and very tedious to write. We therefore choose values for the parameters \( a, b, \) and \( s \) and use numerical values of the integrals \( (v_m, v_n) \) and \( (v_m, v_m) \) in Equation 78. The values of \( (v_m, v_n) \) and \( (v_m, v_m) \) for \( a = 2, b = 4, \) and \( s = 10 \) are shown in Table 8.

<table>
<thead>
<tr>
<th>( m )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tbody>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-0.9716</td>
<td>1.8583</td>
<td></td>
<td></td>
<td></td>
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<td>-0.8976</td>
<td>1.3417</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
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<td>0.3946</td>
<td>-0.8065</td>
<td>1.0259</td>
<td></td>
</tr>
<tr>
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<td>-0.2508</td>
<td>0.4238</td>
<td>-0.7122</td>
<td>0.8248</td>
</tr>
</tbody>
</table>

Using Equation 78 and the values of \( (v_m, v_n) \) and \( (v_m, v_m) \) from Table 8 we find the orthonormal function \( \lambda_m \) in the same way as we did in the first problem where we used Equation 14 and values of \( (u_m, u_n) \) and \( (u_m, u_m) \) from Table 1. For this new problem \( \lambda_0 \) is zero.

Because the process for the generation of \( \lambda_m \) is so similar in the two problems we will present only the results for the first five members of the set \( \lambda_m \) for the problem at hand. The reader is referred to Equations 18 through 27.
for details of the generation of $\lambda_m$. The orthonormal functions $\lambda_1, \lambda_2, \lambda_3, \lambda_4,$ and $\lambda_5$ are given by

\begin{align*}
\lambda_1 &= \frac{v_1}{(2.6215)^{1/2}}, \\
\lambda_2 &= \frac{v_2 + 0.3706v_1}{(1.4982)^{1/2}}, \\
\lambda_3 &= \frac{v_3 + 0.0617v_1 + 0.5153v_2}{(0.9001)^{1/2}}, \\
\lambda_4 &= \frac{v_4 + 0.0564v_1 + 0.1487v_2 + 0.6863v_3}{(0.5177)^{1/2}}, \\
\lambda_5 &= \frac{v_5 + 0.0137v_1 + 0.0821v_2 + 0.2576v_3 + 0.8683v_4}{(0.2972)^{1/2}}
\end{align*}

respectively.

From our development of $\lambda_1, \lambda_2, \lambda_3, \lambda_4,$ and $\lambda_5$ we can see, as we stated earlier, that all we need to know in order to use Equation 78 are values of the integral $(v_m, v_n)$ and $(v_m, v_m)$.

If we use the notation of Equation 28 in the first problem and let $D_m^{1/2}$ be the denominator of each $\lambda_m$ and let $J_{mn}$ be the constant coefficient of the function $v_n$ for each $\lambda_m$, then we can write a general notation for $\lambda_m$ given by

$$\lambda_m = \frac{v_m - \sum_{n=1}^{m-1} J_{mn} v_n}{D_m^{1/2}}.$$
Using the expression for $\lambda_m$ from Equation 89 in the right hand side of Equation 77 we see that $B_m$ is given by

$$B_m = \frac{1}{D_m^{1/2}} \int_0^s [(R+gc)x + (d+ga)] [v_m - \sum_{n=1}^{m-1} J_{mn} v_n] dx .$$

(90)

To get a better picture of the integrals involved in Equation 90 we expand the integral and move the constants $(R+gc)$ and $(d+ga)$ outside the integral sign to yield

$$B_m = \frac{1}{D_m^{1/2}} \left[ \int_0^s x [v_m - \sum_{n=1}^{m-1} J_{mn} v_n] dx - \left( \frac{[v_m - \sum_{n=1}^{m-1} J_{mn} v_n]}{D_m^{1/2}} \right) \right] .$$

(91)

Upon substituting the right hand side of Equation 91 for $B_m$ in Equation 76 we have

$$-Rx = d + g(a + cx) + \sum_{m=1}^N \left( \left( \frac{1}{D_m^{1/2}} \int_0^s x [v_m - \sum_{n=1}^{m-1} J_{mn} v_n] dx - \frac{[v_m - \sum_{n=1}^{m-1} J_{mn} v_n]}{D_m^{1/2}} \right) \right)$$

or

$$-Rx = d + g(a + cx) + \sum_{m=1}^N \left( \frac{1}{D_m^{1/2}} \int_0^s x [v_m - \sum_{n=1}^{m-1} J_{mn} v_n] dx \right)$$

$$\sum_{m=1}^N \left( \left( \frac{1}{D_m^{1/2}} \int_0^s [v_m - \sum_{n=1}^{m-1} J_{mn} v_n] dx \right) \right) \{v_m - \sum_{n=1}^{m-1} J_{mn} v_n \} .$$

(92)
If we denote $q_m$ by

$$q_m = \frac{\int_0^s [v_m - \sum_{n=1}^{m-1} J_{mn} v_n] dx}{D_m}$$

(93)

and $p_m$ by

$$p_m = \frac{\int_0^s [v_m - \sum_{n=1}^{m-1} J_{mn} v_n] dx}{D_m}$$

(94)

we can write Equation 92 as

$$-R x = d + g(a+cx) - \sum_{m=1}^N \left[ (R+g)c q_m + (d+g)a p_m \right] [v_m - \sum_{n=1}^{m-1} J_{mn} v_n].$$

(95)

We will find the values of $q_m$ and $p_m$ later.

Upon recalling the definition of $v_m$ of Equation 75 we can write Equation 95 as

$$-R x = d + g(a+cx) - \sum_{m=1}^N \left[ (R+g)c q_m + (d+g)a p_m \right] \left[ \frac{\sinh m \tau (a+cx)/s}{\sinh m \tau b/s} \cos \frac{m \tau x}{s} - \sum_{n=1}^{m-1} J_{mn} \frac{\sinh n \tau (a+cx)/s}{\sinh n \tau b/s} \cos \frac{n \tau x}{s} \right].$$

(96)

Recalling that $-R x$ is $\psi$ along the boundary $y = a+cx$ we can write Equation 96 as

$$\psi = d + g y - \sum_{m=1}^N \left[ (R+g)c q_m + (d+g)a p_m \right] \left[ \frac{\sinh m \tau y/s}{\sinh m \tau b/s} \cos \frac{m \tau x}{s} - \sum_{n=1}^{m-1} J_{mn} \frac{\sinh n \tau y/s}{\sinh n \tau b/s} \cos \frac{n \tau x}{s} \right].$$

(97)

In our expression for $\psi$ given in Equation 97 we need to
find values for \( q_m, p_m, d, \) and \( g \) for the parameters of \( a = 2, \) 
\( b = 4, \) and \( s = 10. \)

Finding \( g \)

We find \( g \) by finding the potential function \( \phi \) and noting its value at \( x = 0 \) and \( x = s \) in the flow region of Fig. 14. To find \( \phi \) we will use the definition of \( \varphi \) given by Equation 51 and the Cauchy-Riemann relations given in Equations 52 and 53.

Upon differentiating the right hand side of Equation 97 with respect to \( x \) and substituting it into the right hand side of the second Cauchy-Riemann relation (Equation 53) we have

\[
\frac{\partial \psi}{\partial y} = - \sum_{m=1}^{N} \left\{ (R+gc)q_m + (d+ga)p_m \right\} \left[ \frac{\sinh m\pi y/s}{\sinh m\pi b/s} \sin m\pi x/s - \sum_{n=1}^{m-1} J_{mn} \left( \frac{\sinh n\pi y/s}{\sinh n\pi b/s} \sin n\pi x/s \right) \right]. 
\]  
(98)

After integrating Equation 98 with respect to \( y \) we have

\[
\psi = f(x) - \sum_{m=1}^{N} \left\{ (R+gc)q_m + (d+ga)p_m \right\} \left[ \frac{\cosh m\pi y/s}{\sinh m\pi b/s} \sin m\pi x/s - \sum_{n=1}^{m-1} J_{mn} \cosh n\pi y/s \sin n\pi x/s \right]. 
\]  
(99)

To find \( f(x) \) in Equation 99 we use the first Cauchy-Riemann relation (Equation 52). Upon differentiating Equation 99 with respect to \( x \) and Equation 97 with respect to \( y \) and substituting the expressions for \( \partial \psi/\partial x \) and \( \partial \psi/\partial y \) into Equation 52 we have the expression
From Equation 100 we see that \( f'(x) \) is given by

\[
f'(x) = g
\]

or

\[
f(x) = gx + h
\]

where both \( g \) and \( h \) are arbitrary constants.

Upon substituting the right hand side of Equation 101 for \( f(x) \) in Equation 99 and dividing through by \( K \) we have \( \phi \) given by

\[
\phi = h/K + gx/K - \frac{1}{K} \sum_{m=1}^{N} \left[ \sum_{n=1}^{m-1} J_{mn}(n\pi/s) \frac{\cosh mny/s}{\sinh m\eta b/s} \cos n\pi x/s \right] \cos m\pi x/s - \sum_{m=1}^{N} \left[ \sum_{n=1}^{m-1} J_{mn}(n\pi/s) \frac{\cosh mny/s}{\sinh m\eta b/s} \sin n\pi x/s \right].
\]

(102)

From Fig. 14 we see that at \( x = 0 \), \( \phi \) is \( a \) (the height of the water in the left hand ditch) or from Equation 102 we have

\[
a = h/K \text{ or } h = Ka.
\]

(103)

From Fig. 14 we further note that at \( x = s \), \( \phi \) is \( b \) (the height of the water in the right hand ditch) or from
Equation 102 we have
\[ b = \frac{h}{K} + \frac{gs}{K} \quad (104) \]
Solving Equation 104 for \( g \) and using the expression for \( h \) given in Equation 103 we have \( g \) given by
\[ g = K \left( \frac{b-a}{s} \right) \]
or because \( \frac{b-a}{s} \) is given as \( c \) by Equation 72 we have \( g \) given by
\[ g = Kc \quad (105) \]
Thus, we now know \( g \) and can use its value in Equations 97 and 102.

Upon placing the expression for \( g \) and \( h \) from Equations 103 and 105 into Equation 102 we can write \( \phi \) as
\[ \phi = a+cx - \frac{1}{K} \sum_{m=1}^{N} \left[ \left( (R+Kc^2)q_m + (d+Kca)p_m \right) \frac{\cosh mn\pi/s}{\sinh mn\pi b/s} \right. \\
\left. \sin mn\pi x/s - \sum_{n=1}^{m-1} J_{mn} \frac{\cosh mn\pi/s}{\sinh mn\pi b/s} \sin mn\pi x/s \right] \quad (106) \]

Upon placing the value for \( g \) from Equation 105 into Equation 97, \( \psi \) can be written as
\[ \psi = d+Kcy - \sum_{m=1}^{N} \left[ \left( (R+Kc^2)q_m + (d+Kca)p_m \right) \frac{\sinh mn\pi/s}{\sinh mn\pi b/s} \cos mn\pi x/s \right. \\
\left. m-1 \sum_{n=1}^{m} J_{mn} \frac{\sinh mn\pi/s}{\sinh mn\pi b/s} \cos mn\pi x/s \right] \quad (107) \]

We have yet to find the values of \( d, q_m \) and \( p_m \) of Equation 107.
Finding $d$

To find $d$ we assign $\psi$ the value of zero at $x = 0$ and $y = a$. We can assign $\psi$ this value because (as we did in the first problem) we will draw our streamlines on a fractional basis which means we need to know only the relative values of $\psi$ at different points in the medium. We can assign $\psi$ the value of zero at $x = 0$ and $y = a$ without changing the relative values of $\psi$.

Upon substituting the value of $a$ for $y$ and zero for $x$ in the right hand side of Equation 107 and setting the resulting expression equal to zero we have

$$0 = d + Kca - \sum_{m=1}^{N} \left[ \frac{\sinh m\alpha/s}{\sinh m\beta/s} \right] - \sum_{n=1}^{m-1} \frac{J_{mn}}{\sinh m\beta/s}$$

or upon splitting the series in Equation 108 into a sum of two series, one containing $d$ and the other not containing $d$, and rearranging we have

$$0 = d - \sum_{m=1}^{N} p_m \left[ \frac{\sinh m\alpha/s}{\sinh m\beta/s} \right] - \sum_{n=1}^{m-1} \frac{J_{mn}}{\sinh m\beta/s} + Kca$$

$$0 = \sum_{m=1}^{N} \left[ (R + Kc^2)q_m + Kca \right] \left[ \frac{\sinh m\alpha/s}{\sinh m\beta/s} \right] - \sum_{n=1}^{m-1} \frac{J_{mn}}{\sinh m\beta/s}.$$ (109)

Solving Equation 109 for $d$ we have

$$d = \frac{\sum_{m=1}^{N} \left[ (R + Kc^2)q_m + Kca \right] \left[ \frac{\sinh m\alpha/s}{\sinh m\beta/s} \right] - \sum_{n=1}^{m-1} \frac{J_{mn}}{\sinh m\beta/s}}{1 - \sum_{m=1}^{N} p_m \left[ \frac{\sinh m\alpha/s}{\sinh m\beta/s} \right] - \sum_{n=1}^{m-1} \frac{J_{mn}}{\sinh m\beta/s}}.$$ (110)
where $q_m$ and $p_m$ are given by Equations 93 and 94 and are yet to be evaluated.

Finding $q^m$ and $p^m$

To evaluate $q^m$ and $p_m$ we must evaluate the integrals in Equations 93 and 94. Expanding the integral in Equation 93 we have

$$
\int_0^s x[v_m - \sum_{n=1}^{m-1} J_{mn}v_n]dx = \int_0^s xv_m dx - \int_0^s J_{m1}v_1 dx - \int_0^s J_{m2}v_2 dx - \ldots - \int_0^s J_{m(m-1)}v_{m-1} dx.
$$

(111)

To find the integral in the left hand side of Equation 111 all we need to know are the values of the $J_{mn}$ and the integrals

$$
\int_0^s xv_m dx ; \quad (m = 1, 2, \ldots) .
$$

(112)

Expanding the integral in Equation 94 we have

$$
\int_0^s [v_m - \sum_{n=1}^{m-1} J_{mn}v_n]dx = \int_0^s v_m dx - \int_0^s J_{m1}v_1 dx - \int_0^s J_{m2}v_2 dx - \ldots - \int_0^s J_{m(m-1)}v_{m-1} dx.
$$

(113)

To find the integral in the left hand side of Equation 113 all we need to know are the values of the $J_{mn}$ and the integrals

$$
\int_0^s v_m dx ; \quad (m = 1, 2, \ldots) .
$$

(114)

Formulas for the integrals in Expressions 112 and 114
for any \(a\), \(b\), \(c\), and \(s\) have been derived in Appendix VI.

From Equations 299 and 300 of this appendix we have the integrals in Expression 112 for \(m\) odd given by

\[
\int_0^s x v_m \, dx = - \frac{s^2}{m\pi(c^2+1) \sinh m\pi b/s} \left[ \frac{1 - c^2}{m\pi(c^2+1)} (\sinh m\pi(a/s+c) + \sinh m\pi a/s + c \cosh m\pi(a/s+c)) \right]
\]

and for \(m\) even given by

\[
\int_0^s x v_m \, dx = \frac{s^2}{m\pi(c^2+1) \sinh m\pi b/s} \left[ \frac{1 - c^2}{m\pi(c^2+1)} (\sinh m\pi(a/s+c) - \sinh m\pi a/s + c \cosh m\pi(a/s+c)) \right].
\]

From Equations 286 and 287 of Appendix VI we have the integrals in Expression 114 for \(m\) odd given by

\[
\int_0^s v_m \, dx = - \frac{sc}{m\pi(c^2+1) \sinh m\pi b/s} [\cosh m\pi(a/s+c) + \cosh m\pi a/s].
\]

and for \(m\) even given by

\[
\int_0^s v_m \, dx = \frac{sc}{m\pi(c^2+1) \sinh m\pi b/s} [\cosh m\pi(a/s+c) - \cosh m\pi a/s].
\]

Values of \(\int_0^s x v_m \, dx\) and \(\int_0^s v_m \, dx\) for the values of \(a = 2\), \(b = 4\), and \(s = 10\) are given in Table 9.

To find the value of \(q_m\) given by Equation 93 we use Equation 111 and the values for \(\int_0^s x v_m \, dx\) from Table 9. The values of the constants \(J_{mn}\) and \(B_m\) are found from Equations 84 through 88. For example from Equation 93 we have \(q_3\) given by
Table 9. Values of $\int_0^s x v m dx$ and $\int_0^s v m dx$ for $a = 2$, $b = 4$, and $s = 10$

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_0^s x v m dx^a$</td>
<td>-19.9282</td>
<td>4.7574</td>
<td>-3.1901</td>
<td>2.0473</td>
<td>-1.5995</td>
</tr>
<tr>
<td>$\int_0^s v m dx^b$</td>
<td>-1.1765</td>
<td>0.2153</td>
<td>-0.2360</td>
<td>0.1406</td>
<td>-0.1277</td>
</tr>
</tbody>
</table>

$^a$Calculated from Equations 115 and 116.
$^b$Calculated from Equations 117 and 118.

\[ q_3 = \frac{\int_0^s [x v_3 - J_{31} v_1 - J_{32} v_2] dx}{D_3} \tag{119} \]

or using Equation 111 we have

\[ q_3 = \frac{\int_0^s x v_3 dx - J_{31} \int_0^s x v_1 dx - J_{32} \int_0^s x v_2 dx}{D_3} \tag{120} \]

Using the values of $\int_0^s x v_3 dx = -3.1901$, $\int_0^s x v_1 dx = -19.9282$, and $\int_0^s x v_2 dx = 4.7574$ from Table 9 along with the values of $J_{31} = -0.0617$, $J_{32} = -0.5153$, and $D_3 = 0.9001$ from Equation 86 we have from Equation 120 $q_3$ given by

\[ q_3 = \frac{-3.1901 - (0.0617)(19.9282) + (0.5153)(4.7574)}{0.9001} = -2.1872. \tag{121} \]

Similarly we use Equation 113, the values of $\int_0^s v m dx$ from Table 9, and the values of $J_{mn}$ and $D_m$ from Equation 86 to
find $p_3$ given by

$$p_3 = \frac{-0.2360 - (0.0617)(1.765) + (0.5153)(0.2153)}{0.9001} = -0.2196.$$  \hfill (122)

The values for $q_m$ and $p_m$ for $m = 1, 2, 3, 4, 5$, are given in Table 10 for $a = 2, b = 4, \text{ and } s = 10$.

Table 10. Values of $q_m$ and $p_m$ for $a = 2, b = 4, \text{ and } s = 10$

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_m^a$</td>
<td>-7.6017</td>
<td>-1.7546</td>
<td>-2.1872</td>
<td>-1.0800</td>
<td>-1.7666</td>
</tr>
<tr>
<td>$p_m^a$</td>
<td>-0.4488</td>
<td>-0.1473</td>
<td>-0.2196</td>
<td>-0.1077</td>
<td>-0.2182</td>
</tr>
</tbody>
</table>

$^a$Calculated from Equations 93 and 94 as illustrated by Equations 111 through 122.

Now that we have values for $q_m$ and $p_m$ we can use Equation 110 to calculate $d$, but first we must specify values for the rainfall rate $R$ and the hydraulic conductivity $K$. For our example we choose the values of $R = 0.00635 \text{ m/day and } K = 0.2540 \text{ m/day}.$

Upon placing the values of $a = 2, b = 4, c = 0.2, s = 10, R = 0.00635, K = 0.2540$, the values for $q_m$ and $p_m$ from Table 10, and the values of $J_{mn}$ from Equations 84 through 88 into Equation 110 we can calculate $d$. The values of $d$ for $N = 1, 2, 3, 4, \text{ and } 5$ are $-0.1455, -0.1530, -0.1589, -0.1610, \text{ and } -0.1631 \text{ m}^2/\text{day}.$
We now have values for all of the constants in the expressions for \( \varphi \) and \( \psi \) given in Equations 106 and 107. If we let \( E_m \) be given by

\[
E_m = (R+Kc^2)q_m + (d+Kca)p_m
\]

we can write the potential function \( \varphi \) as

\[
\varphi = a + cx - \frac{1}{K} \sum_{m=1}^{N} E_m \frac{\cosh \frac{m\pi y}{s}}{\sinh \frac{m\pi b}{s}} \sin \frac{m\pi x}{s} - \\
\sum_{n=1}^{m-1} \sum_{m=1}^{N} J_{mn} \frac{\cosh \frac{m\pi y}{s}}{\sinh \frac{m\pi b}{s}} \sin \frac{n\pi x}{s}
\]

and the stream function \( \psi \) as

\[
\psi = d + Kcy - \sum_{m=1}^{N} E_m \frac{\sinh \frac{m\pi y}{s}}{\sinh \frac{m\pi b}{s}} \cos \frac{m\pi x}{s} - \\
\sum_{n=1}^{m-1} \sum_{m=1}^{N} J_{mn} \frac{\sinh \frac{m\pi y}{s}}{\sinh \frac{m\pi b}{s}} \cos \frac{n\pi x}{s}
\]

where \( R \) is the rainfall rate, \( K \) is the hydraulic conductivity of the soil, \( c \) is given by Equation 72, and \( d \) is given by Equation 110. The symbols \( q_m \) and \( p_m \) are defined by Equation 93 and 94 and are evaluated by the technique shown in Equations 111 through 122.

We have programmed the computer to give the values for \( q_m, p_m, d \) and \( E_m \).

To determine the value of \( N \) to approximate boundary condition 4 we use the same graphical method of the first problem. We put \( \psi = -Rx \) and \( y = a + cx \) in Equation 125 which then becomes
\[-Rx = d + Ko(a+cx) - \sum_{m=1}^{N} E_m \left[ \frac{\sinh m\pi(a+cx)}{\sinh m\pi b/s} \right] \cos m\pi x/s - \sum_{n=1}^{m-1} J_{mn} \left[ \frac{\sinh n\pi(a+cx)}{\sinh n\pi b/s} \right] \cos n\pi x/s \] \quad (126a)

In Fig. 15 we have drawn \( \psi \) versus \( x \) as given by the left and right hand side of Equation 126a on the graph for \( N = 1 \), \( N = 5 \), and \( N = 10 \). The smooth curve passing through the circles is a plot of the right hand side of Equation 126a. The straight line from \( \psi = 0 \) to \( \psi = -0.0635 \) is a plot of \(-Rx\). In Fig. 16 we have drawn the same graphs but for \( N = 15 \) and \( N = 20 \). In our solution we will use the value \( N = 20 \).

**Determination of the Water Table**

If we replace \( y \) in Equation 124 by \( a + cx \), the water table \( \phi_T \) is given by

\[ \phi_T = a+cx - \frac{1}{K} \sum_{m=1}^{N} E_m \left[ \frac{\cosh m\pi(a+cx)}{\sinh m\pi b/s} \right] \sin m\pi x/s - \sum_{n=1}^{m-1} J_{mn} \left[ \frac{\cosh n\pi(a+cx)}{\sinh n\pi b/s} \right] \sin n\pi x/s \] \quad (126b)

where the symbol \( \phi_T \) represents the height above the barrier at a point \( x \).

We have plotted in Fig. 17 the water table height \( \phi_T \) along with the flow net for our example. The water table shape is true for the assumption that there is no hydraulic head loss across the water table arch (ABCDPEA of Fig. 13).

Kirkham (1960b, 1964) has derived an equation to correct for this assumption. If we let \( w \) be the distance between the
Fig. 15. Approximations of boundary condition \( \psi = -Rx \) by Equation 126a; the smooth curve passing through the circled points is a plot of the right hand side of Equation 126a; the straight line from \( \psi = 0 \) to \( \psi = -0.0635 \) is a plot of \(-Rx\).
On (MO t—I 

10

11

CD

?102

A

N=1

B

N=5

C

N=10

+0.02

0.02

0.02

0.02

+0.02

0.02

0.02

0.02
Fig. 16. Approximations of boundary condition $4(\psi = -Rx)$ by Equation 126a; the smooth curve passing through the circled points is a plot of the right hand side of Equation 126a; the straight line from $\psi = 0$ to $\psi = -0.0635$ is a plot of $-Rx$. 
Diagram with axes labeled with \( \psi \) on the y-axis and \( \theta \) on the x-axis. Two sets of data points are plotted, labeled as B and A, with the following characteristics:

- B: \( N = 20 \)
- A: \( N = 15 \)

The data points are shown as circular markers along the lines extending from the origin to various points on the axes, indicating a linear relationship between the variables.
Fig. 17. Flow net and water table shape for the parameters $a = 2$, $b = 4$, $s = 10$, $R = 0.00635$, and $K = 0.2540$; the arrows on the streamlines indicate the direction of the water flow; the lines without arrows are equipotential lines.
STEADY RAINFALL R
DRAINAGE DITCH
SOIL SURFACE
WATER TABLE

DRAINAGE DITCH

a = 2

ψ = 0

2 4 6 8

0 2 4 6 8 10

DITCH SPACING (s = 10)

b = 4

—BARRIER 6

1.0
water table and the line \( y = a + cx \) when we assume there is no hydraulic head loss, then \( z \), the distance between the water table and the line \( y = a + cx \) when we take into account hydraulic head loss is given by Kirkham as

\[
z = w(l - \frac{R}{K})^{-1}.
\]  (127)

If we choose our units to be meters, then for our example (see Fig. 17) our maximum \( w \) is about 0.35 meters. Using Equation 127 this value is corrected to

\[
z = 0.35(1-0.025)^{-1} = 0.35897 \text{ meters}
\]

or about 0.36 meters.

We have corrected our water table height by about 1 cm. For all practical purposes, this correction is negligible.

We can see from Equation 127 that for small values of \( w \) and \( \frac{R}{K} \) the correction can be ignored in an applied problem. Because our values for \( w \) and \( \frac{R}{K} \) are small we will not adjust our water table heights by the correction factor of Equation 127.

Plotting of Potential and Stream Functions: Flow Nets

To plot the streamlines and the equipotential lines we will use the same technique of drawing auxiliary curves as we did for the first problem (see Fig. 7, 6, and 8), however we will use different definitions for \( \phi' \) and \( \psi' \) for the problem at hand.

To place the stream function \( \psi \) on a fractional basis we define \( \psi' \) by
\[ \psi' = \frac{\psi(x,y) - \psi_{\min}}{\psi(0,a) - \psi_{\min}}. \] 

We have chosen the parameter \( \psi' \) so that \( \psi' \) is zero at \( x = 0 \) and \( y = a \) in Fig. 14 and unity at \( \psi_{\min} \) where \( \psi_{\min} \) is the minimum value of \( \psi \). Later figures will show that \( \psi \) has its minimum value either at the point \( x = s \) and \( y = b \) or \( \psi \) along the line \( y = 0 \). In our example \( \psi_{\min} \) has the value \( d \).

To place the potential function \( \phi \) on a fractional basis we define \( \phi' \) by

\[ \phi' = \frac{\phi(x,y) - \phi(0,a)}{\phi_{\max} - \phi(0,a)}. \] 

We choose the parameter \( \phi' \) because it is zero at \( x = 0 \) and \( y = a \) and unity at \( \phi_{\max} \) (the maximum value of \( \phi \)). \( \phi_{\max} \) is either \( \phi \) at \( x = s \) and \( y = b \) or it is the maximum value of \( \phi \) as determined from the plot of the right hand side of Equation 126b versus \( x \).

We will not show graphs of the auxiliary curves of \( \phi' \) and \( \psi' \) versus \( x \) or \( y \), but refer the reader to Fig. 6, 7, and 8 for an example of how to use auxiliary curves to plot flow nets.

Fig. 17 shows the flow net for our example.

In Fig. 17 we have shown the geometry of the problem including, ditches, the soil surface and the rainfall to remind the reader of the whole physical problem. In the future, we will present the flow nets and water tables using
just the flow medium OABCDPO of Fig. 13.

Theory of Water Movement in Soil in Relation to Rainfall and Relative Water Level Heights

In this problem we can visualize three different water movement situations: (1) the water from the rainfall moves to both ditches and no water moves from the right hand ditch to the left hand ditch, (2) the water from the rainfall moves to both ditches and some water moves from the right hand ditch to the left hand ditch, and (3) all the rain water moves to the left hand ditch and some water moves from the right hand ditch to the left hand ditch.

We are interested in the percent of the rain water which each ditch receives and the percent of the water reaching the left hand ditch which comes from the right hand ditch. If the water table has a maximum height greater than b then water from the rain flows into both ditches. If we plot $\psi$ along $y = a + cx$ and find $\psi$ where $\psi$ is a maximum we then have a value for $\psi$ upon either side of which the water must flow to different ditches. Let us call this value of $\psi, \psi_c$.

The percent of the rain water which moves into the left hand ditch $D_L$ is given by

$$D_L = \frac{\psi_c - \psi(0,a)}{\psi(s,b) - \psi(0,a)} \times 100$$
or because $\psi(0,a)$ has been taken as zero we have

$$D_L = \left[ \frac{\psi_c}{\psi(s,b)} \right] \times 100 . \tag{130}$$

The percent of the rain water which moves into the right hand ditch $D_r$ is given by

$$D_r = \frac{\psi(s,b) - \psi_c}{\psi(s,b) - \psi(0,a)} \times 100$$

or because $\psi(0,a)$ is zero we have

$$D_r = \frac{\psi(s,b) - \psi_c}{\psi(s,b)} \times 100 . \tag{131}$$

The percent $W_r$ of the water reaching the left hand ditch which comes from the right hand ditch is given by

$$W_r = \frac{d - \psi_c}{d} \times 100 . \tag{132}$$

In Equation 132 we must define $W_r$ as zero for $\psi_c$ equal or greater than $d$.

In our example problem we have situation 3. From the water table shape we see that $\varnothing_{\text{max}}$ equals $b$ so $\psi_c$ is $\psi(s,b)$. The value of $d$ is -0.1697 and the value of $\psi_c$ is -0.0614 so we have from Equation 130 that 100 percent of the rainfall water moves to the left hand ditch. From Equation 132 we see that 63.6 percent of the water entering the left hand ditch comes from the right hand ditch.

Results of the Theory for Field Drainage Cases

In this problem we have chosen 9 cases to investigate. The values of the parameters $a$, $b$, $s$, $R$, and $K$ for each case
are shown in Table 11. In the table, under case IV, the parameters a and b are nearly equal, 0.999 and 1.000. Actually we wanted their values equal, but used a = 0.999 rather than 1.000 because for a = b, i.e., for equal ditch water levels, the integral formula of Equation 271 used in the orthonormalization process is not defined. To develop new equations and new a computer program was not worth the difference 1.000 - 0.999. For practical purposes the numbers 0.999 and 1.000 are equal.

Table 11. Values of the parameters a, b, s, R, and K for 9 drainage cases

<table>
<thead>
<tr>
<th>Case</th>
<th>a (meters)</th>
<th>b (meters)</th>
<th>s (meters)</th>
<th>R (m/day)</th>
<th>K (m/day)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0.600</td>
<td>1.100</td>
<td>10</td>
<td>0.00635</td>
<td>0.254</td>
</tr>
<tr>
<td>II</td>
<td>1.000</td>
<td>1.500</td>
<td>10</td>
<td>0.00635</td>
<td>0.254</td>
</tr>
<tr>
<td>III</td>
<td>1.500</td>
<td>2.000</td>
<td>10</td>
<td>0.00635</td>
<td>0.254</td>
</tr>
<tr>
<td>IV</td>
<td>0.999</td>
<td>1.000</td>
<td>10</td>
<td>0.00635</td>
<td>0.254</td>
</tr>
<tr>
<td>V</td>
<td>1.000</td>
<td>2.000</td>
<td>10</td>
<td>0.00635</td>
<td>0.254</td>
</tr>
<tr>
<td>VI</td>
<td>1.000</td>
<td>1.500</td>
<td>10</td>
<td>0.01270</td>
<td>0.254</td>
</tr>
<tr>
<td>VII</td>
<td>1.000</td>
<td>1.500</td>
<td>20</td>
<td>0.00635</td>
<td>0.254</td>
</tr>
<tr>
<td>VIII</td>
<td>1.000</td>
<td>1.500</td>
<td>30</td>
<td>0.00635</td>
<td>0.254</td>
</tr>
</tbody>
</table>

To show how well our solutions for the cases in Table 11 approximate the fourth boundary condition (ψ = −Rx along y = a + cx) we have tabulated in Tables 12, 13, and 14 ψ = −Rx and the right hand side of Equation 126a for N = 0.
Table 12. Comparison of $\psi = -Rx$ and approximated values of $\psi$ along $y = a + cx$ for cases I, II, and III of Table 11 for $N = 20$

<table>
<thead>
<tr>
<th>x/s</th>
<th>Case I $\psi = -Rx$ Appr. Value</th>
<th>Case II $\psi = -Rx$ Appr. Value</th>
<th>Case III $\psi = -Rx$ Appr. Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.00635</td>
<td>0.00546</td>
<td>0.00635</td>
</tr>
<tr>
<td>0.2</td>
<td>0.01270</td>
<td>0.01183</td>
<td>0.01270</td>
</tr>
<tr>
<td>0.3</td>
<td>0.01905</td>
<td>0.01823</td>
<td>0.01905</td>
</tr>
<tr>
<td>0.4</td>
<td>0.02540</td>
<td>0.02462</td>
<td>0.02540</td>
</tr>
<tr>
<td>0.5</td>
<td>0.03175</td>
<td>0.03101</td>
<td>0.03175</td>
</tr>
<tr>
<td>0.6</td>
<td>0.03810</td>
<td>0.03740</td>
<td>0.03810</td>
</tr>
<tr>
<td>0.7</td>
<td>0.04445</td>
<td>0.04378</td>
<td>0.04445</td>
</tr>
<tr>
<td>0.8</td>
<td>0.05080</td>
<td>0.05016</td>
<td>0.05080</td>
</tr>
<tr>
<td>0.9</td>
<td>0.05715</td>
<td>0.05653</td>
<td>0.05715</td>
</tr>
<tr>
<td>1.0</td>
<td>0.06350</td>
<td>0.06247</td>
<td>0.06350</td>
</tr>
</tbody>
</table>

*Calculated from the right hand side of Equation 126a.

Table 13. Comparison of $\psi = -Rx$ and approximated values of $\psi$ along $y = a + cx$ for cases IV, V, and VI of Table 11 for $N = 20$

<table>
<thead>
<tr>
<th>x/s</th>
<th>Case IV $\psi = -Rx$ Appr. Value</th>
<th>Case V $\psi = -Rx$ Appr. Value</th>
<th>Case IV $\psi = -Rx$ Appr. Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.00635</td>
<td>0.00573</td>
<td>0.00635</td>
</tr>
<tr>
<td>0.2</td>
<td>0.01270</td>
<td>0.01206</td>
<td>0.01270</td>
</tr>
<tr>
<td>0.3</td>
<td>0.01905</td>
<td>0.01841</td>
<td>0.01905</td>
</tr>
</tbody>
</table>

*Calculated from the right hand side of Equation 126a.
Table 13 (Continued)

<table>
<thead>
<tr>
<th>x/s</th>
<th>Case IV $\psi = -Rx$ Appr. Value</th>
<th>Case V $\psi = -Rx$ Appr. Value</th>
<th>Case IV $\psi = -Rx$ Appr. Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.02540</td>
<td>0.02476</td>
<td>0.02540</td>
</tr>
<tr>
<td>0.5</td>
<td>0.03175</td>
<td>0.03111</td>
<td>0.03175</td>
</tr>
<tr>
<td>0.6</td>
<td>0.03810</td>
<td>0.03746</td>
<td>0.03810</td>
</tr>
<tr>
<td>0.7</td>
<td>0.04445</td>
<td>0.04381</td>
<td>0.04445</td>
</tr>
<tr>
<td>0.8</td>
<td>0.05080</td>
<td>0.05016</td>
<td>0.05080</td>
</tr>
<tr>
<td>0.9</td>
<td>0.05715</td>
<td>0.05649</td>
<td>0.05715</td>
</tr>
<tr>
<td>1.0</td>
<td>0.06350</td>
<td>0.06222</td>
<td>0.06350</td>
</tr>
</tbody>
</table>

Table 14. Comparison of $\psi = -Rx$ and approximated values of $\psi$ along $y = a + cx$ for cases VII, VIII, and IX of Table 11 for $N = 20$

<table>
<thead>
<tr>
<th>x/s</th>
<th>Case VII $\psi = -Rx$ Appr. Value</th>
<th>Case VIII $\psi = -Rx$ Appr. Value</th>
<th>Case IX $\psi = -Rx$ Appr. Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.01270</td>
<td>0.01113</td>
<td>0.01905</td>
</tr>
<tr>
<td>0.2</td>
<td>0.02540</td>
<td>0.02383</td>
<td>0.03810</td>
</tr>
<tr>
<td>0.3</td>
<td>0.03810</td>
<td>0.03657</td>
<td>0.05715</td>
</tr>
<tr>
<td>0.4</td>
<td>0.05080</td>
<td>0.04931</td>
<td>0.07620</td>
</tr>
<tr>
<td>0.5</td>
<td>0.06350</td>
<td>0.06206</td>
<td>0.09525</td>
</tr>
<tr>
<td>0.6</td>
<td>0.07620</td>
<td>0.07481</td>
<td>0.11430</td>
</tr>
<tr>
<td>0.7</td>
<td>0.08890</td>
<td>0.08756</td>
<td>0.13335</td>
</tr>
<tr>
<td>0.8</td>
<td>0.10160</td>
<td>0.10031</td>
<td>0.15240</td>
</tr>
<tr>
<td>0.9</td>
<td>0.11430</td>
<td>0.11304</td>
<td>0.17145</td>
</tr>
<tr>
<td>1.0</td>
<td>0.12700</td>
<td>0.12488</td>
<td>0.19050</td>
</tr>
</tbody>
</table>

*aCalculated from the right hand side of Equation 126a.
Cases I through VII have been grouped in sets of three to show the influence of the depth of the water in the ditches, the relative depth of the water in the two ditches, and the value of R/K on the flow nets and water table shapes. In parts A, B, and C of Fig. 18 we have presented cases I, II, and III respectively, to show the affect of a changing depth of the water in the ditches. In parts A, B, and C of Fig. 19 we have presented cases IV, II, and V respectively, to show the affect of changing the relative depths of the water levels in the two ditches. In parts A, B, and C of Fig. 20 we have presented cases VI, II, VII respectively, to show the affect of changing the value of R/K. Cases VIII and IX are not presented graphically because the graphs would be too long and thin; these cases are discussed in the next table.

We will delay further discussion of Fig. 18, Fig. 19, and Fig. 20 until we present Tables 16 and 17, except to note that on Figs. 18A, B, and C; 19A and B; and 20B and C, there are stagnation points—points of zero velocity—shown by circles and the notation "Stag. Pt.". The stagnation points are all on the line y = 0 and at the point where $\Phi'$ of Equation 129 and hence $\Phi$ has a maximum.

The influence of increased ditch spacing may be brought out advantageously with reference to a table. In Table 15 the water table heights $\Phi_T$ for cases II ($s = 10$ meters),
Fig. 18. Flow nets and water table shapes for increasing depths of water in the ditches for a constant rain to conductivity ratio $R/K = 0.025$. 
Fig. 19. Flow nets and water table shapes for increasing difference in water levels in the two ditches for a constant rain to conductivity ratio \( R/K = 0.025 \); the dashed line in part C divides the total water entering the left hand ditch into that portion coming from the rain and that coming from the right hand ditch.
Fig. 20. Flow nets and water table shapes for increasing values of \( R/K \); part A, \( R/K = 0 \); part B, \( R/K = 0.025 \); and part C, \( R/K = 0.05 \).
VIII \((s = 20\) meters\), and IX \((s = 30\) meters\) of Table 11 are given. Table 15 shows the affect of increasing the ditch spacing when all other parameters \((a, b, \text{ and } R/K)\) are constant.

Table 15. Influence of spacing \(s\) on water table heights \(\varnothing_T\) above the barrier for \(a = 1\) meter, \(b = 1.5\) meters and \(R/K = 0.025\)

<table>
<thead>
<tr>
<th>(x/s)</th>
<th>(s = 10) meters (\varnothing_T) (meters)</th>
<th>(s = 20) meters (\varnothing_T) (meters)</th>
<th>(s = 30) meters (\varnothing_T) (meters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>1.1729</td>
<td>1.4784</td>
<td>1.9885</td>
</tr>
<tr>
<td>0.2</td>
<td>1.3049</td>
<td>1.8323</td>
<td>2.7124</td>
</tr>
<tr>
<td>0.3</td>
<td>1.4082</td>
<td>2.0807</td>
<td>3.2025</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4857</td>
<td>2.2329</td>
<td>3.4792</td>
</tr>
<tr>
<td>0.5</td>
<td>1.5394</td>
<td>2.2968</td>
<td>3.5600</td>
</tr>
<tr>
<td>0.6</td>
<td>1.5713</td>
<td>2.2794</td>
<td>3.4604</td>
</tr>
<tr>
<td>0.7</td>
<td>1.5831</td>
<td>2.1868</td>
<td>3.1942</td>
</tr>
<tr>
<td>0.8</td>
<td>1.5760</td>
<td>2.0245</td>
<td>2.7734</td>
</tr>
<tr>
<td>0.9</td>
<td>1.5508</td>
<td>1.7972</td>
<td>2.2087</td>
</tr>
<tr>
<td>1.0</td>
<td>1.5000</td>
<td>1.5000</td>
<td>1.5000</td>
</tr>
</tbody>
</table>

The maximum height of the water table and its location along the \(x\) axis for the nine cases of Table 11 are shown in Table 16.
Table 16. Values and x coordinates of the maximum water table heights $\phi_{\text{max}}$ of the nine cases of Table 11

<table>
<thead>
<tr>
<th>Case</th>
<th>Max. Water$^a$</th>
<th>x-coordinate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Table Height $\phi_{\text{max}}$ (meters)</td>
<td>(meters)</td>
</tr>
<tr>
<td>I</td>
<td>1.2900</td>
<td>6.10</td>
</tr>
<tr>
<td>II</td>
<td>1.5850</td>
<td>7.10</td>
</tr>
<tr>
<td>III</td>
<td>2.0355</td>
<td>8.20</td>
</tr>
<tr>
<td>IV</td>
<td>1.3204</td>
<td>5.00</td>
</tr>
<tr>
<td>V</td>
<td>2.0098</td>
<td>9.55</td>
</tr>
<tr>
<td>VI</td>
<td>1.5000</td>
<td>10.00</td>
</tr>
<tr>
<td>VII</td>
<td>1.8200</td>
<td>5.85</td>
</tr>
<tr>
<td>VIII</td>
<td>2.3010</td>
<td>10.55</td>
</tr>
<tr>
<td>IX</td>
<td>3.5700</td>
<td>14.70</td>
</tr>
</tbody>
</table>

$^a$Determined from a plot of $\phi_{\text{T}}$ versus x of Equation 126b.

We are also interested in the flow distribution of the rain water into the two ditches and the amount of water moving from the right hand ditch to the left hand ditch.

We have tabulated in Table 17 the percent $D_L$ of the rain water moving into the left ditch, the percent $D_R$ of the rain water moving into the right ditch, and the percent $W_R$ of the water reaching the left hand ditch that is coming from the right hand ditch for each case. Also included in Table 17 are the values of $\psi_c$, $\psi(s,b)$ and d needed to compute $D_L$, $D_R$, and $W_R$.

Tables 16 and 17 become more meaningful if we compare them with Figs. 18, 19, and 20. We then see the influence that the depth of the water in the ditches, the relative
Table 17. Values of $\psi_c$, $\psi(s,b)$, $d$, $D_L$, $D_r$, and $W_r$ for the nine drainage cases defined in Table 11

<table>
<thead>
<tr>
<th>Case</th>
<th>$\psi_c$</th>
<th>$\psi(s,b)$</th>
<th>$d$</th>
<th>$D_L$</th>
<th>$D_r$</th>
<th>$W_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(m$^2$/day)</td>
<td>(m$^2$/day)</td>
<td>(m$^2$/day)</td>
<td>(Percent)</td>
<td>(Percent)</td>
<td>(Percent)</td>
</tr>
<tr>
<td>I</td>
<td>0.0381</td>
<td>0.0625</td>
<td>0.0381</td>
<td>61.0</td>
<td>39.0</td>
<td>0.0</td>
</tr>
<tr>
<td>II</td>
<td>0.0443</td>
<td>0.0624</td>
<td>0.0443</td>
<td>71.1</td>
<td>28.9</td>
<td>0.0</td>
</tr>
<tr>
<td>III</td>
<td>0.0513</td>
<td>0.0623</td>
<td>0.0513</td>
<td>82.3</td>
<td>17.7</td>
<td>0.0</td>
</tr>
<tr>
<td>IV</td>
<td>0.0311</td>
<td>0.0622</td>
<td>0.0311</td>
<td>50.0</td>
<td>50.0</td>
<td>0.0</td>
</tr>
<tr>
<td>V</td>
<td>0.0601</td>
<td>0.0623</td>
<td>0.0632</td>
<td>95.2</td>
<td>4.8</td>
<td>4.8</td>
</tr>
<tr>
<td>VI</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0156</td>
<td>00.0</td>
<td>00.0</td>
<td>100.0</td>
</tr>
<tr>
<td>VII</td>
<td>0.0731</td>
<td>0.1249</td>
<td>0.0731</td>
<td>58.5</td>
<td>41.5</td>
<td>00.0</td>
</tr>
<tr>
<td>VIII</td>
<td>0.0656</td>
<td>0.1249</td>
<td>0.0656</td>
<td>52.5</td>
<td>47.5</td>
<td>00.0</td>
</tr>
<tr>
<td>IX</td>
<td>0.0920</td>
<td>0.1873</td>
<td>0.0920</td>
<td>49.1</td>
<td>50.9</td>
<td>00.0</td>
</tr>
</tbody>
</table>

Note:

- $a$ The value of $\psi$ where the water table is a maximum; obtained from a plot of the right hand side of Equation 126b.
- $b$ The value of $\psi$ at $x = s$, $y = b$.
- $c$ The value of $\psi$ along the impermeable barrier as computed by Equation 110.
- $d$ Percent of rain water moving into left hand ditch; calculated from Equation 130.
- $e$ Percent of rain water moving into right hand ditch; calculated from Equation 131.
- $f$ Percent of water moving into left hand ditch that comes from right hand ditch; calculated from Equation 132.

height of the water in the two ditches, and the values of $R/K$ have on the flow distribution and the water table shapes. The ditch water heights $a$ and $b$ and the value of $R/K$ are all independent. The ditch water heights represent external water controls which may have nothing to do with the rainfall rate.
If we hold all other parameters constant and let the depth of the water in the ditches in Fig. 18 go from \( a = 0.6 \) m to \( a = 1.5 \) m in the left hand ditch and from \( b = 1.1 \) m to \( b = 2 \) m in the right hand ditch we see from parts A, B, and C of Fig. 18 (cases I, II, and III) and the sequence of cases I, II, and III in Table 17 that we increase the percent of rain water moving to the left hand ditch from about 60 (61.0) to 82.3 percent. In Table 16 we see that the maximum height of the water table is as we go from case I to case III shifted to the right from \( x = 6.1 \) m to \( x = 8.2 \) m.

We see from parts A, B, and C of Fig. 19 (cases IV, II, and V) and the sequence of cases IV, II, and V in Table 17 that by increasing the difference in water levels in the two ditches from zero to one meter we increase the percent of rain water moving to the left hand ditch from 50.0 to 95.2 percent. The dashed line in Fig. 19C shows the streamline which divides the total water moving into the left hand ditch into that portion coming from the rain and that coming from the right hand ditch. From Table 16 we see from the sequence of cases IV, II, and V that the maximum height in the water table shifts from \( x = 5.0 \) m to \( x = 9.55 \) m.

If we increase \( R/K \) from 0 to 0.5 we see from parts A, B, and C of Fig. 19 (cases VI, II, and VII) and the sequence of cases VI, II, and VII in Table 16 that we increase the maximum height of the water table from 1.50 m to 1.82 m and
the maximum of the water table shifts from the right at \( x = 10.00 \, \text{m} \) to the left at \( x = 5.85 \, \text{m} \). From the sequence of cases II and VIII in Table 17 we see that the percent of the rain water reaching the left hand ditch decreases from 71.1 to 58.5 percent.

From the sequence of cases II, VIII and IX in Table 16 we see that by increasing the ditch spacing from 10 m to 30 m that the maximum height of the water table increases from 1.58 m to 3.57 m and the location of the maximum height moves from \( x = 7.1 \, \text{m} \) to \( x = 14.7 \, \text{m} \). The value of \( x = 14.7 \, \text{m} \) for the location of the maximum in the water table height for the 30 m ditch spacing is determined from our formula for \( \phi_T \), Equation 126b. We would expect the location of the maximum to be at some value of \( x \) greater than 15 m. Our value of \( x = 14.7 \, \text{m} \) is probably a result of the harmonic nature of our function \( \phi_T \).

A check on the theory

As a check on our solution to the field-ditch problem we can compare a particular case with one solved by Kirkham.\(^1\) Kirkham solved our problem when the ditches are at equal water levels. Kirkham's expression for \( \phi \) is (we shall prove it in a moment) given by

\[^{1}\text{Kirkham, Don. Department of Agronomy, Iowa State University of Science and Technology, Ames, Iowa. Computations and flow net for steady state ditch drainage of soil overlying an impermeable barrier. Private communication. 1962.}\]
\[ \Theta = a + \frac{4Ra_2}{K} \sum_{m=1}^{\infty} \frac{1}{m} \frac{\cosh mmh/s}{\sinh mmh/s} \sin mmx/s ; \quad (m = 1, 3, \ldots) \]

(133)

where \( a \) is the height of the water in the ditches.

We can see from inspection that Equation 133 satisfies the boundary conditions along the left ditch wall \( (\Theta = a) \), along the impermeable barrier \( (\partial \Theta / \partial y = 0) \) and along the right ditch wall \( (\Theta = a) \). The boundary condition along \( y = a \) is \( K \partial \Theta / \partial y = R \). To show that this condition is also satisfied, we differentiate both sides of Equation 133 with respect to \( y \), multiply through by \( K \), and replace \( y \) with \( a \) to find

\[ K \frac{\partial \Theta}{\partial y} = 4R \frac{\pi}{\sum_{m=1}^{\infty} \frac{1}{m} \sin mmx/s} ; \quad (m = 1, 3, \ldots). \]

From formula 416.01 of Dwight (1962) we see that the series in the right hand side of the above equation is \( \pi/4 \). The above equation then becomes

\[ K \frac{\partial \Theta}{\partial y} = R \]

which is the boundary condition along \( y = a \).

Thus we see that Equation 133 is a solution to our problem for equal water levels in the ditches.

Kirkham's\(^1\) expression for \( \Psi \) is given by

\(^1\)Kirkham, Don. Department of Agronomy, Iowa State University of Science and Technology, Ames, Iowa. Computations and flow net for steady state ditch drainage of soil overlying an impermeable barrier. Private communication. 1962.
Using the values of $a = 1$ m, $s = 10$ m, $R = 0.00276$ m/day, and $K = 0.254$ m/day and using enough terms to get four decimal accuracy, Kirkham drew the flow net and water table shown in part A of Fig. 21.

Using our solution for values of the parameters $a = 0.999$ m, $b = 1.000$ m, and $s = 10$ m, $R = 0.00276$ m/day and $K = 0.254$ m/day we have drawn the flow net in part B of Fig. 21. We let $a$ be 0.999 rather than 1.000 because for $c = 0$, $[c = (b-a)/s]$, Equation 271 used in our orthonormalization process is not defined.

Upon comparing Fig. 21A with Fig. 21B we can see no difference in the flow nets or the shape of the water table, except maybe that difference due to draftsman error. To show even more closely how well our solution agrees with that of Kirkham\textsuperscript{1} we have tabulated in Table 18 the water table height for values of $x$ for both solutions.

Upon comparing a particular case of our solution with that of Kirkham\textsuperscript{1} we see that the two solutions are in very good agreement. The results agree to one part in a thousand.

\textsuperscript{1}Kirkham, Don. Department of Agronomy, Iowa State University of Science and Technology, Ames, Iowa. Computations and flow net for steady state ditch drainage of soil overlying an impermeable barrier. Private communication. 1962.
Fig. 21. Comparison of the case for $a = 0.999$, $b = 1$, $s = 10$, $R = 0.00276$, and $K = 0.254$ to that of Kirkham (1962); part A is Kirkham's solution; part B our solution.
Table 18. Comparison of water table heights with those of Kirkham\textsuperscript{a}

<table>
<thead>
<tr>
<th>x (meters)</th>
<th>$\phi_T^b$ (meters)</th>
<th>$\phi_K^c$ (meters)</th>
<th>x (meters)</th>
<th>$\phi_T^b$ (meters)</th>
<th>$\phi_K^c$ (meters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.9990</td>
<td>1.0000</td>
<td>6</td>
<td>1.1337</td>
<td>1.1340</td>
</tr>
<tr>
<td>1</td>
<td>1.0514</td>
<td>1.0524</td>
<td>7</td>
<td>1.1175</td>
<td>1.1178</td>
</tr>
<tr>
<td>2</td>
<td>1.0899</td>
<td>1.0906</td>
<td>8</td>
<td>1.0903</td>
<td>1.0906</td>
</tr>
<tr>
<td>3</td>
<td>1.1171</td>
<td>1.1178</td>
<td>9</td>
<td>1.0522</td>
<td>1.0524</td>
</tr>
<tr>
<td>4</td>
<td>1.1335</td>
<td>1.1340</td>
<td>10</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>5</td>
<td>1.1390</td>
<td>1.1395</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\textsuperscript{a}Kirkham, Don. Department of Agronomy, Iowa State University of Science and Technology, Ames, Iowa. Computations and flow net for steady state ditch drainage of soil overlying an impermeable barrier. Private communication. 1962.

\textsuperscript{b}Values of $\phi_T$ for our solution calculated from Equation 126b for $N = 20$.

\textsuperscript{c}Value of $\phi$ calculated from Equation 133 for $y = a$ and $m = 1, 3, 5, \ldots 119$. 
GENERAL DISCUSSION AND CONCLUSIONS

In this thesis we have shown how members of a set of generated orthonormal functions can be used to find theoretical solutions for two steady state drainage problems. Using the Gram-Schmidt process we derived a general formula which was used to generate a set of orthonormal functions. Using members of this set we formed a finite series which in combination with other solutions of Laplace's equation was used to derive expressions for the potential function $\phi$ and the stream function $\psi$.

The orthonormalization process and the expressions for $\phi$ and $\psi$ were programmed for the IBM 360 computer. The numerical values for $\phi$ and $\psi$ were determined by the computer and used to draw the flow nets for the two steady state problems.

We obtained theoretical solutions for the seepage of water through soil bedding over an impermeable barrier and for the seepage of steady rainfall through soil into drainage ditches of unequal water level heights. In the first problem we found the minimum rainfall rate necessary to keep the soil saturated and the percent of the rain water which can be expected to move through the soil bedding. In the second problem we found the shape of the water table, the relative amounts of the rain water moving to each ditch, and the amount of water seeping from one ditch to the other. Our
solution for a particular case of the second problem (that of equal water level heights in the ditches) was found to be in very good agreement (one part in a thousand) with a previous solution by Kirkham\(^1\).

Theoretically, the general formula we used to generate the set of orthonormal functions \(\lambda_m\) could be used to generate as many members of the set \(\lambda_m\) as we please. However for our examples, the expression

\[
(u_m, u_m) - \sum_{n=0}^{m-1} (u_m, \lambda_n)^2
\]

in the denominator of Equation 158 (our orthonormalization formula) became negative for \(\lambda_{21}\). If Expression 135 is negative, \(\lambda_{21}\) will be complex function because the denominator of \(\lambda_m\) is the square root of Expression 135.

We believe that Expression 135 should not be negative and that it became negative because of rounding errors. From the development of Equation 158 in Appendix I we see that each succeeding \(\lambda_m\) is generated using the preceding \(\lambda_m\)'s. Therefore any rounding error would tend to "snowball" for each succeeding \(\lambda_m\) generated. Two reasons we believe Expression 135 should not be negative are:

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\(^1\)Kirkham, Don. Department of Agronomy, Iowa State University of Science and Technology, Ames, Iowa. Computations and flow net for steady state ditch drainage of soil overlying an impermeable barrier. Private communication. 1962.
1. When only 8 significant digits were used to generate $\lambda_m$, the Expression 135 became negative a few terms before $\lambda_{21}$. By using double precision (16 significant digits), we extended $\lambda_m$ to $\lambda_{21}$ before Expression 135 became negative. If Expression 135 should be negative, this increased accuracy should not have changed the $\lambda_m$ for which it became negative.

2. The Expression 135 becomes smaller and smaller for each succeeding member generated, for example in $\lambda_{15}$, $\lambda_{16}$, $\lambda_{17}$, $\lambda_{18}$, $\lambda_{19}$, and $\lambda_{20}$ the Expressions 135 of the second example problem are 0.0010975, 0.0006268, 0.0003548, 0.0001958, 0.0000995, and 0.0000340. We can see that a rounding error in the fourth decimal place of any of the terms of Expression 135 could make the expression negative for $\lambda_{18}$, $\lambda_{19}$, and $\lambda_{20}$.

We were not bothered by this situation for our two problems because we could generate enough terms to approximate our boundaries before Expression 135 became negative.

It seems feasible that our technique could be used to solve a number of soil physics problems provided the integrals in the orthonormalization formula can be evaluated and provided enough members of the orthonormal set can be generated to form a finite series capable of closely approximating the boundary conditions.
In conclusion we can say that with the aid of the computer a generated set of orthonormal functions might be used to solve soil physics problems which prior to the present time have not been solved because of the lengthy computations involved. To demonstrate this technique we have solved theoretically two steady state drainage problems which are important for efficient use of poorly drained agricultural lands.
LITERATURE CITED


Kirkham, Don. ca. 1966. Steady state theories for drainage. [To be published in American Society of Civil Engineers Proceedings of Irrigation and Drainage Division.]


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APPENDIX I: DERIVATION OF ORTHONORMALIZATION FORMULA

Before we derive our orthonormalization formula we should recall the definitions for orthonormal and orthogonal functions. A set of functions \( \lambda_n(x) \) orthonormal on the interval \((0,\pi)\) is defined by Equation 4. A set of functions \( \lambda'_n(x) \) orthogonal on \((0,\pi)\) is defined as one with the following property:

\[
\int_0^\pi [\lambda'_m(x)\lambda'_n(x)]dx \begin{cases} = 0; & m \neq n \\ \neq 0; & m = n \end{cases} \quad (m,n = 0,1,2, \ldots) \quad (136)
\]

We now derive a general formula that can be used to form an orthonormal set of functions. To develop this formula we will use the Gram-Schmidt method as described on page 415 of Agnew (1960) and on page 51 of Courant and Hilbert (1953). We will also use their notation.

Suppose we have a set of linearly independent functions \( u_m(x), (m = 0,1,2, \ldots) \). We define the notation \((u_m, u_n)\), \((m,n = 0,1,2, \ldots)\), by

\[
(u_m, u_n) = \int_0^\pi [u_m(x) u_n(x)] dx . \quad (137)
\]

Further we use the notation \((u_m, u_m)^{1/2}\) defined by

\[
(u_m, u_m)^{1/2} = \left\{ \int_0^\pi [u_m(x) u_m(x)] dx \right\}^{1/2} . \quad (138)
\]

From the set \( u_m \) we develop now a set of orthogonal functions \( \lambda'_m \). Each orthonormal function \( \lambda_m \) of the set will be formed by dividing each orthogonal member \( \lambda'_m \) by its normalizing factor \((\lambda'_m, \lambda'_m)^{1/2}\).
We take the first member of the orthogonal set to be $u_0$. Upon dividing $u_0$ by its normalizing factor $(u_0, u_0)^{1/2}$ we have for the first member of the orthonormal set $\lambda_0$ the expression

$$\lambda_0 = \frac{u_0}{(u_0, u_0)^{1/2}}. \quad (139)$$

We see in Equation 139 that the integral $(\lambda_0, \lambda_0)$ is equal to one, as it should be.

We now define a new function $\lambda_1$ by

$$\lambda_1 = c_{10}\lambda_0 + c_{11}u_1 \quad (140)$$

where $c_{10}$ and $c_{11}$ are constants to be determined which make $\lambda_1$ orthogonal to $\lambda_0$, i.e., $(\lambda_1, \lambda_0) = 0$.

By our orthogonal condition, Equation 136, we must have from Equation 140 the equation

$$(\lambda_1, \lambda_0) = 0 = c_{10}(\lambda_0, \lambda_0) + c_{11}(u_1, \lambda_0)$$

or because $(\lambda_0, \lambda_0) = 1$ we have

$$c_{10} = -c_{11}(u_1, \lambda_0). \quad (141)$$

Putting the right hand side of Equation 141 in for $c_{10}$ of Equation 140 we have

$$\lambda_1 = -c_{11}(u_1, \lambda_0)\lambda_0 + c_{11}u_1 \quad (142)$$

or

$$\lambda_1 = c_{11}[u_1 - \lambda_0(u_1, \lambda_0)]. \quad (143)$$

To obtain the orthonormal member $\lambda_1$ we divide $\lambda_1$ by its normalizing factor $(\lambda_1, \lambda_1)^{1/2}$ which yields
\[ \lambda_1 = \frac{\lambda_1^1}{(\lambda_1^1, \lambda_1^1)^{1/2}} \]  (144)

which is the first order member of our orthonormal set \( \lambda_0, \lambda_1, \lambda_2, \ldots \).

We use Equation 142 in the denominator of Equation 144 and find

\[ (\lambda_1^1, \lambda_1^1)^{1/2} = c_{11}[(u_1^1, u_1^1) - 2(u_1^1, \lambda_0^0)(u_1^1, \lambda_0^0) + (\lambda_0^0, \lambda_0^0)(u_1^1, \lambda_0^0)^2]^{1/2}. \]

Noting that \( (\lambda_0, \lambda_0) \) equals 1 and \( 2(u_1^1, \lambda_0^0)(u_1^1, \lambda_0^0) \) equals \( 2(u_1^1, \lambda_0^0)^2 \), the above expression can be reduced to

\[ (\lambda_1^1, \lambda_1^1)^{1/2} = c_{11}[(u_1^1, u_1^1) - (u_1^1, \lambda_0^0)^2]^{1/2}. \]  (145)

Replacing \( \lambda_1^1 \) and \( (\lambda_1^1, \lambda_1^1)^{1/2} \) in Equation 144 by the right hand sides of Equation 143 and 145 respectively we find the expression

\[ \lambda_1 = \frac{c_{11}[(u_1^1 - \lambda_0^0)(u_1^1, \lambda_0^0)]}{c_{11}[(u_1^1, u_1^1) - (u_1^1, \lambda_0^0)^2]^{1/2}} \]

or

\[ \lambda_1 = \frac{u_1^1 - \lambda_0^0(u_1^1, \lambda_0^0)}{[(u_1^1, u_1^1) - (u_1^1, \lambda_0^0)^2]^{1/2}}. \]  (146)

To develop a third function \( \lambda_2^1 \) orthogonal to \( \lambda_0 \) and \( \lambda_1 \) we begin by letting \( \lambda_2^1 \) be given by

\[ \lambda_2^1 = c_{20}^0 \lambda_0 + c_{21}^1 \lambda_1 + c_{22}^2 u_2, \]  (147)

where \( c_{20}, c_{21}, \) and \( c_{22} \) are constants which we shall choose to make \( \lambda_2^1 \) orthogonal to \( \lambda_0 \) and \( \lambda_1 \). Using the property of orthonormality for \( \lambda_0 \) and \( \lambda_1 \) we can write
\( (\lambda'_2, \lambda'_0) = c_{20}(\lambda_0, \lambda_0) + c_{21}(\lambda_1, \lambda_0) + c_{22}(u_2, \lambda_0) \) \hspace{1cm} (148)

or because \((\lambda_0, \lambda_0) = 1\) and \((\lambda_0, \lambda_1) = 0\) we find from Equation 148

\[ (\lambda'_2, \lambda'_0) = c_{20} + c_{22}(u_2, \lambda_0) \] \hspace{1cm} (149)

We also operate on Equation 149 to find

\[ (\lambda'_2, \lambda'_1) = c_{20}(\lambda_0, \lambda_1) + c_{21}(\lambda_1, \lambda_1) + c_{22}(u_2, \lambda_1) \]

or

\[ (\lambda'_2, \lambda'_1) = 0 + c_{21} + c_{22}(u_2, \lambda_1) \] \hspace{1cm} (150)

From the fact that we must have \(c_{20}, c_{21}, \) and \(c_{22}\) such that \(\lambda'_2\) is orthogonal to \(\lambda_0\) and \(\lambda_1\); and since we know that \((\lambda_2, \lambda_0) = 0\) and \((\lambda_2, \lambda_1) = 0\), we see that Equations 149 and 150 can be written as

\[ c_{20} = -c_{22}(u_2, \lambda_0) \] \hspace{1cm} (151)

and

\[ c_{21} = -c_{22}(u_2, \lambda_1) \] \hspace{1cm} (152)

respectively.

We place the expressions for \(c_{20}\) and \(c_{21}\) from Equations 151 and 152 into Equation 147, we rearrange, and factor out \(c_{22}\), and find that Equation 147 can be written as

\[ \lambda'_2 = c_{22}[u_2 - \lambda_0(u_2, \lambda_0) - \lambda_1(u_2, \lambda_1)] \] \hspace{1cm} (153)

By definition the orthonormal function \(\lambda'_2\) is

\[ \lambda'_2 = \frac{\lambda'_2}{(\lambda'_2, \lambda'_2)^{1/2}} \] \hspace{1cm} (154)
Using Equation 153 in the denominator of Equation 154 we find

\[ (\lambda_2^1, \lambda_2^1)^{1/2} = \{c_2 \int_0^s [u_2 - \lambda_0(u_2, \lambda_0) - \lambda_1(u_2, \lambda_1)]^2 \, dx \}^{1/2} \]

or

\[ (\lambda_2^1, \lambda_2^1)^{1/2} = c_2 \frac{\int_0^s [u_2 u_2 + \lambda_0 \lambda_0 (u_2, \lambda_0)^2 + \lambda_1 \lambda_1 (u_2, \lambda_1)^2 \]

\[ - 2u_2 \lambda_0 (u_2, \lambda_0) + 2 \lambda_0 \lambda_1 (u_2, \lambda_0)(u_2, \lambda_1) -
\]

\[ 2u_2 \lambda_1 (u_2, \lambda_1) \, dx \}^{1/2} \]

or using our notation for the indicated integrations

\[ (\lambda_2^1, \lambda_2^1)^{1/2} = c_2 [(u_2, u_2) + (u_2, \lambda_0)^2 + (u_2, \lambda_1)^2 - 2(u_2, \lambda_0)^2 \]

\[ + 0 - 2(u_2, \lambda_1)^2 \}^{1/2} \]

or

\[ (\lambda_2^1, \lambda_2^1)^{1/2} = c_2 [(u_2, u_2) - (u_2, \lambda_0)^2 - (u_2, \lambda_1)^2]^{1/2}. \] (155)

We place the right hand side of Equations 153 and 155 in the right hand side of Equation 154 and find

\[ \lambda_2 = \frac{c_2 \{u_2 - \lambda_0(u_2, \lambda_0) - \lambda_1(u_2, \lambda_1)\}}{c_2 [(u_2, u_2) - (u_2, \lambda_0)^2 - (u_2, \lambda_1)^2]^{1/2}} \]

or

\[ \lambda_2 = \frac{u_2 - \lambda_0(u_2, \lambda_0) - \lambda_1(u_2, \lambda_1)}{[(u_2, u_2) - (u_2, \lambda_0)^2 - (u_2, \lambda_1)^2]^{1/2}}. \] (156)

Carrying through a similar process we see that \( \lambda_3 \) is given by

\[ \lambda_3 = \frac{u_3 - \lambda_0(u_3, \lambda_0) - \lambda_1(u_3, \lambda_1) - \lambda_2(u_3, \lambda_2)}{[(u_3, u_3) - (u_3, \lambda_0)^2 - (u_3, \lambda_1)^2 - (u_3, \lambda_2)^2]^{1/2}}. \] (157)
From Equations 138, 146, 156, and 157 we see that a general formula for $\lambda_m$ can be given by

$$\lambda_m = \frac{u_m - \sum_{n=0}^{m-1} \lambda_n (u_m, \lambda_n)}{\left[ (u_m^2 - \sum_{n=0}^{m-1} (u_m, \lambda_n)^2 \right]^{1/2}}; \quad (m = 0, 1, 2, \ldots).$$

(158)

This completes Appendix I.
APPENDIX II: INTEGRATION FORMULAS FOR THE
INTEGRALS \((u_m, u_n)\) AND \((u_m, u_m)\)

From our definition of \(u_m\), Equation 46, we see that
\((u_m, u_n)\) is defined by
\[
(u_m, u_n) = \int_0^s \left[ \frac{\cosh \frac{m\pi(a+cx)}{s}}{\cosh \frac{n\pi(a+cx)}{s}} \right] \cos \frac{m\pi x}{s} \cos \frac{n\pi x}{s} \, dx. \tag{159}
\]

If we use formula 654.2 of Dwight and define by
\[
G_{mn} = \frac{1}{\cosh \frac{m\pi b}{s} \cosh \frac{n\pi b}{s}} \tag{160}
\]
we can write Equation 159 as
\[
(u_n, u_n) = \frac{G_{mn}}{4} \int_0^s \left[ e^{\frac{m\pi(a+cx)}{s}} + e^{-\frac{m\pi(a+cx)}{s}} \right] \right. \\
\left. \left[ e^{\frac{n\pi(a+cx)}{s}} + e^{-\frac{n\pi(a+cx)}{s}} \right] \cos \frac{m\pi x}{s} \cos \frac{n\pi x}{s} \, dx. \tag{161}
\]

Multiplying out the terms within the bracket and multiplying through by \(\frac{4}{G_{mn}}\) the right hand side of Equation 161 becomes
\[
\int_0^s \left[ e^{(m+n)\frac{\pi(a+cx)}{s}} + e^{-(m-n)\frac{\pi(a+cx)}{s}} + e^{(m-n)\frac{\pi(a+cx)}{s}} \right] \cos \frac{m\pi x}{s} \cos \frac{n\pi x}{s} \, dx
\]

or rearranging the exponential terms, separating the integral of the sum into the sum of the integrals, and placing any constant exponent terms before the integrals we have
\[
\left[ e^{(m+n)\frac{\pi a}{s}} \int_0^s e^{(m+n)\frac{\pi c x}{s}} \cos \frac{m\pi x}{s} \cos \frac{n\pi x}{s} \, dx \right. \tag{162}
\]
\[
+ e^{-(m+n)\frac{\pi a}{s}} \int_0^s e^{-(m+n)\frac{\pi c x}{s}} \cos \frac{m\pi x}{s} \cos \frac{n\pi x}{s} \, dx \tag{163}
\]
\[
+ e^{(m-n)\frac{\pi a}{s}} \int_0^s e^{(m-n)\frac{\pi c x}{s}} \cos \frac{m\pi x}{s} \cos \frac{n\pi x}{s} \, dx \tag{164}
\]
To evaluate the integrals in Expressions 162, 163, 164, and 165 we use formula 315 page 83 of Burington (1955) which is

\[
\int e^{ax} \cos bx \cos cx \, dx =
\frac{e^{ax}[(b-c) \sin (b-c) x + a \cos (b-c)x]}{2[a^2 + (b-c)^2]}
+ \frac{e^{ax}[(b+c) \sin (b+c) x + a \cos (b+c)x]}{2[a^2 + (b+c)^2]}.
\]  

Upon comparing the integrals in Expressions 162, 163, 164, and 165 with Burington's formula we see that his \(b\) and \(c\) are our \(m\pi/s\) and \(n\pi/s\) respectively. The terms \(\sin(b-c)x\) and \(\sin(b+c)x\) of Equation 166 now become \(\sin(m-n\pi x/s)\) and \(\sin(m+n\pi x/s)\) which are zero at the limits of integration \(x = 0\) and \(x = s\). Therefore, in our particular case, the formula given in Equation 166 can be reduced to

\[
\int e^{ax} \cos bx \cos cx \, dx = \frac{ae^{ax}}{2} \left[ \frac{\cos(b-c)x}{a^2+(b-c)^2} + \frac{\cos(b+c)x}{a^2+(b+c)^2} \right].
\] 

Comparing this formula with Expressions 162, 163, 164, and 165 we see that \((b-c)\) is \((m-n)\pi/s\) and \((b+c)\) is \((m+n)\pi/s\) for all four integrals. Further comparison shows that the only symbol we need to change in Equation 167 is \(a\). For the integrals in Expressions 162, 163, 164, and 165 the symbol \(a\)
in Burlington's formula is given by $a = (m+n)\pi c/s$, $a = -(m+n)\pi c/s$, $a = (m-n)\pi c/s$ and $a = -(m-n)\pi c/s$ respectively.

Using Equation 167 and returning to the integral in Expression 162 we see the formula for the integral is

$$\int_{\theta}^{s} e^{(m+n)\pi c x/s} \cos m\pi x/s \cos n\pi x/s \, dx = \frac{(m+n)\pi c e^{(m+n)\pi c x/s}}{2s}$$

$$\left[ \frac{\cos(m-n)\pi x/s}{(m+n)^2 \pi c^2 + (m-n)^2 \pi} + \frac{\cos(m+n)\pi x/s}{(m+n)^2 \pi c^2 + (m+n)^2 \pi} \right]_{0}^{s}.$$  

(168)

After we factor out $s^2/\pi^2$, the right hand side of Expression 168 becomes

$$\frac{(m+n)sc e^{(m+n)\pi c x/s}}{2\pi} \left[ \frac{\cos(m-n)\pi x/s}{(m+n)^2 c^2 + (m-n)^2} + \frac{\cos(m+n)\pi x/s}{(m+n)^2 c^2 + (m+n)^2} \right]_{0}^{s}.$$  

(169)

If $(m-n)$ and $(m+n)$ are odd we will get a different value for Expression 169 than if $(m-n)$ and $(m+n)$ are even. If $(m-n)$ is odd so is $(m+n)$ and if $(m-n)$ is even then so is $(m+n)$.

For $(m+n)$ odd Expression 169 becomes

$$-\frac{(m+n)sc e^{(m+n)\pi c}}{2\pi} \left[ \frac{1}{(m+n)^2 c^2 + (m-n)^2} + \frac{1}{(m+n)^2 c^2 + (m+n)^2} \right] -$$

$$\frac{(m+n)sc}{2\pi} \left[ \frac{1}{(m+n)^2 c^2 + (m-n)^2} + \frac{1}{(m+n)^2 c^2 + (m+n)^2} \right].$$

or

$$-\frac{(m+n)sc e^{(m+n)\pi c} + 1}{(m+n)^2 c^2 + (m-n)^2} + \frac{e^{(m+n)\pi c} + 1}{(m+n)^2 c^2 + (m+n)^2}.$$  

(170)
For \((m+n)\) even, Expression 169 becomes
\[
\frac{(m+n)\text{sc}}{2\pi} \left[ e^{(m+n)\pi c - l} \frac{e^{(m+n)\pi c - l}}{(m+n)^2 c^2 + (m-n)^2} + \frac{e^{(m+n)\pi c - l}}{(m+n)^2 c^2 + (m-n)^2} \right] \cdot (171)
\]

Similarly if we use Equation 167, the integral in Expression 163 becomes
\[
\frac{(m+n)\text{sc}}{2\pi} \left[ e^{-(m+n)\pi c + l} \frac{e^{-(m+n)\pi c + l}}{(m+n)^2 c^2 + (m-n)^2} + \frac{e^{-(m+n)\pi c + l}}{(m+n)^2 c^2 + (m-n)^2} \right] (172)
\]

for \((m+n)\) odd and
\[
\frac{(m+n)\text{sc}}{2\pi} \left[ e^{-(m+n)\pi c - l} \frac{e^{-(m+n)\pi c - l}}{(m+n)^2 c^2 + (m-n)^2} + \frac{e^{-(m+n)\pi c - l}}{(m+n)^2 c^2 + (m-n)^2} \right] (173)
\]

for \((m+n)\) even. The integral in Expression 164 becomes
\[
\frac{(m-n)\text{sc}}{2\pi} \left[ e^{(m-n)\pi c + l} \frac{e^{(m-n)\pi c + l}}{(m-n)^2 c^2 + (m-n)^2} + \frac{e^{(m-n)\pi c + l}}{(m-n)^2 c^2 + (m-n)^2} \right] (174)
\]

for \((m+n)\) odd and
\[
\frac{(m-n)\text{sc}}{2\pi} \left[ e^{(m-n)\pi c - l} \frac{e^{(m-n)\pi c - l}}{(m-n)^2 c^2 + (m-n)^2} + \frac{e^{(m-n)\pi c - l}}{(m-n)^2 c^2 + (m-n)^2} \right] (175)
\]

for \((m+n)\) even. The integral in Expression 165 becomes
\[
\frac{(m-n)\text{sc}}{2\pi} \left[ e^{-(m-n)\pi c + l} \frac{e^{-(m-n)\pi c + l}}{(m-n)^2 c^2 + (m-n)^2} + \frac{e^{-(m-n)\pi c + l}}{(m-n)^2 c^2 + (m-n)^2} \right] (176)
\]

for \((m+n)\) odd and
\[
\frac{(m-n)\text{sc}}{2\pi} \left[ e^{-(m-n)\pi c - l} \frac{e^{-(m-n)\pi c - l}}{(m-n)^2 c^2 + (m-n)^2} + \frac{e^{-(m-n)\pi c - l}}{(m-n)^2 c^2 + (m-n)^2} \right] (177)
\]

for \((m+n)\) even.

Upon replacing the integrals in Expressions 162, 163, 164, and 165 by Expressions 170, 172, 174, and 176 we have for \((m+n)\) and \((m-n)\) odd, the result
\[
\left( u_m, u_n \right) = \frac{G_{mn}}{4} \left( e^{(m+n)\pi a/s} \frac{(m+n)\text{sc}}{2\pi} \left[ e^{(m+n)\pi c + l} \frac{e^{(m+n)\pi c + l}}{(m+n)^2 c^2 + (m-n)^2} + \right] \right)
\]
\[
\frac{e^{(m+n)\pi a/2}}{(m+n)^2(c^2+1)} + \frac{e^{-(m+n)\pi a/2} - e^{(m+n)\pi a/2}}{(m+n)^2(c^2+1)} + \frac{e^{-(m-n)\pi a/2}}{(m-n)^2(c^2+1)} + \frac{e^{(m-n)\pi a/2}}{(m-n)^2(c^2+1)}.
\]

If we factor out \(sc/2\pi\), multiply through by the constant exponent terms, and collect terms with like denominators, the right hand side of Equation 178 becomes

\[
\frac{\partial G_{mn}}{\partial \pi} \frac{sc}{2\pi} \left[ -(m+n) \frac{e^{(m+n)\pi(a/s+c)} + e^{(m+n)\pi a/s} - e^{-(m+n)\pi(a/s+c)} - e^{-(m+n)\pi a/s}}{(m+n)^2c^2 + (m-n)^2} \right.
\]

\[
\left. - (m+n) \frac{e^{(m+n)\pi(a/s+c)} + e^{(m+n)\pi a/s} - e^{-(m+n)\pi(a/s+c)} - e^{-(m+n)\pi a/s}}{(m+n)^2(c^2+1)} \right] . \tag{179}
\]

Upon interchanging the second and third terms in each numerator of Expression 179, putting the first and second fractions over common denominators, and putting the third and fourth fraction over common denominators Expression 179 can be written as
Upon factoring out $1/(c^2+1)$ and combining the first and second terms and combining the third and fourth terms

Expression 180 can be written as

$$\frac{G_{mnsc}}{8\pi} \left\{ -\frac{(m+n)^2(c^2+1)[e^{(m+n)\pi(a/s+c)} - e^{-(m+n)\pi(a/s+c)}] + e^{(m+n)\pi a/s} - e^{-(m+n)\pi a/s}}{(m+n)(c^2 + 1)[(m+n)^2c^2 + (m-n)^2]} \right\} .$$

(180)
or from 654.1 of Dwight (1962) and our definition of \( G_{mn} \), 

\[ (u_m, u_n) \] for \((m+n)\) and \((m-n)\) odd is given by

\[
(u_m, u_n) = -\frac{sc}{4\pi(c^2+1)\cosh m\pi b/s \cosh n\pi b/s}\]

\[
[\frac{2(m+n)^2c^2+(m+n)^2+(m-n)^2}{(m+n)[(m+n)^2c^2+(m-n)^2]}\sinh(m+n)\pi(a/s+c) + \\
\sinh(m+n)\pi a/s] + \frac{2(m-n)^2c^2+(m+n)^2+(m-n)^2}{(m-n)[(m-n)^2c^2+(m+n)^2]} \\
[\sinh(m-n)\pi(a/s+c) - \sinh(m-n)\pi a/s].
\]

(182)

In a similar manner we can obtain an expression for 

\[ (u_m, u_n) \] for \((m+n)\) and \((m-n)\) even if we replace the integrals 
in Expressions 162, 163, 164, and 165 by Expressions 171, 
173, 175, and 177 respectively. The result is

\[
(u_m, u_n) = -\frac{sc}{4\pi(c^2+1)\cosh m\pi b/s \cosh n\pi b/s}\]

\[
[\frac{2(m+n)^2c^2+(m+n)^2+(m-n)^2}{(m+n)[(m+n)^2c^2+(m-n)^2]}\sinh(m+n)\pi(a/s+c) - \\
\sinh(m+n)\pi a/s] + \frac{2(m-n)^2c^2+(m+n)^2+(m-n)^2}{(m-n)[(m-n)^2c^2+(m+n)^2]} \\
[\sinh(m-n)\pi(a/s+c) - \sinh(m-n)\pi a/s].
\]

(183)

We have completed our evaluation of \((u_m, u_n)\) for \(m\) not 
equal to \(n\). We now consider \(m\) equal to \(n\).

To find an integration formula for \((u_m, u_m)\) we use formula 
654.2 of Dwight (1962) and write
\[(u_m^2 u_m) = \frac{G_m^2}{4} \int_0^s (e^{m(a+cx)/s} + e^{-m(a+cx)/s})^2 \cos^2 m \pi x \, dx \quad (184)\]

where \(G_m^2\) is given by
\[G_m^2 = \frac{1}{(\cosh m \pi b/s)^2} \quad (185)\]

Upon squaring the term within the parenthesis in Equation 184 and changing the integration of the sum to the sum of the integration we have the right hand side of given by
\[
\frac{G_m^2}{4} \left\{ \int_0^s e^{2m \pi (a+cx)/s} \cos^2 m \pi x/s \, dx \right. \\
+ \int_0^s 2 \cos^2 m \pi x/s \, dx \\
+ \int_0^s e^{-2m \pi (a+cx)/s} \cos^2 m \pi x/s \, dx \} \\
\quad (186)
\]

Upon interchanging the second and third integrals in Equation 186 and moving the constants outside of each integral, we find
\[
\frac{G_m^2}{4} \left\{ e^{2m \pi a/s} \int_0^s e^{2m \pi cx/s} \cos^2 m \pi x/s \, dx \right. \\
+ e^{-2m \pi a/s} \int_0^s e^{-2m \pi cx/s} \cos^2 m \pi x/s \, dx \\
+ 2 \int_0^s \cos^2 m \pi x/s \, dx \} \\
\quad (187)
\]

\[
+ e^{-2m \pi a/s} \int_0^s e^{-2m \pi cx/s} \cos^2 m \pi x/s \, dx \\
+ 2 \int_0^s \cos^2 m \pi x/s \, dx \} \\
\quad (188)
\]

\[
+ 2 \int_0^s \cos^2 m \pi x/s \, dx \} \\
\quad (189)
\]

To find the integrals in Expressions 187 and 188 we use Burington's (1955) formula 323, page 84, which for a cosine squared in the integral is given by
\[ \int e^{ax} \cos^2 bx \, dx = \frac{e^{ax} \cos bx(a \cos bx + 2b \sin bx)}{a^2 + 4b^2} + \frac{2b^2 e^{ax}}{a(b^2 + 4b^2)}. \]  

(190)

The symbol \( b \) in Burlington's (1955) formula is \( m \pi / s \). Because \( \sin m \pi x / s \) is zero at our limits of integration, \( x = 0 \) and \( x = s \), Equation 190 becomes

\[ \int e^{ax} \cos^2 bx \, dx = \frac{e^{ax}}{a^2 + 4b^2} \left[ a \cos^2 bx + \frac{2b^2}{a} \right]. \]  

(191)

We compare Equation 191 with the integral in Expression 187 and find the expression

\[ \frac{e^{2m \pi cx/s}}{4 \frac{m^2 \pi^2}{s^2} + 4 \frac{m^2 \pi^2}{s^2}} \left[ 2m \pi x / s \cos^2 m \pi x / s + \frac{2}{s} \frac{m^2 \pi}{s} \right] \bigg|_0^s \]

or simplifying we have

\[ \frac{s e^{2m \pi cx/s}}{4m \pi (c^2 + 1)} \left[ 2c \cos^2 m \pi x / s + \frac{1}{c} \right] \bigg|_0^s. \]  

(192)

After applying the limits of integration in Expression 192 we obtain

\[ \frac{s e^{2m \pi c}}{4m \pi (c^2 + 1)} \left[ 2c + \frac{1}{c} \right] - \frac{s}{4m \pi (c^2 + 1)} \left[ 2c + \frac{1}{c} \right] \]

or upon combining \( 2c + \frac{1}{c} \) into \((2c^2 + 1)/c\) and factoring this term out along with \( s/\left[4m \pi (c^2 + 1)\right] \) we have

\[ \frac{s(2c^2 + 1)}{4m \pi (c^2 + 1)} \left( e^{2m \pi c} - 1 \right). \]  

(193)
Similarly the integral in Expression 188 is found to be

\[ \frac{s(2c^2 + 1)}{4m\pi c(c^2 + 1)} (1 - e^{-2m\pi c}) . \]  

(194)

To find the integral in Expression 189 we use formula 202 of Burington (1955) which is

\[ \int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} . \]  

(195)

The symbol \( a \) in this formula is \( m\pi/s \) of Expression 189 and \( \sin 2m\pi x/s \) is zero at both limits of integration \( x = 0 \) and \( x = s \). Therefore the integral in Expression 189 is

\[ \int_0^s \cos^2 m\pi x/s \, dx = s/2 . \]  

(196)

Upon replacing the integrals in Expressions 187, 188, and 189 by Expressions 193 and 194 and the right hand side of Equation 196 we can write \((u_m', u_n')\) as

\[
(u_m', u_m) = \frac{G_m^2}{4} \left\{ \frac{e^{2m\pi a/s} s(2c^2 + 1)}{4m\pi c(c^2 + 1)} (e^{2m\pi c} - 1) + \frac{e^{-2m\pi a/s} s(2c^2 + 1)(1 - e^{-2m\pi c}) + s}{4m\pi c(c^2 + 1)} \right\} .
\]  

(197)

Multiplying the first and second terms through by the exponent terms, combining the first and second terms, and factoring out \( s/2m\pi \) we may write Equation 147 as

\[
(u_m', u_m) = \frac{s G_m^2}{8m\pi} \left\{ \frac{(2c^2 + 1)}{2c(c^2 + 1)} [e^{2m(a/s + c)\pi} - e^{2m\pi a/s} + e^{-2m\pi a/s} - e^{-2m\pi a(s+c)}] + 2m\pi \right\} .
\]  

(198)
Upon recalling our definitions of \( Q^2_m \) and the hyperbolic cosine, we can write \((u_m,u_m)\) as

\[
(u_m,u_m) = \frac{s}{8m\pi (\cosh m\pi b/s)^2} \left\{ \frac{2c^2+1}{c(c^2+1)} \left[ \sinh 2m\pi (a/s+c) \right. \right.
\]

\[
- \left. \sinh 2m\pi a/s \right] + 2m\pi \right\}.
\]  

(199)

If \( m \) is zero \((u_m,u_m)\) becomes

\[
(u_m,u_m) = \int_0^s \left( \frac{\cosh 0}{\cosh g} \right)^2 \cos^2 g \, dx
\]

or

\[
(u_m,u_m) = \int_0^s dx
\]

or

\[
(u_m,u_m) = s.
\]

(200)

This completes Appendix II.
Using our definition of $u_m$ from Equation 46 we see that

$$\int_0^s u_m dx$$

can be written as

$$\int_0^s u_m dx = \int_0^s \frac{\cosh m\pi(a+cx)/s}{\cosh m\pi b/s} \cos m\pi x/s \, dx \quad (201)$$

If we define $G_m$ by

$$G_m = \frac{1}{[\cosh m\pi b/s]} \quad (202)$$

and use formula 654.2 of Dwight (1962), the right hand side of Equation 201 becomes

$$\frac{G_m}{2} \int_0^s \left( e^{m\pi (a+cx)/s} + e^{-m\pi (a+cx)/s} \right) \cos m\pi x/s \, dx$$

or changing the integral of the sum to the sum of the integrals and placing the constant exponent term outside of the integral sign we have

$$\frac{G_m}{2} \left[ e^{m\pi a/s} \int_0^s e^{m\pi cx/s} \cos m\pi x/s \, dx \right] \quad (203)$$

$$+ e^{-m\pi a/s} \int_0^s e^{-m\pi cx/s} \cos m\pi x/s \, dx \right]. \quad (204)$$

To evaluate the integrals in Expressions 203 and 204 we use formula 314 page 83 of Burington (1955) which is

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx) \quad (205)$$
In both integrals of Expressions 203 and 204 we see that

\( b \) of Equation 205 is for our case \( \frac{m\pi}{s} \) so that \( \sin bx \) of

Equation 205 is given by \( \sin \frac{m\pi x}{s} \) which is zero at \( x = 0 \) and

\( x = s \), our limits of integration. Therefore in our particular

case Equation 205 becomes

\[
\int e^{ax} \cos bx \, dx = \frac{a e^{ax} \cos bx}{a^2 + b^2}.
\] (206)

Using Equation 206 by letting \( a \) equal \( \frac{m\pi c}{s} \) and \( b \) equal

\( \frac{m\pi}{s} \) we can write the integral in Expression 203 as

\[
\int \frac{s}{0} e^{\frac{m\pi cx}{s}} \cos \frac{m\pi x}{s} \, dx =
\]

\[
\frac{m\pi c}{s} e^{\frac{m\pi cx}{s}} \left[ \frac{m\pi c^2}{s^2} + \frac{n\pi c^2}{s^2} \right] \cos \frac{m\pi x}{s} \bigg|^{s}_{0}
\] (207)

or simplifying we have the right hand side of Equation 207
given by

\[
\frac{sc}{m\pi(c^2+1)} e^{\frac{m\pi cx}{s}} \cos \frac{m\pi x}{s} \bigg|^{s}_{0}
\] (208)

Upon using the limits of integration for \( m \) odd Expression

208 becomes

\[
- \frac{sc}{m\pi(c^2+1)} \frac{m\pi c}{m\pi(c^2+1)} - \frac{sc}{m\pi(c^2+1)}
\]
or

\[
\frac{sc}{m\pi(c^2+1)} \left( e^{\frac{m\pi c}{s}} + 1 \right)
\] (209)

For \( m \) even, Expression 208 becomes
\[ \frac{sc \ e^{m \pi c}}{m \pi (c^2 + 1)} - \frac{sc}{m \pi (c^2 + 1)} \]

or

\[ \frac{sc \ (e^{m \pi c} - 1)}{m \pi (c^2 + 1)} \] \quad (210)

Similarly for \( m \) odd the integral in Expression 204 becomes

\[ \frac{sc(e^{-m \pi c} + 1)}{m \pi (c^2 + 1)} \] \quad (211)

and for \( m \) even it becomes

\[ -\frac{sc(e^{-m \pi c} - 1)}{m \pi (c^2 + 1)} \] \quad (212)

Upon substituting Expressions 209 and 211 for the integrals in Expressions 203 and 204 we have for \( m \) odd

\[ \int_0^s u_m \, dx = \frac{G_m}{2} \left\{ -\frac{e^{m \pi a/s} \ sc}{m \pi (c^2 + 1)} (e^{m \pi c} + 1) \right. \\
+ \frac{e^{-m \pi a/s} \ sc}{m \pi (c^2 + 1)} (e^{-m \pi c} + 1) \right\}. \] \quad (213)

Upon factoring out \( sc/[m \pi (c^2 + 1)] \) and multiplying each term through by their respective terms with constant exponents the right hand side of Equation 213 becomes

\[ \frac{G_m \ sc}{2 \ m \pi (c^2 + 1)} \left\{ -e^{m \pi (a/s+\pi)} - e^{m \pi a/s} + e^{-m \pi (a/s+\pi)} + e^{-m \pi a/s} \right\} \]

or upon rearranging the second and third terms and multiplying through by minus one we have

\[ -\frac{G_m \ sc}{2 \ m \pi (c^2 + 1)} \left\{ e^{m \pi (a/s+\pi)} - e^{m \pi (a/s+\pi)} + e^{m \pi a/s} - e^{-m \pi a/s} \right\}. \]
Upon recalling the definition of a hyperbolic sine and $G_m$ we see that $\int_0^s u_m dx$ for $m$ odd is given by

$$\int_0^s u_m dx = -\frac{sc}{m(s^2+1)cosh m\eta b/s} [\sinh m\eta (a/s+c)+\sinh m\eta a/s].$$

(214)

In a similar manner if we replace the integrals in Expressions 203 and 204 by Expressions 210 and 212 we have for $m$ even that $\int_0^s u_m dx$ is given by

$$\int_0^s u_m dx = \frac{sc}{m(s^2+1)cosh m\eta b/s} [\sinh m\eta (a/s+c)-\sinh m\eta a/s].$$

(215)

To find $\int_0^s x u_m dx$ we write it as

$$\int_0^s x u_m dx = \int_0^s x \frac{cosh m\eta (a+cx)/s}{cosh m\eta b/s} \cos m\eta x/s dx.$$  

(216)

Upon using the definition of $G_m$ given by

$$G_m = 1/[cosh m\eta b/s]$$

and formulas 654.2 of Dwight (1962), we see that the right hand side of Equation 216 is given by

$$\frac{G_m}{2} \int_0^s x (e^{m\eta (a+cx)/s} + e^{-m\eta (a+cx)/s}) \cos m\eta x/s dx$$

or upon changing the integral of the sum to the sum of the integrals and factoring out any terms with constant exponents, the above expression can be written as

$$\frac{G_m}{2} e^{m\eta a/s} \int_0^s x e^{m\eta cx/s} \cos m\eta x/s dx$$

$$+ e^{-m\eta a/s} \int_0^s x e^{-m\eta cx/s} \cos m\eta x/s dx.$$  

(217)  

(218)
To evaluate the integrals in Expressions 217 and 218 we use formula 322 page 84 of Burlington (1955) which is

\[
\int x e^{ax} \cos bx \, dx = \frac{x e^{ax}}{a^2 + b^2} \left( a \cos bx + b \sin bx \right)
\]

\[
= \frac{e^{ax}}{(a^2 + b^2)^2} \left( (a^2 - b^2) \cos bx + 2ab \sin bx \right).
\]

(219)

In our case we see the \( b \) in the above formula is \( m \pi /s \) or \( \sin bx \) is \( \sin m \pi x/s \) which is zero for our limits of integration \( x = 0 \) and \( x = s \). Equation 219 now becomes, for our case

\[
\int x e^{ax} \cos bx \, dx = \frac{x e^{ax}}{a^2 + b^2} a \cos bx - \frac{e^{ax}(a^2 - b^2) \cos bx}{(a^2 + b^2)^2}
\]

or

\[
\int x e^{ax} \cos bx \, dx = \frac{e^{ax} \cos bx}{a^2 + b^2} \left[ a - \frac{a^2 - b^2}{a^2 + b^2} \right].
\]

(220)

Using Equation 220, we see that the integral in Expression 217 becomes

\[
\frac{e^{m \pi cx/s} \cos m \pi x/s}{m^2 \pi^2 (c^2 + 1)} \left[ \frac{m \pi cx}{s} - \frac{c^2 - 1}{c^2 + 1} \right] \bigg|_0^s
\]

or

\[
\frac{s^2 e^{m \pi cx/s} \cos m \pi x/s}{m^2 \pi^2 (c^2 + 1)} \left[ \frac{m \pi cx}{s} - \frac{c^2 - 1}{c^2 + 1} \right] \bigg|_0^s
\]

or

\[
\frac{s e^{m \pi x/s} \cos m \pi x/s}{m \pi (c^2 + 1)} \left[ \frac{\pi x + s(1 - c^2)}{m \pi (c^2 + 1)} \right] \bigg|_0^s.
\]

(221)
For $m$ odd Expression 221 becomes

\[ - \frac{s \cdot e^{m \pi c}}{m \pi (c^2 + 1)} \left[ \cos + \frac{s(1 - c^2)}{m \pi (c^2 + 1)} \right] - \frac{s^2 (1 - c^2)}{m \pi (c^2 + 1)m \pi (c^2 - 1)} \]

or

\[ - \frac{s^2 (1 - c^2)}{m \pi (c^2 + 1)^2} \left( e^{m \pi c} + 1 \right) + \frac{s^2 c \cdot e^{m \pi c}}{m \pi (c^2 + 1)} \]

(222)

Similarly for $m$ even the integral in Expression 217 becomes

\[ \frac{s^2 (1 - c^2)}{m \pi (c^2 + 1)^2} \left( e^{m \pi c} - 1 \right) + \frac{s^2 c \cdot e^{m \pi c}}{m \pi (c^2 + 1)} \]

(223)

Similarly the integral in Expression 218 becomes for $m$ odd

\[ - \frac{s^2 (1 - c^2)}{m \pi (c^2 + 1)^2} \left( e^{-m \pi c} + 1 \right) + \frac{s^2 c \cdot e^{-m \pi c}}{m \pi (c^2 + 1)} \]

(224)

and becomes for $m$ even

\[ \frac{s^2 (1 - c^2)}{m \pi (c^2 + 1)^2} \left( e^{-m \pi c} - 1 \right) - \frac{s^2 c \cdot e^{-m \pi c}}{m \pi (c^2 + 1)} \]

(225)

Upon replacing the integrals in Expressions 217 and 218 with Expressions 222 and 224 we have \( \int_{\delta}^{s} x \cdot u_{m} \cdot dx \), for $m$ odd, given by

\[ \frac{s}{2} \cdot \left[ \frac{-e^{m \pi a/s} \cdot s^2 (1 - c^2)}{m \pi (c^2 + 1)^2} \left( e^{m \pi c} + 1 \right) + \frac{s^2 c \cdot e^{m \pi c}}{m \pi (c^2 + 1)} \right] - \frac{s^2 c \cdot e^{m \pi c}}{m \pi (c^2 + 1)} \]

(226)

Upon factoring out \( s^2/[m \pi (c^2 + 1)] \) and multiplying through by the terms with constant exponents, the right hand side of Equation 226 becomes
\[
\frac{s^2 G_m}{2 \pi m (c^2 + 1)} \left\{ \frac{1 - c^2}{\pi m (c^2 + 1)} \left[ e^{\pi m (a/s + c)} + e^{\pi m (a/s + c)} \right] - e^{\pi m (a/s + c)} \right\}.
\]

(227)

Upon combining the first and third term of Expression 227 we have

\[
\frac{s^2 G_m}{2 \pi m (c^2 + 1)} \left\{ \frac{1 - c^2}{\pi m (c^2 + 1)} \left[ e^{\pi m (a/s + c)} + e^{\pi m (a/s + c)} \right] - e^{\pi m (a/s + c)} \right\}.
\]

(228)

Upon recalling our definition of \( G_m \), the hyperbolic cosine and the hyperbolic sine, we see that \( \int_0^s x u_m dx \) for \( m \) odd is given by

\[
\int_0^s x u_m dx = - \frac{s^2}{\pi m (c^2 + 1) \cosh m \pi b/s \pi m (c^2 + 1)} \left[ \frac{1 - c^2}{\pi m (c^2 + 1)} \left[ \cosh m \pi (a/s + c) + \cosh m \pi a/s \right] + c \sinh m \pi (a/s + c) \right].
\]

(229)

In a similar manner if we replace the integrals in Expressions 217 and 218 with Expressions 223 and 225 we have that for \( m \) even \( \int_0^s x u_m dx \) is given by

\[
\int_0^s x u_m dx = \frac{s^2}{\pi m (c^2 + 1) \cosh m \pi b/s \pi m (c^2 + 1)} \left[ \frac{1 - c^2}{\pi m (c^2 + 1)} \left[ \cosh m \pi (a/s + c) - \cosh m \pi a/s \right] + c \sinh m \pi (a/s + c) \right].
\]

(230)

For the special case of \( m = 0 \) we have \( \int_0^s u_m dx \) given by

\[
\int_0^s u_m dx = \int_0^s \frac{\cosh 0}{\cosh 0} \cos 0 \, dx
\]
or
\[ \int_0^s u_n \, dx = \int_0^s dx = s. \quad (231) \]

For the special case of \( m = 0 \) we have \( \int_0^s x u_m \, dx \) given by
\[ \int_0^s x u_m \, dx = \int_0^s x \frac{\cosh 0}{\cosh 0} \cos 0 \, dx \quad (232) \]
or
\[ \int_0^s x u_m \, dx = \int_0^s x \, dx = \frac{s^2}{2}. \quad (233) \]

This completes Appendix III.
APPENDIX IV: IBM 360 COMPUTER PROGRAM, FLOW CHART, AND SAMPLE OUTPUT FOR THEORY OF SURFACE WATER MOVING INTO AND THROUGH WATER-SATURATED SOIL BEDDING

In this appendix we present the program, a flow chart of the program, and a sample of the output for the computer program of the first problem.

The program itself is presented first. The letter C in the left hand column of the program denotes a comment which has been added to help the reader follow the program.

A flow chart is included in this appendix to help the reader get an overall picture of the program. Flow charts of the subroutines have been omitted, but the subroutines are given in the program itself.

In the program and the flow chart the dimensioned variables are TOP(30,30), BOT(30), COEFT(30), SC(75), SS(75), BN(30), YY(15), ST(40), and SP(40). Functions determined by subroutines are UMSQ(EM), UMN(EM,EN), SINH H(X), UM(EM), COSH H(XX), XUM(EM), THETA(X,EN,Y), BETA(X,EN,Y) and EVUL(X). The "CHECK(EM)" is also a subroutine.

We do not expect anyone but experienced programmers to study the program and flow charts. Therefore, there will be no further explanation of the program or the flow chart.

The sample of the output is for our example where we used a = 2, b = 4, s = 10 and N = 20. The output shows this as "A = 2.000, B = 4.000, C = 0.200 and S = 10.000"
20 ITERATIONS".

The "FOR LAMBDA ( )" is $\lambda_m$ of Equation 14.

The "THE DENOMINATOR" is $D_m$ of Equation 28. If we round to four decimal places we have from the sample output that for $\lambda_0$, $\lambda_1$, and $\lambda_2$, $D_m$ is 10.0000, 3.2268, and 1.7029.

The "THE NUMERATOR(S)" gives the $J_{mn}$ of Equation 28. From the sample output we have for $\lambda_0$, $J_{00} = 0$; for $\lambda_1$, $J_{10} = -0.0736$; and for $\lambda_2$, $J_{20} = 0.0025$ and $J_{21} = -0.2681$.

From these results we see that for $\lambda_2$ ($m = 2$) Equation 28 becomes

$$\lambda_2 = \frac{u_2 - 0.0025u_0 + 0.2680u_1}{(1.7029)^{1/2}}$$

which agrees with Equation 25.

In the computer output, the "EN" is $E_m$ defined by Equation 47.

The "THE NUMERATOR SQUARED" is a check of $\lambda_m$ and if it is equal to "THE DENOMINATOR" then we have, as a check, the result $\int_0^s \lambda_m \lambda_m \, dx = 1$.

Our sample of computer output shows only the output for five $\lambda_m$'s.

In the second page of our sample output under the heading "CURVE FUNCTION AFTER 20 ITERATIONS", the "THE POTENTIAL CURVE" and "THE STREAM CURVE" are values of $\phi$ and $\psi/K$ for the indicated values of $x$ along $y = a + cx$.

On this second page, under the heading "FUNCTIONS FOR $y =$
1.00 AFTER 20 ITERATION", the "THE STREAM FUNCTION" and "THE POTENTIAL FUNCTION" are values of $\psi/k$ and $\phi$ for the indicated values of $x$ and for $y = 1.00$. The results for other values of $y$ are printed out similarly.
THEORY OF SURFACE WATER MOVING INTO AND THROUGH WATER-SATURATED SOIL BEDDING

DOUBLE PRECISION TOP(30,30), BOT(30), COEFT(30), PI, Y, DUMMY, DODO, DIX
DOUBLE PRECISION A, B, C, S, E1, E2, E3, SC(75)
DOUBLE PRECISION X, ZZ, SUT, SS(75), BN(30), YY(15), ST(40), SP(40), SUK
COMMON A, B, C, S, PI, BOT, TOP

99 WRITE (3,98)
98 FORMAT(1H1)
PI = 3.14159265358979300

READ INPUT PARAMETERS
A, B, C, AND S ARE PROBLEM PARAMETERS
NN IS NUMBER OF ITERATIONS DESIRED
IT - CONTROL PARAMETER
-1 COMPUTE ALL FUNCTIONS
0 COMPUTE CURVE FUNCTIONS ONLY
+1 COMPUTE STREAM AND POTENTIAL ONLY

READ (1,1) A, B, C, S, NN, IT

1 FORMAT(4F5.1, 2F15)
DO 30 I = 1, 75
SC(I) = 0.000
30 SS(I) = 0.000
IF(A - 50.) 97, 96, 96
WRITE PARAMETERS
97 WRITE (3,2) A, B, C, S, NN
2 FORMAT(19H1 RESULTS FOR A =F6.2, 6H B =F7.4, 6H C =F7.4, 9H A
AND S =F6.1, I4, 11H ITERATIONS//)

COMPUTER PROGRAM FOR PROBLEM NUMBER ONE
C COMPUTE NUMERATOR(1) AND DENOMINATOR(1) AND WRITE RESULTS
TOP(1,1) = 0.000
BOT(1) = S
NO = 0
WRITE (3,3) NO,BOT(1),TOP(1,1)
3 FORMAT(12H FOR LAMBDA(12,22H)
THE DENOMINATOR IS E24.16/21H
THE NUMERATOR(S) IS E6F16.10/(E6F16.10)
C COMPUTE BN(1) AND WRITE RESULTS
BN(1) = (A*UM(0.000) + C*UM(0.000))/BOT(1)
WRITE (3,16) BN(1)
16 FORMAT(6H EN = F18.12)
NNP=NN+1
C
C START LOOP TO COMPUTE NUMERATOR, DENOMINATOR AND BN VALUES
C FOR ITERATION 2 THROUGH NN
C
DO 10 II=2,NNP
DUMMY = 0.000
DODO = 0.000
IM = II-1
EI = II-1
DO 11 KK=1,IM
EK = KK-1
KM = KK-1
IF(KM) 4,5,4
5 DODO = 0.000
GO TO 6
4 DO 12 JJ=1,KM
EJ=JJ-1
DODO = DODO + TOP(KK,JJ)*UMN(EJ,EI)
12 CONTINUE

COMPUTER PROGRAM FOR PROBLEM NUMBER ONE (CONTINUED)
6  COEFT(KK) = UMN(EK, EI) - DODO
   DUMMY = DUMMY + \((UMN(EK, EI) - DODO)\times 2)/BOT(KK)
   DODO = 0.0DO
11  CONTINUE
   BOT(II) = (UMSQ(EI) - DUMMY)
   DO 13 LL=1, IM
   LP= LL+1
   DIX = 0.0DO
   IF(LP-IM) 7, 7, 8
7    DO 14 MM=LP, IM
   DIX = DIX + ((COEFT(MM) \times TOP(MM, LL))/BOT(MM))
14   CONTINUE
8    CONTINUE
   TOP(II, LL) = (COEFT(LL)/BOT(LL)) - DIX
13   CONTINUE
   NI = II-1
C   WRITE NUMERATOR VALUES AND DENOMINATOR FOR THIS ITERATION
   WRITE (3, 3) NI, BOT(II), (TOP(II,JI), JI=1, IM)
C   COMPUTE BN FOR THIS ITERATION AND WRITE RESULTS
   BN(II) = EVUL(EI)
   WRITE (3, 16) BN(II)
C   END OF LOOP TO GET NUMERATORS AND DENOMINATORS
   TOP(II, II) = -1.0DO
   CALL CHECK(EI)
10   CONTINUE
C   TEST CONTROL PARAMETER TO DETERMINE IF BOTH CURVES AND
C   STREAM AND POTENTIAL ARE DESIRED

COMPUTER PROGRAM FOR PROBLEM NUMBER ONE (CONTINUED)
COMPUTER PROGRAM FOR PROBLEM NUMBER ONE (CONTINUED)
WRITE (3,60) ILP
60 FORMAT(22H1 CURVE FUNCTION AFTER I4,11H ITERATIONS//)
X = 0.000

C
C LOOP TO WRITE OUT THE STREAM AND POTENTIAL CURVES
C
DO 55 I=1,KTP
WRITE (3,61) X,SS(I),SC(I)
61 FORMAT(/7H AT X =F5.2,23H THE POTENTIAL CURVE IS F20.16,26H AND THE STREAM CURVE IS F20.16/)
55 X = X + .5000

C TEST CONTROL PARAMETER TO DETERMINE IF STREAM AND POTENTIAL ARE DESIRED
C
IF(IT) 62,64,62

C *************
C
C COMPUTE STREAM AND POTENTIAL FUNCTIONS
C
C *************
C
READ NY - NUMBER OF Y'S - AND THE Y VALUES
62 READ (1,65) NY, (YY(I), I=1,NY)
65 FORMAT(I5,15F5.1)

C LOOP TO COMPUTE FUNCTIONS FOR EACH Y
DO 66 IY=1,NY
C WRITE HEADER FOR EACH Y
WRITE (3,67) YY(IY)
67 FORMAT(19H1 FUNCTIONS FOR Y =F6.2//)
Y = YY(IY)
C IF VALUE OF Y READ IN EXCEEDS 50 THEN Y WILL BE SET TO A + CX
IGO = 0

COMPUTER PROGRAM FOR PROBLEM NUMBER ONE (CONTINUED)
IF(Y- 50.) 68,69,69
69  IGO = 1
68  CONTINUE
DO 70 IB=1,40
ST(IB) = 0.000
70  SPCI8) = 0.000
C IS EQUALS NUMBER OF X'S TO BE COMPUTED
IS = S + 1.000
C LOOP TO COMPUTE FUNCTIONS AND SUM FOR NN ITERATIONS
DO 71 II=1,NNP
TOP(II,II) = -1.000
C LOOP TO COMPUTE SUMS FOR EACH X
DO 72 IXX=1,IS
X = IXX-1
IF(IGO) 73,74,73
73  Y=A + C*X
74  CONTINUE
SUT = 0.000
SUK = 0.000
C SUM NUMERATOR * STREAM AND POTENTIAL FUNCTIONS
C COMPLETE LOOP FOR NN ITERATIONS
DO 75 IZ = 1,II
ZZ = IZ-1
SUT = SUT - THETA(X,ZZ,Y)*TOP(II,IZ)
SUK = SUK - BETA(X,ZZ,Y)*TOP(II,IZ)
SP(IXX) = SP(IXX) + SUT*BN(II)
72  ST(IXX) = ST(IXX) + SUK*BN(II)
71  CONTINUE
C WRITE HEADER AND NN
WRITE (3,76) NN
76  FORMAT(16H RESULTS AFTERI4,11H ITERATIONS//)

COMPUTER PROGRAM FOR PROBLEM NUMBER ONE (CONTINUED)
C LOOP TO WRITE FUNCTIONS FOR EACH X
DO 77 IX = 1, IS
    X = IX - 1
    IF (IGO) 78, 79, 78
78  Y = A + C*X
79  CONTINUE
77  WRITE (3, 80) X, ST(IX), SP(IX)
80  FORMAT (' FOR X = F4.0, 24H THE STREAM FUNCTION = F20.12, 32H AND THE POTENTIAL FUNCTION = F20.12/')
66  CONTINUE
64  CONTINUE
GO TO 99
96  CONTINUE
END

C SUBROUTINE TO COMPUTE UM-UM INTEGRAL
C
DOUBLE PRECISION FUNCTION UMSQCEM)
DOUBLE PRECISION A, B, C, S, EM, CCC, CC
COMMON A, B, C, S, PI
CCC = S / (8.0D0*EM*PI*(COSH((EM*PI*B)/S)**2))
CC = (2.0D0*C*C+1.0D0)/(C*(C*C+1.0D0))
UMSQ = CCC*(CC*(SINHH(2.0D0*EM*PI*((A/S)+C))-SINHH((2.0D0*EM*A*PI)/IS)) + 2.0D0*EM*PI)
RETURN
END

COMPUTER PROGRAM FOR PROBLEM NUMBER ONE (CONTINUED)
SUBROUTINE CHECK(EN)
DOUBLE PRECISION TOP(30,30),BOT(30),A,B,C,S,PI,SUM,EN,EI,EJ
COMMON A,B,C,S,PI,BOT,TOP
SUM = 0.000
N = EN + 1.0
SUM = (TOP(N,1)**2)*S
DO 1 I=2,N
EI = I-1
1 SUM = SUM + (TOP(N,I)**2)*UMSQ(EI)
NNN = EN
DO 2 I=1,NNN
JI = I+1
DO 2 J=JI,N
EI = I-1
EJ = J-1
SUM = SUM + (2.0D0*TOP(N,I)*TOP(N,J)*UMN(EI,EJ))
2 CONTINUE
WRITE (3,3) SUM
3 FORMAT(' THE NUMERATOR SQUARED IS','F24.16)
RETURN
END

C SUBROUTINE TO COMPUTE UM-UN INTEGRAL
C
DOUBLE PRECISION FUNCTION UMN(EM,EN)
DOUBLE PRECISION A,B,C,S,EM,EN,Q,D,CCC,C1,C2
COMMON A,B,C,S,PI
Q = EM + EN

COMPUTER PROGRAM FOR PROBLEM NUMBER ONE (CONTINUED)
D = EN - EM 
CCC = (S*C)/(4.0D0*PI*(C*C+1.0D0)*COSH((EM*PI*B)/S)*COSH((EN*PI*B 1)/S)) 
C1 = (2.0D0*Q*Q*C*C +Q*Q +D*D)/(Q*(Q*Q*C*C +D*D)) 
C2 = (2.0D0*D*D*C*C +Q*Q +D*D)/(D*(D*D*C*C +Q*Q)) 
MUG = Q 
4 IF(MUG) 1,2,3 
3 MUG = MUG - 2 
GO TO 4 
1 UMN = -CCC*(C1*SINHH(Q*PI*((A/S)+C))+SINHH((A*Q*PI)/S)) +C2*(SINHH 1 (D*PI*((A/S)+C))+SINHH((D*A*PI)/S)) 
GO TO 5 
2 UMN = CCC*(C1*(SINHH(Q*PI*((A/S)+C))-SINHH((A*Q*PI)/S)) +C2*(SINHH 1 (D*PI*((A/S)+C))-SINHH((D*A*PI)/S))) 
5 RETURN 
END 

C HYPERBOLIC SINE FUNCTION

C DOUBLE PRECISION FUNCTION SINHH(X)
DOUBLE PRECISION X
IF(X-20.0) 1,1,2 
1 SINHH =(DEXP(X)-DEXP(-X)) * 0.5 
GO TO 3 
2 SINHH = 0.5 * DEXP(X) 
3 RETURN 
END 

COMPUTER PROGRAM FOR PROBLEM NUMBER ONE (CONTINUED)
SUBROUTINE TO COMPUTE UM INTEGRAL

DOUBLE PRECISION FUNCTION UM(EM)
DOUBLE PRECISION A, B, C, S, PI, CC, MUM, EM
COMM N A, B, C, S, PI
IF(EM) 1, 2, 1
2 UM = S
GO TO 3
1 CC = (S*C) / (EM*PI*(C*C+1.0D0)*COSH((EM*PI*B)/S))
MUM = EM
7 IF(MUM) 4, 5, 6
6 MUM = MUM - 2
GO TO 7
4 UM = -CC*(SINH(EM*PI*((A/S)+C)) + SINH((EM*A*PI)/S))
GO TO 3
5 UM = CC*(SINH(EM*PI*((A/S)+C)) - SINH((EM*A*PI)/S))
3 RETURN
END

HYPERBOLIC COSINE FUNCTION

DOUBLE PRECISION FUNCTION COSHH(XX)
DOUBLE PRECISION XX
COSHH = 0.5 * (DEXP(XX) + DEXP(-XX))
RETURN
END

COMPUTER PROGRAM FOR PROBLEM NUMBER ONE (CONTINUED)
SUBROUTINE TO COMPUTE XUM INTEGRAL

DOUBLE PRECISION FUNCTION XUM(EM)
DOUBLE PRECISION A, B, C, S, PI, EM, CC, MUM, CCC
COMMON A, B, C, S, PI
IF(EM) 1, 2, 1
2 XUM = (S*S)/2.0D0
GO TO 3
1 CC = (S*S)/(EM*PI*(C+C+1.0D0)*COSH((EM*PI*B)/S))
CCC = (1.0D0-C*C)/(EM*PI*(C+C+1.0D0))
MUM = EM
7 IF(MUM) 4, 5, 6
6 MUM = MUM - 2
GO TO 7
4 XUM = -CC*(C*SINH(EM*PI*((A/S)+C))+CCC*(COSH(EM*PI*((A/S)+C)) +
1*COSH((EM*A*PI)/S)))
GO TO 3
5 XUM = CC*(C*SINH(EM*PI*((A/S)+C))+CCC*(COSH(EM*PI*((A/S)+C)) -
1*COSH((EM*A*PI)/S)))
3 RETURN
END

SUBROUTINE TO COMPUTE STREAM CURVE

STREAM CURVE = (SINH((N*PI*Y)/S) / COSH((N*PI*B)/S)) * SIN((N*PI*X)/S)

DOUBLE PRECISION FUNCTION THETA(X, EN, Y)

COMPUTER PROGRAM FOR PROBLEM NUMBER ONE (CONTINUED)
DOUBLE PRECISION ACOS, ASIN, Y, A, B, C, S, PI, X, EN
COMMON A, B, C, S, PI
IF(EN) 1, 2, 1
2 THETA = 1.000
GO TO 3
1 ACOS = COSHH((EN*PI*Y)/S)
ASIN = COSHH((EN*PI*B)/S)
THETA = (ACOS/ASIN)*DCOS((EN*PI*X)/S)
3 RETURN
END

SUBROUTINE TO COMPUTE POTENTIAL FUNCTION

POTENTIAL CURVE = (COSH((N*PI*Y)/S) / COSH((N*PI*B)/S))
* COS((N*PI*X)/S)

DOUBLE PRECISION FUNCTION BETA(X, EN, Y)
DOUBLE PRECISION ACOS, ASIN, Y, A, B, C, S, PI, X, EN
COMMON A, B, C, S, PI
IF(EN) 1, 2, 1
2 BETA = 0.000
GO TO 3
1 ACOS = SINHH((EN*PI*Y)/S)
ASIN = COSHH((EN*PI*B)/S)
BETA = (ACOS/ASIN)*DSIN((EN*PI*X)/S)
3 RETURN
END

COMPUTER PROGRAM FOR PROBLEM NUMBER ONE (CONTINUED)
SUBROUTINE TO COMPUTE BN
BN = (A * SUM OF NUMERATOR * UM + C * SUM OF NUMERATOR * XUM) / DENOMINATOR

DOUBLE PRECISION FUNCTION EVUL(X)
DOUBLE PRECISION SUM, A, B, C, S, PI, TOP(30,30), BOT(30), X, SS, EI, BN
COMMON A, B, C, S, PI, BOT, TOP
N = X
M = N + 1
SUM = 0.000
SS = 0.000
DO 1 I = 1, N
EI = I - 1
1 SUM = SUM - TOP(M, I) * UM(EI)
SS = SS - TOP(M, I) * XUM(EI)
SUM = SUM + UM(X)
SS = SS + XUM(X)
EVUL = (A * SUM + C * SS) / BOT(M)
RETURN
END
Fig. 22. Flow chart of the computer program for problem number one.
THEORY OF SURFACE WATER

DOUBLE PRECISION ALL VARIABLES

\[ \pi = 3.141592653589793 \]

READ A, B, C, S, NN, IT

SET SC AND SS ARRAYS = 0.0

IF A \geq 50

WRITE A, B, C, S, NN

NO

WRITE NO, BOT(1), TOP(1,1)

BN(1) = (A * UM(0.0) + C * XM(0.0)) / BOT(1)

WRITE BN(1)

NNP = NN + 1

TOP(1,1) = 0.0

BOT(1) = S

NO = 0

END
Fig. 23. Page two of the flow chart of the computer program for problem number one.
Fig. 24. Page three of the flow chart of the computer program for problem number one.
WRITE NI, BOT(IJ), Top(IJ,JI), JI=1,IM

BN(IJ) = EVUL(EI)

WRITE BN(IJ)

TOP(IJ,II) = -1.0

CALL CHECK(EI)

II = II + 1

II = MM + 1

DIX = DIX + (COEFT(MM)*TOP(MM,LL)) / BOT(MM)

MM = MM + 1

LP = LL + 1

DIX = 0.0

IS LP = IM

NO

IS LP = IM

NO

SET MM = LP

IS MM = IM

NO

IS MM = IM

NO

LP = LL + 1

LL = LL + 1

LP = LL + 1

DIX = 0.0

IS LL = IM

NO

TOP(IJ,LL) = (COEFT(LL) / BOT( LL)) - DIX

IS MM = IM

NO

2

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Fig. 25. Page four of the flow chart of the computer program for problem number one.
COMPUTE CURVE FUNCTIONS

IS IT \leq 0

NO

SET IL = 1

ILP = IL - 1
TOP(IL,IL) = -1.0
KTP = (S*2.0) + 1.0
X = 0.0

X = 0.0

Y = A + C*X
SUT = 0.0
SUK = 0.0

SET IXX = 1

IS IXX = KTP

NO

IL = IL + 1

YES

Y = A + C*X
SUT = 0.0
SUK = 0.0

SET IZ = 1

IS IZ = IL

NO

ZZ = IZ - 1
SUK = SUK - BETA(X,ZZ,Y) * TOP(IL,IZ)
SUT = SUT - THETA(X,ZZ,Y) * TOP(IL,IZ)

I Z = IZ + 1

YES

ZZ = IZ - 1
SUK = SUK - BETA(X,ZZ,Y) * TOP(IL,IZ)
SUT = SUT - THETA(X,ZZ,Y) * TOP(IL,IZ)

I Z = IZ + 1
Fig. 26. Page five of the flow chart of the computer program for problem number one.
Fig. 27. Page six of the flow chart of the computer program for problem number one.
Fig. 28. Page seven of the flow chart of the computer program for problem number one.
WRITE NN

SET IX = 1

IF IX = IX

NO

X = IX - 1

YES

Y = A + C * X

WRITE SP(IX), X, ST(IX)

IX = IX + 1

IY = IY + 1

IF IX = 0

NO

YES

IF IX = IX

NO

YES
Sample of computer output for problem number one for
a = 2, b = 4, s = 10, and N = 20

RESULTS FOR A = 2.000  B = 4.000  C = 0.200  AND S = 10.0  20 ITERATIONS

FOR LAMBDA (0) THE DENOMINATOR IS 9.999999999
THE NUMERATOR IS 0.0
EN = 3.000000000

FOR LAMBDA (1) THE DENOMINATOR IS 3.22678552
THE NUMERATOR(S) IS -0.07365129
EN = -1.00506687
THE NUMERATOR SquARED IS 3.22678552

FOR LAMBDA (2) THE DENOMINATOR IS 1.70288240
THE NUMERATOR(S) IS 0.00251186 -0.26805597
EN = -0.10307407
THE NUMERATOR SquARED IS 1.70288240

FOR LAMBDA (3) THE DENOMINATOR IS 0.93520104
THE NUMERATOR(S) IS -0.01556093 -0.03664671 -0.47388277
EN = -0.20798629
THE NUMERATOR SquARED IS 0.93520104

FOR LAMBDA (4) THE DENOMINATOR IS 0.52614744
THE NUMERATOR(S) IS -0.00173670 -0.03894625 -0.12351412 -0.67018556
EN = -0.08188706
THE NUMERATOR SquARED IS 0.52614744
Sample computer output for problem number one (Cont.)

CURVE FUNCTION AFTER 20 INTERATIONS

FOR X = 0  THE POTENTIAL CURVE = 2.03830472   THE STREAM CURVE = 0.00000000
FOR X = 1  THE POTENTIAL CURVE = 2.21190220   THE STREAM CURVE = -0.28396739
FOR X = 2  THE POTENTIAL CURVE = 2.40886961   THE STREAM CURVE = -0.40423428
FOR X = 3  THE POTENTIAL CURVE = 2.6051618   THE STREAM CURVE = -0.47007388
FOR X = 4  THE POTENTIAL CURVE = 2.79691741   THE STREAM CURVE = -0.50682867
FOR X = 5  THE POTENTIAL CURVE = 2.99656038   THE STREAM CURVE = -0.52482796
FOR X = 6  THE POTENTIAL CURVE = 3.19785604   THE STREAM CURVE = -0.52262549
FOR X = 7  THE POTENTIAL CURVE = 3.40047128   THE STREAM CURVE = -0.49245995
FOR X = 8  THE POTENTIAL CURVE = 3.60402199   THE STREAM CURVE = -0.42003394
FOR X = 9  THE POTENTIAL CURVE = 3.80478452   THE STREAM CURVE = -0.28011286
FOR X = 10 THE POTENTIAL CURVE = 3.98715961   THE STREAM CURVE = 0.00000000

FUNCTIONS FOR Y = 1.00 AFTER 20 ITERATIONS

FOR X = 0  THE STREAM FUNCTION = 0.00000000   THE POTENTIAL FUNCTION = 2.25843828
FOR X = 1  THE STREAM FUNCTION = -0.09496940   THE POTENTIAL FUNCTION = 2.32033492
FOR X = 2  THE STREAM FUNCTION = -0.14440603   THE POTENTIAL FUNCTION = 2.45630933
Sample computer output for problem number one (Cont.)

FOR X = 3  THE STREAM FUNCTION = -0.16419360  THE POTENTIAL FUNCTION = 2.61949700
FOR X = 4  THE STREAM FUNCTION = -0.16835036  THE POTENTIAL FUNCTION = 2.79082019
FOR X = 5  THE STREAM FUNCTION = -0.16255111  THE POTENTIAL FUNCTION = 2.95997319
FOR X = 6  THE STREAM FUNCTION = -0.14830417  THE POTENTIAL FUNCTION = 3.11895051
FOR X = 7  THE STREAM FUNCTION = -0.12518842  THE POTENTIAL FUNCTION = 3.25960147
FOR X = 8  THE STREAM FUNCTION = -0.09221977  THE POTENTIAL FUNCTION = 3.37250095
FOR X = 9  THE STREAM FUNCTION = -0.04944313  THE POTENTIAL FUNCTION = 3.44696395
FOR X = 10 THE STREAM FUNCTION = -0.00000000  THE POTENTIAL FUNCTION = 3.47319862
APPENDIX V: INTEGRATION FORMULAS FOR THE INTEGRALS

\((v_m, v_n)\) AND \((v_m, v_m)\)

We have from Equation 75 and 80 that \((v_m, v_n)\) is defined as

\[
(v_m, v_n) = \int_0^\infty \left[ \frac{\sinh m(a+cx)/s \sinh n(a+cx)/s}{\sinh m\pi b/s \sinh n\pi b/s} \cos m\pi x/s \cos n\pi x/s \right] dx. \quad (234)
\]

If we define \(F_{mn}\) by

\[
F_{mn} = \frac{1}{\sinh m\pi b/s \sinh n\pi b/s} \quad (235)
\]

and use formula 654.1 of Dwight (1962) we can write Equation 234 as

\[
(v_m, v_n) = \frac{F_{mn}}{4} \int_0^\infty \left[ \left[ e^{m\pi(a+cx)/s} - e^{-m\pi(a+cx)/s} \right] \left[ e^{n\pi(a+cx)/s} - e^{-n\pi(a+cx)/s} \right] \cos m\pi x/s \cos n\pi x/s \right] dx. \quad (236)
\]

Upon multiplying out the exponential terms, changing the integration of a sum to the sum of an integration, and moving all constant terms outside of the integral signs we have, upon multiplying by \(4/F_{mn}\) the expression

\[
\frac{4}{F_{mn}} (v_m, v_n) = e^{(m+n)\pi s} \int_0^\infty e^{(m+n)cx\pi/s} \cos m\pi x/s \cos n\pi x/s \, dx \quad (237)
\]

\[
+ e^{-(m+n)\pi s} \int_0^\infty e^{-(m+n)cx\pi/s} \cos m\pi x/s \cos n\pi x/s \, dx \quad (238)
\]

\[
- e^{(m-n)\pi s} \int_0^\infty e^{(m-n)cx\pi/s} \cos m\pi x/s \cos n\pi x/s \, dx \quad (239)
\]
To evaluate the integrals in the expressions 237, 238, 239, and 240 we use formula 315, page 83 of Burington (1955) which is

\[ \int [e^{ax} \cos bx \cos cx] \, dx = \]

\[ \frac{e^{ax}[ (b-c) \sin (b-c)x + a \cos (b-c)x ]}{2[a^2 + (b-c)^2]} \]

\[ + \frac{e^{ax}[ (b+c) \sin (b+c)x + a \cos (b+c)x ]}{2[a^2 + (b+c)^2]} \]  

(241)

Upon comparing the integrals in expressions 237, 238, 239, and 240 with Burington's formula we see that his b and c are our \( m \pi / s \) and \( n \pi / s \) respectively. The terms \( \sin (b-c)x \) and \( \sin (b+c)x \) of equation 241 now become \( \sin (m-n)\pi x / s \) and \( \sin (m+n)\pi x / s \) which are zero at the limits of integration \( x = 0 \) and \( x = s \). Therefore, in our particular case, the formula given in equation 241 can be reduced to

\[ \int e^{ax} \cos bx \cos cx \, dx = \frac{ae^{ax}}{2} \left[ \frac{\cos (b-c)x}{a^2 + (b-c)^2} + \frac{\cos (b+c)x}{a^2 + (b+c)^2} \right]. \]

(242)

Comparing this formula with expressions 237, 238, 239, and 240 we see that Burington's \( b-c \) is \( (m-n)\pi / s \) and \( b+c \) is \( (m+n)\pi / s \) for all four integrals. Further comparison shows that the only symbol we need to change in equation 242 is a.
For the integrals in Expressions 237, 238, 239, and 240 the symbol \( a \) in Burington's formula is given by \( a = (m+n)\pi c/s \), \( a = -(m+n)\pi c/s \), \( a = (m-n)\pi c/s \) and \( a = -(m-n)\pi c/s \) respectively.

Using Equation 242 and returning to the integral in Expression 237 we see that the formula for the integral is

\[
\int_0^s e^{(m+n)\pi c/s} \cos \frac{m\pi}{s} \cos \frac{n\pi x}{s} \, dx =
\]

\[
\frac{(m+n)\pi c}{2s} e^{(m+n)\pi c/s} \left[ -\frac{\cos \frac{(m-n)\pi x}{s}}{(m+n)^2 c^2 + (m-n)^2} + \frac{\cos \frac{(m+n)\pi x}{s}}{(m+n)^2 c^2 + (m+n)^2} \right]_0^s.
\]

(243)

After we factor out \( s^2/\pi^2 \), the right hand side of Equation 243 becomes

\[
\frac{(m+n)s c}{2\pi} e^{(m+n)c\pi/s} \left[ -\frac{\cos \frac{(m-n)\pi x}{s}}{(m+n)^2 c^2 + (m-n)^2} + \frac{\cos \frac{(m+n)\pi x}{s}}{(m+n)^2 c^2 + (m+n)^2} \right]_0^s.
\]

(244)

If \( (m-n) \) and \( (m+n) \) are odd we will get a different value for Expression 244 than if \( (m-n) \) and \( (m+n) \) are even. If \( (m-n) \) is odd so is \( (m+n) \) and if \( (m-n) \) is even then so is \( (m+n) \). For \( (m+n) \) odd Expression 244 becomes

\[
\frac{(m+n)s c}{2\pi} \left[ -\frac{1}{(m+n)^2 c^2 + (m-n)^2} + \frac{1}{(m+n)^2 c^2 + (m+n)^2} \right]
\]

or

\[
\frac{(m+n)s c}{2\pi} \left[ -\frac{1}{(m+n)^2 c^2 + (m-n)^2} + \frac{1}{(m+n)^2 c^2 + (m+n)^2} \right]
\]

or
For \((m+n)\) even Expression 244 becomes

\[
\frac{\text{sc}\left[ e^{(m+n)\pi c + 1} \right]}{2\pi} + \frac{e^{(m+n)\pi c + 1}}{(m+n)c^2 + (m-n)^2}.
\]  

\hspace{1cm} (245)

Similarly using Expression 244 the integral in Expression 238 becomes for \((m+n)\) odd

\[
\frac{\text{sc}\left[ e^{-(m+n)\pi c + 1} \right]}{2\pi} + \frac{e^{-(m+n)\pi c + 1}}{(m+n)c^2 + (m-n)^2}.
\]  

\hspace{1cm} (246)

and for \(m+n\) even becomes

\[
-\frac{\text{sc}\left[ e^{-(m+n)\pi c - 1} \right]}{2\pi} + \frac{e^{-(m+n)\pi c - 1}}{(m+n)c^2 + (m-n)^2}.
\]  

\hspace{1cm} (248)

The integral in Expression 239 becomes, for \((m+n)\) odd,

\[
-\frac{\text{sc}\left[ e^{(m-n)\pi c + 1} \right]}{2\pi} + \frac{e^{(m-n)\pi c + 1}}{(m-n)c^2 + (m-n)^2}.
\]  

\hspace{1cm} (249)

and becomes, for \((m+n)\) even,

\[
\frac{\text{sc}\left[ e^{(m-n)\pi c - 1} \right]}{2\pi} + \frac{e^{(m-n)\pi c - 1}}{(m-n)c^2 + (m-n)^2}.
\]  

\hspace{1cm} (250)

The integral in Expression 240 becomes, for \((m+n)\) odd,

\[
\frac{\text{sc}\left[ e^{-(m-n)\pi c + 1} \right]}{2\pi} + \frac{e^{-(m-n)\pi c + 1}}{(m-n)c^2 + (m-n)^2}.
\]  

\hspace{1cm} (251)

and for \((m+n)\) even becomes

\[
-\frac{\text{sc}\left[ e^{-(m-n)\pi c - 1} \right]}{2\pi} + \frac{e^{-(m-n)\pi c - 1}}{(m-n)c^2 + (m-n)^2}.
\]  

\hspace{1cm} (252)

Upon replacing the integrals in Expressions 237, 238, 239,
and \(2^{40}\) with Expressions 245, 247, 249, and 251 we have for \((m+n)\) and \((m-n)\) odd

\[
(v_m, v_n) = \frac{F_{mn}}{4} \left\{ -e^{(m+n)na/s} \frac{e^{(m+n)nc + 1}}{2\pi (m+n)^2c^2 + (m-n)^2} \right. \\
+ e^{(m+n)nc + 1} \left. \right\} + e^{(m+n)na/s} \frac{e^{-(m+n)nc + 1}}{2\pi (m+n)^2c^2 + (m-n)^2} \\
+ e^{-(m+n)nc + 1} \frac{e^{(m-n)na/s} e^{-(m-n)nc + 1}}{2\pi (m-n)^2c^2 + (m+n)^2} \\
\left. - e^{-(m-n)nc + 1} \right\}.
\]  

(253)

If we factor out \(sc/2\pi\), multiply through by the constant exponent terms, and collect terms with like denominators, the right hand side of Equation 253 becomes

\[
\frac{F_{mn} sc}{c, \pi} \left\{ -(m+n) \frac{e^{(m+n)na/s} e^{-(m+n)na/s}}{(m+n)^2c^2 + (m-n)^2} \\
- (m+n) \frac{e^{(m+n)na/s} e^{-(m+n)na/s}}{(m+n)^2c^2 + (m-n)^2} \\
+ (m-n) \frac{e^{(m-n)na/s} e^{-(m-n)na/s}}{(m-n)^2c^2 + (m+n)^2} \\
+ (m-n) \frac{e^{(m-n)na/s} e^{-(m-n)na/s}}{(m-n)^2c^2 + (m+n)^2} \right\}.
\]  

(254)
Upon interchanging the second and third term in each numerator of Expression 254, putting the first and second functions over common denominators, and putting the third and fourth fraction over common denominators we can write

\[
\frac{F_{mn}}{\pi} = \frac{(m+n)^2(c^2+1)(e^{(m+n)\pi(a/s+c)} - e^{-(m+n)\pi(a/s+c)} + e^{(m+n)\pi a/s}_e -(m+n)\pi a/s)}{(m+n)(c^2+1)[(m+n)^2c^2 + (m-n)^2]}
\]

or factoring out \(1/(c^2+1)\) and combining the first and second terms and combining the third and fourth terms we have

\[
\frac{F_{mn}}{\pi} = \frac{2(m+n)^2c^2+(m+n)^2+(m-n)^2}{\pi(c^2+1)(m+n)[(m+n)^2c^2 + (m-n)^2]} \left\{ e^{(m+n)\pi(a/s+c)} - e^{-(m+n)\pi(a/s+c)} + e^{(m+n)\pi a/s}_e -(m+n)\pi a/s \right\}
\]

or

\[
\frac{F_{mn}}{\pi} = \frac{2(m-n)^2c^2+(m-n)^2+(m+n)^2}{\pi(c^2+1)(m-n)[(m-n)^2c^2 + (m+n)^2]} \left\{ e^{(m-n)\pi(a/s+c)} - e^{-(m-n)\pi(a/s+c)} + e^{(m-n)\pi a/s}_e -(m-n)\pi a/s \right\}
\]
Using the definition of the hyperbolic sine and our definition of $F_{mn}$ we have for $(m+n)$ and $(m-n)$ odd that $(v_m,v_n)$ is given by

\[
(v_m,v_n) = \frac{sc}{4\pi(c^2+1)\sinh \eta b/s \sinh n\eta b/s} \left[ \frac{2(m-n)^2c^2+(m+n)^2+(m-n)^2}{(m-n)[(m-n)^2c^2+(m+n)^2]} \right]
\]

\[
[sinh (m-n)\pi(a/s+c)+sinh (m-n)\pi a/s] - \frac{2(m+n)^2c^2+(m+n)^2+(m-n)^2}{(m+n)[(m+n)^2c^2+(m-n)^2]} [sinh(m+n)\pi(a/s+c)+sinh(m+n)\pi a/s].
\]

(255)

In a similar manner we can obtain $(v_m,v_n)$ for $(m+n)$ and $(m-n)$ even if we replace the integrals in Expressions 237, 238, 239, and 240 by Expressions 240, 248, 250, and 252 respectively. The result is

\[
(v_m,v_n) = \frac{sc}{4\pi(c^2+1)\sinh \eta b/s \sinh n\eta b/s} \left[ \frac{2(m+n)^2c^2+(m+n)^2+(m-n)^2}{(m-n)[(m-n)^2c^2+(m+n)^2]} \right]
\]

\[
[sinh (m+n)\pi(a/s+c)- sinh (m+n)\pi a/s] - \frac{2(m-n)^2c^2+(m+n)^2+(m-n)^2}{(m-n)[(m-n)^2c^2+(m+n)^2]} [sinh (m-n)\pi(a/s+c)-sinh (m-n)\pi a/s].
\]

(256)

To find a formula for $(v_m,v_m)$ we first recall that it is given by

\[
(v_m,v_m) = \int_0^s v_m^2 dx = \int_0^s \left[ \frac{(\sinh m\pi(a+cx)/s)^2}{(\sinh m\eta b/s)^2} \cos^2 m\eta x/s \right] dx.
\]

(257)

If we define $F_m^2$ by

\[
F_m^2 = \frac{1}{(\sinh m\eta b/s)^2}
\]

(258)
and use formula 654.1 of Dwight (1962) we can write Equation 275 as

$$ (v_m, v_m) = \frac{F_m^2}{4} \int_0^s \left[ (e^{m\pi(a+cx)/s} - e^{-m\pi(a+cx)/s})^2 \cos^2 \frac{m\pi x}{s} \right] dx \tag{259} $$

Upon expanding the right hand side of Equation 259, changing the integration of a sum to the sum of the integrations, and moving the constant terms outside of the integral signs, we have

$$ \frac{F_m^2}{4} \left[ e^{2m\pi a/s} \int_0^s e^{2m\pi cx/s} \cos^2 \frac{m\pi x}{s} \, dx \right] \tag{260} $$

$$ + e^{-2m\pi a/s} \int_0^s e^{-2m\pi cx/s} \cos^2 \frac{m\pi x}{s} \, dx \tag{261} $$

$$ - 2 \int_0^s \cos^2 \frac{m\pi x}{s} \, dx \right] \tag{262} $$

To find the integrals in Expressions 260 and 261 we use Burington's (1955) formula 323 page 84 which for a cosine squared in the integrand is given by

$$ \int e^{ax} \cos^2 bx \, dx = \frac{e^{ax} \cos bx(a \cos bx + 2b \sin bx)}{a^2 + 4b^2} $$

$$ + \frac{2b^2 e^{ax}}{a(a^2+4b^2)} \tag{263} $$

The symbol $b$ in Equation 263, Burington's formula 323, is $m\pi/s$. Because $\sin m\pi x/s$ is zero at our limits of integration of $x = 0$ and $x = s$, Equation 263 becomes

$$ \int e^{ax} \cos^2 bx \, dx = \frac{e^{ax}}{a^2+4b^2} \left[ a \cos bx + \frac{2b^2}{2} \right] \tag{264} $$

In view of Equation 264 we see that the integral in
Expression 260 can be written as

\[
\frac{e^{2m\pi c x/s}}{4m\pi (c^2+1)} \left[ 2m\pi c/s \cos^2 m\pi x/s + \frac{2m^2 \pi^2}{s^2} \right] \bigg|_0^s
\]

or simplifying we have

\[
\frac{s e^{2m\pi c x/s}}{4m\pi (c^2+1)} \left[ 2c \cos^2 m\pi x/s + \frac{1}{c} \right] \bigg|_0^s . \quad (265)
\]

After applying the limits of integration we have

\[
\frac{s e^{2m\pi c}}{4m\pi (c^2+1)} \left[ 2c + \frac{1}{c} \right] - \frac{s}{4m\pi (c^2+1)} \left[ 2c + \frac{1}{c} \right]
\]

or upon combining \(2c + \frac{1}{c}\) into \((2c^2 + 1)/c\) and factoring the term out along with \(s/[4m\pi (c^2 + 1)]\) we have Expression 265 given by

\[
\frac{s(2c^2 + 1)}{4m\pi c (c^2 + 1)} \left( e^{2m\pi c} - 1 \right) . \quad (266)
\]

Similarly for the integral in Expression 261 we find

\[
\frac{s(2c^2 + 1)}{4m\pi c (c^2 + 1)} \left( 1 - e^{-2m\pi c} \right) . \quad (267)
\]

To find the integral in Expression 262 we use formula 202 of Burington (1955) which is

\[
\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a} . \quad (268)
\]

The symbol \(a\) in this formula is \(m\pi/s\) of Expression 262 and \(\sin 2m\pi x/s\) is zero at both limits of integration \(x = 0\) and \(x = s\). Therefore the integral in Expression 262 can be written as
Replacing the integrals in Expressions 260, 261, and 262 by Expressions 266, 267, and 269 we can write \((v_m', v_m)\) as

\[
(v_m', v_m) = \frac{F_m^2}{8m} \left[ \frac{e^{2m\pi a/s}}{s(c^2 + 1)} (e^{2m\pi c} - 1) + \frac{e^{-2m\pi a/s}}{s(c^2 + 1)} (1 - e^{-2m\pi c}) - 2 \frac{s}{2} \right].
\]

(270)

Multiplying the first and second term through by the constant exponential term, combining the first and second terms, and factor out \(s/2m\pi\) we have \((v_m', v_m)\) given by

\[
(v_m', v_m) = \frac{sF_m^2}{8m\pi} \left[ \frac{2c^2 + 1}{2c(c^2 + 1)} \left[ e^{2m\pi(a/s+c)} - e^{-2m\pi(a/s+c)} \right. \right.

\[ \left. \left. - e^{2m\pi a/s} + e^{-2m\pi a/s} \right] - 2m\pi \right]
\]

or recalling our definitions of \(F_m^2\) and the hyperbolic sine we have

\[
(v_m', v_m) = \frac{s}{8m\pi \sinh m\pi b/s} \left[ \frac{2c^2 + 1}{c(c^2 + 1)} \left[ \sinh 2m\pi(a/s+c) \right. \right.

\[ \left. \left. - \sinh 2m\pi a/s \right] - 2m\pi \right].
\]

(271)

In Equation 271 we note that the right hand side is undefined for \(c = 0\). Physically \(c\) equals zero when the water levels in the two ditches are equal. For this special case we can let \(c\) be finite, but small enough for the ditches to have
negligible differences in their water levels. We can also use the alternate theory given by Equations 133 and 134.

This concludes Appendix V.
APPENDIX VI: INTEGRATION FORMULAS FOR THE INTEGRALS
\[ \int_0^s v_m(x) \, dx \quad \text{and} \quad \int_0^s x v_m(x) \, dx \]

First we will obtain a formula for \( \int_0^s v_m(x) \, dx \). Recalling our symbolism for \( v_m \) we can write
\[ \int_0^s v_m(x) \, dx = \int_0^s \frac{\sinh m \pi(a+cx)/s}{\sinh m \pi b/s} \cos m \pi x/s \, dx . \quad (272) \]

If we use formulas 654.1 of Dwight (1962) and let \( F_m \) be given by
\[ F_m = \frac{1}{\sinh m \pi b/s} \quad (273) \]
we can write the right hand side of Equation 272 as
\[ \frac{F_m}{2} \int_0^s \left[ e^{m \pi (a+cx)/s} - e^{-m \pi (a+cx)/s} \right] \cos m \pi x/s \, dx \]
or as
\[ \frac{F_m}{2} \left[ e^{m \pi a/s} \int_0^s e^{m \pi c x/s} \cos m \pi x/s \, dx \quad - e^{-m \pi a/s} \int_0^s e^{-m \pi c x/s} \cos m \pi x/s \, dx \right] . \quad (274) \]

To evaluate the integrals in Expressions 274 and 275 we use formula 314 page 83 of Burington (1955) which is
\[ \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) . \quad (276) \]

In both integrals of Expressions 274 and 275 we see that \( b \) of Equation 276 is \( m \pi /s \) so that \( \sin bx \) is given by \( \sin m \pi x/s \) which is zero at \( x = 0 \) and \( x = s \), our limits of integration. Therefore in our particular problem, we see that Equation 276 becomes
\[ \int e^{ax} \cos bx \, dx = \frac{ae^{ax} \cos bx}{a^2 + b^2}. \]  

(277)

Using Equation 277 and letting \(a = \frac{m \pi c}{s}\) and \(b = \frac{m \pi}{s}\), we see that the integral in Expression 274 can be written as

\[ \int_0^s e^{m\pi cx/s} \cos m\pi x/s \, dx = \frac{\frac{m \pi c}{s} e^{m\pi cx/s}}{\frac{m^2 \pi^2 c^2}{s^2} + \frac{m^2 \pi^2}{s^2}} \cos m\pi x/s \bigg|_0^s \]  

(278)

or simplifying we have the right hand side of Equation 278 given by

\[ \frac{\frac{m \pi c}{s} e^{m\pi cx/s}}{m\pi(c^2 + 1)} \cos m\pi x/s \bigg|_0^s \]  

(279)

which upon using the limits of integration for \(m\) odd is given by

\[ -\frac{\frac{m \pi c}{s} e^{m\pi c}}{m\pi(c^2 + 1)} - \frac{\frac{m \pi c}{s}}{m\pi(c^2 + 1)} \]  

(280)

or by

\[ \frac{-\frac{m \pi c}{s} e^{m\pi c} + 1}{m\pi(c^2 + 1)}. \]  

(280)

For \(m\) even Expression 279 becomes

\[ \frac{\frac{m \pi c}{s} e^{m\pi c}}{m\pi(c^2 + 1)} - \frac{\frac{m \pi c}{s}}{m\pi(c^2 + 1)} \]

or

\[ \frac{-\frac{m \pi c}{s} e^{m\pi c} - 1}{m\pi(c^2 + 1)}. \]  

(281)

Similarly the integral in Expression 276 becomes, for \(m\) odd,
\[
\frac{\text{sc}(e^{-m\pi c} + 1)}{m\pi (c^2 + 1)}
\]

(282)

and becomes for \( m \) even

\[
- \frac{\text{sc}(e^{-m\pi c} - 1)}{m\pi (c^2 + 1)}.
\]

(283)

Upon substituting Expressions 280 and 282 for the integrals in Expressions 274 and 275, we have, for \( m \) odd, the expression

\[
\frac{F_m}{2} \left[ -e^{m\pi a} \text{sc} \left( \frac{e^{m\pi c}}{m\pi (c^2 + 1)} \right) - e^{-m\pi a} \text{sc} \left( \frac{e^{-m\pi c}}{m\pi (c^2 + 1)} \right) \right]
\]

or multiplying through by the exponential term and factoring out \( \text{sc}/[m\pi (c^2 + 1)] \) we have

\[
- \frac{\text{sc} F_m}{2m\pi (c^2 + 1)} \left[ e^{m\pi (a/s+c)/s} e^{m\pi a/s} - e^{-m\pi (a/s+c)/s} e^{-m\pi a/s} \right].
\]

(284)

From the definition of hyperbolic cosine and \( F_m \) we see that Expression 284 can be written as

\[
- \frac{\text{sc}}{m\pi (c^2 + 1) \sinh m\pi b/s} [\cosh m\pi (a/s+c) + \cosh m\pi a/s].
\]

(285)

Because Expression 285 is the right hand side of Equation 272 we have for \( m \) odd

\[
\int_0^s v_m(x) \, dx = - \frac{\text{sc}}{m\pi (c^2 + 1) \sinh m\pi b/s} [\cosh m\pi (a/s+c) + \cosh m\pi a/s].
\]

(286)

Similarly for \( m \) even we have

\[
\int_0^s v_m(x) \, dx = \frac{\text{sc}}{m\pi (c^2 + 1) \sinh m\pi b/s} [\cosh m\pi (a/s+c) - \cosh m\pi a/s].
\]

(287)
To find \( \int_0^s x v_m(x) \, dx \) we write
\[
\int_0^s x v_m(x) \, dx = \int_0^s \frac{x \sinh m(x+a)/s}{\sinh m(b)/s} \cos m x/s \, dx. \tag{288}
\]

Using our symbolism for \( F_m \) given in Equation 273 and the formula 654.1 of Dwight (1962) we can write the right hand side of Equation 288 as
\[
\frac{F_m}{\pi^2} \left\{ \int_0^s x [e^{m(x+a)/s} - e^{-m(x+a)/s}] \cos m x/s \, dx \right\}
\]
or as
\[
\frac{F_m}{\pi^2} \left\{ e^{m(x+a)/s} \int_0^s x e^{m(x+a)/s} \cos m x/s \, dx - e^{-m(x+a)/s} \int_0^s x e^{-m(x+a)/s} \cos m x/s \, dx \right\}. \tag{289}
\]

To find the integrals in Expressions 289 and 290 we use formula 322 of Burington (1955) which is
\[
\int x e^{ax} \cos bx \, dx = \frac{xe^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)
- \frac{e^{ax}}{(a^2 + b^2)^2} [(a^2 - b^2) \cos bx + 2ab \sin bx]. \tag{291}
\]

In our case we see that the \( b \) in the above formula is \( m x/s \) or \( \sin bx \) is \( \sin m x/s \) which is zero for our limits of integration \( x = 0 \) and \( x = s \). Equation 291 now becomes, for our case,
\[
\int x e^{ax} \cos bx \, dx = \frac{xe^{ax}}{a^2 + b^2} a \cos bx - \frac{e^{ax}(a^2 - b^2) \cos bx}{(a^2 + b^2)^2}
\]
\[ \int x e^{ax} \cos bx \, dx = \frac{e^{ax} \cos bx}{a^2 + b^2} \left[ ax - \frac{a^2 - b^2}{a^2 + b^2} \right]. \quad (292) \]

Upon comparing Equation 292 with the integral in Expression 289 we have

\[ \frac{e^{m\pi c \pi /s}}{m\pi c \pi /s + m\pi c \pi /s} \left[ \frac{\pi c \pi /s}{s} - \frac{\pi c \pi /s}{s} \right] \bigg|_0^s \]

or

\[ \frac{s^2 e^{m\pi c \pi /s}}{m\pi (c^2 + 1)} \left[ \cos + \frac{s(1 - c^2)}{m\pi (c^2 + 1)} \right] \bigg|_0^s \quad (293) \]

For \( m \) odd Expression 293 becomes

\[ - \frac{s e^{m\pi c}}{m\pi (c^2 + 1)} \left[ \cos + \frac{s(1 - c^2)}{m\pi (c^2 + 1)} \right] \quad (294) \]

or becomes

\[ - \frac{s^2(1 - c^2)}{m^2 \pi^2 (c^2 + 1)^2} \left( e^{m\pi c} + 1 \right) + \frac{s^2 c e^{m\pi c}}{m\pi (c^2 + 1)} \].

Similarly for \( m \) even the integral in Expression 289 becomes

\[ \frac{s^2(1 - c^2)}{m^2 \pi^2 (c^2 + 1)^2} \left( e^{m\pi c} - 1 \right) + \frac{s^2 c e^{m\pi c}}{m\pi (c^2 + 1)} \].

Similarly the integral in Expression 290, for \( m \) odd, becomes
and, for \( m \) even, becomes

\[
\frac{s^2(1 - c^2)}{m^2 \pi^2 (c^2 + 1)^2} \left( e^{-m \pi c} - 1 \right) - \frac{s^2 c e^{-m \pi c}}{m \pi (c^2 + 1)} .
\]  

(297)

If we replace the integrals in Expressions 289 and 290 by Expressions 294 and 296 we have for \( m \) odd

\[
\frac{F_m}{2} \left[ - e^{m \pi a/s} \left[ \frac{s^2(1 - c^2)}{m^2 \pi^2 (c^2 + 1)^2} \left( e^{m \pi c} + 1 \right) + \frac{s^2 c e^{m \pi c}}{m \pi (c^2 + 1)} \right] - e^{-m \pi a/s} \left[ - \frac{s^2(1 - c^2)}{m^2 \pi^2 (c^2 + 1)^2} \left( e^{-m \pi c} + 1 \right) + \frac{s^2 c e^{-m \pi c}}{m \pi (c^2 + 1)} \right] \right].
\]

(298)

Placing Expression 298 in the right hand side of Equation 288 and recalling our definitions for \( F_m \), the hyperbolic sine and the hyperbolic cosine, we have for \( m \) odd

\[
\int_0^s x v_m(x) dx = - \frac{s^2}{m \pi (c^2 + 1)} \left[ \frac{1 - c^2}{\sinh m \pi (a/s + c)} \left( \sinh m \pi (a/s + c) \right) \right]
\]

\[
+ \sinh m \pi a/s + c \cosh m \pi (a/s + c)].
\]

(299)

Similarly we have for \( m \) even
\[ \int_0^s x v_m(x) \, dx = \frac{s^2}{m_\eta(c^2+1) \sinh \, m_\eta b/s \, m_\eta(c^2+1)} \left[ 1 - \frac{c^2}{m_\eta(c^2+1) \sinh \, m_\eta a/s} \right] \cosh \, m_\eta(a/s+c). \] (300)

This completes Appendix VI.
APPENDIX VII: IBM 360 COMPUTER PROGRAM, FLOW CHART, AND SAMPLE OUTPUT FOR THEORY OF STEADY RAINFALL SEEPING THROUGH SOIL INTO DRAINAGE DITCHES OF UNEQUAL WATER LEVELS

In this appendix we present the program, a flow chart of the program, and a sample of the computer output for the second problem.

The program itself is presented first. The letter C in the left hand column of the program denotes a comment which has been added to help the reader follow the program.

A simplified flow chart is included in this appendix to help the reader get a better overall picture of the program. The subroutines of the program are so simple that they were not included in the flow chart.

In the program and the flow chart, the dimensioned variables are BB(25), QQ(25), PP(25), EM(25), SS(80), TOP(30, 30), BOT(30), COEFT(30), SP(25), ST(25), SC(80), and YY(15). Functions determined by subroutines are COSHH(XX), SINHH(XX), VMN(EM,EN), FBB(EM), FQQ(EM), VM(EM), PHETA(X,EN,Y), XVM(EM), FPP(EM), THETA(X,EN,Y), and the CHECK(EN).

We do not expect anyone but experienced programmers to study the program and flow charts. Therefore, there will be no further explanation of the program or the flow chart.

The sample output is for a = 2, b = 4, s = 10, R = 0.00635, K = 0.254 and N = 20. The output shows this as "A = 2.000  B = 4.000  C = 0.200  R = 0.00635  and K =
The "FOR LAMBDA( )" is \( \lambda_m \) of Equation 78.

The "THE DENOMINATOR" is \( D_m \) of Equation 78. If we round to four places we have from the sample output that for \( \lambda_1, \lambda_2, \) and \( \lambda_3, D_m \) is 2.6716, 1.4982, and 0.9001.

The "THE NUMERATOR(S)" gives the \( J_{mn} \) of Equation 89.

From the sample output we have for \( \lambda_1, J_{11} = 0 \); for \( \lambda_2, J_{21} = -0.3706 \); and for \( \lambda_3, J_{31} = -0.6172 \) and \( J_{32} = -0.5153 \).

From these results we see that for \( \lambda_3 \) \((m = 3)\) Equation 89 is given by

\[
\lambda_3 = \frac{v_3 + 0.0617v_1 + 0.5153v_2}{(0.9001)^{1/2}}
\]

which agrees with Equation 86.

The "THE NUMERATOR SQUARED" is a check of \( \lambda_m \) and if it is equal to "THE DENOMINATOR" then we have, as a check, the result

\[
\int_0^s \lambda_m \lambda_m \, dx = 1.
\]

Our sample of computer output shows only the output for five \( \lambda_m \)'s.

The term "BETA" is

\[
[\frac{\sinh m_n a/s}{\sinh m_b /s} - \sum_{n=1}^{m-1} \frac{\sinh n_n a/s}{\sinh n_b /s}]
\]

in \( d \) as given by Equation 110.

The letters \( Q, P, \) and \( D \) denote \( q_m, p_m, \) and \( d \) of Equations 93, 94, and 110.

The "EM" is \( E_m \) of Equation 123.
On the second page of our sample output under the heading "CURVE FUNCTION AFTER 20 ITERATIONS", the "THE PSI LINE" and the "THE PHI LINE" are values of \( \psi \) and \( \phi \) for various \( x \) along the line \( y = a + cx \), i.e., "THE PSI LINE" is boundary condition \( 4 \) and the "THE PHI LINE" is the water table \( \phi_T \).

On this second page, under the heading "FUNCTIONS FOR \( y = 1.00 \)"; "THE STREAM FUNCTION" and the "THE POTENTIAL FUNCTION" are values of \( \psi \) and \( \phi \) for indicated values of \( x \) and for \( y = 1.00 \). The results for other values of \( y \) are printed similarly.
THEORY OF STEADY RAINFALL SEEPING THROUGH SOIL INTO DRAINAGE DITCHES OF UNEQUAL WATER LEVELS

DOUBLE PRECISION BB(25), QQ(25), PP(25), DD(25), EI, EK, EJ, EM(25), GK
DOUBLE PRECISION SS(80), A, B, C, S, BN, HH, X, DUMMY, DODO, DIX, SUT, ZZ, PI, Y
DOUBLE PRECISION TOP(30, 30), BOT(30), COEFT(30), SP(25)
DOUBLE PRECISION ST(25), SC(80), YY(15), SUK, D, CC1, CC2, YT, YP
COMMON A, B, C, S, PI, BOT, TOP
PI = 3.141592653589793D00

DO 86 I = 1, 80
SC(I) = 0.0D0
SS(I) = 0.0D0
DO 87 I = 1, 25
SP(I) = 0.0D0
ST(I) = 0.0D0

READ INPUT PARAMETERS
A, B, C AND S ARE PROBLEM PARAMETERS
NN IS NUMBER OF ITERATIONS DESIRED
R AND K ARE ALSO PROBLEM PARAMETERS

READ (1, 1) A, B, C, S, NN, R, GK
1 FORMAT(4F5.1, 15, 2F5.1)
776 IF (A-50.0) 777, 778, 778
C WRITE PARAMETERS
777 WRITE (3, 33) A, B, C, S, R, GK
33 FORMAT(19H1 RESULTS FOR A =F6.3, 6H B =F6.1, 6H C =F6.4, 10H

COMPUTER PROGRAM FOR PROBLEM NUMBER TWO
1  S =F6.1.8H  R =F6.4.8H AND K =F5.1//
C COMPUTE NUMERATOR(1) AND DENOMINATOR(1) AND WRITE RESULTS
.TOP(1,1) = O.ODO
.BOT(1) = ( VMSQ(1.0))
.NO = 1
WRITE (3,3) NO,BOT(1),TOP(1,1)
3 FORMAT(1H ' FOR LAMBDA(',I2,') THE DENOMINATOR IS 'F24.16/'
1 THE NUMERATOR(S) IS '6F16.10/(6F16.10))
.TOP(1,1) = -1.ODO
C COMPUTE BETA(1), Q(1), P(1), D(1) AND WRITE RESULTS
.BB(1) = FBB(1.ODO)
.QQ(1) = FQQ(1.ODO)
.PP(1) = FPP(1.ODO)
.DD(1) = (-GK*C*A-((-R-GK*C*C)*QQ(1)-GK*C*A*PP(1)))*BB(1))/(1.ODO-
1BB(1)*PP(1))
WRITE (3,21) BB(1),QQ(1),PP(1),DD(1)
21 FORMAT(7H BETA =F12.8,7H Q =F12.8,7H P =F12.8,7H D =F15.8
1//)
C C START LOOP TO GET ITERATION 2 THROUGH NN
C
DO 10 II=2,NN
DUMMY = O.ODO
.DO = O.ODO
.IM = II-1
.EI = II
.DO 11 KK=1,IM
.EK = KK
.KM = KK - 1
.1F(KM) 4,5,4
5.DO = O.ODO

COMPUTER PROGRAM FOR PROBLEM NUMBER TWO (CONTINUED)
GO TO 6
4 DO 12 JJ=1,KM
  EJ = JJ
  DODO = DODO + TOP(KK,JJ)*VMN(EJ, EI)
12 CONTINUE
6 COEFT(KK) = VMN(EK, EI) - DODO
  DUMMY = DUMMY + ((VMN(EK, EI) - DODO)**2)/BOT(KK)
  DODO=0.000
11 CONTINUE
  BOT(II) = (VMSQ(EI)-DUMMY)
DO 13 LL=1,IM
  LP=LL+1
  DIX = 0.000
  IF(LP=IM) 7,7,8
7 DO 14 MM=LP,IM
  DIX = DIX + (COEFT(MM)*TOP(MM, LL))/(BOT(MM))
14 CONTINUE
8 CONTINUE
  TOP(II,LL) = (COEFT(LL)/BOT(LL)) - DIX
13 CONTINUE
C WRITE NUMERATOR VALUES AND DENOMINATOR FOR THIS ITERATION
WRITE (3,44)
44 FORMAT(///)
WRITE (3,3) II,BOT(II),(TOP(II,JI),JI=1,IM)
  TOP(II,II) = -1.000
C COMPUTE BETA, Q, AND P FOR THIS ITERATION
  BB(II) = FBB(EI)
  QQ(II) = FQQ(EI)
  PP(II) = FPP(EI)
C OBTAIN SUMS NECESSARY FOR COMPUTING D
  DODO = 0.000

COMPUTER PROGRAM FOR PROBLEM NUMBER TWO (CONTINUED)
DUMMY = 0.000
DO 20 IQ=1,II
   DODO = DODO + ((-R-GK*C*C)*QQ(IQ)-GK*C*A*PP(IQ))*BB(IQ)
20   DUMMY = DUMMY + PP(IQ)*BB(IQ)
C COMPUTE D AND WRITE BETA, Q, P, AND D FOR THIS ITERATION
   DD(II) = (-GK*C*A-D0D0)/(1.0D0-DUMMY)
   WRITE (3,21) BB(II),QQ(II),PP(II),DD(II)
C
C END OF LOOP TO GET NUMERATORS AND DENOMINATORS
   TOP(I,II) = -1.0DO
   CALL CHECK(EI)
10 CONTINUE
C SET D EQUAL TO FINAL COMPUTED D
   D = DD(NN)
C COMPUTE AND WRITE VALUES FOR EM
DO 366 II=1,NN
   EM(II) = (-R-GK*C*C)*QQ(II) -(D +GK*C*A)*PP(II)
366 WRITE (3,368) (I,EM(I),I=1,NN)
368 FORMAT(5H1 EM /(15,F15.10))
C ********
C
C COMPUTE PSI LINE AND PHI LINE
C
C ********
C LOOP TO COMPUTE CURVES AND SUM FOR NN ITERATIONS
62 DO 63 II=1,NN
   TOP(I,II) = -1.0DO
   KTP IS NUMBER OF X'S TO BE COMPUTED
   KTP = (S*2.0) +1.0
   X=0.0DO
C LOOP TO COMPUTE SUMS FOR EACH X

COMPUTER PROGRAM FOR PROBLEM NUMBER TWO {CONTINUED}
DO 102 IXX = 1, KTP
Y = A + C*X
C SUM NUMERATOR * STREAM AND POTENTIAL CURVES
SUK = 0.000
SUT = 0.000
DO 50 IZ = 1, II
ZZ = IZ
SUK = SUK - PHETA(X, ZZ, Y)*TOP(II, IZ)
50 SUT = SUT - THETA(X, ZZ, Y)*TOP(II, IZ)
SS(IXX) = SS(IXX) +( EM(II)*SUT)
SC(IXX) = SC(IXX) + (EM(II)*SUK)
X = X + 0.500
102 CONTINUE
63 CONTINUE
C WRITE HEADER AND ITERATION NUMBER
301 WRITE (3, 305) II
305 FORMAT(22H CURVE FUNCTION AFTER I4, 11H ITERATIONS//)
C LOOP TO ADD THE NECESSARY CONSTANTS AND OUTPUT
C THE PSI LINE AND THE PHI LINE FOR EACH X
X = 0.000
DO 80 I = 1, KTP
Y = A + C*X
CC2 = A + C*X
CCI = D + GK*C*Y
YT = CCI + SS(I)
YP = CC2 + (1.000/GK)*SC(I)
WRITE (3, 81) X, YT, YP
81 FORMAT(/7H AT X = F5.1, 17H THE PSI LINE IS F20.16, 22H AND THE PHI
1 LINE IS F20.16)
80 X = X + 0.500

COMPUTER PROGRAM FOR PROBLEM NUMBER TWO (CONTINUED)
**COMPUTE STREAM AND POTENTIAL FUNCTIONS**

---

**read ny - number of y's - and the y values**

read (1,66) ny,(yy(i),i=1,ny)

66 format(i5,15f5.2).

**loop to compute functions for each y**

do 76 iy=1,ny

**write header for each y**

write (3,91) yy(iy)

91 format(19h1 functions for y =f6.2/)

Y = YY(IY)

do 92 ib=1,25

ST(IB)= 0.000

92 SP(IB)= 0.000

**loop to compute functions and sum for nn iterations**

do 64 ii=1,nN

TOP(II,II) = -1.000

**is1 equals number of x's to be computed**

IS1 = S + 1.0

**loop to compute sums for each x**

do 112 ixx=1,IS1

X = IXX - 1

**sum numerator * stream and potential functions**

SUT = 0.000

SUK = 0.000

do 51 iz=1,ii

ZZ = IZ

SUT = SUT - THETA(X,ZZ,Y)*TOP(II,IZ)

---

**computer program for problem number two (continued)**
SUH = SUH - PHETA(X,ZZ,Y)*TOP(II,IZ)
ST(IXX) = ST(IXX) + EM(II)*SUT
SP(IXX) = SP(IXX) + EM(II)*SUH
CONTINUE
C WRITE HEADER AND ITERATION NUMBER
WRITE (3,315) II
315 FORMAT(17H FUNCTIONS AFTERI4,11H ITERATIONS//)
C LOOP TO ADD CONSTANTS AND OUTPUT FUNCTIONS
CC1 = D + GK*C*Y
DO 93 IX=1,IS1
X = IX-1
CC2 = A + C*X
YT = CC1 + ST(IX)
YP = CC2 + (1.0D0/GK)*SP(IX)
93 WRITE (3,94) X,YT,YP
94 FORMAT(9H FOR X =F4.0,24H THE STREAM FUNCTION =F20.12,32H AN
1D THE POTENTIAL FUNCTION =F20.12/)
CONTINUE
GO TO 999
CONTINUE
END

SUBROUTINE CHECK(EN)
DOUBLE PRECISION TOP(30,30),BOT(30),A,B,C,S,PI,SUM,EN,EI,EJ
COMMON A,B,C,S,PI,BOT,TOP
SUM = 0.000
N = EN
DO 1 I=1,N

COMPUTER PROGRAM FOR PROBLEM NUMBER TWO (CONTINUED):
EI = I
1 SUM = SUM + (TOP(N,I)**2)*VMSQ(EI)
NM = N-1
DO 2 I=1,NM
JI = I+1
DO 2 J=JI,N
EI = I
EJ = J
SUM = SUM + (2.0D0*TOP(N,I)*TOP(N,J)*VMN(EI,EJ))
2 CONTINUE
WRITE (3,3) SUM
3 FORMAT(H1H ' THE NUMERATOR SQUARED IS'F24.16)
RETURN
END

C HYPERBOLIC COSINE FUNCTION
C
DOUBLE PRECISION FUNCTION COSHH(XX)
DOUBLE PRECISION XX
COSHH = 0.5* (DEXP(XX) + DEXP(-XX))
RETURN
END

C HYPERBOLIC SINE FUNCTION
C
COMPUTER PROGRAM FOR PROBLEM NUMBER TWO (CONTINUED)
DOUBLE PRECISION FUNCTION SINHHCX(X)
DOUBLE PRECISION X
IF(X-20.0) 1,1,2
1 SINH = (DEXP(X)-DEXP(-X)) * 0.5
GO TO 3
2 SINH = 0.5 * DEXP(X)
3 RETURN
END

C SUBROUTINE TO COMPUTE VM-VN INTEGRAL
C
DOUBLE PRECISION FUNCTION VMN(EM,EN)
DOUBLE PRECISION A,B,C,S,EM,EN,Q,D,CCC,C1,C2
COMMON A,B,C,S,PI
Q = EM + EN
D = EN - EM
CCC = (S*C)/(4.0D0*PI*(C*C+1.0D0)*SINHH((EM*PI*B)/S)*SINHH((EN*PI*B)/S))
C1 = (2.0D0*Q*Q*C*C + Q*Q + D*D)/(Q*(Q*Q*C*C + D*D))
C2 = (2.0D0*D*D*C*C + Q*Q + D*D)/(D*(D*D*C*C + Q*Q))
MUG = Q
4 IF(MUG).I. 1,2,3
3 MUG = MUG - 2
GO TO 4
1 VMN = -CCC*C1*{SINHH(Q*PI*{(A/S)+C})+SINHH((A*Q*PI)/S)} - C2*{SINHH(D*PI*{(A/S)+C})+SINHH((D*A*PI)/S))}
GO TO 5
2 VMN = CCC*C1*{SINHH(Q*PI*{(A/S)+C})-SINHH((A*Q*PI)/S)} - C2*{SINHH(D*PI*{(A/S)+C})-SINHH((D*A*PI)/S))}

COMPUTER PROGRAM FOR PROBLEM NUMBER TWO (CONTINUED)
C SUBROUTINE TO COMPUTE THE BETA FUNCTION

C

C BETA = SUM OF NUMERATOR *(SINH((M*PI*A)/S)/SINH((M*PI*B)/S))

C

DOUBLE PRECISION FUNCTION FBB(EM)
DOUBLE PRECISION TOP(30,30),BOT(30),EM,SUT,SUB,EI,A,B,C,S,PI
COMMON A,B,C,S,PI,BOT,TOP

M = EM
SUB= 0.0DO
DO 1 I=1,M
EI=I
1 SUB = SUB - TOP(M,I)» (SINHH((EI*PI*A)/S)/SINHH((EI*PI*B)/S))
FBB = SUB
RETURN
END

C SUBROUTINE TO COMPUTE THE Q FUNCTION

C

C Q = (SUM OF NUMERATOR * XVM(M) )/ DENOMINATOR

C

DOUBLE PRECISION FUNCTION FQQ(EM)

COMPUTER PROGRAM FOR PROBLEM NUMBER TWO (CONTINUED)
DOUBLE PRECISION TOP(30,30), BOT(30), EM, A, B, C, S, PI, EI, SUB
COMMON A, B, C, S, PI, BOT, TOP
M = EM
SUB = 0.0DO
DO 1 I=1,M
EI = I
1 SUB = SUB - TOP(M, I)*XVM(EI)
FQQ = SUB/BOT(M)
RETURN
END

C SUBROUTINE TO COMPUTE VM INTEGRAL
C
DOUBLE PRECISION FUNCTION VM(EM)
DOUBLE PRECISION A, B, C, S, PI, CC, MUM, EM
COMMON A, B, C, S, PI
IF(EM) 1, 2, 1
2 UM = S
GO TO 3
1 CC = (S*C)/ (EM*PI*(C*C+1.0D0)*SINH((EM*PI*B)/S))
MUM = EM
7 IF(MUM) 4, 5, 6
6 MUM = MUM - 2
GO TO 7
4 VM = -CC*(COSH(EM*PI*((A/S)+C)) + COSH((EM*A*PI)/S))
GO TO 3
5 VM = CC*(COSH(EM*PI*((A/S)+C)) - COSH((EM*A*PI)/S))

COMPUTER PROGRAM FOR PROBLEM NUMBER TWO (CONTINUED)
C SUBROUTINE TO COMPUTE VM-VM INTEGRAL
C
DOUBLE PRECISION FUNCTION VMSQ(EM)
DOUBLE PRECISION A,B,C,S,EM,CCC,CC
COMMON A,B,C,S,PI
CCC = S/ (8.0D0*EM*PI*(SINH((EM*PI*B)/S)**2))
CC = (2.0D0*C*C+1.0D0)/(C*(C*C+1.0D0))
VMSQ = CCC*(CC*(SINH(2.0D0*EM*PI*((A/S)+C))-SINH((2.0D0*EM*A*PI)/
1S)) -2.0D0*EM*PI)
RETURN
END

C SUBROUTINE TO COMPUTE THE POTENTIAL CURVE
C
POTENTIAL CURVE = (COSH((N*PI*Y)/S)/SINH((N*PI*B)/S))
  * SIN((N*PI*X)/S)
C
DOUBLE PRECISION FUNCTION PHETA(X,EN,Y)
DOUBLE PRECISION ACOS,ASIN,Y,A,B,C,S,PI,X,EN
COMMON A,B,C,S,PI
ACOS = COSHH((EN*PI*Y)/S)
ASIN = SINHH((EN*PI*B)/S)

COMPUTER PROGRAM FOR PROBLEM NUMBER TWO (CONTINUED)
PHETA = (ACOS/ASIN)*DSIN((EN*PI*X)/S)
RETURN
END

SUBROUTINE TO COMPUTE XVM INTEGRAL

DOUBLE PRECISION FUNCTION XVM(EM)
DOUBLE PRECISION A,B,C,S,PI,EM,CC,MUM,CCC
COMMON A,B,C,S,PI
IF(EM) 1,2,1
2 XUM = (S*S)/2.0D0
GO TO 3
1 CC = (S*S)/(EM*PI*(C*C+1.0D0)*SINHH((EM*PI*B)/S))
CCC = (1.0D0-C*C)/(EM*PI*(C*C+1.0D0))
MUM = EM
7 IF(MUM) 4,5,6
6 MUM = MUM - 2
GO TO 7
4 XVM = -CC*(C*COSH(EM*PI*((A/S)+C))+CCC*(SINHH(EM*PI*((A/S)+C)) +
1*SINHH((EM*A*PI)/S)))
GO TO 3
5 XVM = CC*(C*COSH(EM*PI*((A/S)+C))+CCC*(SINHH(EM*PI*((A/S)+C)) -
1*SINHH((EM*A*PI)/S)))
3 RETURN
END

COMPUTER PROGRAM FOR PROBLEM NUMBER TWO (CONTINUED)
C SUBROUTINE TO COMPUTE THE P FUNCTION

C P = (SUM OF NUMERATOR * VM(M)) / DENOMINATOR
DOUBLE PRECISION FUNCTION FPP(EM)
DOUBLE PRECISION TOP(30,30),BOT(30),EM,A,B,C,S,PI,EI,SUB
COMMON A,B,C,S,PI,BOT, TOP
M = EM
SUB = 0.000
DO 1 I=1,M
EI = I.
1 SUB = SUB - TOP(M,I)*VM(EI)
FPP = SUB/BOT(M)
RETURN
END

C SUBROUTINE TO COMPUTE THE STREAM CURVE

C STREAM CURVE = (SIN((N*PI*Y)/S)/SIN((N*PI*B)/S))
* COS((N*PI*X)/S)

C DOUBLE PRECISION FUNCTION THETA(X,EN,Y)
DOUBLE PRECISION ACOS,ASIN,Y,A,B,C,S,PI,X,EN
COMMON A,B,C,S,PI
3 ACOS = SINH((EN*PI*Y)/S)
ASIN = SINH((EN*PI*B)/S)
1 THETA = (ACOS/ASIN) * DCOS((EN*PI*X)/S)
11 CONTINUE
RETURN
END

COMPUTER PROGRAM FOR PROBLEM NUMBER TWO (CONTINUED)
Fig. 29. Flow chart of the computer program for problem number two.
THEORY OF STEADY RAINFALL

DOUBLE PRECISION ALL VARIABLES

PI = 3.141592653589793

SET SC, SS, SP, AND ST ARRAYS = 0

READ A, B, C, S, NN, R, GK

IS A ≥ 50

NO

WRITE A, B, C, S, R, GK

TOP(I,I) = 0.0
BOT(I) = VMSQ(I)
NO = I

WRITE NO, BOT(I), TOP(I, I)

YES

TOP(I, I) = -1.0
BB(I) = FBB(I, 0)
QQ(I) = FQQ(I, 0)
PP(I) = FPP(I, 0)

WRITE BB(I), QQ(I), PP(I, DD(I))

DD(I) = (-GK * C * A -((-R - GK * C * C) * QQ(I) - GK * C * A * PP(I) + B * BB(I)) / (1.0 - BB(I) * PP(I)))

WRITE

END
Fig. 30. Page two of the flow chart of the computer program for problem number two.
Fig. 31. Page three of the flow chart of the computer program for problem number two.
Fig. 32. Page four of the flow chart of the computer program for problem number two.
\[ D = DD(\text{NN}) \]

\[ \text{SET } I = 1 \]

\[ \text{COMPUTE PSI LINE AND PHI LINE} \]

\[ \text{WRITE } \begin{cases} I, EM(I), II & I = 1, \text{NN} \\ \text{SET } I = 1 & \end{cases} \]

\[ \text{IF } I = \text{NN} \]

\[ \begin{align*}
EM(I) &= (-R - G*K*C*C)*Q(KII) - (D + G*K*C*A)*PP(I) \\
Y &= A + C*X \\
SUK &= 0.0 \\
SUT &= 0.0 \\
TOP(I, II) &= -1.0 \\
KTP &= S*2.0 + 1.0 \\
X &= 0.0
\end{align*} \]

\[ \text{SET } I = 1 \]

\[ \text{IF } I = \text{NN} \]

\[ \begin{align*}
II &= II + 1 \\
\text{SET } I = 1 \\
\text{IF } I = \text{NN} \\
\end{align*} \]

\[ \begin{align*}
\text{IF } I = \text{II} \\
X &= X + 0.5 \\
ZZ &= ZZ + KTP \\
\text{SUK} &= \text{SUK} - \text{PHETA}(X, ZZ, Y) * TOP(I, II, ZZ) \\
\text{SUT} &= \text{SUT} - \text{THETA}(X, ZZ, Y) * TOP(I, II, ZZ)
\end{align*} \]
Fig. 33. Page five of the flow chart of the computer program for problem number two.
\[ X = 0.0 \]

\[ Y = A + C \times X \]
\[ CC2 = A + C \times X \]
\[ CC1 = D + Gk \]
\[ C \times Y \]

\[ Y = CC1 + SS(I) \]
\[ YP = CC2 + (1.0/Gk) \times SC(I) \]

\[ X = X + 0.5 \]

\[ SET \ I = 1 \]

\[ READ NY, YY(I), I = 1, NY \]

\[ SET IY = 1 \]

\[ WRITE YY(IY) \]

\[ Y = YY(IY) \]
\[ SET st AND sp arrays = 0 \]

\[ i = i + 1 \]

\[ WRITE X, YT, YP \]

\[ SET IY = NY \]

\[ i = i + 1 \]

[Diagram of computational flow and logic]
Fig. 34. Page six of the flow chart of the computer program for problem number two.
Sample of Computer Output for \( a = 2, \ b = 4, \ s = 10, \ R = 0.0635, \ K = 0.254, \ N = 20 \)

RESULTS FOR \( A = 2.00 \ B = 4.00 \ C = 0.2000 \ S = 10.00 \ R = 0.00635 \) AND \( K = 0.254 \)

FOR LAMBDA(1) THE DENOMINATOR IS 2.62155247
  THE NUMERATOR(S) IS 0.0
  BETA = 0.41529208 \ Q = -7.60166359 \ P = -0.44879162 \ D = -0.14553248

FOR LAMBDA(2) THE DENOMINATOR IS 1.49817808
  THE NUMERATOR(S) IS -0.37063775
  BETA = 0.41720581 \ Q = -1.75459957 \ P = -0.14734179 \ D = -0.15305356
  THE NUMERATOR SQUARED IS 1.49817847

FOR LAMBDA(3) THE DENOMINATOR IS 0.9000752
  THE NUMERATOR(S) IS -0.61719878 \ -0.51528731
  BETA = 0.30971229 \ Q = -2.18721771 \ P = -0.21957779 \ D = -0.15889375
  THE NUMERATOR SQUARED IS 0.90007520

FOR LAMBDA(4) THE DENOMINATOR IS 0.51765646
  THE NUMERATOR(S) IS -0.05640597 \ -0.14865558 \ -0.68633735
  BETA = 0.24490017 \ Q = -1.07996857 \ P = -0.10768455 \ D = -0.16102137
  THE NUMERATOR SQUARED IS 0.51765647

FOR LAMBDA(5) THE DENOMINATOR IS 0.29723335
  THE NUMERATOR(S) IS -0.01366163 \ -0.08213079 \ -0.25755209 \ -0.86825555
  BETA = 0.17852718 \ Q = -1.76658154 \ P = -0.2181558 \ D = -0.16311575
  THE NUMERATOR SQUARED IS 0.29723335

EM
1  0.09496204
2  0.01894145
3  0.02116813
4  0.01050237
5  0.01432019
Sample computer output for problem number two (cont.)

CURVE FUNCTION AFTER 20 ITERATIONS

AT X = 0.0  THE PSI LINE IS  0.00000000  THE PHI LINE IS  2.00000000
AT X = 1.0  THE PSI LINE IS  -0.00471122  THE PHI LINE IS  2.37548276
AT X = 2.0  THE PSI LINE IS  -0.01102397  THE PHI LINE IS  2.67022607
AT X = 3.0  THE PSI LINE IS  -0.01665687  THE PHI LINE IS  2.92362107
AT X = 4.0  THE PSI LINE IS  -0.02278611  THE PHI LINE IS  3.14588919
AT X = 5.0  THE PSI LINE IS  -0.02925186  THE PHI LINE IS  3.34382617
AT X = 6.0  THE PSI LINE IS  -0.03588458  THE PHI LINE IS  3.52064373
AT X = 7.0  THE PSI LINE IS  -0.04265612  THE PHI LINE IS  3.67775272
AT X = 8.0  THE PSI LINE IS  -0.04953411  THE PHI LINE IS  3.81458624
AT X = 9.0  THE PSI LINE IS  -0.05613068  THE PHI LINE IS  3.92736361
AT X =10.0  THE PSI LINE IS  -0.06133926  THE PHI LINE IS  4.00000000

FUNCTIONS FOR Y = 1.00

AT X = 0.0  THE STREAM FUNCTION IS  -0.09113117  THE POTENTIAL FUNCTION IS  2.00000000
AT X = 1.0  THE STREAM FUNCTION IS  -0.09515860  THE POTENTIAL FUNCTION IS  2.31185090
AT X = 2.0  THE STREAM FUNCTION IS  -0.10259801  THE POTENTIAL FUNCTION IS  2.59219623
AT X = 3.0  THE STREAM FUNCTION IS  -0.10988910  THE POTENTIAL FUNCTION IS  2.84079726
Sample computer output for problem number two (cont.)

AT X = 4.0 THE STREAM FUNCTION IS -0.11635636 THE POTENTIAL FUNCTION IS 3.06195762
AT X = 5.0 THE STREAM FUNCTION IS -0.12198554 THE POTENTIAL FUNCTION IS 3.25938876
AT X = 6.0 THE STREAM FUNCTION IS -0.12684483 THE POTENTIAL FUNCTION IS 3.43625596
AT X = 7.0 THE STREAM FUNCTION IS -0.13095090 THE POTENTIAL FUNCTION IS 3.59542097
AT X = 8.0 THE STREAM FUNCTION IS -0.13420990 THE POTENTIAL FUNCTION IS 3.73981866
AT X = 9.0 THE STREAM FUNCTION IS -0.13638526 THE POTENTIAL FUNCTION IS 3.87303840
AT X = 10.0 THE STREAM FUNCTION IS -0.13716347 THE POTENTIAL FUNCTION IS 4.00000000