On univariate and bivariate extreme value theory

Jose Aurelio Villasenor
Iowa State University

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On univariate and bivariate extreme value theory

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Jose Aurelio Villasenor

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I. INTRODUCTION

For a sequence \( \{X_i\} \) of independent identically distributed (iid) random variables (rv's) with cumulative distribution function (cdf) \( F(x) \), define the sequences of rv's

\[
M_n = \max(X_1, X_2, \ldots, X_n)
\]

\[
W_n = \min(X_1, X_2, \ldots, X_n)
\]

\( n = 1, 2, \ldots \)

Their respective cdf's are

\[
P(M_n \leq x) = F^n(x)
\]

\[
P(W_n \leq x) = 1 - [1 - F(x)]^n.
\]

Since \( W_n = -\max(-X_1, -X_2, \ldots, -X_n) \), extreme value theory is concerned mainly with the maxima.

A distribution function \( F(x) \) belongs to the domain of attraction of a non-degenerate distribution function \( G(x) \) (notation \( F \in \mathcal{D}(G) \)), if there exist real sequences \( \{a_n > 0\} \) and \( \{b_n\} \) such that

\[
\lim_{n \to \infty} F^n(a_n x + b_n) = G(x)
\]

for all \( x \) in the continuity set of \( G(x) \). The cdf \( G(x) \) is called an (asymptotic) extreme distribution and the sequences \( \{a_n > 0\} \) and
\[ \{b_n \} \text{ are called norming constants.} \]

In the above, we use the term non-degenerate to avoid distribution functions \( H(x) \) of the form

\[
H(x) = \begin{cases} 
0, & x < x_0 \\
1, & x \geq x_0 
\end{cases}
\]

The limit condition is usually denoted by

\[
f^n(a_n x + b_n) \xrightarrow{\text{w}} G(x).
\]

Several authors have studied the problem of finding conditions for \( F(x) \) to belong to the domain of attraction of an extreme distribution. Gnedenko (1943) developed a complete solution following the work of Fisher and Tippett (1928), who showed that an extreme distribution must be one of the following types:

\[
\varphi_\alpha(x) = \begin{cases} 
0, & x \leq 0 \\
\exp(-x^{-\alpha}), & x > 0, \quad \alpha > 0 
\end{cases}
\]

\[
\psi_\alpha(x) = \begin{cases} 
\exp\{-(x)^\alpha\}, & x \leq 0, \quad \alpha > 0 \\
1, & x > 0 
\end{cases}
\]

\[
\Lambda(x) = \exp(-e^{-x}), \quad -\infty < x < +\infty.
\]
However, as Gnedenko himself stated it, his results concerning the domain of attraction of \( \Lambda(x) \) cannot be considered very satisfactory from a practical point of view. A related result which is easier to apply was given by de Mises (1939), but still it is sometimes difficult to use in practice since the norming constants rely on the inverse function of the function \( 1 - F(x) \). In Chapter II, Theorem 2.15, we give a criterion which in some way is included in a result by Balkema and de Haan (1972), but nevertheless is presented here since the norming constants are plainly stated without going about finding any inverse function. We believe this result is useful because it includes a wide class of functions in the domain of attraction of \( \Lambda(x) \).

Chapter III is devoted to the problem of determining the domain of attraction to which the convolution of two given cdf's belongs. We search for the extreme distribution that attracts \( F * G(x) \) when \( F(x) \) and \( G(x) \) are each attracted to any of the three different extreme distributions. This problem is analogous to the one studied by Tucker (1968) on attraction of convolutions to stable laws for sums of random variables.

In Chapters IV through VII we treat the bivariate extreme value problem: let \( \{(X_1, Y_1)\} \) be a sequence of bivariate rv's, and define the sequences of rv's

\[
\begin{align*}
M_n^{(1)} &= \max(X_1, X_2, \ldots, X_n) \\
M_n^{(2)} &= \max(Y_1, Y_2, \ldots, Y_n) \quad , \quad n = 1, 2, \ldots
\end{align*}
\]

(1.1)
then the problem is to find the limit distribution of \((M^{(1)}_n, M^{(2)}_n)\), properly normalized as \(n\) tends to infinity.

Along these lines, Geffroy (1959) and Sibuya (1960) proved that if the bivariate rv's \((X_i, Y_i)'\) are iid and their distribution function \(F_{X,Y}(x, y)\) is a bivariate normal distribution with correlation coefficient \(\rho\) such that \(|\rho| < 1\), then \(M^{(1)}_n\) and \(M^{(2)}_n\) are asymptotically independent.

It can be shown (Lemma 5.1) that if \(F_{X,Y}(x, y)\) is a bivariate normal distribution with \(|\rho| < 1\), then there exist independent normally distributed rv's \(U, V\) and \(W\) such that \(F_{X,Y}(x, y)\) is the cdf of \((U + W, V + \beta W)\) for some real \(\beta\). That is, if \(U, V\) and \(W\) have cdf's \(G(u), J(v)\) and \(F(w)\), respectively, then we can express

\[
F_{X,Y}(x, y) = \int_{-\infty}^{+\infty} G(x - \theta) H(y, \theta) \, dF(\theta) \tag{1.2}
\]

where

\[
H(y, \theta) = J(y - \beta \theta).
\]

These properties of the bivariate normal distributions suggested the study of finding conditions to obtain the limit distribution of \((M^{(1)}_n, M^{(2)}_n)\), when \(F_{X,Y}(x, y)\) can be expressed as (1.2) with the cdf's \(G(u), H(v)\) and \(F(w)\) not necessarily of the same form. The rv's \(X\) and \(Y\) are called conditionally independent on a location parameter when their joint distribution \(F_{X,Y}(x, y)\) admits representation (1.2).

In a more general context, we consider bivariate distributions with cdf
\[ F_{X,Y}(x, y) = \int_{-\infty}^{+\infty} G(x, \theta) H(y, \theta) \, dF(\theta) \]  

(1.3)

where \( F(\theta) \) is any cdf, \( G(x, \theta) \) and \( H(y, \theta) \) are cdfs in \( x \) and \( y \), respectively, for every \( \theta \) in the support of \( F(\theta) \), with \( G(x, \theta) \) and \( H(y, \theta) \) Borel measurable functions in \( \theta \) for every \( x \) and \( y \). The random variables \( X \) and \( Y \) are called conditionally independent when representation (1.3) holds (Loève, 1963).

Chapter IV is concerned with iid bivariate rv's \((X_i, Y_i)^t\) with cdf \( F_{X,Y}(x, y) \). It is verified in this chapter that for certain classes of bivariate distributions, as well as for bivariate distributions of the form (1.3) under some conditions, \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically independent. Hence, the limit distribution of \((M_n^{(1)}, M_n^{(2)})^t\) properly normalized is the product of two extreme distributions.

Chapter V is aimed towards finding conditions to obtain the asymptotic independence of \( M_n^{(1)} \) and \( M_n^{(2)} \) where the underlying distribution is of the form (1.2). Applying the results of Chapter III, we are able to get the limit distribution of the maxima properly normalized. We apply the results to a problem approached by David (1973).

For a sequence \{\((X_i, Y_i)^t\)\} of iid bivariate rv's with distribution function \( F_{X,Y}(x, y) \), we can see that the joint distribution function of \( M_n^{(1)} \) and \( M_n^{(2)} \) is
The bivariate distribution function \( F_{X,Y}(x, y) \) is in the domain of attraction of a non-degenerate bivariate distribution function \( G(x, y) \) if there exist real sequences \( \{a_n > 0\} \), \( \{c_n > 0\} \), \( \{b_n\} \) and \( \{d_n\} \) such that

\[
\lim_{n \to \infty} F_{X,Y}^n(a_n x + b_n, c_n y + d_n) = G(x, y)
\]

for all \((x, y)\) in the continuity set of \( G(x, y) \). The cdf \( G(x, y) \) is called a bivariate extreme distribution, and the sequences \( \{a_n > 0\} \), \( \{c_n > 0\} \), \( \{b_n\} \) and \( \{d_n\} \) are called norming constants.

As in the univariate case, the above limit condition is denoted by

\[
\lim_{n \to \infty} F_{X,Y}^n(a_n x + b_n, c_n y + d_n) \overset{w}{\longrightarrow} G(x, y)
\]

Unlike the univariate case, in the bivariate case we do not have only three types of extreme distributions, but an infinite number (de Oliveira, 1959; Sibuya, 1960). Perhaps this is why the domain of attraction aspect of the theory in the bivariate case has not received much attention. We present in Chapter VI sufficient conditions for attraction to some known bivariate extreme distributions, as well as characterizations of the domains of attraction of some others. It is also shown that the bivariate negative exponential distribution of
Marshall and Olkin (1967) is a bivariate extreme distribution, and sufficient conditions for attraction to it are given. Furthermore, a bivariate negative Weibull distribution is found as an extreme distribution.

Chapter VII deals with the case when the bivariate rv's \((X_i, Y_i)\)' are not iid but are exchangeable. That is, the sequence \(\{(X_i, Y_i)\}'\) of pairs of rv's defined on a probability space \((\Omega, G, P)\) is such that the joint distribution function of any \(m\) of these pairs can be represented as

\[
\int_{\Omega} H_u(x_1, y_1) H_u(x_2, y_2) \cdots H_u(x_m, y_m) \, dP(u)
\]

where for fixed \((x, y)\) \(H_u(x, y)\) is a random variable, and for each \(u\) \(H_u(x, y)\) is a bivariate distribution function. Conditions are given to obtain the limit distribution of \((M_n^{(1)}, M_n^{(2)})\) properly normalized. The results are extensions of some of the findings of Berman (1962) in the univariate case. We apply the extended results to the solution of a general problem formulated by David (1973).

The relationship among the chapters is indicated by the following diagram:

```
I
/  \\
II --- VI
/
/
III IV
/
/
/
V

We indicate the end of a proof with the symbol \(\Box\).
II. SURVEY OF LITERATURE AND PRELIMINARIES

Many authors have contributed to the development of the Extreme Value Theory since the early 1920's. Among the relevant pioneer works are those of Fréchet (1927), who found the limit laws \( \xi_\alpha(x) \) and \( \psi_\alpha(x) \), and Fisher and Tippett (1928) who established that the limit laws for the maximum \( M_n \) properly normalized are reduced to only the types of \( \xi_\alpha(x) \), \( \psi_\alpha(x) \) and \( \Lambda(x) \).

In 1939 de Mises studied the problem systematically and found the following sufficient conditions for the parent distribution to be attracted to each of the three limit laws.

The (right) endpoint of a cdf \( F(x) \) is defined by

\[
x_0(F) = \sup \{x : F(x) < 1\}
\]

**Theorem 2.1.** Suppose that \( F(x) \) is a cdf with \( x_0(F) = \infty \) and derivative \( F'(x) \) for all \( x \) greater than some value \( x_2 \). If

\[
\lim_{x \to \infty} \frac{x F'(x)}{1 - F(x)} = \alpha, \quad 0 < \alpha < \infty
\]

then

\[
F^R(s_n x) \xrightarrow{w} \xi_\alpha(x)
\]

where
\[ a_n = \inf \{ x : 1 - F(x) \leq \frac{1}{n} \} , \quad n = 1, 2, \ldots \]

**Theorem 2.2.** Suppose that \( F(x) \) is a cdf with \( x_0(F) = x_0 < \infty \) and derivative \( F'(x) \) for all \( x \) in some interval \((x_2, x_0)\). If

\[
\lim_{x \to x_0^-} \frac{(x_0 - x) F'(x)}{1 - F(x)} = \alpha , \quad 0 < \alpha < \infty
\]

then \( F(x) \) belongs to the domain of attraction of \( \psi\alpha(x) \).

**Theorem 2.3.** Suppose that \( F(x) \) is a cdf with \( x_0(F) = \infty \) and second derivative \( F''(x) \) for all \( x \) greater than some value \( x_2 \). If

\[
\lim_{x \to \infty} \frac{d}{dx} \left( \frac{1 - F(x)}{F'(x)} \right) = 0
\]

then

\[
F^n \left( \frac{x}{n F'(b_n)} + b_n \right) \xrightarrow{w} \Lambda(x)
\]

where

\[ b_n = \inf \{ x : 1 - F(x) \leq \frac{1}{n} \} , \quad n = 1, 2, \ldots \]

Gnedenko (1943) made the remark that a theorem analogous to Theorem 2.3 can be stated when \( x_0(F) = x_0 < \infty \): Suppose that \( F(x) \) is a cdf
with $x_0(F) = x_0 < \infty$ and second derivative $F''(x)$ for all $x$ in some interval $(x_2, x_0)$. If

$$\lim_{x \to x_0^-} \frac{d}{dx} \left( \frac{1 - F(x)}{F'(x)} \right) = 0$$

then $F(x)$ belongs to the domain of attraction of $\Lambda(x)$.

It was in 1943 when Gnedenko presented the first characterizations of the domains of attraction of the three limit laws. He also gave some results which would become very helpful in the development of the theory. Some of his results which will be needed later are quoted here.

The next theorem was formulated originally in Fisher and Tippett (1928). Here we quote it in the form in which Gnedenko (1943) presented it.

**Theorem 2.4.** The class of the limit laws for $F^n(a_n x + b_n)$, where $a_n > 0$ and $b_n$ are conveniently chosen constants, consists only of the laws of the types $\xi_\alpha(x)$, $\phi_\alpha(x)$ and $\Lambda(x)$.

The following well-known theorem of Gnedenko (1943) is presented in its extended form as given by Feller (1966).

**Theorem 2.5.** Let $G_1(x)$ and $G_2(x)$ be two non-degenerate cdf's. If for a sequence $\{F_n\}$ of cdf's and constants $a_n > 0$, $b_n$ and $\alpha_n > 0$, $\beta_n$

$$F_n(a_n x + b_n) \xrightarrow{w} G_1(x), \quad F_n(\alpha_n x + \beta_n) \xrightarrow{w} G_2(x). \quad (2.1)$$
Then

\[ \frac{\alpha_n}{a_n} \rightarrow A > 0, \quad \frac{\beta_n - b_n}{a_n} \rightarrow B \]  

(2.2)

and

\[ G_2(x) = G_1(Ax + B). \]  

(2.3)

Conversely, if (2.2) holds then each of the two relations (2.1) implies the other and (2.3).

This theorem leads to the following formal definition of the concept of type.

**Definition 2.1.** The cdf \( G_1(x) \) is said to be of the same type as the cdf \( G_2(x) \) if there exist two constants \( a > 0 \) and \( b \) such that

\[ G_2(x) = G_1(ax + b) \quad \text{for all real } x. \]

This relation between \( G_1(x) \) and \( G_2(x) \) is symmetric, reflexive and transitive. Hence, it provides us with equivalence classes of distribution functions. These classes are called types. Sometimes a type is indicated by one representative of the equivalence class.

The subsequent lemma is used in several places in this dissertation.

**Lemma 2.1.** Suppose that \( \{a_n > 0\} \) and \( \{b_n\} \) are sequences of
real numbers. For cdf's $F(x)$ and $G(x)$ we have for a real $x$ with $0 < G(x) < 1$

$$\lim_{n \to \infty} F^{n}(a_n x + b_n) = G(x)$$

if and only if

$$\lim_{n \to \infty} n [1 - F(a_n x + b_n)] = -\log G(x) .$$

In 1970 de Haan approaches the works of Gnedenko and de Mises by means of regular variation theory and adds other characterizations of the domain of attraction of $\Lambda(x)$. Here we recall several results in the way he presented them since they will be very useful in the development of this thesis.

Let $\mathbb{R}^+$ be the set of positive real numbers, let $\mathbb{R}$ be the set of all real numbers and let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.

For $x > 0$ the following convention is adopted

$$x^\infty = \begin{cases} 0 & \text{for } x < 1 \\ 1 & \text{for } x = 1 \\ \infty & \text{for } x > 1 \end{cases}$$

$$x^{-\infty} = \begin{cases} \infty & \text{for } x < 1 \\ 1 & \text{for } x = 1 \\ 0 & \text{for } x > 1 \end{cases} .$$
Definition 2.2. A function \( U : \mathbb{R}^+ \to \mathbb{R}^+ \) varies regularly at infinity if there exists a \( \rho \in \mathbb{R} \) such that for all \( x \in \mathbb{R}^+ \)

\[
\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^\rho .
\]

This number \( \rho \) is called the exponent of regular variation for \( U \).

In the particular case when \( \rho = 0 \), \( U \) is often called slowly varying at infinity.

Definition 2.3. A function \( U : \mathbb{R}^+ \to \mathbb{R}^+ \) varies rapidly at infinity if for all \( x \in \mathbb{R}^+ \)

\[
\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^\rho
\]

where \( \rho = +\infty \) or \( \rho = -\infty \).

For brevity the expression \( \rho \)-varying (at infinity) will be used also for functions satisfying Definition 2.2 or Definition 2.3 (hence \( \rho \in \mathbb{R} \)).

Theorem 2.6. a) If a function \( U : \mathbb{R}^+ \to \mathbb{R}^+ \) is Lebesgue-summable on finite intervals and regularly varying with exponent \( \rho \), then there exist functions \( a : \mathbb{R}^+ \to \mathbb{R} \) and \( c : \mathbb{R}^+ \to \mathbb{R}^+ \) with

\[
\begin{align*}
\lim_{x \to \infty} a(x) &= \rho , \\
\lim_{x \to \infty} c(x) &= c_0 \quad (0 < c_0 < \infty)
\end{align*}
\] (2.4)
such that for all positive $x$

$$U(x) = c(x) \exp \left\{ \int_1^x \frac{a(t)}{t} \, dt \right\}.$$  \hspace{1cm} (2.5)

b) Every function of the form (2.5) where the auxiliary functions $c(x)$ and $a(x)$ satisfy (2.4) with finite or infinite $\rho$, is $\rho$-varying.

Corollary 2.1. If $U$ is $\rho$-varying at infinity ($-\infty < \rho < +\infty$) then

$$\lim_{x \to \infty} \frac{\log U(x)}{\log x} = \rho$$

and hence

$$\lim_{x \to \infty} U(x) = \begin{cases} 0 & \text{if } \rho < 0 \\ \infty & \text{if } \rho > 0 \end{cases}.$$

The next theorem was first formulated in Feller (1966).

Theorem 2.7. A distribution function $F(x)$ belongs to the domain of attraction of $\xi_\alpha(x)$ if and only if $1 - F(x)$ is $(-\alpha)$-varying at infinity.

Corollary 2.2. If $F \in \mathcal{B}(\xi_\alpha)$, then

$$F^\alpha(a_n x) \xrightarrow{w} \xi_\alpha(x)$$
with

\[ a_n = \inf \{ x : 1 - F(x) \leq \frac{1}{n} \}, \quad n = 1, 2, 3, \ldots \]

**Theorem 2.8.** A distribution function \( F(x) \) belongs to the domain of attraction of \( \psi_{\alpha}(x) \) if and only if \( F(x) \) has a finite endpoint \( x_0 \) and the function \( U \) defined by \( U(x) = 1 - F(x_0 - x^{-1}) \) for all \( x \in \mathbb{R}^+ \) is \((-\alpha)\)-varying at infinity.

**Corollary 2.3.** If \( F \in \mathcal{F}(\psi_{\alpha}) \), then

\[ F^n(\{x_0 - a_n\} x + x_0) \xrightarrow{w} \psi_{\alpha}(x) \]

with

\[ a_n = \inf \{ x : 1 - F(x) \leq \frac{1}{n} \}, \quad n = 1, 2, 3, \ldots \]

**Theorem 2.9.** A distribution function \( F(x) \) is in the domain of attraction of \( \Lambda(x) \) if and only if there exist functions \( a : \mathbb{R} \rightarrow \mathbb{R}^+ \) and \( b : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[ \lim_{s \rightarrow \infty} s \left\{ 1 - F(a(s)x + b(s)) \right\} = e^{-x} \quad \text{for all } x \in \mathbb{R}. \quad (2.6) \]

Moreover, then (2.6) holds with
\[
\begin{align*}
\begin{cases}
  b(s) = U\left(\frac{1}{s}\right) \\
  a(s) = U\left(\frac{1}{es}\right) - U\left(\frac{1}{s}\right) & \text{for all } s \in \mathbb{R}^+ 
\end{cases}
\end{align*}
\]  

(2.7)

where \( U: \mathbb{R}^+ \rightarrow \mathbb{R} \) is defined by

\[
U(x) = \inf \{y: 1 - F(y) \leq x\}.
\]

**Theorem 2.10.** A distribution function \( F(x) \) belongs to the domain of attraction of \( \Lambda(x) \) if and only if there exists a function \( f: \mathbb{R} \rightarrow \mathbb{R}^+ \) such that

\[
\lim_{t \to x^-} \frac{1 - F(t + xf(t))}{1 - F(t)} = e^{-x} \text{ for all } x \in \mathbb{R}.
\]

Here \( x^- \) is the endpoint of \( F(x) \).

**Theorem 2.11.** A distribution function \( F(x) \) belongs to the domain of attraction of \( \Lambda(x) \) if and only if

\[
\lim_{t \to x^-} \frac{1 - F(t + xf(t))}{1 - F(t)} = e^{-x} \text{ for all } x \in \mathbb{R}
\]

with

\[
f(t) = \frac{\int_{x_0}^t (1 - F(s)) \, ds}{1 - F(t)} \text{ for all real } t < x^-.
\]
Here $x_0$ is the endpoint of $F(x)$.

**Corollary 2.4.** If $F \in \mathcal{B}(\Lambda)$, then

$$F_n(a_n x + b_n) \xrightarrow{w} \Lambda(x)$$

with

$$
\begin{align*}
    b_n &= \inf \{ x : 1 - F(x) \leq \frac{1}{n} \} \\
    a_n &= \frac{b_n}{1 - F(b_n)}, \quad n = 1, 2, \ldots
\end{align*}
$$

**Theorem 2.12.** A distribution function $F(x)$ belongs to $\mathcal{B}(\Lambda)$ if and only if there exist a real constant $c_1$ and real-valued functions $c, a$ and $f$ defined on $(-\infty, x_0)$ with

$$
\begin{align*}
    c(x) > 0 \quad &\text{for all } x < x_0, \quad \lim_{x \to x_0^-} c(x) = c_1 > 0 \\
    \lim_{x \to x_0^-} a(x) = 1 \\
    f(x) \text{ positive and differentiable for all } x < x_0 \quad &\text{and} \quad \lim_{x \to x_0^-} f'(x) = 0 \\
    \text{moreover } \lim_{x \to x_0^-} f(x) = 0 \quad &\text{if } x_0 < \infty
\end{align*}
$$
such that for all $x < x_0$

$$1 - F(x) = c(x) \exp \left\{ - \int_{x_1}^{x} \frac{a(t)}{f(t)} \, dt \right\}. $$

Here $x_0$ is the endpoint of $F(x)$ and

$$x_1 = \begin{cases} 
1 & \text{if } x_0 = \infty \\
 x_0 - 1 & \text{if } x_0 < \infty.
\end{cases}$$

**Corollary 2.5.** If $F \in \mathcal{B}(\Lambda)$, then

$$\lim_{x \to \infty} \frac{\log \{1 - F(x)\}}{\log x} = -\infty \quad \text{if } x_0 = \infty$$

and

$$\lim_{x \to x_0} \frac{\log \{1 - F(x)\}}{\log (x_0 - x)} = \infty \quad \text{if } x_0 < \infty$$

where $x_0$ is the endpoint of $F(x)$. (Hence, if $x_0 = \infty$,

$$\lim_{x \to \infty} x^\alpha \{1 - F(x)\} = 0 \quad \text{for all } \alpha > 0 \quad \text{and thus} \quad \int_0^\infty x^\alpha dF(x) < \infty$$

for all $\alpha > 0$.)

Balkema and de Haan (1972) presented an additional characterization
of the domain of attraction of \( \Lambda(x) \). It is given in terms of de Misès' functions.

Suppose that \( F(x) \) is a cdf with density \( f(x) \) which is positive and differentiable on a left neighborhood of \( x_\circ = \sup \{x: F(x) < 1\} \).

If

\[
\lim_{x \to x_\circ^-} \frac{\frac{d}{dx} \left( \frac{1 - F(x)}{f(x)} \right)}{1 - F(x)} = 0 ,
\]

then \( F(x) \) is called a de Misès' function.

**Theorem 2.13.** A cdf \( F(x) \) belongs to the domain of attraction of \( \Lambda(x) \) if and only if there exists a de Misès function \( F_*(x) \) such that \( x_\circ(F) = x_\circ(F_*) = x_\circ \) and

\[
\lim_{x \to x_\circ^-} \frac{1 - F(x)}{1 - F_*(x)} = 1 .
\] (2.8)

Next we quote a result of Resnick (1971) that has to do with tail equivalence and domains of attraction.

Two cdfs \( F(x) \) and \( G(x) \) are tail equivalent if and only if \( x_\circ(F) = x_\circ(G) = x_\circ \) and

\[
\lim_{x \to x_\circ^-} \frac{1 - F(x)}{1 - G(x)} = a , \quad 0 < a < \infty .
\]
Resnick (1971) makes the remark that for two arbitrary distributions the ratio of the tails need not have a limit as $x \to x_0$.

**Theorem 2.1.** Let $F(x), G(x)$ be distribution functions and let $\xi(x)$ be an extreme value distribution. Suppose $F \in B(\xi)$ and let $F_n(a_n x + b_n) \xrightarrow{w} \xi(x)$ for norming constants $a_n > 0$ and $b_n$, $n \geq 1$. Then $G_n(a_n x + b_n) \xrightarrow{w} \xi(x), \xi(x)$ non-degenerate, if and only if for some $A > 0, B$:

\[ \xi(x) = \xi(Ax + B), \]

\[ x_0(F) = x_0(G) = x_0, \]

\[ \lim_{x \to x_0} \frac{1 - F(x)}{1 - G(x)} \text{ exists}, \]

and if

(i) $\xi(x) = \xi(x), \alpha > 0$, then $B = 0$ and

\[ \lim_{x \to \infty} \frac{1 - F(x)}{1 - G(x)} = A^\alpha; \]

(ii) $\xi(x) = \xi(x), \alpha > 0$, then $B = 0$ and

\[ \lim_{x \to x_0^-} \frac{1 - F(x)}{1 - G(x)} = A^{-\alpha}; \]

(iii) $\xi(x) = \Lambda(x)$, then $A = 1$ and

\[ \lim_{x \to x_0^-} \frac{1 - F(x)}{1 - G(x)} = e^B. \]
Notice that if a cdf $F_*(x)$ is a de Mise's function, then by Theorem 2.3, $F_* \in \mathcal{S}(\Lambda)$. Therefore, by Theorem 2.14, the limit relation (2.8) implies that for the convergence of the cdf's $F^N(x)$ and $F^N_*(x)$ the same norming constants $a_n > 0$ and $b_n$ may be used.

So far we have seen that all the characterizations of the domain of attraction of $\Lambda(x)$ and sufficient conditions for a distribution to belong to such a domain make use of the inverse function of the tail function $1 - F(x)$ to construct the appropriate norming constants $a_n > 0$ and $b_n$ for

$$F^n(a_n x + b_n) \xrightarrow{w} \Lambda(x).$$

In some instances, however, the inverse function of $1 - F(x)$ is practically impossible to obtain. As an example, consider the cdf defined by

$$F(x) = \begin{cases} 
1 - (x + 1) e^{-x^2} & \text{if } x \geq 1 \\
0 & \text{if } x < 1.
\end{cases} \quad (2.9)$$

We can see that the conditions of Theorem 2.3 are satisfied, so $F(x)$ belongs to the domain of attraction of $\Lambda(x)$. However, we cannot know which is the linear function of $M_n$ (defined in Chapter I) whose distribution tends to $\Lambda(x)$ as $n$ tends to infinity.
This problem is solved for a wide class of distributions in our following result.

**Theorem 2.15.** Let $F(x)$ be a cdf with endpoint at infinity.

For some real constants $\alpha > 0$ and $\beta$

$$
\lim_{x \to \infty} x^\beta e^{\gamma x} [1 - F(x)] = c, \quad 0 < c < \infty
$$

(2.10)

if and only if

$$
F^n(a_n x + b_n) \xrightarrow{w} \Lambda(x)
$$

(2.11)

with

$$
a_n = (\alpha \log n c)^{\frac{\alpha-1}{\alpha}} - 1
$$

and

$$
b_n = (\log n c)^{\frac{1}{\alpha}} - \frac{\beta \log \log n c}{\alpha^2 (\log n c)^{\frac{\alpha-1}{\alpha}}}
$$

(2.12)

For the proof of this theorem, see Part A of the Appendix.

**Corollary 2.6.** Let $F(x)$ be a cdf with endpoint at infinity.

For some real constants $\alpha > 0$, $k > 0$, $\beta$ and $\gamma$

$$
\lim_{x \to \infty} (kx + \beta) e^{(kx+\beta)x} [1 - F(x)] = b, \quad 0 < b < \infty
$$
if and only if

$$F^n(a_n x + b_n) \xrightarrow{w} \Lambda(x)$$

with

$$\begin{align*}
  a_n &= (ck \{\log n\} \alpha) \\
  b_n &= \left[\left\{\log n\right\}\alpha \right] k \left(1 - \frac{\log\log n}{\log n}\right) + \frac{\log\log n}{4} \left\{\log n\right\}^{1/2}.
\end{align*}$$

For the distribution defined in (2.9) we can easily see that

$$\lim_{x \to \infty} x^{-1} e^{x^2 \{1 - F(x)\}} = 1.$$

Then, from Theorem 2.15, the norming constants are

$$\begin{align*}
  a_n &= \frac{1}{2} \{\log n\} - \frac{1}{2} \\
  b_n &= \left[\left\{\log n\right\}\frac{1}{2}\right] + \frac{\log\log n}{4} \left\{\log n\right\}^{1/2}.
\end{align*}$$

hence

$$\lim_{n \to \infty} P(2 \sqrt{\log n} M_n - 2 \log n - \log \sqrt{\log n} \leq x) = \Lambda(x).$$
for all $x \in \mathbb{R}$.

As applications of Corollary 2.6, we furnish the following examples.

Example 2.1. Let $F(x)$ be the normal distribution with mean $\mu$ and variance $\sigma^2$.

If $x > \mu$ then

$$
\frac{\sigma}{x - \mu} \left( 1 - \left[ \frac{\sigma}{x - \mu} \right]^2 \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2} \leq 1 - F(x) \leq \frac{\sigma}{x - \mu} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2}.
$$

Hence

$$
\lim_{x \to \infty} \frac{(x - \mu)^2}{\sqrt{2\sigma}} e^{\frac{1}{\sqrt{2\sigma}}} \{1 - F(x)\} = \frac{1}{2\sqrt{\pi}} > 0.
$$

Then, by Corollary 2.6 with $\alpha = 2$, $k = \frac{1}{\sqrt{2\sigma}}$, $\beta = 1$, $\ell = -\frac{\mu}{\sqrt{2\sigma}}$

$$
F_n(a_n x + b_n) \xrightarrow{k} \Lambda(x)
$$

with

$$
\begin{align*}
\begin{cases}
ea_n = \sigma \left( 2 \log \frac{n}{2\sqrt{\pi}} \right)^{-\frac{1}{2}} \\
b_n = \sigma \left( 2 \log \frac{n}{2\sqrt{\pi}} \right)^{-\frac{1}{2}} - \frac{\log \log \frac{n}{2\sqrt{\pi}}}{\sqrt{2\sigma} \left( \log \frac{n}{2\sqrt{\pi}} \right)^{\frac{1}{2}}} + \mu.
\end{cases}
\end{align*}
$$
That is

$$\lim_{n \to \infty} P\left( \frac{1}{\sigma} \left[ 2 \log \frac{n}{2 \sqrt{\pi}} \right]^2 \log \frac{n}{2 \sqrt{\pi}} - 2 \log \frac{n}{2 \sqrt{\pi}} + \frac{1}{2} \log \log \frac{n}{2 \sqrt{\pi}} \right) = N \left( \sigma \sqrt{2 \log \frac{n}{2 \sqrt{\pi}}}, 1 \right)$$

for all $x \in \mathbb{R}$.

**Example 2.2.** Let $F(x)$ be the gamma($\nu$, $\gamma$)-distribution with $\nu$ an integer.

Notice that using integration by parts

$$1 - F(x) = \gamma F'(x)(1 + \frac{\nu-1}{x/\gamma} + \frac{(\nu-1)(\nu-2)}{(x/\gamma)^2} + \ldots + \frac{(\nu-1)!}{(x/\gamma)^{\nu-1}})$$

hence

$$\lim_{x \to \infty} x^{1-\nu} e^{x/\gamma} \left[ 1 - F(x) \right] = \frac{1}{\gamma^\nu \Gamma(\nu)} > 0$$

Then by Corollary 2.6 with $\alpha = 1$, $k = 1/\gamma$, $\beta = 1-\nu$, $\lambda = 0$

$$F^n(a_n x + b_n) \stackrel{w}{\longrightarrow} \Lambda(x)$$

where
\[ a_n = \gamma \]
\[ b_n = \gamma \log \frac{n}{\gamma^\nu \Gamma(\nu)} + \gamma(\nu-1) \log \log \frac{n}{\gamma^\nu \Gamma(\nu)}. \]

That is

\[ \lim_{n \to \infty} P\left( \frac{1}{\gamma} M_n - \log \frac{n}{\gamma^\nu \Gamma(\nu)} - (\nu - 1) \log \log \frac{n}{\gamma^\nu \Gamma(\nu)} \leq x \right) = \Lambda(x) \]

for all \( x \in \mathbb{R} \).

Next we state two lemmas suggested by Lamperti (1966). These lemmas are useful since the norming constants are given without going about determining the inverse function of the tail function.

**Lemma 2.2.** Let \( F(x) \) be a cdf with endpoint at infinity. For a real constant \( \alpha > 0 \)

\[ \lim_{x \to \infty} x^\alpha \{1 - F(x)\} = b, \quad 0 < b < \infty \]

if and only if

\[ F^n(a_n x) \xrightarrow{\mathcal{L}} \bar\varphi_\alpha(x) \]

with
To illustrate the practical use of Lemma 2.2 in comparison to Theorem 2.1 and Corollary 2.2, consider the cdf defined by

$$F(x) = \begin{cases} 
1 - \frac{1}{x} - \frac{2}{x^2} & \text{if } x > 2 \\
0 & \text{if } x \leq 2.
\end{cases}$$

We can see that to obtain the inverse function of $1 - F(x) = \frac{1}{x} + \frac{2}{x^2}$ is not an easy task, while applying Lemma 2.2, since $\lim_{x \to \infty} x \{1 - F(x)\}$ 1, we have that

$$\lim_{n \to \infty} F\left(\frac{1}{n} \cdot \frac{1}{x} \leq x\right) = \frac{1}{n} \cdot \frac{1}{x} \leq x$$

for all $x \in \mathbb{R}$.

**Lemma 2.3.** Let $F(x)$ be a cdf with $x_0(F) = x_0 < \infty$. For a real constant $\alpha > 0$

$$\lim_{x \to x_0} \frac{1 - F(x)}{(x_0 - x)^\alpha} = b, \quad 0 < b < \infty$$

if and only if

$$F^n(a_n x + x_0) \xrightarrow{w} \Phi(x)$$
with 

\[ a_n = \left( \frac{\alpha}{n} \right)^{-\frac{1}{\alpha}}. \]

For the case when the random variables of the sequence \{X_i\} are dependent, several authors have studied the problem for different kinds of dependence.

Watson (1954) and then Newell (1964) showed that if the sequence \{X_i\} is m-dependent (i.e., \(X_i\) and \(X_j\) are independent when \(|i - j| > m\)) then the limit distribution of \(M_n\) suitably normalized is the same as in the case when the \(X_i\)'s are independent.

For mixing sequences of random variables, Loynes (1965) found similar results.

In a recent paper, Galambos (1972) proved a more general result for dependent sequences. Under weaker conditions on dependence and without stationarity, he obtained the same limit distribution as if the \(X_i\)'s were independent. Consequently, Galambos' result includes the previous ones along the same lines.

For another sort of dependence, Berman (1962) found the class of limit distributions that can be obtained by using the norming constants for iid random variables in sequences of exchangeable random variables. Unlike the above cases of dependence, these limit distributions are mixtures of the limit distributions for the case when the random variables are iid.

For further references, see Gumbel (1958) who collected most of the important results in univariate extreme value theory subsequent to
the works of Fisher and Tippett (1928) and Gnedenko (1943). Also see
the book by David (1970) for a more recent bibliography.

The latest studies of the theory have been mainly on the multi­
variate case, particularly on the bivariate extreme value problem.

A general form of the limit distribution of \( (M_n^{(1)}, M_n^{(2)}) \),
(defined in Chapter I), properly normalized, has been obtained by
Finkelstein (1953), Geffroy (1959), de Oliveira (1959) and Sibuya
(1960). These forms include all possible bivariate extreme distribu­
tions. Gumbel (1962) has shown that all these generalized forms are
mutually consistent.

Geffroy (1959) also proved that \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically
independent when the parent bivariate distribution is normal. This same
result was found by Sibuya (1960).

Berman (1961) presents necessary and sufficient conditions for a
bivariate cdf to be attracted to a bivariate asymptotic extreme dis­
tribution of the form given by Sibuya (1960). He imposes the assump­
tion that the marginal distributions are attracted to some univariate
asymptotic extreme distributions. In the same article, Berman presents
a sufficient condition for the asymptotic independence of the maxima
in the multivariate case. This condition is the generalization of the
condition given by Geffroy (1959) for the bivariate case.

de Oliveira (1961) gives another representation of his general
form of the bivariate asymptotic extreme distribution. This representa­
tion is unique and depends on a distribution function and a bounded
parameter; in the absolutely continuous case the representation is
given by an almost arbitrary density function and a bounded parameter.

In 1962 de Oliveira makes a review of the theory of bivariate asymptotic extreme distributions. He gives a unification of the theory up to that time, presenting more simple proofs and adding some properties of the extreme distributions.

Campbell and Tsokos (1973) found the form of the bivariate asymptotic extreme distribution when the parent bivariate distribution admits a certain canonical expansion and under the assumption that its marginal distributions are attracted to some univariate asymptotic extreme distributions. They also proved that when the parent distribution is the Kibble (1941) bivariate gamma distribution or the compound correlated Poisson distribution, then $M_n^{(1)}$ and $M_n^{(2)}$ are asymptotically independent.

Galambos (1973) gives a necessary and sufficient condition for a multivariate distribution to be attracted to his generalized asymptotic extreme distribution which is similar to that of Finkelstein (1953). Galambos' condition relies heavily on the norming constants of the univariate marginal distributions; from this we deduce that it is necessary for the univariate marginal distributions to be attracted to some asymptotic extreme distributions. Because of the involvement of the norming constants, we might say that this is not a domain of attraction type condition. In the same article he presents a sufficient condition for the asymptotic independence. This condition also relies on the norming constants.
Some more references can be found in the book by Johnson and Kotz (1972).
III. CONVOLUTIONS AND DOMAINS OF ATTRACTION

Suppose that we are observing a sequence of random variables \( \{X_n\} \), and let \( \{Y_n\} \) be a sequence of random errors associated with \( \{X_n\} \). That is, instead of observing \( \{X_n\} \) we are actually observing \( \{X_n + Y_n\} \). This chapter is concerned with determining the asymptotic extreme distribution of the sequence \( \{\max(X_1 + Y_1, \ldots, X_n + Y_n)\} \).

We assume that the \( X_n \)'s are independent identically distributed rv's and independent of the independent identically distributed rv's \( \{Y_n\} \). Under these assumptions, if the \( \{X_n\} \) and \( \{Y_n\} \) have distributions \( F(x) \) and \( H(x) \), respectively, then we know that the convolution of \( F(x) \) and \( H(x) \), denoted by \( F * H(x) \), is the common distribution of the independent identically distributed rv's \( \{X_n + Y_n\} \). Then the problem is reduced to establishing the domain of attraction in which the convolution lies.

We investigate the problem for all different cases when \( F(x) \) and \( H(x) \) belong to the domain of attraction of each of the three asymptotic extreme distributions. The results of this chapter will be applied to the bivariate case later.

The first lemma is helpful in proving a result with some applications of interest through its corollary.

**Lemma 3.1.** Suppose that \( F(x) \) and \( H(x) \) are cdf's with common endpoint \( x_0 = \infty \). Let \( G(x) \) be an asymptotic extreme distribution. If \( F(x) \) belongs to the domain of attraction of \( G(x) \) and there exist real constants \( a > 0 \) and \( b \) such that
then $H(x)$ belongs to the domain of attraction of $G(x)$.  

Proof: Since $F \in \mathcal{F}(G)$, there exist sequences of real constants $\{a_n > 0\}$ and $\{b_n\}$ such that

$$F^n(a_n x + b_n) \xrightarrow{w} G(x).$$

Define the distribution $F_*(x)$ by

$$F_*(x) = F(ax + b).$$

Then, if we let

$$\alpha_n = \frac{a_n}{a} \quad \text{and} \quad \beta_n = \frac{b_n - b}{a} \quad \text{for} \quad n = 1, 2, \ldots$$

we have that

$$F_*^{n}(\alpha_n x + \beta_n) = F^n(a_n x + b_n) \xrightarrow{w} G(x) \quad \text{i.e.} \quad F_* \in \mathcal{F}(G).$$

On the other hand

$$\lim_{x \to \infty} \frac{1 - H(x)}{1 - F(x)} = \frac{1}{L}, \quad \lim_{x \to \infty} \frac{1 - F(ax + b)}{1 - H(x)} = \frac{1}{L}, \quad 0 < L < \infty.$$
Thus, by Theorem 2.1, \( H(x) \) belongs to the domain of attraction of \( G(x) \). []

**Theorem 3.1.** Suppose that \( F(x) \) and \( H(x) \) are cdf's with common endpoint \( x_\infty = \infty \). Let \( G(x) \) be an asymptotic extreme distribution. If (1) \( F(x) \) belongs to the domain of attraction of \( G(x) \), (2) there exist constants \( a > 0, \alpha > 0, b, \beta \) and a cdf \( J(x) \) with endpoint at \( x_\infty = \infty \) such that

\[
1 - F * H(x) = K \{1 - F(ax + b)\} J(\alpha x + \beta) + R(x) \tag{3.1}
\]

with \( 0 < K < \infty \) and (3)

\[
\lim_{x \to \infty} \frac{R(x)}{1 - F(ax + b)} = 0 .
\]

Then \( F * H(x) \) belongs to the domain of attraction of \( G(x) \).

**Proof:** Since \( F(x) \) and \( H(x) \) have endpoint at infinity, it follows that the endpoint of \( F * H(x) \) is at infinity. From (2) and (3) we have that

\[
\lim_{x \to \infty} \frac{1 - F * H(x)}{1 - F(ax + b)} = \lim_{x \to \infty} \left\{ K J(\alpha x + \beta) + \frac{R(x)}{1 - F(ax + b)} \right\}
\]

\[
= K \lim_{x \to \infty} J(\alpha x + \beta) + \lim_{x \to \infty} \frac{R(x)}{1 - F(ax + b)}
\]

\[
= K .
\]
From this and hypothesis (1), we have by Lemma 3.1 that \( F \ast H(x) \) belongs to the domain of attraction of \( G(x) \).

As a direct application of this theorem consider the following example.

**Example 3.1.** Suppose that \( F(x) \) is a stable distribution with endpoint \( x_0 = \infty \) and \( F(x) \) belongs to the domain of attraction of an asymptotic extreme distribution \( G(x) \). Then there exist constants \( a > 0 \) and \( b \) such that

\[
1 - F \ast F(x) = 1 - F(ax + b).
\]

Therefore, taking \( \beta = \infty \) we have by Theorem 3.1 that \( F \ast F(x) \) belongs to the domain of attraction of \( G(x) \).

**Corollary 3.1.** Suppose that \( F(x) \) is a gamma(\( \nu \), \( \beta \))-distribution with \( \nu \) an integer and \( H(x) \) is a cdf with endpoint at \( x_0 = \infty \). If the density function \( h(x) \) of \( H(x) \) is such that

\[
e^{x/\beta} h(x) = K j(x)
\]

where \( K \) is a positive constant and \( j(x) \) is a probability density with \( (\nu-1) \)-st finite moment, then \( F \ast H(x) \) belongs to the domain of attraction of \( A(x) \).

**Proof:** (1) By Example 2.2 \( F \in \mathcal{S}(\lambda) \). (2) By integration by parts we have for \( x > 0 \)
\[ 1 - F(x) = \beta F'(x) \{1 + \frac{(v-1)\beta}{x} + \frac{(v-1)(v-2)\beta^2}{x^2} + \cdots + \frac{(v-1)!\beta^{v-1}}{x^{v-1}}\} \]

\[ = e^{-x/\beta} \sum_{\ell=0}^{v-1} \frac{(x/\beta)^\ell}{\ell!} . \]

Hence

\[ 1 - F \star H(x) = \int_{-\infty}^{+\infty} [1 - F(x-y)] dH(y) \]

\[ = \int_{-\infty}^{x} [1 - F(x-y)] dH(y) + \int_{x}^{+\infty} [1 - F(x-y)] dH(y) \]

\[ = \int_{-\infty}^{x} e^{-y/\beta} \sum_{\ell=0}^{v-1} \frac{(x-y)^\ell}{\beta^\ell \ell!} dH(y) + \{1 - H(x)\} \]

\[ = e^{-x/\beta} \sum_{\ell=0}^{v-1} \frac{(x-y)^\ell}{\beta^\ell \ell!} \int_{-\infty}^{x} e^{y/\beta} h(y) dy + \{1 - H(x)\} \]

\[ = Ke^{-x/\beta} \sum_{\ell=0}^{v-1} \frac{(x-y)^\ell}{\beta^\ell \ell!} \int_{-\infty}^{x} j(y) dy + \{1 - H(x)\} . \]

Let \( J(y) \) be the cdf of \( j(y) \), and since

\[ (x-y)^\ell = \sum_{k=0}^{\ell} \binom{\ell}{k} x^{\ell-k} (-1)^k y^k \]

we have

$$1 - F \ast H(x) = Ke^{-x/\beta} \int_{-\infty}^{\infty} \sum_{l=0}^{v-1} \frac{1}{\beta^l l!} \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} x^{\ell-k} y^k dJ(y) + \{1 - H(x)\}$$

$$= Ke^{-x/\beta} \sum_{l=0}^{v-1} \frac{x^l}{\beta^l l!} \int_{-\infty}^{x} dJ(y)$$

$$+ Ke^{-x/\beta} \sum_{l=1}^{v-1} \frac{1}{\beta^l l!} \sum_{k=1}^{\ell} (-1)^k \binom{\ell}{k} x^{\ell-k} \int_{-\infty}^{x} y^k dJ(y)$$

$$+ \{1 - H(x)\}$$

$$= K \{1 - F(x)\} J(x) + R(x)$$

where

$$R(x) = Ke^{-x/\beta} \sum_{l=1}^{v-1} \frac{1}{\beta^l l!} \sum_{k=1}^{\ell} (-1)^k \binom{\ell}{k} x^{\ell-k} \int_{-\infty}^{x} y^k dJ(y) + \{1 - H(x)\}.$$ 

(3) Notice that by L'Hospital's rule

$$\lim_{x \to \infty} \frac{1 - H(x)}{1 - F(x)} = \lim_{x \to \infty} \frac{h(x)}{F'(x)} = \lim_{x \to \infty} \frac{K \cdot j(x) e^{-x/\beta}}{x^{v-1} e^{-x/\beta}} = 0.$$ 

Then
\[
\frac{R(x)}{1 - F(x)} = \frac{K e^{-x/\beta} \sum_{k=1}^{\nu-1} \frac{1}{\beta^\nu} \sum_{\ell=1}^{\nu} (-1)^k (\frac{\ell}{k}) x^{\ell-k} \int_{-\infty}^{x} y^{k-1} g(y) \, dy}{x^{\nu-1} e^{-x/\beta} \frac{1}{\Gamma(\nu)} \frac{1 + (\nu-1)B}{x} + \frac{(\nu-1)(\nu-2)B^2}{x^2} + \cdots + \frac{(\nu-1)!B^{\nu-1}}{x^{\nu-1}}} + \frac{1 - H(x)}{1 - F(x)} \quad x \to \infty \to 0
\]

since the largest power of \( x \) in the numerator of the first term is \( \nu-2 \), and

\[
\lim_{x \to \infty} \int_{-\infty}^{x} y^k \, dJ(y) = \int_{-\infty}^{\infty} y^k \, dJ(y) < \infty
\]

for all \( k \leq \nu-1 \).

Therefore, by Theorem 3.1, \( F \ast H(x) \) belongs to the domain of attraction of \( A(x) \). \( \Box \)

As an illustration of the last corollary, consider the following examples for the classes of normal and gamma distributions.

**Example 3.2.** Let \( F(x) \) be a gamma(\( \nu, \rho \))-distribution and let \( H(x) \) be the normal(\( \mu, \sigma^2 \))-distribution. Then since
where \( j(x) \) is the probability density of the \( \text{normal}(\mu + \frac{\sigma^2}{\beta}, \sigma^2) \)-distribution and it has all finite moments. Then, by Corollary 3.1, \( F \ast H(x) \) belongs to the domain of attraction of \( \Lambda(x) \).

**Example 3.3.** Let \( F(x) \) be the \( \text{gamma}(\nu, \beta) \)-distribution with \( \nu \) an integer and let \( H(x) \) be the \( \text{gamma}(\alpha, \delta) \)-distribution with \( \delta < \beta \). Then

\[
e^{x/\beta} h(x) = e^{x/\beta} \frac{x^{\alpha-1} e^{-x/\delta}}{\Gamma(\alpha) \delta^\alpha}
\]

\[
= \left( \frac{\beta}{\beta-\delta} \right)^\alpha x^{\alpha-1} \exp \left\{ -x/(\frac{\delta\beta}{\beta-\delta}) \right\} \frac{1}{\Gamma(\alpha) \left( \frac{\delta\beta}{\beta-\delta} \right)^\alpha}
\]

\[
= K j(x)
\]

where the \( j(x) \) is the probability density of the \( \text{gamma}(\alpha, \frac{\delta\beta}{\beta-\delta}) \)-distribution and it has all finite moments. Therefore, by Corollary
3.1, $F \ast H(x)$ belongs to the domain of attraction of $\Lambda(x)$.

**Remark 3.1.** In Example 3.3 when $\delta = \beta$ we know that $F \ast H(x)$ is the gamma($\nu + \alpha$, $\beta$)-distribution which by Example 2.2, belongs to the domain of attraction of $\Lambda(x)$ provided $\nu + \alpha$ is an integer.

Next we present a lemma which provides us with bounds for the tail function of the convolution of two distributions. These bounds were found by Feller (1966). Here we present a slightly different proof from his.

**Lemma 3.2.** If $F(x)$ and $H(x)$ are two distribution functions, then for $t > 0$ and $\varepsilon > 0$

$$\{1 - F(t(l+\varepsilon))\} \{H(te) - H(-te)\} + \{1 - H(t(l+\varepsilon))\} \{F(te) - F(-te)\}$$

$$\leq 1 - F \ast H(t) \leq \{1 - F(t(l-\varepsilon))\} + \{1 - H(t(l-\varepsilon))\} + [1 - F(te)] [1 - H(te)].$$

**Proof:** Suppose that $X$ and $Y$ are two independent random variables with distribution $F(x)$ and $H(x)$, respectively. Consider the events

$$A = [X + Y > t], \quad B = [X > t(l+\varepsilon), |Y| < te] \quad \text{and} \quad C = [Y > t(l+\varepsilon), |X| < te].$$

Suppose that $\omega \in B \cap C$. Then $X(\omega) > t(l+\varepsilon)$ and $-te < X(\omega) < te$ which is impossible since $t > 0$. Therefore $B$ and $C$ are disjoint.
Now suppose that \( \omega \in B \cup C \). Then \( \omega \in B \) or \( \omega \in C \). If \( \omega \in B \) then \( X(\omega) > t(1+\varepsilon) \) and \( -t\varepsilon < Y(\omega) < t\varepsilon \). This implies \( X(\omega) + Y(\omega) > t \). That is, \( \omega \in A \). Similarly, if \( \omega \in C \) then \( \omega \in A \). Therefore, \( B \cup C \subseteq A \). Hence, since \( B \) and \( C \) are disjoint,

\[
P(A) \geq P(B) + P(C).
\]

Therefore, since \( X \) and \( Y \) are independent,

\[
P(X + Y > t) \geq P(X > t(1+\varepsilon)) P(|Y| < t\varepsilon) + P(Y > t(1+\varepsilon)) P(|X| < t\varepsilon).
\]  

(3.2)

On the other hand, notice that

\[
[X \in Y > t] = [X + Y > t, X > t(1-\varepsilon)] \cup [X + Y > t, Y > t(1-\varepsilon), X \leq t(1-\varepsilon)]
\]

\[
U [X + Y > t, Y \leq t(1-\varepsilon), X \leq t(1-\varepsilon)],
\]

where it is clear that the events on the right side are disjoint. Now, since

\[
[X + Y > t, X > t(1-\varepsilon)] \subseteq [X > t(1-\varepsilon)]
\]

and

\[
[X + Y > t, Y > t(1-\varepsilon), X \leq t(1-\varepsilon)] \subseteq [Y > t(1-\varepsilon)],
\]

\[
P(X + Y > t) \leq P(X > t(1-\varepsilon)) + P(Y > t(1-\varepsilon)) + P(X + Y > t, Y \leq t(1-\varepsilon), X \leq t(1-\varepsilon)).
\]
Consider the events
\[ D = [X + Y > t, Y \leq t(1-\varepsilon), X \leq t(1-\varepsilon)] \]
and
\[ E = [X > t\varepsilon, Y > t\varepsilon] . \]

Suppose that \( \omega \in E^c \cap D \). Then \( X(\omega) \leq t\varepsilon \) and \( Y(\omega) \leq t(1-\varepsilon) \). This implies \( X(\omega) + Y(\omega) \leq t \) which is impossible since \( \omega \in D \) implies \( X(\omega) + Y(\omega) > t \). Therefore \( E^c \cap D = \emptyset \). This implies \( E^c \subset D^c \) and hence \( D \subset E \). Therefore, since \( X \) and \( Y \) are independent,

\[
P(X + Y > t) \leq P(X > t(1-\varepsilon)) + P(Y > t(1-\varepsilon)) + P(X > t\varepsilon)P(Y > t\varepsilon). \tag{3.3}
\]

Using the independence of \( X \) and \( Y \) once more, we have that

\[
1 - F*H(t) = P(X + Y > t) .
\]

Thus, from (3.2) and (3.3) we have the desired conclusion. 

We are interested in knowing about the behaviour of the tail function of the convolution related to the convolutant functions. The subsequent theorem gives insight into such a relation.

**Theorem 3.2.** Suppose that \( F(x) \) and \( H(x) \) are distribution functions. If \( F(x) \) belongs to the domain of attraction of \( \varphi_\alpha(x) \) and

\[
\lim_{x \to \infty} \frac{1 - H(x)}{1 - F(x)} = L , \quad 0 \leq L < \infty
\]
then \( F \ast H(x) \) belongs to the domain of attraction of \( \bar{\phi}_\alpha(x) \).

Proof: Since \( F \in \mathcal{B}(\bar{\phi}_\alpha) \), by Corollary 2.2, there exists a sequence \( \{a_n > 0\} \) with \( a_n \rightarrow \infty \) and such that

\[
F^n(a_n x) \rightarrow \bar{\phi}_\alpha(x).
\]

For any \( 0 < \varepsilon < 1 \) let \( \alpha = a_n(1 + \varepsilon) \) and \( \alpha' = a_n(1 - \varepsilon) \). Then

\[
\frac{\alpha}{a_n} \rightarrow 1 + \varepsilon \quad \text{and} \quad \frac{\alpha'}{a_n} \rightarrow 1 - \varepsilon.
\]

Hence, by Theorem 2.5

\[
F^n(\alpha x) \rightarrow \bar{\phi}_\alpha((1+\varepsilon)x) \quad \text{and} \quad F^n(\alpha' x) \rightarrow \bar{\phi}_\alpha((1-\varepsilon)x). \quad (3.4)
\]

Note that by Lemma 3.2

\[
n \{1 - F \ast H(a_n t)\} \geq n \{1 - F(a_n t(1+\varepsilon))\} \left\{ [H(ta_n \varepsilon) - H(-ta_n \varepsilon)] + \frac{1 - H(ta_n (1+\varepsilon))}{1 - F(ta_n (1+\varepsilon))} \cdot [F(ta_n \varepsilon) - F(-ta_n \varepsilon)] \right\}
\]

and

\[
n \{1 - F \ast H(a_n t)\} \leq n \{1 - F(a_n t(1-\varepsilon))\} \left\{ 1 + \frac{1 - H(ta_n (1-\varepsilon))}{1 - F(ta_n (1-\varepsilon))} \right\}
\]
From the first relation in (3.4) and Lemma 2.1 we have that for all $x$ for which $0 < \beta_{(1+\epsilon)x} < 1$,

$$
\lim_{n \to \infty} n \{1 - F(\alpha_{n}(1+\epsilon)x)\} = - \log \beta_{(1+\epsilon)x} \cdot (1-L) \log \phi_{\alpha}(1+\epsilon)t.
$$

Since, by hypothesis,

$$
\lim_{n \to \infty} \frac{1 - H(\alpha_{n}(1+\epsilon))}{1 - F(\alpha_{n}(1+\epsilon))} = L
$$

and the fact that

$$
\lim_{n \to \infty} \{F(\alpha_{n} - \epsilon) - F(-\alpha_{n} - \epsilon)\} = 1
$$

and

$$
\lim_{n \to \infty} \{H(\alpha_{n} - \epsilon) - H(-\alpha_{n} - \epsilon)\} = 1,
$$

we have, from the first inequality, that

$$
\lim_{n \to \infty} \frac{n \{1 - F(\alpha_{n}t)\}}{1 - F(\alpha_{n}(1+\epsilon))} \geq - (1+L) \log \phi_{\alpha}(1+\epsilon)t.
$$
On the other hand, from the second relation in (3.4) and Lemma 2.1 we have that for all $x$ for which $0 < s^{((1-\varepsilon)x)} < 1$,

$$\lim_{n \to \infty} n \{1 - F(a_n(1-\varepsilon)x)\} = -\log s^{((1-\varepsilon)x)}.$$

Since $F \in S(\check{\alpha})$, by Theorem 2.7, $1 - F(x)$ is $(-\alpha)$-varying, hence

$$\lim_{n \to \infty} \frac{1 - F(ta_n(1-\varepsilon))}{1 - F(ta_n(1-\varepsilon))} = \frac{1 - F(ta_n(1-\varepsilon) + \frac{\varepsilon}{1-\varepsilon})}{1 - F(ta_n(1-\varepsilon))} = \left(\frac{e}{1-\varepsilon}\right)^{-\alpha}.$$

Also, by hypotheses,

$$\lim_{n \to \infty} \frac{1 - H(ta_n(1-\varepsilon))}{1 - F(ta_n(1-\varepsilon))} = L$$

and

$$\lim_{n \to \infty} \{1 - H(ta_n\varepsilon)\} = 0.$$

Therefore, from the second inequality, we have that

$$\lim_{n \to \infty} n \{1 - F * H(a_n t)\} \leq -(1+L) \log s^{\check{\alpha}((1-\varepsilon)t)}.$$

Since $\lim \leq \lim$ and $s^{\check{\alpha}(x)}$ is a continuous function, letting $\varepsilon$ tend to zero we conclude that
\[ \lim_{n \to \infty} n \left\{ 1 - F^*H(a^n) \right\} = -(1+L) \log \tilde{\phi}_\alpha(t). \]

By Lemma 2.1,

\[ \lim_{n \to \infty} (F^*H)^n (a^n) = \tilde{\phi}_\alpha^{1+L}(t). \]

Since \( \tilde{\phi}_\alpha^{1+L}(x) \) and \( \tilde{\phi}_\alpha(x) \) are of the same type, \( F^*H(x) \) belongs to the domain of attraction of \( \tilde{\phi}_\alpha(x) \).

This result can be stated in other words by an application of Theorem 2.7. That is, for two distributions \( F(x) \) and \( H(x) \), if \( 1 - F(x) \) is \((-\alpha)\)-varying at infinity with \( \alpha > 0 \) and

\[ \lim_{x \to \infty} \frac{(1 - H(x))}{(1 - F(x))} = L, \quad 0 \leq L < \infty \]

then \( 1 - F^*H(x) \) is \((-\alpha)\)-varying.

**Remark 3.2.** Feller (1966) proved that if \( 1 - F(x) \) and \( 1 - H(x) \) are \( \rho \)-varying at infinity then so is \( 1 - F^*H(x) \). This implies Theorem 3.2 when \( L \neq 0 \).

The following corollaries to Theorem 3.2 give a solution to part of the proposed problem.

**Corollary 3.2.** If \( F(x) \) belongs to the domain of attraction of \( \tilde{\phi}_\alpha(x) \) and \( H(x) \) belongs to the domain of attraction of \( \tilde{\phi}_\beta(x) \) then \( F^*H(x) \) belongs to the domain of attraction of \( \tilde{\phi}_\gamma \) where \( \gamma = \min(\alpha, \beta) \).

**Proof:** a) Suppose \( \alpha < \beta \). By Theorem 2.7, \( 1 - F(x) \) is \((-\alpha)\)-varying at infinity and \( 1 - H(x) \) is \((-\beta)\)-varying at infinity. Then
if we let

\[ R(x) = \frac{1 - H(x)}{1 - F(x)} \]

we have that for all \( x > 0 \)

\[ \frac{R(tx)}{R(t)} = \frac{1 - H(tx)}{1 - F(tx)} \frac{1 - H(t)}{1 - F(t)} = \frac{1 - H(tx)}{1 - H(t)} \frac{1 - F(tx)}{1 - F(t)} \xrightarrow{t \to \infty} x^{-(\beta - \alpha)}. \]

That is, \( R(x) \) is \( \{-(\beta - \alpha)\}\)-varying at infinity. Hence, by Corollary 2.1

\[ \lim_{x \to \infty} R(x) = \lim_{x \to \infty} \frac{1 - H(x)}{1 - F(x)} = 0. \]

Therefore, by Theorem 3.2, \( F \ast H(x) \) belongs to the domain of attraction of \( \dot{\gamma}(\alpha) \).

b) Suppose that \( \alpha = \beta \). By Remark 3.2, \( 1 - F \ast H(x) \) is \( (-\alpha) \)-varying at infinity. Then by Theorem 2.7, \( 1 - F \ast H(x) \) belongs to the domain of attraction of \( \dot{\gamma}(\alpha) \). \[ \]

Corollary 3.3. If \( F(x) \) belongs to the domain of attraction of \( \dot{\gamma}(\alpha) \) and \( H(x) \) belongs to the domain of attraction of \( \Lambda(x) \) then \( F \ast H(x) \) belongs to the domain of attraction of \( \dot{\gamma}(\alpha) \).

Proof: a) Suppose that \( x_\alpha(H) = \sup \{x: H(x) < 1\} < \infty \). Then it is clear that
b) Suppose that \( x_o(H) = \infty \). By Theorem 2.7, \( 1 - F(x) \) is \((-\alpha)\)-varying at infinity. Hence, by Corollary 2.1

\[
\lim_{x \to \infty} \frac{\log \{1 - F(x)\}}{\log x} = -\alpha .
\]

On the other hand, by Corollary 2.5

\[
\lim_{x \to \infty} \frac{\log \{1 - H(x)\}}{\log x} = -\infty .
\]

Therefore,

\[
\frac{1 - H(x)}{1 - F(x)} = \exp \left( \log \{1 - H(x)\} - \log \{1 - F(x)\} \right)
\]

\[
= \exp \left\{ - \log x \left( \frac{\log \{1 - F(x)\}}{\log x} - \frac{\log \{1 - H(x)\}}{\log x} \right) \right\}
\]

\[
\to 0 .
\]

Thus, from a) and b) and Theorem 3.2, \( F * H(x) \) belongs to the domain of attraction of \( \hat{\xi}_\alpha(x) \). \[\square\]
Corollary 3.4. If $F(x)$ belongs to the domain of attraction of $\psi_\alpha(x)$ and $H(x)$ belongs to the domain of attraction of $\psi_\beta(x)$ then $F \ast H(x)$ belongs to the domain of attraction of $\psi_\alpha(x)$.

Proof: By Theorem 2.8, $x_0(H) = \sup \{x: H(x) < 1\} < \infty$. Then it is clear that

$$\lim_{x \to \infty} \frac{1 - H(x)}{1 - F(x)} = 0.$$ 

Therefore, by Theorem 3.2, we have the desired conclusion.

In order to develop further the solution of our problem, we need to define the concept of regular variation at the origin.

Definition 3.1. A function $U: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ varies regularly at zero if there exists $\rho \in \mathbb{R}$ such that for all $x \in \mathbb{R}^+$

$$\lim_{t \to 0^+} \frac{U(tx)}{U(t)} = x^\rho.$$ 

For brevity the expression $\rho$-varying at zero will be used for functions satisfying Definition 3.1.

The concepts of regular variation at infinity and regular variation at zero are very related. In fact, consider the following lemma.

Lemma 3.3. A function $U: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is $\rho$-varying at zero if and only if the function $V: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by $V(x) = U \left( \frac{1}{x} \right)$ is $(-\rho)$-varying at infinity.

Proof: a) Suppose that $U(x)$ is $\rho$-varying at zero. Then
That is, \( V(x) \) is \((-\rho)\)-varying at infinity.

b) Suppose that \( V(x) = U\left(\frac{1}{x}\right) \) is \((-\rho)\)-varying at infinity. Then

\[
\frac{U(tx)}{U(t)} = \frac{V\left(\frac{1}{t} \cdot \frac{1}{x}\right)}{V\left(\frac{1}{t}\right)} \xrightarrow{t \to 0^+} \left(\frac{1}{x}\right)^{-\rho} = x^\rho \quad \text{for all } x \in \mathbb{R}^+.
\]

That is, \( U(x) \) is \(\rho\)-varying at zero.

Under a rather weak condition, it is possible to determine the extreme distribution to which the convolution is attracted, when one of the convolutants is attracted to \( \psi_\alpha(x) \) and the other one to \( \psi_\beta(x) \).

**Theorem 3.3.** Suppose that \( F(x) \) belongs to the domain of attraction of \( \psi_\alpha(x) \) and \( H(x) \) belongs to the domain of attraction of \( \psi_\beta(x) \) with right-endpoints \( x_\circ \) and \( y_\circ \) respectively. If

\[
\lim_{x \to 0^+} \frac{1 - F * H(x_\circ + y_\circ - x)}{\left[1 - F(x_\circ - x)\right]\left[1 - H(y_\circ - x)\right]}
\]

exists, then \( F * H(x) \) belongs to the domain of attraction of \( \psi_{\alpha + \beta}(x) \).

**Proof:** Let

\[
x_2 = \inf \{x: F(x) > 0\} \geq -\infty
\]

and

\[
y_2 = \inf \{x: H(x) > 0\} \geq -\infty.
\]
If $X'$ and $Y'$ are rv's with cdf's $F(x)$ and $H(x)$, then $X = x_0 - X'$ and $Y = y_0 - Y'$ have cdf's $F_1(x)$ and $H_1(x)$ where

$$F_1(x) = \begin{cases} 
0, & x < 0 \\
1 - F(x_0 - x), & 0 < x < x_0 - x_2 \\
1, & x > x_0 - x_2
\end{cases}$$

$$H_1(x) = \begin{cases} 
0, & x < 0 \\
1 - H(y_0 - x), & 0 < x < y_0 - y_2 \\
1, & x > y_0 - y_2
\end{cases}$$

Then note that for all $x$ such that $0 < x < \min(x_0 - x_2, y_0 - y_2)$,

$$F_1 \ast H_1(x) = \int_{0}^{x} H_1(x - \Theta) dF_1(\Theta) = \int_{0}^{x} \{1 - H(y_0 - x + \Theta)\} dF_1(\Theta)$$

$$= \{1 - F(x_0 - x)\} + \int_{0}^{x} \{1 - H(y_0 - x + x_0 - t)\} dF(x_0 - \Theta)$$

let $\Theta = x_0 - t$

$$= \{1 - F(x_0 - x)\} + \int_{x_0}^{x_0 - x} H(y_0 - x + x_0 - t) dF(t)$$

$$= \{1 - F(x_0 - x)\} - \int_{x_0 - x}^{x_0} H(y_0 + x_0 - x - t) dF(t)$$

$$= 1 - \int_{x_2}^{x_0 - x} dF(t) - \int_{x_0}^{x_0 - x} H(x_0 + y_0 - x - t) dF(t)$$
Suppose that $X$ and $Y$ are two independent random variables with distributions $F_1(x)$ and $H_1(x)$, respectively. Then, since $P(X > 0) = P(Y > 0) = 1$, for all $t > 0$

$$P(X \leq \frac{1}{2} t) P(Y \leq \frac{1}{2} t) \leq P(X + Y \leq t) \leq P(X \leq t) P(Y \leq t).$$

Hence, since $F_1 \ast H_1(t) = P(X + Y \leq t)$,

$$\frac{F_1 \left( \frac{1}{2} t \right)}{F_1(t)} \frac{H_1 \left( \frac{1}{2} t \right)}{H_1(t)} \leq \frac{F_1 \ast H_1(t)}{F_1(t) H_1(t)} \leq 1.$$

By Theorem 2.8 and Lemma 3.3, $F_1(x)$ is $\alpha$-varying at zero and $H_1(x)$ is $\beta$-varying at zero. Then

$$\lim_{t \to 0^+} \frac{F_1 \left( \frac{1}{2} t \right)}{F_1(t)} = \left( \frac{1}{2} \right)^{\alpha} \text{ and } \lim_{t \to 0^+} \frac{H_1 \left( \frac{1}{2} t \right)}{H_1(t)} = \left( \frac{1}{2} \right)^{\beta}.$$
\[ \lim_{t \to 0^+} \frac{F_1 \ast H_1(t)}{F_1(t)H_1(t)} = \lim_{t \to 0^+} \frac{1 - F \ast H(x_o + y_o - t)}{(1 - F(x_o - t))(1 - H(y_o - t))} \]

exists, say equal to \( L \), we conclude that

\[ 0 < \left( \frac{1}{2} \right)^{\alpha+\beta} \leq L \leq 1 \quad \text{that is} \quad 0 < L < 1. \]

Therefore, it is easy to see that

\[ F_1 \ast H_1(t) = 1 - F \ast H(x_o + y_o - t) \]

is \((\alpha+\beta)\)-varying at zero. Then by Lemma 3.3, \( 1 - F \ast H(x_o + y_o - \frac{1}{t}) \)

is \(\{-(\alpha+\beta)\}\)-varying at infinity. Thus, by Theorem 2.8, \( F \ast H(x) \)

belongs to the domain of attraction of \( \psi_{\alpha+\beta}(x) \).

As an application of Theorem 3.3, consider the following example.

**Example 3.** For real numbers \( a, b, c \) and \( d \) where \( a < b \) and \( c < d \), let \( F(x) \) be the uniform\((a, b)\)-distribution and let \( H(x) \) be the uniform\((c, d)\)-distribution. Then

\[
F(x) = \begin{cases} 
0 & \text{if } x < a \\
\frac{x - a}{b - a} & \text{if } a < x < b \\
1 & \text{if } x > b
\end{cases}
\]

\[
H(x) = \begin{cases} 
0 & \text{if } x < c \\
\frac{x - c}{d - c} & \text{if } c < x < d \\
1 & \text{if } x > d
\end{cases}
\]
Note that
\[
\lim_{x \to b^-} \frac{1 - F(x)}{b - x} = \frac{1}{b - a} \quad \text{and} \quad \lim_{x \to d^-} \frac{1 - H(x)}{d - x} = \frac{1}{d - c}.
\]

Then, by Lemma 2.3, \( F(x) \) and \( H(x) \) are members of \( \mathcal{L}(\psi_1) \).

Now, without loss of generality, assume \( b - a \leq d - c \). Then, for all \( x \) such that \( a + d < x < b + d \)

\[
F \ast H(x) = \int_{a}^{b} H(x - \theta) dF(\theta) = \int_{a}^{x-d} dF(\theta) + \int_{x-d}^{b} \frac{x-\theta-c}{d-c} dF(\theta)
\]

\[
= \frac{1}{b-a} \int_{a}^{x-d} d\theta + \frac{1}{(b-a)(d-c)} \int_{x-d}^{b} (x-\theta-c) d\theta
\]

\[
= \frac{x-d-a}{b-a} + \frac{b+d-x}{(b-a)(d-c)} \left\{ \frac{1}{2} (x+d-a) - c \right\}.
\]

Hence, for all \( x \) such that \( 0 < x < b - a \),

\[
1 - F \ast H(b + d - x) = \frac{x^2}{2(b-a)(d-c)}.
\]

On the other hand, for \( 0 < x < b - a \)

\[
\left[ 1 - F(b-x) \right] \left[ 1 - H(d-x) \right] = \frac{x^2}{(b-a)(d-c)}.
\]
Therefore

\[ \lim_{x \to 0^+} \frac{1 - F * H(b + d - x)}{[1 - F(b - x)] [1 - H(d - x)]} = \frac{1}{2}. \]

Thus, by Theorem 3.3, \( F * H(x) \) belongs to the domain of attraction of \( \psi_2(x) \).

In seeking the extreme distribution which attracts the convolution of two given distributions, each attracted to a \( \psi_\rho \)-type distribution, we obtained the following result.

**Theorem 3.4.** If \( F(x) \) belongs to the domain of attraction of \( \psi_\alpha(x) \) and \( H(x) \) belongs to the domain of attraction of \( \psi_\beta(x) \), then if \( F * H(x) \) belongs to any domain of attraction it is that of \( \psi_{\alpha+\beta}(x) \).

Proof: By Theorem 2.8, the endpoints \( x_o = \sup \{x: F(x) < 1\} \) and \( y_o = \sup \{x: H(x) < 1\} \) are finite. Hence \( F * H(x) \) has finite endpoint \( x_o + y_o \). Therefore, by Theorem 2.7, \( F * H \not\in \mathcal{S}(\psi_\rho) \) for all \( \rho > 0 \).

Now suppose that \( F * H \in \mathcal{S}(\Lambda) \). Then, by Corollary 2.5

\[ \lim_{x \to (x_o + y_o)^-} \frac{\log \{1 - F * H(x)\}}{\log \{x_o + y_o - x\}} = \infty. \]

On the other hand, if \( X' \) and \( Y' \) are rv's with cdf's \( F(x) \) and \( H(x) \), then \( X = X' - x_o \) and \( Y = Y' - y_o \) have cdf's \( F_1(x) \) and \( H_1(x) \) where
\[
F_1(x) = \begin{cases} 
0 & x < x_2 - x_o \\
F(x_o + x) & x_2 - x_o \leq x < 0 \\
1 & x \geq 0 
\end{cases}
\]

\[
H_1(x) = \begin{cases} 
0 & x < y_2 - y_o \\
H(y_o + x) & y_2 - y_o \leq x < 0 \\
1 & x \geq 0 
\end{cases}
\]

where

\[
x_2 = \inf \{x: F(x) > 0\} \geq -\infty
\]

and

\[
y_2 = \inf \{x: H(x) > 0\} \geq -\infty
\]

Note that for all \(x\) such that \(\max(x_2 - x_o, y_2 - y_o) < x < 0\)

\[
F_1 \ast H_1(x) = \int_{x_2 - x_o}^x H_1(x-t) dF_1(t) = \int_{x_2 - x_o}^0 dF_1(t) + \int_0^x H(y_o + x - t) dF_1(t)
\]

\[
= F_1(x) + \int_0^x H(y_o + x - t) dF_1(t)
\]

\[
= F(x_o + x) + \int_0^x H(y_o + x - t) dF(x_o + t)
\]
let \( t = \theta - x_0 \)

\[
\begin{align*}
&= F(x_0 + x) + \int_{x_0+x}^{x_0} H(y_0 + x - \theta + x_0) \, dF(\theta) \\
&= \int_{x_2}^{x_0+x} H(x_0 + y_0 + x - \theta) \, dF(\theta) + \int_{x_0+x}^{x_0} H(y_0 + x - \theta + x_0) \, dF(\theta) \\
&= F \ast H(x_0 + y_0 + x).
\end{align*}
\]

Also note that by Theorem 2.8 and Corollary 2.1

\[
\begin{align*}
\lim_{x \to 0^-} \frac{\log \{1 - F_1(x)\}}{\log (-x)} &= \lim_{x \to 0^-} \frac{\log \{1 - F(x_0 + x)\}}{\log (-x)} \\
&= \lim_{t \to \infty} \frac{\log \{1 - F(x_0 - t^{-1})\}}{- \log t} \\
&= \alpha,
\end{align*}
\]

similarly

\[
\lim_{x \to 0^-} \frac{\log \{1 - H_1(x)\}}{\log (-x)} = \beta.
\]

Suppose that \( X \) and \( Y \) are two independent random variables, with distribution \( F_1(x) \) and \( H_1(x) \), respectively. Since \( P(X < 0) = \)
$P(Y < O) = 1$, we have that for all $t < 0$

\[ P(X > \frac{1}{2} t) \cdot P(Y > \frac{1}{2} t) \leq P(X + Y > t) \leq P(X > t) \cdot P(Y > t). \]

Since $P(X + Y > t) = 1 - F_1 \ast H_1(t)$, we then have that for all $t < 0$

\[
\log \{1 - F_1(\frac{1}{2} t)\} + \log \{1 - H_1(\frac{1}{2} t)\} \leq \log \{1 - F_1 \ast H_1(t)\} \leq \log \{1 - F_1(t)\} + \log \{1 - H_1(t)\}.
\]

For $-1 < t < 0$ we have that $\log(-t) < 0$, hence

\[
\frac{\log \{1 - F_1(t)\}}{\log (-t)} + \frac{\log \{1 - H_1(t)\}}{\log (-t)} \leq \frac{\log \{1 - F_1 \ast H_1(t)\}}{\log (-t)} \leq \frac{\log \{1 - F_1(\frac{1}{2} t)\}}{\log (-\frac{1}{2} t)} \cdot \frac{\log (-\frac{1}{2} t)}{\log (-t)} + \frac{\log \{1 - H_1(\frac{1}{2} t)\}}{\log (-\frac{1}{2} t)} \cdot \frac{\log (-\frac{1}{2} t)}{\log (-t)}.
\]

Since

\[
\frac{\log (-\frac{1}{2} t)}{\log (-t)} = 1 + \frac{\log (\frac{1}{2})}{\log (-t)} \xrightarrow{t \to 0^-} 1,
\]

we conclude that
Since for all $x$ such that $\max(x_2 - x_o, y_2 - y_0) < x < 0$, $F_1 * H_1(x) = F * H(x_0 + y_0 + x)$,

\[
\lim_{x \to (x_0+y_0)^-} \frac{\log \{1 - F * H(x)\}}{\log (x_0 + y_0 - x)} = \lim_{t \to 0^-} \frac{\log \{1 - F * H(x_0 + y_0 + t)\}}{\log (-t)} = \lim_{t \to 0^-} \frac{\log \{1 - F_1 * H_1(t)\}}{\log (-t)} = \alpha + \beta.
\]

This contradicts the supposition that $F * H \in \mathcal{B}(\Lambda)$. Therefore, by Theorem 2.4, if $F * H(x)$ belongs to any domain of attraction, it is that of $\psi_\gamma(x)$ for some $\gamma > 0$. Hence, by Theorem 2.8 and Corollary 2.1

\[
\lim_{x \to (x_0+y_0)^-} \frac{\log \{1 - F * H(x)\}}{\log (x_0 + y_0 - x)} = \gamma.
\]

By the above, we conclude that $\gamma = \alpha + \beta$. [□]

We proceed to prove theorems similar to the last one for the convolution of $F(x)$ and $H(x)$ in the cases when $F \in \mathcal{B}(\Lambda)$, $H \in \mathcal{B}(\psi_\alpha)$ and $F \in \mathcal{B}(\Lambda)$, $H \in \mathcal{B}(\Lambda)$. 
Theorem 3.5. If $F(x)$ belongs to the domain of attraction of $\Lambda(x)$ and $H(x)$ belongs to the domain of attraction of $\Psi_\alpha(x)$ then if $F \ast H(x)$ belongs to any domain of attraction, it is that of $\Lambda(x)$.

Proof: a) Assume that $x_0 = \sup \{x: F(x) < 1\} = \infty$. Since $F \ast H(x)$ has endpoint at infinity, $F \ast H \not\in \mathcal{S}(\Psi_\gamma)$ for all $\gamma > 0$ (see Theorem 2.8).

Suppose that $F \ast H \in \mathcal{S}(\Psi_\beta)$ for some $\beta > 0$. Then by Theorem 2.7 and Corollary 2.1,

$$\lim_{x \to \infty} \frac{\log \{1 - F \ast H(x)\}}{\log x} = -\beta.$$  

On the other hand, suppose that $X$ and $Y$ are two independent random variables, with distribution $F(x)$ and $H(x)$, respectively. Consider the following events

$$A = [X + Y > x], \ B = [X > \frac{1}{2} x], \ C = [Y > \frac{1}{2} x].$$

Assume $w \in A$; that is, $X(w) + Y(w) > x$. Then either $X(w) > \frac{1}{2} x$ or $Y(w) > \frac{1}{2} x$, because if $X(w) < \frac{1}{2} x$ and $Y(w) < \frac{1}{2} x$ then $X(w) + Y(w) < t$ which contradicts the assumption that $w \in A$.

Therefore

$$A \subset B \cup C.$$  

Hence
\begin{align*}
P(A) & \leq P(B \cup C) \leq P(B) + P(C). \\
Since \ 1 - F \ast H(x) = P(A), \ we \ have \ that \\
1 - F \ast H(x) & \leq (1 - F\left(\frac{1}{2}x\right)) + (1 - H\left(\frac{1}{2}x\right)) \\
\leq (1 - F\left(\frac{1}{2}x\right)) \left(1 + \frac{1 - H\left(\frac{1}{2}x\right)}{1 - F\left(\frac{1}{2}x\right)}\right). \\

Therefore, \ since \ for \ x > 2, \ \frac{1}{\log x} \leq \frac{1}{\log \frac{1}{2} x}, \\
\frac{\log \{1 - F \ast H(x)\}}{\log x} & \leq \frac{\log \{1 - F\left(\frac{1}{2}x\right)\}}{\log \left(\frac{1}{2} x\right)} + \frac{\log \left\{1 + \frac{1 - H\left(\frac{1}{2}x\right)}{1 - F\left(\frac{1}{2}x\right)}\right\}}{\log \left(\frac{1}{2} x\right)}.
\end{align*}

Since, \ by \ Corollary \ 2.5 \\
\lim_{x \to \infty} \frac{\log \{1 - F(x)\}}{\log x} = -\infty,

and, \ by \ Theorem \ 2.8, \ y_\circ = \sup \{x: H(x) < 1\} \ is \ finite \ and \ hence \\
\lim_{x \to \infty} \frac{1 - H\left(\frac{1}{2}x\right)}{1 - F\left(\frac{1}{2}x\right)} = 0,

we \ conclude \ that
This contradicts the supposition that $F \ast H \in \mathcal{S}(\xi_{\beta})$. Thus, by Theorem 2.4, if $F \ast H(x)$ belongs to any domain of attraction, it is that of $A(x)$.

b) Assume $x_0 < \infty$. By Theorem 2.8, $y_0 = \sup \{x : H(x) < 1\}$ is finite. Then $F \ast H(x)$ has finite endpoint $x_0 + y_0$. Hence, $F \ast H \notin \mathcal{S}(\xi_{\gamma})$ for all $\gamma > 0$ (see Theorem 2.7).

Suppose that $F \ast H \in \mathcal{S}(\xi_{\gamma})$ for some $\gamma > 0$. Then, by Theorem 2.8 and Corollary 2.1,

$$\lim_{x \to (x_0 + y_0)^{-}} \frac{\log (1 - F \ast H(x))}{\log (x_0 + y_0 - x)} = \lim_{t \to \infty} \frac{\log (L H(x_0 + y_0 - t^{-l}))}{- \log (t)} = \gamma .$$

On the other hand, define the distributions $F_1(x)$ and $H_1(x)$ as in the proof of Theorem 3.4. Note that by Corollary 2.5

$$\lim_{x \to 0^+} \frac{\log (1 - F_1(x))}{\log (-x)} = \lim_{x \to 0^+} \frac{\log (1 - F(x_0 + x))}{\log (-x)} = \lim_{t \to x_0^+} \frac{\log (1 - F(t))}{\log (x_0 - t)} = \infty .$$
Also, note that by Theorem 2.8 and Corollary 2.1,

\[
\lim_{x \to 0^-} \frac{\log \{1 - H(x)\}}{\log(-x)} = \lim_{x \to 0^-} \frac{\log \{1 - H(y_0 + x)\}}{\log(-x)} = \lim_{t \to 0^-} \frac{\log \{1 - H(y_0 - t)\}}{-\log t} = \alpha.
\]

Now let \( X \) and \( Y \) be two independent random variables with distribution \( F_1(x) \) and \( H_1(x) \), respectively. Since \( P(X < 0) = P(Y < 0) = 1 \), for all \( t < 0 \)

\[
P(X + Y > t) \leq P(X > t) P(Y > t).
\]

Hence, since \( 1 - F_1 * H_1(t) = P(X + Y > t) \)

\[
\log \{1 - F_1 * H_1(t)\} \leq \log \{1 - F_1(t)\} + \log \{1 - H_1(t)\}.
\]

For \(-1 < t < 0\) we have that \( \log(-t) < 0 \), hence

\[
\frac{\log \{1 - F_1 * H_1(t)\}}{\log(-t)} \geq \frac{\log \{1 - F_1(t)\}}{\log(-t)} + \frac{\log \{1 - H_1(t)\}}{\log(-t)}.
\]

Since the right-hand side tends to infinity as \( t \) tends to zero from the left, we conclude that
Therefore, since we saw in the proof of Theorem 3.4 that for
\[
\max(x_2 - x_0, y_2 - y_0) < x < 0 \text{ we have } F_1 \ast H_1(x) = F \ast H(x_0 + y_0 + x),
\]

\[
\lim_{x \to (x_0 + y_0)^-} \frac{\log \{1 - F \ast H(x)\}}{\log(x_0 + y_0 - x)} = \lim_{t \to 0^-} \frac{\log \{1 - F \ast H(x_0 + y_0 + t)\}}{\log (-t)} = \lim_{t \to 0^-} \frac{\log \{1 - F_1 \ast H_1(t)\}}{\log (-t)} = \infty.
\]

This contradicts the supposition that \( F \ast H \in \mathcal{E}(\gamma) \). Thus, by Theorem 2.4, if \( F \ast H(x) \) belongs to any domain of attraction, it is that of \( \Lambda(x) \).

Next, we prove a mathematical result which will be used to show that, if the convolution of two distributions in \( \mathcal{E}(\Lambda) \) is attracted to any asymptotic extreme distribution, it has to be that of \( \Lambda(x) \).

**Lemma 3.4.** Let \( f: A \to B \) and \( g: B \to C \) be two functions with \( A, B \) and \( C \) intervals of the real line. If \( g(x) \) is a strictly increasing homeomorphism (\( g(x) \) and its inverse \( g^{-1}(x) \) are continuous) then

\[
(a) \quad \lim_{x \to y} g(f(x)) = g(\lim_{x \to y} f(x))
\]

and
(b) \[ \lim_{x \to y} g(f(x)) = g\left( \lim_{x \to y} f(x) \right). \]

Proof: (a) \[ \lim_{x \to y} f(x) = L \] then there exists a sequence \([x_n]\) such that \(x_n \to y\) as \(n \to \infty\) and \(\lim_{n \to \infty} f(x_n) = L\). Since \(g(x)\) is continuous, \(\lim_{n \to \infty} g(f(x_n)) = g(L)\). Therefore \(L_* = \lim_{x \to y} g(f(x)) \leq g(L)\). On the other hand, there exists a sequence \([y_n]\) such that \(y_n \to y\) as \(n \to \infty\) and \(\lim_{n \to \infty} g(f(y_n)) = L_*\). Since \(g^{-1}(x)\) is continuous, \(\lim_{n \to \infty} g^{-1}(g(f(y_n))) = g^{-1}(L_*);\) that is \(\lim_{n \to \infty} f(y_n) = g^{-1}(L_*)\). Therefore \(g^{-1}(L_*) \geq L\). Since \(g(x)\) is increasing, \(g(g^{-1}(L_*)) \geq g(L)\); that is \(L_* \geq g(L)\). Thus, we have that \(L_* = g(L)\).

(b) The proof is similar to that of (a). \[ \]

We conclude this chapter with the following theorem.

**Theorem 3.6.** If \(F(x)\) and \(H(x)\) belong to the domain of attraction of \(\Lambda(x)\) then if \(F \ast H(x)\) belongs to any domain of attraction, it is that of \(\Lambda(x)\).

Proof: a) Assume that \(x_0 = \sup \{x: F(x) < 1\} = \infty\) and \(y_0 = \sup \{x: H(x) < 1\} < \infty\). This case is identical to part a) of the proof of Theorem 3.5.

b) Assume that \(x_0 < \infty, y_0 < \infty\). This case is similar to part b) of the proof of Theorem 3.5 except for...
\[
\lim_{x \to 0^-} \frac{\log [1 - H(x)]}{\log (-x)} = \infty
\]

which is obtained in the same way as that for \( F_1(x) \).

c) Assume that \( x_0 = y_0 = \infty \). As in part a), \( F^*H \notin \mathcal{F} (\psi_\gamma) \) for all \( \gamma > 0 \) since \( F^*H(x) \) has endpoint at \( \infty \).

Suppose that \( F^*H \in \mathcal{F} (\psi_\beta) \) for some \( \beta > 0 \). Then by Theorem 2.7 and Corollary 2.1,

\[
\lim_{x \to \infty} \frac{\log [1 - F^*H(x)]}{\log x} = -\beta.
\]

On the other hand, as in part a),

\[
1 - F^*H(x) \leq \left[1 - F\left(\frac{1}{2} x\right)\right] \left\{1 + \frac{1 - H\left(\frac{1}{2} x\right)}{1 - F\left(\frac{1}{2} x\right)}\right\}.
\]

Since for \( x \geq 2e \), \( \frac{1}{\log x} \leq \frac{1}{\log \left(\frac{1}{2} x\right)} \leq 1 \) and \( \frac{1 - H\left(\frac{1}{2} x\right)}{1 - F\left(\frac{1}{2} x\right)} \geq 0 \),
we have that for \( x \geq 2e \)

\[
\frac{\log [1 - F^*H(x)]}{\log x} \leq \frac{\log [1 - F\left(\frac{1}{2} x\right)]}{\log \left(\frac{1}{2} x\right)} + \log \left\{1 + \frac{1 - H\left(\frac{1}{2} x\right)}{1 - F\left(\frac{1}{2} x\right)}\right\}.
\]

(3.5)

(1) Consider the case when \( \lim_{x \to \infty} \frac{1 - H\left(\frac{1}{2} x\right)}{1 - F\left(\frac{1}{2} x\right)} = L < \infty \).
Note that if we let \( g(t) = \log t \) and \( f(s) = 1 + \frac{1 - H(\frac{1}{2} s)}{1 - F(\frac{1}{2} s)} \),
then by Lemma 3.4

\[
\lim_{x \to \infty} \log \left( 1 + \frac{1 - H(\frac{1}{2} x)}{1 - F(\frac{1}{2} x)} \right) = \log (1 + L).
\]

Since by Corollary 2.5

\[
\lim_{y \to \infty} \frac{\log \{1 - F(y)\}}{\log y} = -\infty.
\]

we have that

\[
\lim_{x \to \infty} \frac{\log \{1 - F(x)\}}{\log x} \leq \lim_{x \to \infty} \left( \frac{\log \{1 - F(\frac{1}{2} x)\}}{\log \left( \frac{1}{2} x \right)} \right) + \log \left( 1 + \frac{1 - H(\frac{1}{2} x)}{1 - F(\frac{1}{2} x)} \right)
\]

\[
\leq \lim_{x \to \infty} \frac{\log \{1 - F(\frac{1}{2} x)\}}{\log \left( \frac{1}{2} x \right)} + \lim_{x \to \infty} \log \left( 1 + \frac{1 - H(\frac{1}{2} x)}{1 - F(\frac{1}{2} x)} \right)
\]

\[
\leq -\infty.
\]
This contradicts the supposition that \( F \ast H \in \mathcal{F}(\beta_\beta) \).

(2) Consider the case when \( \lim_{x \to \infty} \frac{1 - H(\frac{1}{2}x)}{1 - F(\frac{1}{2}x)} = \infty \). Then there exists a sequence \( \{x_n\} \) such that \( x_n \to \infty \) and

\[
\lim_{n \to \infty} \frac{1 - H(\frac{1}{2}x_n)}{1 - F(\frac{1}{2}x_n)} = \infty .
\]

This implies that \( \lim_{n \to \infty} \frac{1 - F(\frac{1}{2}x_n)}{1 - H(\frac{1}{2}x_n)} = 0 \).

Since \( \frac{1 - F(\frac{1}{2}x)}{1 - H(\frac{1}{2}x)} \geq 0 \), we have that \( \lim_{x \to \infty} \frac{1 - F(\frac{1}{2}x)}{1 - H(\frac{1}{2}x)} = 0 \).

Hence, if we let \( g(t) = \log t \) and \( f(s) = 1 + \frac{1 - F(\frac{1}{2}s)}{1 - H(\frac{1}{2}s)} \), then by Lemma 3.4,

\[
\lim_{x \to \infty} \log \left( 1 + \frac{1 - F(\frac{1}{2}x)}{1 - H(\frac{1}{2}x)} \right) = 0 .
\]

Interchanging \( F(x) \) and \( H(x) \) in the steps to obtain the inequality (3.5), we have that for \( x \geq 2e \)

\[
\frac{\log \{1 - F \ast H(x)\}}{\log x} \leq \frac{\log \{1 - H(\frac{1}{2}x)\}}{\log (\frac{1}{2}x)} + \log \left( 1 + \frac{1 - F(\frac{1}{2}x)}{1 - H(\frac{1}{2}x)} \right) .
\]

Since by Corollary 2.5
\[
\lim_{y \to \infty} \frac{\log \{1 - H(y)\}}{\log y} = -\infty.
\]

we have that

\[
\lim_{x \to \infty} \frac{\log \{1 - F \cdot H(x)\}}{\log x} \leq \lim_{x \to \infty} \left( \frac{\log \{1 - H\left(\frac{1}{2}x\right)\}}{\log\left(\frac{1}{2}x\right)} + \log \left[1 + \frac{1 - F\left(\frac{1}{2}x\right)}{1 - H\left(\frac{1}{2} x\right)}\right]\right).
\]

\[
\leq \lim_{x \to \infty} \frac{\log \{1 - H\left(\frac{1}{2} x\right)\}}{\log \left(\frac{1}{2} x\right)} + \lim_{x \to \infty} \log \left[1 + \frac{1 - F\left(\frac{1}{2} x\right)}{1 - H\left(\frac{1}{2} x\right)}\right] \leq -\infty.
\]

This also contradicts the supposition that \( F \cdot H \in A(\mathcal{B}) \). Therefore, by Theorem 2.4, if \( F \cdot H(x) \) belongs to any domain of attraction, it is that of \( A(x) \).
IV. ASYMPTOTIC INDEPENDENCE

Let \( \{(X_n, Y_n)\}' \) be a sequence of bivariate random variables with joint distributions \( \{F_n(x, y)\} \).

**Definition 4.1.** The random variables \( X_n \) and \( Y_n \) are asymptotically independent if

\[
F_n(x, y) \overset{w}{\to} F(x, y) = F_1(x) F_2(y)
\]

where \( F_1(x) \) and \( F_2(y) \) are distribution functions.

Here we consider sequences \( \{(X_i, Y_i)\}' \) of independent identically distributed bivariate random variables with common cumulative distribution \( F_{X,Y}(x, y) \). Suppose that \( F_{X,Y}(x, y) \) has marginal distributions \( F_X(x) \) and \( F_Y(y) \) each attracted to some extreme distribution.

We define the random variables

\[
M_n^{(1)} = \max(X_1, X_2, \ldots, X_n) \\
M_n^{(2)} = \max(Y_1, Y_2, \ldots, Y_n), \quad n = 1, 2, \ldots
\]

A sufficient condition for the asymptotic independence of \( M_n^{(1)} \) and \( M_n^{(2)} \) was found by Geffroy (1959) even though \( F_{X,Y}(x, y) \neq F_X(x) F_Y(y) \). We will use this result to identify some classes of bivariate distributions for which \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically independent.
Theorem 4.1. Suppose that $x_o$ and $y_o$ are the right endpoints of $F_X(x)$ and $F_Y(y)$, respectively. If

$$
\lim_{x \to x_o^-} \lim_{y \to y_o^-} \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)}{1 - F_{X,Y}(x, y)} = 0 \quad (4.1)
$$

then $M_{n}^{(1)}$ and $M_{n}^{(2)}$ are asymptotically independent.

For the proof of this theorem, see either Geffroy (1959) or Berman (1961) or de Oliveira (1962).

We first consider the class of bivariate distributions of bounded dependence.

Definition 4.2. The random vector $(X, Y)'$ or its distribution $F_{X,Y}(x, y)$ is of bounded dependence if

$$
\lim_{x \to x_o^-} \frac{P(X > x, Y > y)}{P(X > x) P(Y > y)} < \infty
$$

where $x_o$ and $y_o$ are the essential supremums of $X$ and $Y$, respectively.

Theorem 4.2. If $F_{X,Y}(x, y)$ is a distribution of bounded dependence, then $M_{n}^{(1)}$ and $M_{n}^{(2)}$ are asymptotically independent.

Proof: Let $x_o$ and $y_o$ be the essential supremums of $X$ and $Y$. Note that $F_{X,Y}(x, y) = P(X \leq x, Y \leq y) \leq P(X \leq x)$. Hence
\[
\frac{1}{1 - F_{X,Y}(x, y)} \leq \frac{1}{P(X > x)} \quad \text{for all } x < x_0, y < y_0.
\]

Then

\[
\frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)}{1 - F_{X,Y}(x, y)} \leq \frac{P(X > x, Y > y)}{P(X > x)}
\]

\[
= \frac{P(X > x, Y > y)}{P(X > x) P(Y > y)} \cdot P(Y > y).
\]

This implies that

\[
0 \leq \lim_{\substack{x \to x_0^- \\ y \to y_0^-}} \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)}{1 - F_{X,Y}(x, y)} \leq \lim_{\substack{x \to x_0^- \\ y \to y_0^-}} \left( \frac{P(X > x, Y > y)}{P(X > x) P(Y > y)} \cdot P(Y > y) \right)
\]

\[
\leq \lim_{\substack{x \to x_0^- \\ y \to y_0^-}} \frac{P(X > x, Y > y)}{P(X > x) P(Y > y)} \lim_{\substack{x \to x_0^- \\ y \to y_0^-}} P(Y > y) = 0.
\]

Hence, Condition (4.1) of Theorem 4.1 holds, thus \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically independent. \( \square \)
Now we consider the class of bivariate distributions which are negatively quadrant dependent. This class was defined by Lehmann (1966).

**Definition 4.3.** The random vector \((X, Y)'\) or its distribution \(F_{X,Y}(x,y)\) is negatively quadrant dependent if

\[
P(X < x, Y < y) < P(X < x) P(Y < y) \quad \text{for all } x,y . \tag{4.2}
\]

**Corollary 4.1.** If \(F_{X,Y}(x, y)\) is negatively quadrant dependent, then \(M_n^{(1)}\) and \(M_n^{(2)}\) are asymptotically independent.

**Proof:** Note that every negatively quadrant dependent distribution is of bounded dependence. Then by Theorem 4.2, the result follows. \[\]

Next we present three more classes of bivariate distributions.

**Definition 4.4.** The random vector \((X, Y)'\) or its distribution \(F_{X,Y}(x, y)\) is negatively associated if

\[
\text{cov} \{f(X, Y), g(X, Y)\} \leq 0 \tag{4.3}
\]

for all functions \(f(x, y)\) and \(g(x, y)\) which are non-decreasing in each argument, and for which the covariance is defined.

**Definition 4.5.** For a random vector \((X, Y)'\), \(Y\) is negatively regression dependent on \(X\) if

\[
P(Y \leq y \mid X = x) \tag{4.4}
\]
is non-decreasing in \( x \).

**Definition 4.6.** The random vector \((X, Y)'\) or its distribution \(F_{X,Y}(x, y)\) is negatively likelihood ratio dependent if

\[
f(x,y) f(x',y') \leq f(x,y') f(x',y) \quad \text{for all } x < x', y < y' (4.5)
\]

where \(f(x,y)\) is the density function.

In the papers of Lehmann (1966) and Esary, Proschan and Walkup (1967), it was shown for bivariate distributions, that negative quadrant dependence, negative association, negative regression dependence, and negative likelihood ratio dependence are successively stronger properties. That is, \((4.5) \implies (4.4) \implies (4.3) \implies (4.2)\).

Therefore, using Corollary 4.1, we have the following remark.

**Remark 4.1.** \(M_n^{(1)}\) and \(M_n^{(2)}\) are asymptotically independent for all classes of bivariate distributions defined above.

By reversing the inequalities in (4.2), (4.3), (4.5) and replacing non-decreasing in \( x \) by non-increasing in \( x \) in (4.4), we have the definitions of positive quadrant dependence, positive association, positive regression dependence and positive likelihood ratio dependence, respectively. These properties are also successively stronger. However, it is not true in general that if \(F_{X,Y}(x, y)\) is a positive quadrant dependent distribution then \(M_n^{(1)}\) and \(M_n^{(2)}\) are asymptotically independent. Consider the following counterexample.

**Counterexample 4.1.** Suppose that \( \phi(x) \) is a univariate asymptotic extreme distribution. Let \( X \) be a random variable with distribution
Consider the vector \((X, X)'\). Then its distribution is given by

\[
F_{X,X}(x, y) = P(X \leq x, X \leq y) = P(X \leq \min(x, y)) .
\]

Without loss of generality, assume that \(x \leq y\). Then, since

\[
P(X \leq x, X \leq y) = P(X \leq x) = P(X \leq x) P(X \leq y) .
\]

That is, \(F_{X,X}(x, y)\) is positive quadrant dependent. Now, note that if \([(X_1, Y_1)']\) is a sequence of independent identically distributed bivariate random variables with common cumulative distribution \(F_{X,Y}(x, y)\) then

\[
P(M^{(1)}_n \leq x, M^{(2)}_n \leq y) = P(X_1 \leq x, Y_1 \leq y; X_2 \leq x, Y_2 \leq y; \ldots; X_n \leq x, Y_n \leq y)
\]

\[
= \left[ P(X_1 \leq x, Y_1 \leq y) \right]^n = F^n_{X,Y}(x, y) .
\]

Therefore, if we let \(F_{X,X}(x, y)\) be the underlying distribution, then we have

\[
P(M^{(1)}_n \leq x, M^{(2)}_n \leq y) = F^n_{X,X}(x, y) = P^n(\min(x, y)) .
\]
Since $F \in \mathcal{F}(\phi)$, there exist sequences $\{a_n > 0\}$ and $\{b_n\}$ of real constants such that

$$F_n(a_n z + b_n) \xrightarrow{w} \phi(z).$$

Hence, since $\min(a_n x + b_n, a_n y + b_n) = a_n \min(x, y) + b_n$,

$$F(M_n^{(1)} \leq a_n x + b_n, M_n^{(2)} \leq a_n y + b_n) = F_n(\min(a_n x + b_n, a_n y + b_n))
= F_n(a_n \min(x, y) + b_n))
\xrightarrow{w} \phi(\min(x, y)) .$$

Thus, since the limit distribution is not the product of two univariate distributions, we conclude that $M_n^{(1)}$ and $M_n^{(2)}$ are not asymptotically independent.

Whether the other forms of positive dependence imply the asymptotic independence of $M_n^{(1)}$ and $M_n^{(2)}$ still remains to be investigated.

In what follows, we concentrate our attention on the class of bivariate distributions of the form

$$F_{X,Y}(x, y) = \int G(x, \theta) H(y, \theta) \, d\Phi(\theta) \quad (4.6)$$
where $F(\theta)$ is any distribution and $G(x, \theta)$ and $H(y, \theta)$ are distributions in $x$ and $y$ for every $\theta$ in the support of $F(\theta)$, with $G(x, \theta)$ and $H(y, \theta)$ Borel measurable in $\theta$ for every $x$ and $y$.

The random variables $X$ and $Y$ will be called conditionally independent when the representation (4.6) holds (see Loève, 1963).

It will be shown that for these bivariate distributions, $M_n^{(1)}$ and $M_n^{(2)}$ are asymptotically independent when $F(\theta)$ has a compact support. The following lemma is needed in the proof of this assertion.

**Lemma 4.1.** Suppose that the distribution $F(x)$ has a compact support $I$, and let

$$\{\varphi_t : I \rightarrow (0, +\infty) \mid t \in T \subset R\}$$

be a family of continuous functions which are non-increasing in $t$.

If

$$\lim_{t \rightarrow \sup T} \varphi_t(\theta) = 0 \quad \text{for all} \quad \theta \in I,$$

then

$$\lim_{t \rightarrow \sup T} \frac{\int_I \varphi_t^2(\theta) \, dF(\theta)}{\int_I \varphi_t(\theta) \, dF(\theta)} = 0.$$

Proof: By Dini's theorem (see Royden, 1968, p. 162), we have that
as \( t \to \sup T \), \( \varphi_t \to 0 \) uniformly on \( I \). Hence

\[
M_t = \sup_{\theta \in I} \varphi_t(\theta) \to 0
\]
as \( t \to \sup T \). Therefore

\[
0 \leq \frac{\int_I \varphi_t^2(\theta) \mathrm{d}P(\theta)}{\int_I \varphi_t(\theta) \mathrm{d}P(\theta)} \leq M_t \to 0
\]
as \( t \to \sup T \). Then, the desired result follows.

At this stage, we remind the reader that we are under the assumption that \( F_{X,Y}(x, y) \) has marginal distributions \( F_X(x) \) and \( F_Y(y) \) each attracted to some extreme distribution.

**Theorem 4.3.** Suppose that \( F_{X,Y}(x, y) \) is a bivariate distribution of the form (4.6) and \( x_0 \) and \( y_0 \) are the right endpoints of \( F_X(x) \) and \( F_Y(y) \). Assume \( F(\theta) \) has compact support \( I \) and for each \( \theta \in I \), \( G(x, \theta) \) is a distribution. If \( G(x, \theta) \) is continuous in \( \theta \) on \( I \) for each \( x \), then \( M_n(1) \) and \( M_n(2) \) are asymptotically independent.

**Proof:** Notice that \( F_X(x) = \int G(x, \theta) \mathrm{d}F(\theta) \) and \( F_Y(y) = \int H(y, \theta) \mathrm{d}F(\theta) \), hence

\[
0 \leq \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)}{1 - F_{X,Y}(x, y)} =
\]
\[ \frac{\int \{1 - G(x, \theta)\} \{1 - H(y, \theta)\} \, dF(\theta)}{\int \{1 - G(x, \theta)\} \{1 - H(y, \theta)\} \, dF(\theta)} . \] 

(4.7)

By Cauchy-Schwarz inequality

\[ \int \{1 - G(x, \theta)\} \{1 - H(y, \theta)\} \, dF(\theta) \]

\[ \leq \left( \int \{1 - G(x, \theta)\}^2 \, dF(\theta) \right)^{1/2} \left( \int \{1 - H(y, \theta)\}^2 \, dF(\theta) \right)^{1/2} . \]

On the other hand, \(1 - G(x, \theta) H(y, \theta) \geq 1 - G(x, \theta)\) and \(1 - G(x, \theta) H(y, \theta) \geq 1 - H(y, \theta)\), hence

\[ \int \{1 - G(x, \theta) H(y, \theta)\} \, dF(\theta) \]

\[ \geq \left( \int \{1 - G(x, \theta)\} \, dF(\theta) \right)^{1/2} \left( \int \{1 - H(y, \theta)\} \, dF(\theta) \right)^{1/2} . \]

Therefore, the right-hand side of (4.7) is dominated by

\[ \left( \frac{\int \{1 - G(x, \theta)\}^2 \, dF(\theta)}{\int \{1 - G(x, \theta)\} \, dF(\theta)} \right)^{1/2} \left( \frac{\int \{1 - H(y, \theta)\}^2 \, dF(\theta)}{\int \{1 - H(y, \theta)\} \, dF(\theta)} \right)^{1/2} \leq \]
\[
L \leq \left[ \frac{\int \{1 - G(x, \theta)\}^2 \, dF(\theta)}{\int \{1 - G(x, \theta)\} \, dF(\theta)} \right]^{1/2}
\]

since \( \{1 - H(y, \theta)\}^2 \leq 1 - H(y, \theta) \).

If we put \( \varphi_t(\theta) = 1 - G(t, \theta) \), we have that all the conditions of Lemma 4.1 are satisfied for the family of functions \( \{\varphi_t : t < x_0\} \).

Therefore

\[
\frac{\int \{1 - G(x, \theta)\}^2 \, dF(\theta)}{\int \{1 - G(x, \theta)\} \, dF(\theta)} \xrightarrow{x \to x_0^-} 0 .
\]

Thus

\[
\lim_{\substack{x \to x_0^- \\ y \to y_0^-}} \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)}{1 - F_{X,Y}(x, y)} = 0 .
\]

Then, by Theorem 4.1, \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically independent.

We might be tempted to believe that Theorem 4.3 holds without the compactness of the support of \( F(\theta) \). However, it is not so, since consider the following.

Counterexample 4.2. Take the conditionally independent random variables \( X \) and \( Y \) with joint distribution function

\[
F_{X,Y}(x, y) = \int_0^1 (1 - e^{-\Theta x})(1 - e^{-\Theta y}) \, d\Theta , \quad x, y > 0 .
\]
Here \( G(x, \theta) = \frac{1}{(0,1)} \), \( x > 0 \) and \( F(\theta) \) is the uniform 
\((0,1)\)-distribution. Notice that

\[
P(X > x, Y > y) = \int_0^1 e^{-\theta x} e^{-\theta y} d\theta = \frac{-1}{x+y} e^{-\theta (x+y)}
\]

Then, as we will see in Example 6.1, \( M_n^{(1)} \) and \( M_n^{(2)} \) are not asymptotically independent.

If we have some knowledge on the behavior of the mixing parameter \( \theta \) in distributions of the form (4.6), we still obtain the asymptotic independence of \( M_n^{(1)} \) and \( M_n^{(2)} \).

\textbf{Theorem 4.4.} Suppose that \( F_{X,Y}(x,y) \) is a bivariate distribution of the form (4.6). If \( \theta_0 = \sup \{ \theta : F(\theta) < 1 \} \) is finite, \( G(x, \theta_0) \) is a distribution with \( x_0(G(\cdot, \theta_0)) = x_0(F_X) \) and \( G(x, \theta) \) is a non-increasing function in \( \theta \), then \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically independent.

Proof: As we saw in the proof of Theorem 4.3,

\[
0 \leq \frac{1 - F_{X,Y}(x,y) + F_{X,Y}(x,y)}{1 - F_{X,Y}(x,y)} \leq \left[ \int_{\theta_0}^{\theta_0} [1 - G(x, \theta)]^2 dF(\theta) \right]^{1/2}
\]
\[ \leq (1 - G(x, \theta_o))^{1/2} \]

since \( G(x, \theta) \) is non-increasing in \( \theta \).

Let \( x_o = x_o(F_X) \) and \( y_o = x_o(F_Y) \). Then, since \( 1 - G(x, \theta_o) \to 0 \) as \( x \to x_o \), we conclude that

\[
\lim_{x \to x_o^-} \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)}{1 - F_{X,Y}(x, y)} = 0.
\]

Then, by Theorem 4.1, \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically independent.

Now, we look into a different class of bivariate distributions. Let \( U, V \) and \( W \) be independent random variables with distribution \( G(u) \), \( H(v) \) and \( F(w) \), respectively. Define the pair of random variables

\[
X = \max (U, W)
\]

\[ Y = \max (V, W) . \]

Then their joint distribution is
\[ F_{X,Y}(x, y) = P(\max(U, W) \leq x, \max(V, W) \leq y) \]

\[ = P(U \leq x, V \leq y, W \leq \min(x, y)) . \]

By independence, we obtain

\[ F_{X,Y}(x, y) = G(x) H(y) F(\min(x, y)) . \tag{4.8} \]

Bivariate distributions of this form are constructed in Arnold (1967).

Recall that

\[ X^F = \sup \{x : F(x) < 1\} . \]

Theorem 4.5. Suppose that \( F_{X,Y}(x, y) \) is a bivariate distribution of the form (4.8). If

\[ \lim_{x \rightarrow x^G} \frac{1 - F(x)}{1 - G(x)} = 0 , \tag{4.9} \]

then \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically independent.

Proof: Note that \( F_X(x) = G(x) F(x) \), \( F_Y(y) = H(y) F(y) \) and

\[ x_o = x_o(F_X) = \max(x_o(G), x_o(F)), \quad y_o = x_o(F_Y) = \max(x_o(H), x_o(F)) . \]

Also, note that (4.9) implies \( x_o(F) \leq x_o(G) \); hence, \( x_o = x_o(G) \).

Then
\[ 1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y) \]
\[ = 1 - G(x) F(x) - H(y) F(y) + G(x) H(y) F(\min(x, y)) \]
\[ \leq 1 - G(x) F(x) - H(y) F(y) + G(x) H(y) F(y) \]
\[ \leq \{1 - G(x)\} \{1 + G(x) \frac{1 - F(x)}{1 - G(x)}\} - H(y) F(y) \{1 - G(x)\} \]
\[ \leq \{1 - G(x)\} \{1 - H(y) F(y) + G(x) \frac{1 - F(x)}{1 - G(x)}\} . \]

On the other hand, \( G(x) H(y) F(\min(x, y)) \leq G(x) \) implies

\[ \frac{1 - G(x)}{1 - G(x) H(y) F(\min(x, y))} \leq 1 . \]

Therefore

\[ 0 \leq \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)}{1 - F_{X,Y}(x, y)} \]
\[ \leq \frac{1 - G(x)}{1 - G(x) H(y) F(\min(x, y))} \{1 - H(y) F(y) + G(x) \frac{1 - F(x)}{1 - G(x)}\} \]
\[ \leq 1 - H(y) F(y) + G(x) \frac{1 - F(x)}{1 - G(x)} \xrightarrow{y \rightarrow y_0} 0 \]
\[ x \rightarrow x_0 \]
That is
\[
\lim_{x \to x_0^-} \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)}{1 - F_{X,Y}(x, y)} = 0 ,
\]

thus, by Theorem 4.1, $M_n^{(1)}$ and $M_n^{(2)}$ are asymptotically independent. 

The result of Theorem 4.5 was obtained under the supposition that the marginal distributions $F_X(x)$ and $F_Y(y)$ are each attracted to an extreme distribution. On this, we make the following remark.

**Remark 4.2.** If $F_{X,Y}(x, y)$ is a bivariate distribution of the form (4.8), where $G(x)$, $H(y)$ and $F(z)$ are each attracted to an extreme distribution, then the marginal distributions $F_X(x)$ and $F_Y(y)$ are each attracted to an extreme distribution.

In fact, this can be seen by using Lemma 2.1 and the equalities
\[
1 - F_X(x) = [1 - G(x)] \left\{ 1 + G(x) \frac{1 - F(x)}{1 - G(x)} \right\}
\]
\[
1 - F_Y(y) = [1 - H(y)] \left\{ 1 + H(y) \frac{1 - F(y)}{1 - H(y)} \right\}
\]
\[
= [1 - F(y)] \left\{ 1 + F(y) \frac{1 - H(y)}{1 - F(y)} \right\} .
\]
V. CONDITIONAL INDEPENDENCE ON A LOCATION PARAMETER

In this chapter we investigate the asymptotic independence of \( M_n^{(1)} \) and \( M_n^{(2)} \) as defined in Chapter IV, for the case when \( \theta \) is a location parameter in distributions of the form (4.6). More specifically, these distributions are of the form

\[
F_{X,Y}(x, y) = \int G(x - \theta) H(y, \theta) \, dF(\theta). \tag{5.1}
\]

A pair of random variables \((X, Y)\) with joint distribution of the form (5.1) will be called conditionally independent on a location parameter.

The corresponding marginal distributions of (5.1) are

\[
F_X(x) = \int G(x - \theta) \, dF(\theta) = G \ast F(x)
\]

and

\[
F_Y(y) = \int H(y, \theta) \, dF(\theta).
\]

It is assumed throughout this chapter that, \( F_X(x) \) and \( F_Y(y) \) are each attracted to an extreme distribution. In some cases, as we will point out, this assumption is superfluous.

Observation 5.1. If \( U, V \) and \( W \) are independent random variables with distributions \( G(u) \), \( J(v) \) and \( F(w) \), respectively, then the bivariate random variable
\[
\begin{pmatrix}
X \\
Y
\end{pmatrix} = \begin{pmatrix}
U + W \\
V + W
\end{pmatrix}
\]

has a distribution function of the form (5.1) with \( H(y, \Theta) = J(y - \Theta) \).

It was shown in Lehmann (1966) that the pair of random variables (\( U + W, V + W \)) is positive quadrant dependent (see Chapter IV). Hence, in general, we cannot obtain the asymptotic independence of \( M^{(1)}_n \) and \( M^{(2)}_n \) from the results in Chapter IV. In fact, it will not be true, in general, that \( M^{(1)}_n \) and \( M^{(2)}_n \) are asymptotically independent when the underlying distribution is of the form (5.1). Consider the following counterexample.

**Counterexample 5.1.** Suppose that \( U, V \) and \( W \) are independent identically distributed with distribution

\[
F(x) = \begin{cases}
1 - \frac{1}{x}, & x > 1 \\
0, & x \leq 1
\end{cases}
\]

By Observation 5.1, \((U + W, V + W)'\) has distribution of the form (5.1) with identical marginals

\[
F_X(x) = F \ast F(x).
\]

Notice that for \( x > 2 \)}
\[ F \ast F(x) = \int_{-\infty}^{+\infty} [1 - F(x - \theta)] dF(\theta) \]

\[ = \int_{-1}^{x-1} \frac{d\theta}{(x - \theta)\sigma^2} + \int_{x-1}^{+\infty} dF(\theta) \]

\[ = \left[-\frac{1}{x\theta} - \frac{1}{x^2} \log \frac{x - \theta}{\theta}\right]_{x-1}^{x} + \frac{1}{x - 1} \]

\[ = \frac{1}{x} \left[-\frac{1}{x - 1} + \frac{2}{x} \log (x - 1) + 1 + \frac{x}{x - 1}\right] . \]

Hence

\[ \lim_{x \to \infty} x \{1 - F_X(x)\} = 2 . \]

Therefore, by Lemma 2.2 and Lemma 2.1

\[ \lim_{n \to \infty} nP(X > x_n) = \frac{1}{x} \]

where

\[ x_n = 2nx . \]

On the other hand, from (5.1) we have that
\[ P(X > x, Y > y) = \int \left[ 1 - F(x - \theta) \right] \left[ 1 - F(y - \theta) \right] dF(\theta). \]

Without loss of generality, assume that \( x \leq y \). Then, when \( x < y \) and \( x > 2 \)

\[
P(X > x, Y > y) = \int_{1}^{x-1} \frac{d\theta}{(x - \theta)(y - \theta) \theta^2} + \int_{x-1}^{y-1} \frac{d\theta}{(y - \theta) \theta^2} + \int_{y-1}^{\infty} dF(\theta).
\]

Using integration tables (Gradshteyn and Ryzhik, 1965)

\[
\int_{1}^{x-1} \frac{d\theta}{(x - \theta)(y - \theta) \theta^2} = \frac{x + y}{2x^2y^2} \log \left( \frac{\theta^2}{(x-\theta)(y-\theta)} \right) - \frac{1}{xy\theta} \\
+ \frac{(x+y)^2}{2x^2y^2} \left( \frac{1}{y-x} \log \frac{y-\theta}{x-\theta} \right)_{1}^{x-1} \\
= \frac{x + y}{2x^2y^2} \log \left( \frac{(x-1)^2}{(y-x+1)} \right) - \frac{1}{xy(x-1)} \\
+ \frac{(x+y)^2}{2x^2y^2(y-x)} \log(y-x+1) - \frac{x + y}{2x^2y^2} \log \frac{1}{(x-1)(y-1)} \\
+ \frac{1}{xy} - \frac{(x+y)^2}{2x^2y^2(y-x)} \log \frac{y-1}{x-1}. \]
Now

\[ \int_{y-1}^{y} \frac{d\theta}{(y-\theta) e^2} = -\frac{1}{y^2} - \frac{1}{y} \log \frac{y-\theta}{\theta} \int_{x-1}^{y-1} \]

\[ = -\frac{1}{y(y-1)} - \frac{1}{y^2} \log \frac{1}{y-1} + \frac{1}{y(x-1)} + \frac{1}{y^2} \log \frac{y-x+1}{x-1}. \]

And

\[ \int_{y-1}^{\infty} dF(\theta) = \frac{1}{y-1}. \]

Therefore, for \( x_n = 2nx \) and \( y_n = 2ny \)

\[ nP(X > x_n, Y > y_n) = \frac{(x+y)}{n^2 16x^2 y^2} \log n \frac{(2x - \frac{1}{n})^2}{2y - 2x + \frac{1}{n}} - \frac{1}{4n^2 xy(2x - \frac{1}{n})} \]

\[ + \frac{(x+y)^2 - 2xy}{8n^2 x^2 y^2(2y-2x)} \log n(2y - 2x + \frac{1}{n}) \]

\[ + \frac{x+y}{16n^2 x^2 y^2} \log n^2(2x - \frac{1}{n})(2y - \frac{1}{n}) + \frac{1}{4nxy} \]
\[-\frac{(x+y)^2 - 2xy}{8n^2x^2y^2(2y - 2x)} \log \frac{2y - \frac{1}{n}}{2x - \frac{1}{n}} - \frac{1}{2ny(2y - \frac{1}{n})} \]

\[+ \frac{1}{4ny^2} \log n \left(2y - \frac{1}{n}\right) + \frac{1}{2ny(2x - \frac{1}{n})} \]

\[+ \frac{1}{4n^2y^2} \log \frac{2y - 2x + \frac{1}{n}}{2x - \frac{1}{n}} + \frac{1}{2y - \frac{1}{n}} \quad n \to \infty \Rightarrow \frac{1}{2y}.\]

Now, when \(x = y > 2\)

\[P(X > x, Y > y) = \int_1^{y-1} \frac{d\theta}{(y-\theta)^2 \theta^2} + \int_{y-1}^{\infty} dF(\theta) \]

\[= - \left\{ \frac{1}{y\theta - \frac{2}{y^2}} \right\} \frac{1}{y-\theta} - \frac{2}{y^3} \log \frac{y-\theta}{\theta} \Bigg|_1^{y-1} + \frac{1}{y-1} \]

\[= - \left\{ \frac{1}{y(y-1)} - \frac{2}{y^2} \right\} + \frac{2}{y^3} \log (y-1) + \left\{ \frac{1}{y} - \frac{2}{y^2} \right\} \frac{1}{y-1} \]

\[+ \frac{2}{y^3} \log (y-1) + \frac{1}{y-1} \cdot\]

Therefore, for \(y_n = 2ny\)
Thus

\[
\lim_{n \to \infty} nP(X > x_n, Y > y_n) = \frac{1}{2 \max(x, y)}.
\]

It follows from a theorem by Galambos (1973) (see Theorem 6.1) that

\[
\lim_{n \to \infty} F^n_{X,Y}(x_n, y_n) = \exp \left\{ -\frac{1}{x} - \frac{1}{y} + \frac{1}{2 \max(x, y)} \right\}.
\]

That is, \( M_n^{(1)} \) and \( M_n^{(2)} \) are not asymptotically independent.

Under some conditions on \( F(x) \) or on the behavior of the tail functions \( 1 - F(x) \) and \( 1 - G(x) \) we will show that the asymptotic independence of \( M_n^{(1)} \) and \( M_n^{(2)} \) can be obtained.

We first consider the case when \( F(\theta) \) has a finite right endpoint.

**Theorem 5.1.** Suppose that \( F_{X,Y}(x, y) \) is a bivariate distribution of the form (5.1). If \( \Theta_0 = \sup \{ \theta : F(\theta) < 1 \} \) is finite and \( G(x) \) and
$H(y, \Theta)$ are any distributions, with $G(x)$ non-degenerate, then $M_n^{(1)}$ and $M_n^{(2)}$ are asymptotically independent.

Proof: Since $\Theta_0 < \infty$, notice that $G(x - \Theta_0)$ is a distribution. Also, we have that $G(x - \Theta)$ is non-increasing in $\Theta$. Therefore, by Theorem 4.4, $M_n^{(1)}$ and $M_n^{(2)}$ are asymptotically independent.

Corollary 5.1. Let $X_1, X_2, \ldots$ be iid rv's with distribution $F(x)$ and $Y_1, Y_2, \ldots$ be iid rv's with distribution $G(y)$. Suppose that the $X_i$'s are independent of the $Y_i$'s. If $x_0 = \sup \{x: F(x) < 1\}$ is finite and $G(y)$ is non-degenerate, then $M_n = \max(X_1, X_2, \ldots, X_n)$ and $M^*_n = \max(X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n)$ are asymptotically independent.

Proof: If we consider the degenerate distribution

$$H(z) = \begin{cases} 
1 & \text{for } z \geq 0 \\
0 & \text{for } z < 0
\end{cases}$$

then the common distribution of the bivariate random variables $(X_i + Y_i, X_i)'$ is

$$F_{X_i, Y_i}(x, y) = \int G(x - \Theta) H(y - \Theta) \, dF(\Theta).$$

Since the bivariate random variables $(X_i + Y_i, X_i)'$ are independent, we have by Theorem 5.1 that $M_n$ and $M^*_n$ are asymptotically independent.
Remark 5.1. If $F(x)$ is a member of either $\mathcal{N}(\psi_\alpha)$ or $\mathcal{B}(\Lambda)$ with finite right endpoint, and $G(y)$ is a member of either $\mathcal{N}(\psi_\beta)$ or $\mathcal{B}(\psi_\gamma)$ or $\mathcal{B}(\Lambda)$ in Corollary 5.1, then, from the results in Chapter III, the limit distribution of $(M_n, M_n^*)$, properly normalized is given by the entries of the following table

<table>
<thead>
<tr>
<th></th>
<th>$\mathcal{N}(\psi_\beta)$</th>
<th>$\mathcal{B}(\psi_\gamma)$</th>
<th>$\mathcal{B}(\Lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{N}(\psi_\alpha)$</td>
<td>$\psi_\alpha(x)\psi_\alpha(y)$</td>
<td>$\psi_\alpha(x)\psi_\gamma(y)$</td>
<td>$\psi_\alpha(x)\Lambda(x)$</td>
</tr>
<tr>
<td>$\mathcal{B}(\Lambda)$</td>
<td>$\Lambda(x)\Lambda(y)$</td>
<td>$\Lambda(x)\psi_\gamma(y)$</td>
<td>$\Lambda(x)\Lambda(y)$</td>
</tr>
</tbody>
</table>

The assumption that $G \ast F(x)$ is attracted to an extreme distribution is superfluous.

Example 5.1. Suppose that $F(x)$ is a uniform(a, b) - distribution ($a < b$), and $G(y)$ is a uniform(c, d) - distribution ($c < d$). Note that

$$\lim_{x \to b^-} \frac{1 - F(x)}{b - x} = \frac{1}{b - a} > 0 \quad \text{and} \quad \lim_{y \to d^-} \frac{1 - G(y)}{d - y} = \frac{1}{d - c} > 0.$$  

Then, by Lemma 2.3, $F,G \in \mathcal{B}(\psi_1)$. Therefore, from the table above, the limit distribution of $(M_n, M_n^*)$, properly normalized is $\psi_1(x)\psi_2(y)$.

The result of this example was first obtained by David (1973) using a different approach.
In order to have an idea of the kind of results we should expect, we refer to Observation 5.1. We notice that to obtain the asymptotic independence of $M_n^{(1)}$ and $M_n^{(2)}$, it would be sufficient that for large values of $x$, the weight of the tail $1 - G(x)$ should be heavier than that of $1 - F(x)$. Because in this case, intuitively for large values of $x$ the tail of the distribution of $U + W$ will be explained by the tail of the distribution of $U$ (see Lemma 3.2). Hence, since $U$ and $V + W$ are independent, we expect the asymptotic independence of $M_n^{(1)}$ and $M_n^{(2)}$.

In fact, as seen in the theorem below, even for the general case of a distribution of the form (5.1), the asymptotic independence follows under certain conditions.

**Theorem 5.2.** Suppose that $F_{X,Y}(x, y)$ is a bivariate distribution of the form (5.1), where $G(x)$, $H(y, \theta)$ and $F(\theta)$ have endpoint at infinity. If $G(x)$ and $F(\theta)$ are non-degenerate distributions such that for some $0 < \varepsilon < 1$

$$
\lim_{x \to \infty} \frac{1 - F(x \varepsilon)}{1 - G(x)} = 0
$$

and $H(y, \theta)$ is any distribution, then $M_n^{(1)}$ and $M_n^{(2)}$ are asymptotically independent.

**Proof:** As in the proof of Theorem 4.3, we can see that
\[0 \leq \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x,y)}{1 - F_{X,Y}(x,y)}\]

\[
\left\{ \int \frac{(1 - G(x - \theta))^2 \, dF(\theta)}{\int (1 - G(x - \theta)) \, dF(\theta)} \right\}^{1/2} = \left\{ \int \frac{(1 - G(x - \theta))^2 \, dF(\theta)}{\int (1 - G(x - \theta)) \, dF(\theta)} \right\}^{1/2}.
\]

(5.3)

Notice that for all real \( \varepsilon \)

\[
\int_{-\infty}^{+\infty} [1 - G(x - \theta)]^2 \, dF(\theta)
\]

\[
= \int_{-\infty}^{x_{\varepsilon}} [1 - G(x - \theta)]^2 \, dF(\theta) + \int_{x_{\varepsilon}}^{+\infty} [1 - G(x - \theta)]^2 \, dF(\theta)
\]

\[
\leq [1 - G(x(1 - \varepsilon))] \int_{-\infty}^{x_{\varepsilon}} [1 - G(x - \theta)] \, dF(\theta) + \int_{x_{\varepsilon}}^{+\infty} \, dF(\theta) .
\]

On the other hand

\[
[1 - G(x)] [1 - F(0)] \leq \int_{0}^{\infty} [1 - G(x - \theta)] \, dF(\theta) \leq \int_{-\infty}^{+\infty} [1 - G(x - \theta)] \, dF(\theta)
\]

and
\[ \int_{-\infty}^{\infty} \{1 - G(x - \theta)\} \, dF(\theta) \leq \int_{-\infty}^{+\infty} \{1 - G(x - \theta)\} \, dF(\theta). \]

Hence

\[ \frac{\int_{-\infty}^{+\infty} [1 - G(x - \theta)]^2 \, dF(\theta)}{\int_{-\infty}^{+\infty} [1 - G(x - \theta)] \, dF(\theta)} \leq \frac{\int_{-\infty}^{+\infty} [1 - G(x - \theta))] \, dF(\theta)}{\int_{-\infty}^{+\infty} [1 - G(x - \theta)] \, dF(\theta)} \]

\[ + \frac{1 - F(x)}{1 - G(x)} \frac{1 - F(\theta)}{1 - G(\theta)} \]

\[ \leq [1 - G(x(1 - \epsilon))] + \frac{1}{1 - F(\theta)} \cdot \frac{1 - F(x)}{1 - G(x)}. \]

Therefore, by (5.2) and inequalities (5.3), we conclude that

\[ \lim_{x \to \infty} \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)}{1 - F_{X,Y}(x, y)} = 0. \]

Thus, by Theorem 4.1, \( M_{n1}^{(1)} \) and \( M_{n2}^{(2)} \) are asymptotically independent.
**Corollary 5.2.** Let \( X_1, X_2, \ldots \) be iid rv's with distribution \( F(x) \) and \( Y_1, Y_2, \ldots \) be iid rv's with distribution \( G(x) \). Suppose that the \( X_i \)'s are independent of the \( Y_i \)'s. If for some \( 0 < \epsilon < 1 \)

\[
\lim_{x \to \infty} \frac{1 - F(x\epsilon)}{1 - G(x)} = 0
\]

then \( M_n = \max(X_1, X_2, \ldots, X_n) \) and \( M_n^* = \max(X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n) \) are asymptotically independent.

**Proof:** Similar to the proof of Corollary 5.1 applying Theorem 5.2.

Related to conditions on domains of attraction, we have the subsequent result.

**Theorem 5.3.** Suppose that \( F_{X,Y}(x, y) \) is a bivariate distribution of the form (5.1). If \( G \in \mathcal{L}(\xi_A) \), either \( F \in \mathcal{L}(\xi_B) \), \( \alpha < \beta \) or \( F \in \mathcal{L}(\Lambda) \) and \( H(y, \Theta) \) is any distribution, then \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically independent.

**Proof:** By Theorem 2.7, \( 1 - G(x) \) is \((-\alpha)\)-varying at infinity. Assume \( F \in \mathcal{L}(\xi_B) \), \( \alpha < \beta \). Then \( 1 - F(x) \) is \((-\beta)\)-varying at infinity. Hence, \( \frac{1 - F(x)}{1 - G(x)} \) is \((-\beta - \alpha)\)-varying at infinity. Since \( \beta - \alpha > 0 \), by Corollary 2.1,

\[
\lim_{x \to \infty} \frac{1 - F(x)}{1 - G(x)} = 0.
\]
Therefore, for all $\epsilon > 0$

$$\frac{1 - F(x\epsilon)}{1 - G(x)} = \frac{1 - F(x\epsilon)}{1 - F(x)} \cdot \frac{1 - F(x)}{1 - G(x)} \xrightarrow{x \to \infty} \epsilon^{-B} \cdot 0 = 0.$$  

Thus, by Theorem 5.2, $M_n^{(l)}$ and $M_n^{(2)}$ are asymptotically independent.

Assume $F \in \mathcal{B}(A)$. If $F(x)$ has a finite right endpoint, the result follows from Theorem 5.1. If $F(x)$ has right endpoint at infinity, then by Corollary 2.5

$$\lim_{x \to \infty} \frac{\log \{1 - F(x)\}}{\log x} = -\infty.$$  

Since $1 - G(x)$ is $(-\alpha)$-varying at infinity, by Corollary 2.1

$$\lim_{x \to \infty} \frac{\log \{1 - G(x)\}}{\log x} = -\alpha.$$  

Therefore, for all $\epsilon > 0$

$$\frac{1 - F(x\epsilon)}{1 - G(x)} = \exp \left[ \log \left\{ 1 - F(x\epsilon) \right\} - \log \left\{ 1 - G(x) \right\} \right]$$

$$= \exp \left\{ - \log x \epsilon \left( \frac{\log \{1 - F(x\epsilon)\}}{\log x \epsilon} + \frac{\log \{1 - G(x)\}}{\log x} \right) \right\} \cdot \frac{\log x}{\log x \epsilon}$$
Thus, by Theorem 5.2, \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically independent.

Corollary 5.3. Let \( X_1, X_2, \ldots \) be iid rv's with distribution \( F(x) \) and \( Y_1, Y_2, \ldots \) be iid rv's with distribution \( G(y) \). Suppose that the \( X_i \)'s are independent of the \( Y_i \)'s. If \( G \in \mathcal{B}(\hat{\beta}) \) and either \( F \in \mathcal{B}(\hat{\alpha}) \) or \( F \in \mathcal{B}(\lambda) \), then \( M_n = \max(X_1, X_2, \ldots, X_n) \) and \( M^* = \max(X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n) \) are asymptotically independent.

Proof: Similar to the proof of Corollary 5.1 applying Theorem 5.3.

Remark 5.2. From the results in Chapter III, the assumption that \( G * F(x) \) is attracted to an extreme distribution is superfluous in Corollary 5.3; and the limit distribution of \( \binom{M_n}{M^*} \) properly normalized is given by the entries of the following table

\[
\begin{array}{ccc}
G & F & \mathcal{B}(\hat{\beta}) \\
\mathcal{B}(\hat{\alpha}) & \hat{\beta}(x)\hat{\alpha}(y) & \Lambda(x)\hat{\alpha}(y)
\end{array}
\]

If the distribution \( H(y, \theta) \) is such that \( H(y, \theta) = F(y - \theta) \) and \( G(x) = F(x) \) in a bivariate distribution of the form (5.1), Theorem 5.2 gives no information about the asymptotic independence of \( M_n^{(1)} \) and \( M_n^{(2)} \) when \( F \in \mathcal{B}(\hat{\alpha}) \) or \( F \in \mathcal{B}(\lambda) \). This is because:
If $F \in \mathcal{S}(\lambda)$ then, by Theorem 2.7, $1 - F(x)$ is $(-\alpha)$-varying at infinity, hence for $\varepsilon > 0$

$$\lim_{x \to \infty} \frac{1 - F(x\varepsilon)}{1 - F(x)} = \varepsilon^{-\alpha} > 0.$$ 

If $F \in \mathcal{S}(\Lambda)$, de Haan (1970) proved that $1 - F(x)$ is $(-\infty)$-varying at infinity, hence for $0 < \varepsilon < 1$

$$\lim_{x \to \infty} \frac{1 - F(x\varepsilon)}{1 - F(x)} = \infty.$$ 

Therefore, Condition (5.3) of Theorem 5.2 does not hold.

In fact, when $F \in \mathcal{S}(\lambda)$, the asymptotic independence of $M_n^{(1)}$ and $M_n^{(2)}$ does not hold for all $\alpha > 0$. This is seen by Counterexample 5.1 where the $F(x)$ considered there is in $\mathcal{S}(\lambda_1)$.

In the following we will see that in some cases, even though $H(y, \Theta) = F(y - \Theta)$ and $G = F$ in (5.1), the asymptotic independence of $M_n^{(1)}$ and $M_n^{(2)}$ can be verified.

**Theorem 5.4.** Suppose that $F_{X,Y}(x, y)$ is a bivariate distribution of the form (5.1). If $G(x)$ is a normal $(\nu, \tau^2)$-distribution, $F(\theta)$ is a normal $(\mu, \sigma^2)$-distribution and $H(y, \Theta)$ is any distribution, then $M_n^{(1)}$ and $M_n^{(2)}$ are asymptotically independent.

Although for some cases Theorem 5.4 is included in Theorem 5.2, here we present a proof without applying Theorem 5.2 to illustrate,
once more, the good behavior of the normal distributions.

Proof: As in the proof of Theorem 4.3, we have that

\[
0 \leq \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)}{1 - F_{X,Y}(x, y)}.
\]

\[
\leq \left\{ \frac{\int [1 - G(x - \theta)]^2 dF(\theta)}{\int [1 - G(x - \theta)] dF(\theta)} \right\}^{1/2}.
\]

(5.4)

Note that \( G * F(x) \) is a normal(\( \mu + \nu, \sigma^2 + t^2 \))-distribution; hence

\[
\int G'(x - \theta) dF(\theta) = (G * F)'(x) = c \exp \left\{ -\frac{1}{2} \cdot \frac{(x - \mu - \nu)^2}{\sigma^2 + t^2} \right\}
\]

where \( c \) is a constant. Consider

\[
\frac{\int [1 - G(x - \theta)] G'(x - \theta) dF(\theta)}{\int G'(x - \theta) dF(\theta)}
\]

\[
= \frac{c_1 \int [1 - G(x - \theta)] \exp \left\{ -\frac{1}{2} \left( \frac{(x - \theta - \nu)^2}{t^2} + \frac{(\theta - \mu)^2}{\sigma^2} \right) \right\} d\theta}{\exp \left\{ -\frac{1}{2} \frac{(x - \mu - \nu)^2}{\sigma^2 + t^2} \right\}}
\]

(5.5)

where \( c_1 \) is a constant. Notice that
\[- \frac{1}{2} \left( \frac{(x-\theta-v)^2}{t^2} + \frac{(\theta-\mu)^2}{\sigma^2} \right) \]

\[= - \frac{1}{2} \left\{ \frac{x^2}{t^2} - \frac{2x(\theta+v)}{t^2} + \frac{(\theta+v)^2}{t^2} + \frac{(\theta+v)^2}{\sigma^2} - 2(\theta+v)(\mu+v) + \frac{(\mu+v)^2}{\sigma^2} \right\} \]

\[= - \frac{1}{2} \left\{ \frac{x^2}{t^2} + \frac{(\mu+v)^2}{\sigma^2} \right\} - \frac{1}{2} \left\{ (\theta+v)^2 \left( \frac{1}{t^2} + \frac{1}{\sigma^2} \right) - 2(\theta+v)(\frac{x}{t^2} + \frac{\mu+v}{\sigma^2}) \right\} \]

\[= - \frac{1}{2} \left\{ \frac{x^2}{t^2} + \frac{(\mu+v)^2}{\sigma^2} \right\} - \frac{1}{2} \left\{ (\theta+v)^2 \left( \frac{1}{t^2} + \frac{1}{\sigma^2} \right) - 2(\theta+v)(\frac{x}{t^2} + \frac{\mu+v}{\sigma^2}) \right\} \]

\[+ \frac{(\frac{x}{t^2} + \frac{\mu+v}{\sigma^2})^2}{\frac{1}{\sigma^2} + \frac{1}{t^2}} + \frac{(\frac{x}{t^2} + \frac{\mu+v}{\sigma^2})^2}{2(\frac{1}{\sigma^2} + \frac{1}{t^2})} \]

\[- \frac{1}{2} \left\{ \frac{x^2}{t^2} + \frac{(\mu+v)^2}{\sigma^2} \right\} - \frac{(\frac{x}{t^2} + \frac{\mu+v}{\sigma^2})^2}{\frac{1}{\sigma^2} + \frac{1}{t^2}} \]

\[= - \frac{1}{2} \left\{ \frac{1}{t^2} + \frac{1}{\sigma^2} \right\}^2 (\theta+v) - \frac{1}{t^2} + \frac{\mu+v}{\sigma^2} \left( \frac{1}{t^2} + \frac{1}{\sigma^2} \right)^{1/2} \]
Hence, expression (5.5) is equal to

\[
\exp \left\{ -\frac{1}{2} \left( \frac{x^2}{t^2} + \frac{(\mu + \nu)^2}{\sigma^2} - \frac{\left( \frac{x}{t^2} + \frac{\mu + \nu}{\sigma^2} \right)^2}{\frac{1}{t^2} + \frac{1}{\sigma^2}} - \frac{(x - \mu - \nu)^2}{\sigma^2 + t^2} \right) \right\}
\]

\[
\times \int \{1 - G(x - \Theta)\} \exp \left\{ -\frac{1}{2} \left( \Theta + \nu - \frac{x}{t^2} \frac{\mu + \nu}{\sigma^2} \right) \left( \frac{1}{t^2} + \frac{1}{\sigma^2} \right)^{-\frac{1}{2}} \right\} d\Theta
\]

\[
= c_2 \exp \left\{ -\frac{1}{2} \left( \frac{x^2}{t^2} + \frac{(\mu + \nu)^2}{\sigma^2} - \frac{\left( \frac{x}{t^2} + \frac{\mu + \nu}{\sigma^2} \right)^2}{\frac{1}{t^2} + \frac{1}{\sigma^2}} - \frac{(x - \mu - \nu)^2}{\sigma^2 + t^2} \right) \right\}
\]

\[
\times \int \{1 - G(x - \Theta)\} dF_x(\Theta)
\]

where \( c_2 \) is a constant and \( F_x(\Theta) \) is a normal \( -\nu + \frac{x}{t^2} \frac{\mu + \nu}{\sigma^2} \left( \frac{1}{t^2} + \frac{1}{\sigma^2} \right)^{-1} \)-distribution. Now, notice that the argument of \( \exp \{ \cdot \} \) is equal to
\[
\frac{1}{2} \left\{ - \frac{x^2}{t^2} \frac{(\mu + \nu)^2}{\sigma^2} + \left( \frac{1}{\sigma^2} + \frac{1}{t^2} \right)^{-1} \left( \frac{x^2}{t^4} + \frac{2x(\mu + \nu)}{t^2 \sigma^2} + \frac{(\mu + \nu)^2}{\sigma^4} \right) \\
+ \frac{1}{\sigma^2 + t^2} \left( x^2 - 2x(\mu + \nu) + (\mu + \nu)^2 \right) \right\}
\]

\[
= - \frac{(\mu + \nu)^2}{2\sigma^2} + \frac{x^2}{2} \left( - \frac{1}{t^2} + \frac{\left( \frac{1}{\sigma^2} + \frac{1}{t^2} \right)^{-1}}{t^4} + \frac{1}{\sigma^2 + t^2} \right) + x \left( \frac{\mu + \nu)(\frac{1}{\sigma^2} + \frac{1}{t^2})^{-1}}{t^2 \sigma^2} \right)
\]

\[
- \frac{\mu + \nu}{\sigma^2 + t^2} + \frac{(\mu + \nu)^2(\frac{1}{\sigma^2} + \frac{1}{t^2})^{-1}}{2\sigma^4} + \frac{(\mu + \nu)^2}{2(\sigma^2 + t^2)}
\]

\[
= \frac{(\mu + \nu)^2}{2} \left( - \frac{1}{\sigma^2} + \left( \frac{1}{\sigma^2} + \frac{1}{t^2} \right)^{-1} \cdot \frac{1}{\sigma^2} \frac{1}{\sigma^2 + t^2} \right)
\]

since

\[
\frac{t^2}{\sigma^2} + 1 = 1 + \frac{t^2}{\sigma^2} \iff t^2 \left( \frac{1}{\sigma^2} + \frac{1}{t^2} \right) = \frac{\sigma^2 + t^2}{\sigma^2}
\]

\[
\iff \frac{t^2}{\sigma^2 + t^2} \left( \frac{1}{\sigma^2} + \frac{1}{t^2} \right) = \frac{1}{\sigma^2} \iff \frac{t^2}{\sigma^2 + t^2} \left( \frac{1}{\sigma^2} + \frac{1}{t^2} \right) + \frac{1}{\sigma^2 + t^2} = \frac{1}{\sigma^2} + \frac{1}{t^2}
\]
Therefore, (5.5) is equal to

$$c_3 \int [1 - G(x - \theta)] \, dF_x(\theta)$$

where $c_3$ is a constant. Since the convolution operation is commutative, (5.5) equals

$$c_3 \int [1 - F_x(x - \theta)] \, dG(\theta) .$$

If we let $\alpha = (1 + \frac{t^2}{\sigma^2})^{-1}$, $\beta = (\frac{1}{\sigma^2} - \frac{\nu}{t^2})(\frac{1}{\sigma^2} + \frac{1}{t^2})^{-1}$ and

$$\zeta^2 = (\frac{1}{\sigma^2} + \frac{1}{t^2})^{-1} ,$$

then $F_x(\theta)$ is a normal($\alpha \xi + \beta$, $\zeta^2$) with $\alpha \neq 1$ . Then by Example 2.1 we have that

$$\frac{(x - \alpha \xi - \beta)^2}{\sqrt{2} \zeta} e^{-\sqrt{2} \zeta} \quad \{1 - F_x(x)\} \xrightarrow{x \to \infty} \frac{1}{2 \sqrt{\pi}} .$$

Hence
Thus, by the Bounded Convergence Theorem

\[
\lim_{x \to \infty} c_3 \int [1 - F_x(x)] dG(\theta) = 0.
\]

By L'Hospital's rule, we have that

\[
\lim_{x \to \infty} \frac{\int [1 - G(x - \theta)]^2 dF(\theta)}{\int [1 - G(x - \theta)] dF(\theta)} = \lim_{x \to \infty} \frac{2\int [1 - G(x - \theta)] G'(x - \theta) dF(\theta)}{\int G'(x - \theta) dF(\theta)} = \lim_{x \to \infty} 2c_3 \int [1 - F_x(x)] dG(\theta) = 0.
\]

By inequalities (5.4), we conclude that

\[
\lim_{x \to \infty} \frac{1 - F_x(x) - F_y(y) + F_{x,y}(x, y)}{1 - F_{x,y}(x, y)} = 0.
\]
Therefore, by Theorem 4.1, $M_n^{(1)}$ and $M_n^{(2)}$ are asymptotically independent. 

As a corollary, we obtain a well-known result for the bivariate normal distributions. The following lemma is needed.

**Lemma 5.1.** If $F_{X,Y}(x, y)$ is a bivariate normal$(\mu, \Sigma)$-distribution with correlation coefficient $\rho$ such that $|\rho| < 1$ and $\rho \neq 0$, then there exist normally distributed independent random variables $U$, $V$, and $W$ such that $(U + W, V + \beta W)'$ has distribution $F_{X,Y}(x, y)$ for some $\beta \in \mathbb{R}$.

For the proof, see Part B of the appendix.

**Corollary 5.4.** If $F_{X,Y}(x, y)$ is a bivariate normal distribution with correlation coefficient $\rho$ such that $|\rho| < 1$, then $M_n^{(1)}$ and $M_n^{(2)}$ are asymptotically independent.

**Proof:** Suppose that $F_{X,Y}(x, y)$ has mean $\mu' = (\mu_1, \mu_2)$ and variance-covariance matrix $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$. Note that the marginal distributions $F_X(x)$ and $F_Y(y)$ are normal and hence, by Example 2.1, are attracted to the extreme distribution $\Lambda(x)$.

If $\sigma_{12} = 0$ then, as we know, $F_{X,Y}(x, y) = F_X(x) F_Y(y)$ and the result follows obviously.

If $\sigma_{12} \neq 0$ then, by Lemma 5.1, we can see that $F_{X,Y}(x, y)$ admits the representation (5.1) where $G(x)$, $H(y, \theta)$ and $F(\theta)$ are normal distributions. Therefore, by Theorem 5.4, the result follows. 

Corollary 5.5. Let $X_1, X_2, \ldots$, be iid rv's with distribution $F(x)$ and $Y_1, Y_2, \ldots$, be iid rv's with distribution $G(y)$. Suppose that the $X_i$'s are independent of the $Y_i$'s. If $F(x)$ is a normal $(\mu, \sigma^2)$-distribution and $G(y)$ is a normal($\nu, \tau^2$)-distribution, then $M_n = \max(X_1, X_2, \ldots, X_n)$ and $M_n^* = \max(X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n)$ are asymptotically independent.

Proof: Similar to the proof of Corollary 5.1, applying Theorem 5.4.

Remark 5.3. Since $G * F(x)$ is a normal distribution in Corollary 5.5, by Example 2.1, $G * F(x)$ is attracted to the extreme distribution $A(x)$. That is, the assumption that $G * F(x)$ is attracted to an extreme distribution is superfluous. And, clearly, the limit distribution of $(M_n, M_n^*)'$ properly normalized is $A(x)A(y)$ since, by Example 2.1, $F \in \mathcal{B}(A)$.

Under the additional condition that $\tau^2 < 3\sigma^2$, Corollary 5.5 was proved in David (1973) using a different method.

Now we look at the class of gamma distributions.

Theorem 5.5. Suppose that $F_{X,Y}(x, y)$ is a bivariate distribution of the form (5.1). If $G(x)$ and $F(\theta)$ are the same gamma($\nu, \beta$)-distribution with $\nu$ a positive integer, and $H(y, \theta)$ is any distribution, then $M_n^{(1)}$ and $M_n^{(2)}$ are asymptotically independent.

Proof: As in the proof of Theorem 4.3, we have that...
\[ \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x,y)}{1 - F_{X,Y}(x,y)} \leq \left\{ \frac{\int [1 - F(x - \theta)]^2 \, dF(\theta)}{\int [1 - F(x - \theta)] \, dF(\theta)} \right\}^{\frac{1}{2}}. \] (5.6)

Integrating by parts, we have for \( x \geq 0 \)

\[ 1 - F(x) = \beta F'(x) \left\{ 1 + \frac{(v-1)\beta}{x} + \frac{(v-1)(v-2)\beta^2}{x^2} + \ldots + \frac{(v-1)! \beta^{v-1}}{x^{v-1}} \right\} \]

\[ = e^{-\frac{x}{\beta}} \sum_{l=0}^{v-1} \frac{x^l}{\beta^l l!}. \]

(a) Consider the distribution function

\[ J(x) = \begin{cases} 1 - [1 - F(x)]^2, & x \geq 0 \\ 0, & x < 0 \end{cases}. \]

Then \( dJ(x) = 2 [1 - F(x)] \, dF(x) \). Now, for \( x \geq 0 \)
1 - F \ast J(x) = \int \{1 - F(x - y)\} \, dJ(y) = \int_{-\infty}^{x} \{1 - F(x - y)\} \, dJ(y) + \int_{x}^{+\infty} \, dJ(y)

= \int_{-\infty}^{x} e^{\frac{x-y}{\beta}} \sum_{\ell=0}^{\nu-1} \frac{(x-y)^{\ell}}{\beta^{\ell} \ell!} \, d\bar{J}(y) + \{1 - \bar{J}(x)\}

= e^{\frac{x}{\beta}} \int_{-\infty}^{x} \sum_{\ell=0}^{\nu-1} \frac{(x-y)^{\ell}}{\beta^{\ell} \ell!} e^{\frac{y}{\beta}} \, d\bar{J}(y) + \{1 - \bar{J}(x)\}.

Since

\( (x - y)^{\ell} = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k x^{\ell-k} y^k, \)

\[ 1 - F \ast J(x) = e^{\frac{x}{\beta}} \int_{-\infty}^{x} \sum_{\ell=0}^{\nu-1} \frac{1}{\beta^{\ell} \ell!} \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} x^{\ell-k} y^k e^{\frac{y}{\beta}} \, d\bar{J}(y) + \{1 - \bar{J}(x)\} \]

= e^{\frac{x}{\beta}} \sum_{\ell=0}^{\nu-1} \frac{x^{\ell}}{\beta^{\ell} \ell!} \int_{-\infty}^{x} e^{\frac{y}{\beta}} \, d\bar{J}(y)

+ e^{\frac{x}{\beta}} \sum_{\ell=1}^{\nu-1} \frac{1}{\beta^{\ell} \ell!} \sum_{k=1}^{\ell} (-1)^k \binom{\ell}{k} x^{\ell-k} y^k e^{\frac{y}{\beta}} \, d\bar{J}(y) + \{1 - \bar{J}(x)\} \]
\begin{equation}
\{1 - F(x)\} ! \int_{-\infty}^{x} e^{\beta y} \, dJ(y) + R(x) + \{1 - F(x)\}^{2}
\end{equation}

where \( R(x) \) is the second term in the preceding expression. Hence

\begin{equation}
\frac{1 - P \cdot J(x)}{1 - F(x)} = \int_{-\infty}^{x} e^{\beta y} \, dJ(y) + \frac{R(x)}{1 - F(x)} + \{1 - F(x)\}.
\end{equation}

Notice that

\begin{equation}
\int_{-\infty}^{x} e^{\beta y} \, dJ(y) = \int_{-\infty}^{x} e^{\beta y} 2 \{1 - F(y)\} \, dF(y) = 2 \int_{-\infty}^{x} \sum_{\ell=0}^{\nu-1} \frac{y^\ell}{\beta^\ell \ell!} \, dF(y)
\end{equation}

\begin{equation}
= 2 \sum_{\ell=0}^{\nu-1} \frac{1}{\beta^\ell \ell!} \int_{-\infty}^{x} y^\ell \, dF(y) \xrightarrow{x \to \infty} 2 \sum_{\ell=0}^{\nu-1} \frac{\mu_{\ell}^{*}}{\beta^\ell \ell!}
\end{equation}

where

\begin{equation}
\mu_{\ell}^{*} = \int_{-\infty}^{+\infty} y^\ell \, dF(y) < \infty.
\end{equation}

Also notice that
\[
\frac{R(x)}{1 - F(x)} = \frac{- \frac{\beta}{\beta} \sum_{\ell=1}^{v-1} \frac{1}{\beta^\ell \ell! \cdot} \sum_{k=1}^\ell (-1)^k \binom{\ell}{k} x^{\ell-k} \int_x^{\infty} y^k e^\beta dJ(y) - \int_x^{\infty} y^k e^\beta dJ(y)}{x^{v-1} e^{-x/\beta}} \frac{1 + (v-1)\beta + (v-1)(v-2)\beta^2 + \ldots + (v-1)!\beta^{v-1}}{\Gamma(v) \beta^{v-1}} 
\]

\[x \to \infty \]

since the largest power of \(x\) in the numerator is \(v-2\) and

\[
\int_x^{\infty} y^k e^\beta dJ(y) = \int_x^{\infty} y^k e^\beta [1 - F(x)] dF(y) = 2 \int_x^{\infty} y^k \sum_{\ell=0}^{v-1} \frac{1}{\beta^\ell \ell!} y^\ell dF(y) 
\]

\[
= 2 \sum_{\ell=0}^{v-1} \frac{1}{\beta^\ell \ell!} \int_x^{\infty} y^{k+\ell} dF(y) \quad \to \quad 2 \sum_{\ell=0}^{v-1} \frac{\mu_{k+\ell}}{\beta^\ell \ell!} 
\]

where

\[
\mu_{k+\ell} = \int_0^{\infty} y^{k+\ell} dF(y) < \infty \ .
\]

Therefore
\[
\frac{1 - F(x)}{1 - F(x)} \rightarrow x \rightarrow \infty \quad 2 \sum_{l=0}^{v-1} \frac{\mu_l^l }{\beta^l l!};
\]

that is

\[
\frac{\int [1 - F(x - \theta)]^2 d\theta}{1 - F(x)} \rightarrow x \rightarrow \infty \quad 2 \sum_{l=0}^{v-1} \frac{\mu_l^l }{\beta^l l!}.
\]

(b) Since \( F \ast F(x) \) is a gamma\((2v, \beta)\) -distribution,

\[
\frac{1 - F \ast F(x)}{1 - F(x)}
\]

\[
\frac{\beta x^{2v-1} e^{-\frac{x}{\beta}}}{\Gamma(2v) \beta^{2v}} \left\{ 1 + \frac{(2v-1)\beta}{x} + \frac{(2v-1)(2v-2)\beta^2}{x^2} + \ldots + \frac{(2v-1)! \beta^{2v-1}}{x^{2v-1}} \right\}
\]

\[
= \frac{\beta x^{2v-1} e^{-\frac{x}{\beta}}}{\Gamma(v) \beta^{v-1}} \left\{ 1 + \frac{(v-1)\beta}{x} + \frac{(v-1)(v-2)\beta^2}{x^2} + \ldots + \frac{(v-1)! \beta^{v-1}}{x^{v-1}} \right\}
\]

\[
= \frac{\Gamma(v) x^v \left\{ 1 + \frac{(2v-1)\beta}{x} + \ldots + \frac{(2v-1)! \beta^{2v-1}}{x^{2v-1}} \right\}}{\Gamma(2v) \beta^v \left\{ 1 + \frac{(v-1)\beta}{x} + \ldots + \frac{(v-1)! \beta^{v-1}}{x^{v-1}} \right\}} \rightarrow \infty; \quad x \rightarrow \infty.
\]

that is
\[
\int \frac{[1 - F(x - \theta)] dF(\theta)}{1 - F(x)} \quad \xrightarrow{x \to \infty} \quad \infty .
\]

Thus, from (a) and (b)

\[
\int \frac{[1 - F(x - \theta)]^2 dF(\theta)}{\int [1 - F(x - \theta)] dF(\theta)} \quad \xrightarrow{x \to \infty} \quad 0 .
\]

Hence, by inequalities (5.6), we conclude that

\[
\lim_{x \to \infty} \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)}{1 - F_{X,Y}(x, y)} = 0 .
\]

Therefore, by Theorem 4.1, \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically independent.

Corollary 5.6. Let \( X_1, X_2, \ldots \) and \( Y_1, Y_2, \ldots \) be two independent sequences of iid rv's with distribution \( F(x) \). If \( F(x) \) is a gamma(\( \nu \), \( \beta \) )-distribution with \( \nu \) a positive integer then \( M_n = \max(X_1, X_2, \ldots, X_n) \) and \( M^*_n = \max(X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n) \) are asymptotically independent.

Proof: Similar to the proof of Corollary 5.1 applying Theorem 5.5.

Remark 5.4. The assumption that \( F \ast F(x) \) is attracted to an
extreme distribution is superfluous since \( F^*F(x) \) is a \( \text{gamma}(2\nu, \beta) \)-distribution, and by Example 2.2 it is attracted to \( \Lambda(x) \). The limit distribution of \( (M_n^*, M_n^*)' \) properly normalized is \( \Lambda(x) \Lambda(y) \) since, by Example 2.2, \( F \in \mathcal{J}(\Lambda) \).

We consider now the class of increasing hazard rate (IHR) distributions. That is, distribution functions \( F(x) \) with derivative \( F'(x) \) for which the function \( q(x) \), defined for \( F(x) < 1 \) by \( q(x) = \frac{F'(x)}{1 - F(x)} \), is non-decreasing (see Johnson and Kotz, 1970, Vol. 3).

Theorem 5.6. Suppose \( F_{X,Y}(x, y) \) is a bivariate distribution of the form (5.1). If \( H(y, \Theta) \) is any distribution and \( G(x) = F(x) \) is an IHR distribution such that \( q(x) \xrightarrow{\theta} \infty \) as \( x \xrightarrow{\infty} \), then \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically independent.

Proof: We have the inequalities

\[
0 \leq \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x,y)}{1 - F_{X,Y}(x,y)} \leq \left\{ \frac{\int [1-F(x-\Theta)]^2 dF(\Theta)}{\int [1-F(x-\Theta)] dF(\Theta)} \right\}^{1/2} \tag{5.7}
\]

Note that for any \( \varepsilon > 0 \) there exists \( M = M(\varepsilon) > 0 \) such that

\[
0 < 1 - F(M) < \varepsilon.
\]

On the other hand, we have that
\[
\int_{-\infty}^{+\infty} (1 - F(x-\Theta))^2 dF(\Theta) = \int_{-\infty}^{x-M} (1 - F(x-\Theta))^2 dF(\Theta) + \int_{x-M}^{+\infty} (1 - F(x-\Theta))^2 dF(\Theta)
\]

\[
\leq (1 - F(M)) \int_{-\infty}^{x-M} (1 - F(x-\Theta)) dF(\Theta) + \int_{x-M}^{+\infty} dF(\Theta)
\]

\[
\leq (1 - F(M)) \int_{-\infty}^{x-M} (1 - F(x-\Theta)) dF(\Theta) + (1 - F(x-M))
\]

On the other hand, since \(q(x)\) is non-decreasing, for \(x > 2M\)

\[
\int_{-\infty}^{+\infty} (1 - F(x-\Theta)) dF(\Theta) \geq \int_{x/2}^{x-M} (1 - F(x-\Theta)) F'(\Theta) d\Theta
\]

\[
\geq \int_{x/2}^{x-M} (1 - F(x-\Theta)) q(\Theta) d\Theta
\]

\[
\geq (1 - F(x-M)) q(x/2) \int_{x/2}^{x-M} (1 - F(x-\Theta)) d\Theta
\]

\[
\geq (1 - F(x-M)) q(x/2) \int_{x/2}^{+\infty} (1 - F(t)) dt
\]

by the change of variable \(t = x - \Theta\). Therefore, since
\[
\int_{-\infty}^{+\infty} (1 - F(x - \theta)) \, dF(\theta) \geq \int_{-\infty}^{x-M} (1 - F(x - \theta)) \, dF(\theta) ,
\]

\[
\int_{-\infty}^{+\infty} (1 - F(x - \theta))^2 \, dF(\theta) \leq \int_{-\infty}^{x-M} (1 - F(x - \theta)) \, dF(\theta) \leq \int_{-\infty}^{+\infty} (1 - F(x - \theta)) \, dF(\theta)
\]

\[
+ \frac{1 - F(x-M)}{\int_{-\infty}^{+\infty} (1 - F(x - \theta)) \, dF(\theta)}
\]

\[
\leq \{1 - F(M)\} + \frac{1 - F(x-M)}{\{1 - F(x-M)\}q(x/2) \int_{M}^{x/2} [1 - F(t)] \, dt}
\]

\[
< \varepsilon + \frac{1}{q(x/2) \int_{M}^{x/2} [1 - F(t)] \, dt}
\]

Since \( q(x) \to \infty \) as \( x \to \infty \) and \( \lim_{x \to \infty} \int_{M}^{x/2} [1 - F(t)] \, dt > 0 \)

since \( F \) is continuous and \( 1 - F(M) > 0 \), we have by inequalities (5.7) that
\[
0 \leq \lim_{x \to \infty} \lim_{y \to \infty} \frac{1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)}{1 - F_{X,Y}(x, y)} < \sqrt{\varepsilon}
\]

for any \( \varepsilon > 0 \). Therefore, condition (4.1) of Theorem 4.1 holds, and the desired result follows. \( \square \)

We finish this chapter with the corresponding corollary of Theorem 5.6 for \((M_n, M_n^*)\).

**Corollary 5.7.** Let \( X_1, X_2, \ldots \) and \( Y_1, Y_2, \ldots \) be two independent sequences of iid rv's with distribution \( F(x) \). If \( F(x) \) is an IHR distribution such that the first moment exists and \( q(x) \to \infty \) as \( x \to \infty \), then \( M_n = \max(X_1, X_2, \ldots, X_n) \) and \( M_n^* = \max(X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n) \) are asymptotically independent.

**Proof:** Similar to the proof of Corollary 5.1 applying Theorem 5.6. \( \square \)
VI. NO ASYMPTOTIC INDEPENDENCE: DOMAINS OF ATTRACTION

Domains of attraction in the bivariate case have not received much attention. It is known that the class of types of bivariate extreme distributions is infinite (see de Oliveira, 1959; Sibuya, 1960), so we will restrict our attention to some subclasses of interest. In this chapter we present sufficient conditions for a bivariate distribution to be attracted to some bivariate extreme distributions, as well as characterizations of the domains of attraction of some others. These results are extensions of some of those in the univariate case.

Let \( \{(X_i, Y_i)\}' \) be a sequence of independent identically distributed bivariate random variables with distribution \( F_{X,Y}(x, y) \). For \( (M_n^{(1)}, M_n^{(2)}) \), as defined in Chapter IV, we can see that

\[
P(M_n^{(1)} \leq x, M_n^{(2)} \leq y) = F_{X,Y}^n(x, y) .
\]

**Definition 6.1.** A bivariate distribution function \( F_{X,Y}(x, y) \) belongs to the domain of attraction of a non-degenerate bivariate distribution function \( G(x, y) \), (notation \( F_{X,Y} \in \mathcal{B}(G) \)) if there exist sequences of real numbers \( \{a_n > 0\}, \{c_n > 0\}, \{b_n\} \) and \( \{d_n\} \) such that

\[
F_{X,Y}^n(a_n x + b_n, c_n y + d_n) \xrightarrow{w} G(x, y) .
\] (6.1)

\( G(x, y) \) is called a bivariate extreme distribution.
Let $x$ be fixed and let $y$ tend to infinity in (6.1), then the sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ are to be determined by the corresponding univariate limit distribution. From the work of Smirnov (1952) we have that the sequences $\{a_n > 0\}$ and $\{b_n\}$ are such that

$$\lim_{n \to \infty} n P(X > a_n x + b_n) = g(x)$$

exists.

Galambos (1973) proved a multivariate result whose bivariate version is as follows.

**Theorem 6.1.** The limit distribution

$$\lim_{n \to \infty} P_{X,Y}(a_n x + b_n, c_n y + d_n) = G(x, y)$$

exists if and only if

1. $\lim_{n \to \infty} n P(X > a_n x + b_n) = g(x)$,
2. $\lim_{n \to \infty} n P(Y > c_n y + d_n) = h(y)$ and
3. $\lim_{n \to \infty} n P(X > a_n x + b_n, Y > c_n y + d_n) = w(x, y)$ exist. When the limits (1), (2) and (3) exist,

$$G(x, y) = \exp \{-g(x) - h(y) + w(x, y)\}.$$

The following convention will help to ease the notation in some places.
Convention 6.1. Under the assumption that the marginal distributions of $F_{X,Y}(x,y)$ are such that $F_X \in \mathcal{B}(\hat{\xi}_1)$ and $F_Y \in \mathcal{B}(\hat{\xi}_2)$ where $\hat{\xi}_i$ $i = 1, 2$ are extreme distributions, we have, by Lemma 2.1, that the limits (1) and (2) of Theorem 6.1 hold with $g(x) = -\log \hat{\xi}_1(x)$ and $h(y) = -\log \hat{\xi}_2(y)$, respectively. Then in this case, we will say that $F_{X,Y}(x,y)$ is in the domain of attraction of $w(x,y)$ (notation $F_{X,Y} \in \mathcal{B}(w(x,y))$) if limit (3) of Theorem 6.1 holds.

Note that $F_{X,Y} \in \mathcal{B}(w(x,y))$ if and only if $F_{X,Y} \in \mathcal{B}(G)$ where $G(x,y) \equiv \exp \{-g(x)-h(y)+w(x,y)\}$. Hence, if $w(x,y) \equiv 0$ then $M_n^{(1)}$ and $M_n^{(2)}$ are asymptotically independent.

We proceed to give sufficient conditions for attraction to a class of bivariate extreme distributions introduced by de Oliveira (1959). These extreme distributions are also described in Gumbel (1965). The following lemma helps to present the result.

Lemma 6.1. Let $\hat{\xi}_i$ $i = 1, 2$ be two extreme distributions and let $F_{X,Y}(x,y)$ be a bivariate distribution with $H(x,y) = P(X > x, Y > y)$. Suppose that $F_{X,Y}(x,y)$ has marginal distributions $F_X(x)$ and $F_Y(y)$ with right endpoints $x_0$ and $y_0$, respectively and such that:

\begin{align*}
(1) \quad & \lim_{x \to x_0} \alpha(x) \{1 - F_X(x)\} = 1 \quad \text{and} \quad F_X \in \mathcal{B}(\hat{\xi}_1) . \\
(2) \quad & \lim_{y \to y_0} \beta(y) \{1 - F_Y(y)\} = 1 \quad \text{and} \quad F_Y \in \mathcal{B}(\hat{\xi}_2) .
\end{align*}
If

$$\lim_{x \to x_0} (\alpha(x) + \beta(y)) H(x, y) = c < \infty,$$

then for all $x$ and $y$ such that $0 < \hat{f}(1)(x) < 1$ and $0 < \hat{g}(2)(y) < 1$,

$$\lim_{n \to \infty} n H(x_n, y_n) = -c \left( \frac{1}{\log \hat{f}(1)(x)} + \frac{1}{\log \hat{g}(2)(y)} \right)$$

where $x_n = a_n x + b_n$ and $y_n = c_n y + d_n$ for some real sequences

$\{a_n > 0\}$, $\{c_n > 0\}$, $\{b_n\}$ and $\{d_n\}$.

Proof: From (1) $F_X \in \mathscr{L}(\hat{f}(1))$. Then there exist real sequences

$\{a_n > 0\}$ and $\{b_n\}$ such that $F_X^n(x_n) \xrightarrow{w} \hat{f}(1)(x)$ where $x_n = a_n x + b_n$. Then by Lemma 2.1, for $0 < \hat{f}(1)(x) < 1$

$$\lim_{n \to \infty} n \{1 - F_X(x_n)\} \to -\log \hat{f}(1)(x).$$

Hence, from (1),

$$\frac{\alpha(x_n)}{n} = \frac{\alpha(x_n) \{1 - F_X(x_n)\}}{n \{1 - F_X(x_n)\}} \to \frac{1}{-\log \hat{f}(1)(x)}.$$

Similarly, there exist real sequences $\{c_n > 0\}$ and $\{d_n\}$ such that
for $0 < \phi_2(y) < 1$,

$$\frac{\beta(y_n)}{n} = \frac{\beta(y_n) \{1 - F_Y(y_n)\}}{n \{1 - F_Y(y_n)\}} \xrightarrow{n \to \infty} -\log \phi_2(y)$$

where $y_n = c_n y + d_n$.

Therefore, for all $x$ and $y$ such that $0 < \phi_1(x) < 1$ and $0 < \phi_2(y) < 1$,

$$n H_n(x, y_n) = (\alpha(x_n) + \beta(y_n)) H(x_n, y_n) \left( \frac{\alpha(x_n)}{n} + \frac{\beta(y_n)}{n} \right)$$

$$\xrightarrow{n \to \infty} -c \left( \frac{1}{\log \phi_1(x)} + \frac{1}{\log \phi_2(y)} \right)^{-1}$$

where $x_n = a_n x + b_n$ and $y_n = c_n y + d_n$.

**Theorem 6.2.** Let $\phi_i$, $i = 1, 2$ be two extreme distributions and let $F_{X,Y}(x, y)$ be a bivariate distribution with $H(x, y) = P(X > x, Y > y)$. Suppose that $F_{X,Y}(x, y)$ has marginal distributions $F_X(x)$ and $F_Y(y)$ with right endpoints $x_0$ and $y_0$, respectively and satisfying (1) and (2) of Lemma 6.1 with the functions $\alpha(x)$ and $\beta(y)$. If
\[
\lim_{x \to x_0^-} (\alpha(x) + \beta(y)) H(x, y) = c < \infty
\]
\[
y \to y_0^-
\]

then

\[
F_{X,Y} \in \mathcal{B} \left( -c \left( \frac{1}{\log \hat{\phi}(1)(x)} + \frac{1}{\log \hat{\phi}(2)(y)} \right)^{-1} \right)
\]

Proof: From (1) and (2) of Lemma 6.1 we have, by Lemma 2.1, that there exist real sequences \( \{a_n > 0\}, \{c_n > 0\}, \{b_n\} \) and \( \{d_n\} \) such that for \( 0 < \hat{\phi}(1)(x) < 1 \) and \( 0 < \hat{\phi}(2)(y) < 1 \),

\[
n P(X > a_n x + b_n) = n \left[ 1 - F_X(a_n x + b_n) \right] \xrightarrow{n \to \infty} -\log \hat{\phi}(1)(x)
\]

\[
n P(Y > c_n y + d_n) = n \left[ 1 - F_Y(c_n y + d_n) \right] \xrightarrow{n \to \infty} -\log \hat{\phi}(2)(y)
\]

That is, limits (1) and (2) of Theorem 6.1 hold. By Lemma 6.1, the limit (3) of Theorem 6.1 holds also. Therefore, by Convention 6.1, the desired result follows. \( \square \)

Notice that when \( c = 0 \) in Theorem 6.2, we get the asymptotic independence of \( M_n^{(1)} \) and \( M_n^{(2)} \).

In many cases, Lemma 2.2, Lemma 2.3 and Corollary 2.6 are very useful to find the sequences \( \{a_n > 0\}, \{c_n > 0\}, \{b_n\} \) and \( \{d_n\} \).
provided we know the functions $\alpha(x)$ and $\beta(y)$ of Lemma 6.1, and thereby obtain the extreme distributions $\hat{\Phi}(1)(x)$ and $\hat{\Phi}(2)(y)$.

We next illustrate this procedure.

**Example 6.1.** Consider the bivariate distribution $F_{x,y}(x, y)$ with

$$H(x, y) = P(X > x, Y > y) = \frac{1 - e^{-(x+y)}}{x + y}, \quad x, y > 0.$$ 

$F_{x,y}(x, y)$ has identical marginal distributions

$$F(x) = 1 - \frac{1 - e^{-x}}{x}, \quad x > 0.$$ 

Note that $\lim_{x \to \infty} x \{1 - F(x)\} = \lim_{x \to \infty} (1 - e^{-x}) = 1$. That is $\alpha(x) = \beta(x) = x$. Therefore, by Lemma 2.2, we have that

$$F(nx) \xrightarrow{n \to \infty} e^{-x^{-1}}.$$ 

Then (1) and (2) of Lemma 6.1 hold with $\alpha(x) = \beta(x) = x$ and $g(x) = h(x) = x^{-1}$.

Now note that

$$(\alpha(x) + \beta(y)) H(x, y) = 1 - e^{-(x+y)} \quad \xrightarrow{x \to \infty, y \to \infty} 1.$$
Thus, by Theorem 6.2 we have that \( F_{X,Y} \in \mathcal{S}(x + y)^{-1} \).

Now we deal with the case when the underlying distribution has identical marginal distributions which belong to the domain of attraction of \( \Lambda(x) \). We present a characterization of the domain of attraction of the bivariate extreme distributions of the form

\[
G(x, y) = \exp \{-e^{-x} - e^{-y} + w(x, y)\}.
\]  
(6.2)

**Theorem 6.3.** Suppose that the bivariate distribution \( F_{X,Y}(x, y) \) with \( H(x, y) = P(X > x, Y > y) \) has identical marginal distributions \( F \in \mathcal{S}(\Lambda) \) with right endpoint \( x_0 \). \( F_{X,Y}(x, y) \) belongs to the domain of attraction of \( w(x, y) \) if and only if there exists a function \( f: \mathbb{R} \rightarrow \mathbb{R}^+ \) such that

\[
\lim_{t \to x_0^-} \frac{H(t + xf(t), t + yf(t))}{1 - F(t)} = w(x, y) \quad \text{for all } x, y.
\]

(6.3)

Moreover, then (6.3) holds with

\[
f(t) = \frac{\int_{x_0}^{x} [1 - F(s)] \, ds}{1 - F(t)}.
\]

**Proof:** Suppose that (6.3) holds. Substitution of \( t(s) = U(\frac{1}{s}) \) for \( t \) in (6.3) where \( U(x) = \inf \{y : 1 - F(y) \leq x\} \), gives
\[
\lim_{s \to \infty} \frac{1 - F(t(s))}{1 - F(t)} = w(x, y) \text{ for all } x, y.
\]

Since \( F \in \mathcal{L}(\Lambda) \), by Theorem 2.10, there exists a function \( f_1 : \mathbb{R} \to \mathbb{R}^+ \) such that

\[
\lim_{t \to x_0^{-}} \frac{1 - F(t + xf_1(t))}{1 - F(t)} = e^{-x} \text{ for all } x.
\]

By definition of \( t(s) \) we have for all \( \varepsilon > 0 \)

\[
l - F(t(s)) \leq \frac{1}{s} \leq l - F(t(s) - \delta) \leq l - F(t(s) - \varepsilon f_1(t))
\]

or

\[
\frac{l - F(t(s))}{l - F(t(s) - \varepsilon f_1(t))} \leq s \{l - F(t(s))\} \leq 1.
\]

Hence, for all \( \varepsilon > 0 \), \( e^{-\varepsilon} \leq \lim_{s \to \infty} s \{l - F(t(s))\} \leq 1 \), that is

\[
s \{l - F(t(s))\} \xrightarrow{s \to \infty} 1.
\]

Therefore
s \ H(f(t(s))x + t(s), f(t(s))y + t(s))

= \frac{H(f(t(s))x + t(s), f(t(s))y + t(s))}{1 - F(t(s))} \cdot s \{1 - F(t(s))\}

\xrightarrow{s \to \infty} w(x, y) \quad \text{for all } x, y.

Thus

n \ H(a_n x + b_n, a_n y + b_n) \xrightarrow{n \to \infty} w(x, y) \quad \text{for all } x, y

with \( a_n = f(t(n)) \) and \( b_n = t(n) \). Thus, by Convention 6.1, \( F_{X,Y} \in S(w(x, y)) \). Conversely, suppose that \( F_{X,Y} \in S(w(x, y)) \).

By Theorem 2.11, \( F \in S(\Lambda) \) implies that

\[ \lim_{t \to x^-} \frac{1 - F(t + x f(t))}{1 - F(t)} = e^{-x} \quad \text{for all } x \]

with \( f(t) \) defined as in (6.4). Hence, substituting \( \frac{1}{1 - F(t)} \) for \( n \), \( f(t) \) for \( a_n \) and \( t \) for \( b_n \) in Lemma 2.1, we have that

\[ \{F(t + x f(t))\} \frac{1}{1 - F(t)} \xrightarrow{t \to x^-} \exp \{ - e^{-x} \} = \Lambda(x). \]
Therefore, since the norming constants for \( F_{X,Y}(x, y) \) are determined by the corresponding univariate limit distributions, and \( F_{X,Y} \in \mathcal{L}(w(x, y)) \), we have by Convention 6.1, that

\[
\lim_{t \to x_0} \frac{\mathbb{H}(t + x f(t), t + y f(t))}{1 - F(t)} \to w(x, y) \text{ for all } x, y.
\]

Consider the following applications of Theorem 6.3.

**Example 6.2.** Look at the bivariate distribution

\[
F_{X,Y}(x, y) = 1 - e^{-x} - e^{-y} + (e^{x} + e^{y} - 1)^{-1}, \quad x, y > 0.
\]

Note that \( F_{X,Y}(x, y) \) has identical marginal distributions

\[
F(x) = 1 - e^{-x}, \quad x > 0
\]

and

\[
H(x, y) = P(X > x, Y > y) = (e^{x} + e^{y} - 1)^{-1}.
\]

By Theorem 2.15, \( F \in \mathcal{L}(\lambda) \). Also notice that

\[
f(t) = \frac{\int_0^\infty (1 - F(s)) \, ds}{1 - F(t)} = \frac{\int_0^\infty e^{-s} \, ds}{e^{-t}} = 1.
\]
Therefore we have that

\[
H(t + x f(t), t + y f(t)) = \frac{(e^{t+x} + e^{t+y} - 1)}{e^{-t}} = (e^X + e^Y - e^{-t})^{-1}
\]

Then, by Theorem 6.3 we have that \( F_{X,Y} \in \mathcal{S}((e^X + e^Y)^{-1}) \).

**Example 6.3.** Consider Gumbel's bivariate exponential distribution (Gumbel, 1960)

\[
F_{X,Y}(x, y) = 1 - e^{-x} - e^{-y} + e^{-(x+y+\Theta xy)}, \quad 0 \leq \Theta \leq 1; \quad x, y > 0.
\]

\( F_{X,Y}(x, y) \) has identical marginal distributions

\[
F(x) = 1 - e^{-x}, \quad x > 0.
\]

Notice that

\[
H(x, y) = P(X > x, Y > y) = e^{-(x+y+\Theta xy)}.
\]

As in Example 6.2, \( F \in \mathcal{S}(\lambda) \) and \( f(t) = 1 \). Then

\[
\frac{H(t + x f(t), t + y f(t))}{1 - F(t)} = \exp \left\{ (t + x + t + y + \Theta(t + x)(t + y)) \right\} \exp \{-t\}
\]
\[ = \exp \{- (t + x + y + \Theta(t + x)(t + y))\}\]

\[ \rightarrow 0. \quad \text{as} \quad t \rightarrow \infty. \]

Therefore, by Theorem 6.3, \( M_n^{(1)} \) and \( M_n^{(2)} \) are asymptotically independent.

For the case when \( F_{X,Y}(x, y) \) has marginal distributions \( F_1(x) \) and \( F_2(y) \) (not necessarily identical) which are attracted to \( \Lambda(x) \), we have another characterization of the domain of attraction of the distributions of the form (6.2).

**Theorem 6.4.** Suppose that the bivariate distribution \( F_{X,Y}(x, y) \) with \( H(x, y) = P(X > x, Y > y) \) has marginal distributions \( F_1, F_2 \in B(\Lambda) \). \( F_{X,Y}(x, y) \) belongs to the domain of attraction of \( w(x, y) \) if and only if there exists functions \( a, c : \mathbb{R} \rightarrow \mathbb{R}^+ \) and \( b, d : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
\lim_{s \to \infty} s H(a(s)x + b(s), c(s)y + d(s)) = w(x, y) \quad \text{for all } x, y.
\]  

(6.5)

Moreover, then (6.5) holds with

\[
\begin{align*}
a(s) &= U_1\left(\frac{1}{se}\right) - U_1\left(\frac{1}{s}\right) \\
c(s) &= U_2\left(\frac{1}{se}\right) - U_2\left(\frac{1}{s}\right) \\
b(s) &= U_1\left(\frac{1}{s}\right) \\
d(s) &= U_2\left(\frac{1}{s}\right)
\end{align*}
\]  

(6.6)
where for \( i = 1, 2, \) \( U_i : \mathbb{R}^+ \to \mathbb{R} \) such that

\[
U_i(y) = \inf \{x : 1 - F_i(x) \leq y\}
\]

Proof: Suppose that \( F_{x,y} \in \mathcal{B}(w(x, y)) \). By Theorem 2.9,

\( F_1, F_2 \in \mathcal{B}(A) \) implies

\[
s \lim_{s \to \infty} \{1 - F_1(a(s)x + b(s))\} \to e^{-x} \text{ for all } x
\]

\[
s \lim_{s \to \infty} \{1 - F_2(c(s)y + d(s))\} \to e^{-y} \text{ for all } y
\]

where \( a(s), b(s), c(s) \) and \( d(s) \) are given by (6.6). Hence, substituting first \( s \) for \( n \), \( a(s) \) for \( a_n \) and \( b(s) \) for \( b_n \) and then \( c(s) \) for \( a_n \) and \( d(s) \) for \( b_n \) in Lemma 2.1, we have that

\[
F_1^s(a(s)x + b(s)) \to \exp \{-e^{-x}\} = \Lambda(x)
\]

\[
F_2^s(c(s)y + d(s)) \to \exp \{-e^{-y}\} = \Lambda(y).
\]

Therefore, since the norming constants for \( F_{x,y}(x, y) \) are determined by the corresponding univariate limit distributions, and \( F_{x,y} \in \mathcal{B}(w(x, y)) \), we have, by Convention 6.1, that
The converse is simple. \( \square \)

**Corollary 6.1.** Suppose that the bivariate distribution 

\[ F_{X,Y}(x, y) \] with \( H(x, y) = P(X > x, Y > y) \) has marginal distributions 

\[ F_1, F_2 \in \mathcal{B}(\Lambda) \]. \( F_{X,Y}(x, y) \) belongs to the domain of attraction of \( \Theta(e^x + e^y)^{-1} \) if and only if

\[
\lim_{s \to \infty} \left( \frac{1}{1 - F_1(a(s)x + b(s))} + \frac{1}{1 - F_2(c(s)y + d(s))} \right) \\
\cdot H(a(s)x + b(s), c(s)y + d(s)) = \Theta \quad (6.7)
\]

for all \( x, y \) with the functions \( a, c: \mathbb{R} \to \mathbb{R}^+ \) and \( b, d: \mathbb{R} \to \mathbb{R} \) as defined in (6.6).

**Proof:** By Theorem 2.9, \( F_1, F_2 \in \mathcal{B}(\Lambda) \) implies

\[
\lim_{s \to \infty} s \{1 - F_1(a(s)x + b(s))\} \quad \xrightarrow{s \to \infty} \quad e^{-x} \quad \text{for all } x
\]

\[
\lim_{s \to \infty} s \{1 - F_2(c(s)y + d(s))\} \quad \xrightarrow{s \to \infty} \quad e^{-y} \quad \text{for all } y
\]

where \( a(s), b(s), c(s) \) and \( d(s) \) are given by (6.6). Suppose that (6.7) holds with (6.6). Then
\[ s \overline{H(a(s)x + b(s), c(s)y + d(s))} \]

\[ = \left( \frac{1}{1 - F_1(a(s)x + b(s))} + \frac{1}{1 - F_2(c(s)y + d(s))} \right) H(a(s)x + b(s), c(s)y + d(s)) \]

\[ \times \left( \frac{1}{s \{1 - F_1(a(s)x + b(s))\}} + \frac{1}{s \{1 - F_2(c(s)y + d(s))\}} \right) \]

\[ \xrightarrow{s \to \infty} \theta(e^x + e^y)^{-1}. \]

Thus, by Theorem 6.4, \( F_{X,Y} \in \mathcal{L}(\theta(e^X + e^Y)^{-1}) \).

Now suppose that \( F_{X,Y} \in \mathcal{L}(\theta(e^X + e^Y)^{-1}) \). Then, by Theorem 6.4,

\[ s \overline{H(a(s)x + b(s), c(s)y + d(s))} \xrightarrow{s \to \infty} \theta(e^X + e^Y) \]

where \( a(s), b(s), c(s) \) and \( d(s) \) are given by (6.6). Therefore

\[ \left( \frac{1}{1 - F_1(a(s)x + b(s))} + \frac{1}{1 - F_2(c(s)y + d(s))} \right) H(a(s)x + b(s), c(s)y + d(s)) \]

\[ \times \left( \frac{1}{s \{1 - F_1(a(s)x + b(s))\}} + \frac{1}{s \{1 - F_2(c(s)y + d(s))\}} \right) \]

\[ \xrightarrow{s \to \infty} \theta(e^X + e^Y) \]
As an illustration of Corollary 6.1, we take the distribution of Example 6.2.

Example 6.4. Consider the bivariate distribution of Example 6.2. 

\[ F_{X,Y}(x, y) = \begin{cases} \frac{1}{1 - F(x + \log s)} + \frac{1}{1 - F(y + \log s)} & H(x + \log s, y + \log s) \\
\frac{1}{e^{-(x + \log s)}} + \frac{1}{e^{-(y + \log s)}} & \left(e^{x + \log s} + e^{y + \log s} - 1\right)^{-1} \\
(e^x + e^y)(e^x + e^y - \frac{1}{s})^{-1} & s \to \infty \end{cases} \]

Then by Corollary 6.5, \( F_{X,Y} \in \mathcal{B}\left( (e^x + e^y)^{-1} \right) \) as we expected from Example 6.2.

Now we search for the extreme distributions that attract
distributions of the form

\[ F_{X,Y}(x, y) = F_1(x) F_2(y) F_3(\min(x, y)) \]  

(6.8)

where \( F_i(x), i = 1, 2, 3 \) are distribution functions. These distributions have already been treated in Chapter IV.

We have seen in Theorem 4.5 that for distributions of the form (6.8), if

\[ \lim_{x \to x_0(F_i)^-} \frac{1 - F_i(x)}{1 - F_i(x)} = 0, \quad i = 1 \text{ or } 2 \]

then asymptotic independence of \( M_n^{(1)} \) and \( M_n^{(2)} \) is obtained. The following theorem shows that when the \( F_i(x), i = 1, 2, 3 \) are tail equivalent (see Chapter II), the asymptotic independence is not obtained.

**Theorem 6.5.** Let \( \xi(x) \) be an extreme distribution. Suppose that \( F_{X,Y}(x, y) \) is of the form (6.8) with \( x_0 = x_0(F_i) \) \( i = 1, 2, 3 \). If \( F_1 \in \mathcal{F}(\xi) \) and

\[ \lim_{x \to x_0} \frac{1 - F_1(x)}{1 - F_2(x)} = a, \quad 0 < a < \infty \]

\[ \lim_{x \to x_0} \frac{1 - F_2(x)}{1 - F_3(x)} = b, \quad 0 < b < \infty \]
then

\[ F_{X,Y} \in \mathcal{B}(\psi(x) \cdot \psi(Ay + B) \cdot \psi(C \min(x, y) + D)) \]

where if

(i) \( \psi(x) = \frac{1}{Q(x)} \) then \( A = a^{1/\alpha}, \ B = 0, \ C = (ab)^{1/\alpha} \) and \( D = 0 \)

(ii) \( \psi(x) = \frac{1}{\psi(x)} \) then \( A = a^{-1/\alpha}, \ B = 0, \ C = (ab)^{-1/\alpha} \) and \( D = 0 \)

(iii) \( \psi(x) = A(x) \) then \( A = 1, \ B = \log a, \ C = 1 \) and \( D = \log ab \).

Proof: Since \( F_1 \in \mathcal{B}(\psi) \) there exist sequences \( \{a_n > 0\} \) and \( \{b_n\} \) such that

\[ F_1^n(a_n x + b_n) \xrightarrow{w} \psi(x). \]

Therefore, since

\[ \lim_{x \to x_0^+} \frac{1 - F_1(x)}{1 - F_2(x)} = a, \quad \text{and} \quad \lim_{x \to x_0^+} \frac{1 - F_1(x)}{1 - F_3(x)} = ab, \]

by Theorem 2.14, there exist constants \( A > 0, \ C > 0, \ B \) and \( D \) given by either (i) or (ii) or (iii) such that

\[ F_2^n(a_n x + b_n) \xrightarrow{w} \psi(Ax + B) \quad \text{and} \quad F_3^n(a_n x + b_n) \xrightarrow{w} \psi(Cx + D). \]
Thus, since $\min(a_n x + b_n, a_n y + b_n) = a_n \min(x, y) + b_n$, we have that

$$F^n_{X,Y}(a_n x + b_n, a_n y + b_n) = \{F_1(a_n x + b_n) F_2(a_n y + b_n) F_3(a_n \min(x, y) + b_n)\}^n$$

The following corollaries give more insight into the result of Theorem 6.5.

**Corollary 6.2.** Suppose that $F_{X,Y}(x, y)$ is of the form (6.8) where the $F_i(x)$ $i = 1, 2, 3$ have finite common right endpoint $x_\circ$.

If for some $\alpha > 0$

$$\lim_{x \to x_\circ} \frac{1 - F_i(x)}{(x_\circ - x)^\alpha} = \lambda_i, \quad 0 < \lambda_i < \infty, \quad i = 1, 2, 3$$

then $F_{X,Y}(x, y)$ belongs to the domain of attraction of

$$G(x, y) = \exp \left\{ -(-x)\alpha + \frac{\lambda_2}{\lambda_1} (-y)\alpha - \frac{\lambda_3}{\lambda_1} (-\min(x, y))\alpha \right\}, \ x, y < 0.$$ 

**Proof:** By Lemma 2.3, $F_1 \in \mathcal{S}(\psi_\alpha)$. Now

$$\lim_{x \to x_\circ} \frac{1 - F_1(x)}{1 - F_2(x)} = \frac{\lambda_1}{\lambda_2} \quad \text{and} \quad \lim_{x \to x_\circ} \frac{1 - F_2(x)}{1 - F_3(x)} = \frac{\lambda_2}{\lambda_3}.$$
Therefore, by Theorem 6.5, \( F_{X,Y}(x, y) \) belongs to the domain of attraction of

\[
\psi_\alpha(x) \psi_\alpha\left( \frac{\lambda_1}{\lambda_2} x \right) \psi_\alpha\left( \frac{\lambda_1}{\lambda_3} \min(x, y) \right)
\]

\[
= \exp \left\{ -(x)\alpha - \frac{\lambda_2}{\lambda_1} (-y)\alpha - \frac{\lambda_3}{\lambda_1} \left( -\min(x, y) \right)^\alpha \right\}, \quad x, y < 0. \quad \square
\]

**Remark 6.1.** If \( \alpha = 1 \) in Corollary 6.2, then \( F_{X,Y}(x, y) \) belongs to the domain of attraction of the well-known bivariate negative exponential distribution of Marshall and Olkin (Marshall and Olkin, 1967).

That is, this distribution is a bivariate extreme distribution.

---

**Corollary 6.3.** Suppose that \( F_{X,Y}(x, y) \) is of the form (6.8) where the \( F_i(x) \) \( i = 1, 2, 3 \) have common right endpoint \( x_0 = \infty \).

If for some \( \alpha > 0 \)

\[
\lim_{x \to \infty} x^\alpha \left( 1 - F_i(x) \right) = \lambda_i, \quad 0 < \lambda_i < \infty, \quad i = 1, 2, 3
\]

then \( F_{X,Y}(x, y) \) belongs to the domain of attraction of

\[
G(x, y) = \exp \left\{ -x^{-\alpha} - \frac{\lambda_2}{\lambda_1} y^{-\alpha} - \frac{\lambda_3}{\lambda_1} \left( \min(x, y) \right)^{-\alpha} \right\}, \quad x, y > 0.
\]

**Proof:** Applying Lemma 2.2 and Theorem 6.5. \( \square \)
Corollary 6.4. Suppose that \( F_{x,y}(x, y) \) is of the form (6.8) where the \( F_i(x) \) \( i = 1, 2, 3 \) have common right endpoint \( x_o = \infty \).

If for some \( \alpha > 0 \) and \( \beta \)

\[
\lim_{x \to \infty} x^\alpha e^x \{1 - F_i(x)\} = \lambda_i, \quad 0 < \lambda_i < \infty, \quad i = 1, 2, 3
\]

then \( F_{x,y}(x, y) \) belongs to the domain of attraction of

\[
G(x, y) = \exp \left\{ -e^{-x} - \frac{\lambda_1}{\lambda_3} e^{-y} - \frac{\lambda_1}{\lambda_2} e^{-\min(x,y)} \right\}, \quad x, y \in \mathbb{R}.
\]

Proof: Applying Theorem 2.15 and Theorem 6.5.

For the other possibilities regarding the behavior of the tails of \( F_i(x) \) \( i = 1, 2, 3 \) where \( x_o = x_o(F_i) \) \( i = 1, 2, 3 \) and \( \check{e}(x) \) is an extreme distribution, we have:

(I) If \( F_3 \in \mathcal{E}(\check{e}) \) and

\[
\lim_{x \to x_o} \frac{1 - F_1(x)}{1 - F_3(x)} = 0, \quad \lim_{x \to x_o} \frac{1 - F_2(x)}{1 - F_3(x)} = b, \quad 0 < b < \infty,
\]

then \( F_{x,y} \in \mathcal{B}(\check{e}(A y + B) \check{e}(\min(x,y))) \) for some real constants \( A > 0 \) and \( B \).

In fact, since \( F_3 \in \mathcal{E}(\check{e}) \), there exist real sequences \( \{a_n > 0\} \) and \( \{b_n\} \) such that \( F_3^n(x_n) \xrightarrow{w} \check{e}(x) \) where \( x_n = a_n x + b_n \). Then
by Lemma 2.1, \( n \{1 - F_3(x_n)\} \xrightarrow{n \to \infty} -\log \phi(x) \). On the other hand, by Theorem 2.14, there exist real constants \( A > 0 \) and \( B \) such that \( F_2^n(y_n) \xrightarrow{w} \phi(Ay + B) \) where \( y_n = a_n y + b_n \). Now note that

\[
F_1^n(x_n) = (1 - \{1 - F_1(x_n)\})^n = \left(1 - \frac{n \{1 - F_3(x_n)\} \frac{1 - F_1(x_n)}{1 - F_3(x_n)}}{n} \right)^n \xrightarrow{n \to \infty} 1
\]

since \( n \{1 - F_3(x_n)\} \frac{1 - F_1(x_n)}{1 - F_3(x_n)} \xrightarrow{n \to \infty} \{- \log \phi(x)\} \cdot 0 = 0 \).

Therefore, for \( x_n \) and \( y_n \) as above and \( z_n = a_n \min(x, y) + b_n \)

\[
F_{X,Y}^n(x_n, y_n) = F_1^n(x_n) F_2^n(y_n) F_3^n(z_n) \xrightarrow{w} \phi(Ay + B) \phi(\min(x, y)).
\]

(II) If \( F_3 \in \mathcal{B}(\phi) \) and

\[
\lim_{x \to x_0} \frac{1 - F_1(x)}{1 - F_3(x)} = 0, \quad \lim_{x \to x_0} \frac{1 - F_2(x)}{1 - F_3(x)} = 0
\]

then \( F_{X,Y} \in \mathcal{B}(\phi(\min(x, y))) \).

In fact, as in (I), \( F_1^n(x_n) \xrightarrow{n \to \infty} 1 \) and \( F_2^n(y_n) \xrightarrow{n \to \infty} 1 \),
therefore, \( F_{X,Y}(x_n, y_n) = F_1^n(x_n) F_2^n(y_n) F_3^n(z_n) \xrightarrow{w} \delta(\min(x, y)) \).

We have seen in Remark 6.1 that the bivariate negative exponential distribution of Marshall and Olkin shows up as an extreme distribution. For a better understanding of this distribution, we provide in Theorem 6.6 a characterization which is the counterpart to the following characterization of the univariate negative exponential distribution.

Let \( X \) and \( Y \) be independent identically distributed random variables with distribution \( F(x) \) for \( x < 0 \). The conditions

\[
2 \text{ max}(X, Y) \text{ has distribution } F(x),
\]

\[
\lim_{x \to 0^-} \frac{1 - F(x)}{-x} = \lambda, \quad 0 < \lambda < \infty
\]

are necessary and sufficient for

\[
F(x) = e^{\lambda x}, \quad x < 0, \quad \lambda > 0.
\]

**Theorem 6.6.** Suppose that \( X' = (X_1, X_2) \) and \( Y' = (Y_1, Y_2) \) are independent identically distributed bivariate random variables with distribution \( F_{X,Y}(x, y) \) for \( x, y < 0 \). The conditions:

\[
(2 \text{ max}(X_1, Y_1), 2 \text{ max}(X_2, Y_2))' \text{ has distribution } F_{X,Y}(x, y) \quad (6.9)
\]
are necessary and sufficient for

\[ F_{X,Y}(x, y) = \exp \{\lambda (\delta_1 x + \delta_2 y + \delta_3 \min(x,y))\}, \quad x,y < 0 \]  

(6.11)

\[ \lambda, \delta_1, \delta_2, \delta_3 > 0 \]

Proof: Suppose that (6.9) and (6.10) hold. Then

\[ F_{X,Y}(x, y) = P(2 \max (X_1, Y_1) < x, 2 \max (X_2, Y_2) < y) \]

\[ = P(X_1 \leq \frac{x}{2}, X_2 \leq \frac{y}{2}, Y_1 \leq \frac{x}{2}, Y_2 \leq \frac{y}{2}) \]

\[ = P(X_1 \leq \frac{x}{2}, X_2 \leq \frac{y}{2}) P(Y_1 \leq \frac{x}{2}, Y_2 \leq \frac{y}{2}) \]

\[ = \left[F_{X,Y}(\frac{x}{2}, \frac{y}{2})\right]^2. \]

Hence, for any positive integer \( k \)

\[ F_{X,Y}(x, y) = \left[F_{X,Y}(\frac{x}{2^k}, \frac{y}{2^k})\right]^{2^k}. \]
\[
\left(1 + \frac{-2^k \left[1 - F_{x,y}(\frac{x}{2^k}, \frac{y}{2^k})\right]}{2^k}\right)^{2^k} = \frac{1 - F_{x,y}(\frac{x}{2^k}, \frac{y}{2^k})}{2^k} \frac{\delta_1 x + \delta_2 y + \delta_3 \min(x, y)}{2^k} - \frac{\delta_1 x + \delta_2 y}{2^k} + \delta_3 \min(\frac{x}{2^k}, \frac{y}{2^k})
\]

\[
\lambda \left(\delta_1 x + \delta_2 y + \delta_3 \min(x, y)\right) \quad \text{as } k \to \infty
\]

Therefore, as \( k \to \infty \) expression (6.12) tends to

\[
\exp \left[\lambda \left(\delta_1 x + \delta_2 y + \delta_3 \min(x, y)\right)\right], \quad x, y < 0; \quad \delta_1, \delta_2, \delta_3, \lambda > 0.
\]

Conversely: Note that

\[
F_{x,y}(x, y) = \exp \left\{\lambda (\delta_1 x + \delta_2 y + \delta_3 \min(x, y))\right\}
\]

\[
= \exp \left\{2\lambda (\delta_1 \frac{x}{2} + \delta_2 \frac{y}{2} + \delta_3 \min(\frac{x}{2}, \frac{y}{2}))\right\}
\]

\[
= P(X_1 \leq \frac{x}{2}, X_2 \leq \frac{y}{2}) P(Y_1 \leq \frac{x}{2}, Y_2 \leq \frac{y}{2})
\]

\[
= P(X_1 \leq \frac{x}{2}, Y_1 \leq \frac{x}{2}, X_2 \leq \frac{y}{2}, Y_2 \leq \frac{y}{2})
\]
\[ P(2 \max(X_1, Y_1) \leq x, 2 \max(X_2, Y_2) \leq y) \]

That is, (6.9) holds.

Now, by L'Hospital's rule

\[ \lim_{x \to 0^-} \frac{1 - \exp\{\lambda(\delta_1 x + \delta_2 y + \delta_3 \min(x, y))\}}{\delta_1 x + \delta_2 y + \delta_3 \min(x, y)} = \lim_{z \to 0^-} \frac{1 - e^{\lambda z}}{-z} \]

\[ = \lim_{z \to 0^-} \lambda e^{\lambda z} = \lambda. \]

That is, (6.10) holds.

**Remark 6.2.** Let \( F_X(x) \) and \( F_Y(y) \) be the marginal distributions of \( F_{X,Y}(x, y), x, y < 0 \). Then the conditions:

\[ \lim_{x \to 0^-} \frac{1 - F_X(x)}{-\delta_1 x - \delta_3 x} = \lambda, \quad 0 < \lambda < \infty \]  

(6.13)

\[ \lim_{x \to 0^-} \frac{1 - F_{X,Y}(x, y)}{-\delta_1 x - \delta_2 y - \delta_3 \min(x, y)} \text{ exists} \]  

(6.14)

are equivalent to Condition (6.10). Therefore, Conditions (6.9), (6.13) and (6.14) provide another characterization for (6.11), which is in terms of the marginal distributions.
In the forthcoming theorems, and as a by-product of the characterization in Theorem 6.6, we find a sufficient condition for attraction to extreme distributions of the form

\[ G(x, y) = \exp \left\{-2^{-\alpha}(-x - y - \min(x, y))^\alpha \right\}, \quad x, y < 0, \quad \alpha > 0. \]  

(6.15)

The marginal distributions of \( G(x, y) \) are identical to \( \psi_{\alpha}(x) \), which is the negative Weibull distribution. And, if \( \alpha = 1 \) then \( G(x, y) \) is the bivariate negative exponential distribution of Marshall and Olkin. So, as in the univariate case, the distribution (6.15) might be called "the bivariate negative Weibull distribution."

**Theorem 6.7.** Suppose that \( F_{X,Y}(x, y) \) is a bivariate distribution which has identical marginal distributions \( F(x) \) with finite right endpoint \( x_0 \). If for \( \alpha > 0 \)

\[
\lim_{x \to x_0^-} \frac{1 - F_{X,Y}(x, y)}{\{3x_0 - x - y - \min(x, y)\}^\alpha} = \lambda, \quad 0 < \lambda < \infty, \quad (6.16)
\]

then \( F_{X,Y}(x, y) \) belongs to the domain of attraction of \( G(x, y) = \exp \left\{-2^{-\alpha}(-x - y - \min(x, y))^\alpha \right\}, \quad x, y < 0, \quad \alpha > 0. \)

Proof: Since we have that for \( y < x_0 \)
\[ \lim_{x \to x_o^-} \frac{1 - F_{x,y}(x, y)}{[3x_o - x - y - \min(x, y)]^\alpha} = \frac{1 - F(y)}{2^\alpha(x_o - y)^\alpha}, \quad (6.17) \]

then

\[ \lim_{y \to x_o^-} \frac{1 - F(y)}{2^\alpha(x_o - y)^\alpha} = \lim_{y \to x_o^-} \lim_{x \to x_o^-} \frac{1 - F_{x,y}(x, y)}{[3x_o - x - y - \min(x, y)]^\alpha} = \lim_{x \to x_o^-} \frac{1 - F_{x,y}(x, y)}{[3x_o - x - y - \min(x, y)]^\alpha} = \lambda. \]

That is

\[ \lim_{y \to x_o^-} \frac{1 - F(y)}{(x_o - y)^\alpha} = 2^\alpha \lambda. \]

Then, by Lemma 2.3

\[ F^n(a_n y + x_o) \xrightarrow{w} \psi_\alpha(y) \]

with

\[ a_n = (n \ 2^\alpha \lambda)^{-1/\alpha}. \]

Now, by (6.16), we have for \( x, y < 0 \) that
\[ n \left\{ 1 - F_{X,Y}(a_x + x_o, a_y + y_o) \right\} \]
\[ = n \left\{ 3x_o - (a_n x + x_o) - (a_n y + x_o) - \min(a_n x + x_o, a_n y + x_o) \right\} \]
\[ \times \frac{1 - F_{X,Y}(a_n x + x_o, a_n y + x_o)}{\left\{ 3x_o - (a_n x + x_o) - (a_n y + x_o) - \min(a_n x + x_o, a_n y + x_o) \right\}^\alpha} \]
\[ \longrightarrow 2^{-\alpha} (-x - y - \min(x, y))^\alpha, \quad n \to \infty \]

since

\[ n \left\{ 3x_o - (a_n x + x_o) - (a_n y + x_o) - \min(a_n x + x_o, a_n y + x_o) \right\} \]
\[ = n a_n^\alpha (-x - y - \min(x, y))^\alpha = \frac{2^{-\alpha}}{\lambda} (-x - y - \min(x, y))^\alpha. \]

Therefore, for \( x, y < 0 \)

\[ F_{X,Y}^n(a_n x + x_o, a_n y + y_o) = \left( 1 - \frac{n \left\{ 1 - F_{X,Y}(a_n x + x_o, a_n y + y_o) \right\}}{n} \right)^n \]
\[ \longrightarrow \exp\{-2^{-\alpha} (-x - y - \min(x, y))^\alpha\}, \quad n \to \infty \]
Remark 6.3. Let $F_{X,Y}(x, y)$ have identical marginal distributions $F(x)$ with finite right endpoint $x_0$. Then, the conditions:

$$\lim_{y \to x_0^-} \frac{1 - F(y)}{(x_0 - y)^\alpha} = 2^\alpha \lambda, \quad 0 < \lambda < \infty \quad (6.18)$$

$$\lim_{x \to x_0^-} \frac{1 - F_{X,Y}(x, y)}{[3x_0 - x - y - \min(x, y)]^\alpha} \text{ exists} \quad (6.19)$$

are equivalent to (6.16) by (6.17). Therefore, Conditions (6.18) and (6.19) are also sufficient conditions for attraction to (6.15).

We illustrate Theorem 6.7 with the following examples.

Example 6.5. Consider any bivariate distribution of the form

$$F_{X,Y}(x, y) = \exp \left\{ \frac{x + y + \min(x, y)}{1 + v(x, y)} \right\}, \quad x, y < 0 \quad (6.20)$$

with identical marginal distributions, and where $v: \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}^+$ is such that

$$\lim_{x \to 0^-} v(x, y) = 0 \quad \lim_{y \to 0^-} v(x, y) = 0$$

Then
\[
\frac{1 - F_{X,Y}(x, y)}{\{x + y + \min(x, y)\}} = \frac{1 - \exp \left\{ \frac{x + y + \min(x, y)}{1 + v(x, y)} \right\}}{\{x + y + \min(x, y)\}} \cdot \frac{1}{1 + v(x, y)}
\]

\[
x \rightarrow 0^- \\
y \rightarrow 0^-
\]

since by L'Hospital's rule \( \frac{1 - e^z}{-z} \xrightarrow{z \rightarrow 0^-} 1 \).

Therefore, by Theorem 6.7, \( F_{X,Y}(x, y) \) belongs to the domain of attraction of

\[
G(x, y) = \exp \left\{ 2^{-1}(x + y + \min(x, y)) \right\}, \quad x, y < 0 . \tag{6.21}
\]

**Example 6.5.1.** Consider the bivariate negative exponential distribution of Marshall and Olkin

\[
F_{X,Y}(x, y) = \exp \left\{ x + y + \min(x, y) \right\}, \quad x, y < 0 .
\]

Then \( F_{X,Y}(x, y) \) is of the form (6.20) with \( v(x, y) \equiv 0 \). Therefore, \( F_{X,Y}(x, y) \) is attracted to (6.21) which is a Marshall and Olkin exponential distribution also.

**Example 6.5.2.** Consider the function in two variables \( F_{X,Y}(x, y) \) of the form (6.20) where
\[ v(x, y) = \sqrt{-(x + y)} \]

\[ F_{X,Y}(x, y) \] has identical marginal functions

\[ F(x) = \begin{cases} 
\exp \left\{ \frac{2x}{1 + \sqrt{-x}} \right\}, & x < 0 \\
1, & x \geq 0
\end{cases} \]

which is a well-defined distribution function. Notice that:

(a) \( F_{X,Y}(x, y) \rightarrow 1 \) as \( \min(x, y) \rightarrow 0^- \),

(b) Since for \( x, y < 0 \), \( x + y + \min(x, y) < x + y \),

\[ \frac{x + y + \min(x, y)}{1 + \sqrt{-(x + y)}} < \frac{x + y}{\sqrt{-(x + y)} (1 + \{-x + y\}^{-1/2})} \]

\[ \leq \frac{-\sqrt{-(x + y)}}{1 + \{-x + y\}^{-1/2}} \rightarrow -\infty \]

as \( x \rightarrow -\infty \) or \( y \rightarrow -\infty \), hence

\[ \frac{x + y + \min(x, y)}{1 + \sqrt{-(x + y)}} \rightarrow -\infty \]
as \( x \to -\infty \) or \( y \to -\infty \). Therefore, \( F_{X,Y}(x, y) \to 0 \) as \( x \) or \( y \to -\infty \).

(c) It is clear that \( F_{X,Y} \) is continuous from the right in each argument.

(d) It can be seen that the measure \( \mu_{F_{X,Y}} \), associated with \( F_{X,Y}(x, y) \), assigns positive measure to every set of the form \( \{(x, y): a < x \leq b, c < y \leq d\} \). Therefore \( F_{X,Y}(x, y) \) is a well-defined bivariate distribution; and by Example 6.5, \( F_{X,Y}(x, y) \) is attracted to (6.21) since \( v(x, y) \to 0 \).

Example 6.6. Consider the function of two variables

\[
F_{X,Y}(x, y) = \frac{1}{1 - (x + y + \min(x, y))}, \quad x, y < 0.
\]

\( F_{X,Y}(x, y) \) has identical marginal functions given by

\[
F(x) = \begin{cases} 
\frac{1}{1 - 2x}, & x < 0 \\
1, & x \geq 0
\end{cases}
\]

which is a well-defined distribution function. Also notice that

(a) \( F_{X,Y}(x, y) \to 1 \) as \( \min(x, y) \to 0^- \)

(b) \( F_{X,Y}(x, y) \to 0 \) as \( x \) or \( y \to -\infty \).
(c) $F_{X,Y}(x, y)$ is continuous from the right in each argument.

(d) It can be seen that the measure $\mu_{F_{X,Y}}$ associated with $F_{X,Y}(x, y)$, assigns positive measure to every set of the form 

$$\{(x, y): a < x \leq b, c < y \leq d\}.$$ 

Therefore, $F_{X,Y}(x, y)$ is a well-defined bivariate distribution.

Now notice that

$$\frac{1 - F_{X,Y}(x, y)}{- (x + y + \min(x, y))} = \frac{1 - [1 - (x + y + \min(x, y))]^{-1}}{- (x + y + \min(x, y))}$$

$$= \frac{-(x + y + \min(x, y)) [1 - (x + y + \min(x, y))]^{-1}}{- (x + y + \min(x, y))}$$

$$= [1 - (x + y + \min(x, y))]^{-1} \rightarrow 1 \quad \text{as} \quad x \rightarrow 0^- \quad \text{and} \quad y \rightarrow 0^-.$$ 

Hence, by Theorem 6.7, $F_{X,Y}(x, y)$ is attracted to

$$G(x, y) = \exp \{2^{-1}(x + y + \min(x, y))\}, \quad x, y < 0.$$
This chapter is devoted to finding the limit distribution of properly normalized in sequences \( \{(X_i, Y_i)\}' \) of bivariate random variables subject to a kind of dependence. That is, now the random vectors \( (X_i, Y_i)' \) are not independent identically distributed but exchangeable. The problem was studied by Berman (1962) in the univariate case. Here we will present an extension of some of his results to the bivariate case. The extended results are applied in the solution of a general problem which was treated by David (1973) under particular distributional assumptions.

Let \((\Omega, G, P)\) be a probability space: \( \Omega \) is a set of points \( \omega \), \( G \) is a Borel field of subsets of \( \Omega \) and \( P \) is a probability measure on \( G \).

**Definition 7.1.** A sequence \( \{(X_i, Y_i)\}' \) of pairs of random variables defined on \((\Omega, G, P)\) is called exchangeable if the joint distribution function \( H_m(x_1, y_1, \ldots, x_m, y_m) \) of any \( m \) of these pairs can be represented as

\[
H_m(x_1, y_1, \ldots, x_m, y_m) = \int_{\Omega} H_w(x_1, y_1) \cdots H_w(x_m, y_m) \, dP(\omega) (7.1)
\]

where for fixed \((x, y)\), \( H_w(x, y) \) is a random variable, and for each \( w \), \( H_w(x, y) \) is a bivariate distribution function in \((x, y)\).

From the representation (7.1), the distribution function of \( (M_n^{(1)}, M_n^{(2)})' \) is
The limit distributions considered are those that can be obtained by using the same norming constants as the ones used to obtain the limit distribution in the iid case. In other words, the norming constants we will use are the sequences \( \{a_n > 0\}, \{c_n > 0\}, \{b_n\} \) and \( \{d_n\} \) for which there exists a bivariate distribution \( F_{X,Y}(x, y) \) such that

\[
F_{X,Y}(a_n x + b_n, c_n y + d_n) \xrightarrow{w} G(x, y)
\]  

(7.2)

where \( G(x, y) \) is a bivariate extreme distribution.

The following theorem provides sufficient conditions for attraction to bivariate extreme distributions described in the preceding paragraph.

**Theorem 7.1.** Let \( \{(X_i, Y_i)\}' \) be a sequence of exchangeable pairs of random variables on \( (\Omega, G, P) \) such that the joint distributions have the representation (7.1). Suppose that there exist real sequences \( \{a_n > 0\}, \{c_n > 0\}, \{b_n\} \) and \( \{d_n\} \), and a distribution \( F_{X,Y}(x, y) \) in the domain of attraction of a bivariate extreme distribution \( G(x, y) \) such that (7.2) holds. If there exists a non-degenerate distribution \( A(z) \) concentrated on \( (0, \infty) \) and such that for all \( z \) in its continuity set
\[
\lim_{u \to x^-_0, v \to y^-_0} P \left\{ \frac{\log H_w(u, v)}{\log F_{X,Y}(u,v)} \leq z \right\} = A(z) \quad (7.3)
\]

\[x_0, y_0 : F_{X,Y}(x_0, y_0) = 1; F_{X,Y}(u, v) < 1 \text{ for } u < x_0, v < y_0,\]
\[x_0 \leq \infty, y_0 \leq \infty\]

then

\[
\lim_{n \to \infty} P(M_n^{(1)} \leq a_n x + b_n, M_n^{(2)} \leq c_n y + d_n) = \int_0^\infty \{G(x, y)\}^2 \, dA(z) .
\]

Proof: Following the proof of Berman (1962) for the univariate case, we have that if (7.3) holds then, by the extended Helly-Bray Lemma (see Loève, 1963, page 181), for every \( s > 0 \),

\[
\lim_{u \to x^-_0, v \to y^-_0} E \left\{ \exp \left\{ - s \frac{\log H_w(u, v)}{\log F_{X,Y}(u,v)} \right\} \right\} = \int_0^\infty e^{-sz} \, dA(z) . \quad (7.4)
\]

The left side of (7.4) is the limit of a double sequence of monotone functions in \( s \) and the right side is a continuous function in \( s \); hence, the convergence is uniform in \( s \) on each closed and bounded interval (see Rudin, 1964, page 156). For \( (x, y) \) such that \( 0 < G(x,y) < 1 \), we have from (7.2) that for all sufficiently large \( n \)

\[0 < -\log F_{X,Y}(a_n x + b_n, c_n y + d_n) < \infty .\]
It follows from these facts that

\[ \lim_{n \to \infty} E \{ \mathcal{N}(a_n x + b_n, c_n y + d_n) \} \]

\[ = \lim_{n \to \infty} E \left( \exp \left\{ \log F_{X,Y}^n(a_n x + b_n, c_n y + d_n) \right. \right. \]

\[ \times \left. \log H_{X,Y}(a_n x + b_n, c_n y + d_n) \right. \right) \]

\[ \cdot \int_0^\infty \exp \{ z \log G(x, y) \} dA(z) = \int_0^\infty [G(x, y)]^z dA(z). \]

In the univariate case we have that, if \( \{X_i\} \) is a sequence of exchangeable random variables on (\( \Omega, G, P \)), then according to the theorem of de Finetti (see Loève, 1963, page 365), the joint distribution \( G_m(x_1, \ldots, x_m) \) of any number \( m \) of these random variables has the representation

\[ G_m(x_1, \ldots, x_m) = \int_{\Omega} G_w(x_1) \cdots G_w(x_m) dP(w) \quad (7.5) \]

where for fixed \( x \), \( G_w(x) \) is a random variable, and for each \( w \), \( G_w(x) \) is a distribution function in \( x \).

The following lemma is needed in the proof of a corollary to
Theorem 7.1.

Lemma 7.1. If \( \{X_i\} \) is a sequence of exchangeable random variables such that the joint distributions have the representation (7.5), and \( \{Y_i\} \) is a sequence of independent identically distributed random variables with distribution \( F(y) \), then the sequence \( \{(X_i + Y_i, X_i)\}' \) of bivariate random variables is exchangeable and the joint distributions have the representation

\[
H_m(x_1, y_1, \ldots, x_m, y_m) = \int_{\Omega} H_w(x_1, y_1) \cdots H_w(x_m, y_m) \, dP(w),
\]

where

\[
H_w(x, y) = \int_{-\infty}^{+\infty} G_w(\min(x - t, y)) \, dF(t)
\]

with \( G_w(x) \) of representation (7.5).

Proof:

\[
H_m(x_1, y_1, \ldots, x_m, y_m)
\]

\[
= P(X_1 + Y_1 \leq x_1, X_1 \leq y_1, \ldots, X_m + Y_m \leq x_m, X_m \leq y_m)
\]

\[
= \int_{\mathbb{R}^m} P(X_1 + t_1 \leq x_1, X_1 \leq y_1, \ldots, X_m + t_m \leq x_m, X_m \leq y_m)
\]

\[
\times dF(t_1) \cdots dF(t_m)
\]
\[ \int_{\mathbb{R}^m} P(\min(x_1 - t_1, y_1), \ldots, x_m \leq \min(x_m - t_m, y_m)) \times dF(t_1) \ldots dF(t_m) \]

\[ = \int_{\Omega} \int_{\mathbb{R}^m} G_w(\min(x_1 - t_1, y_1)) \ldots G_w(\min(x_m - t_m, y_m)) \times dF(t_1) \ldots dF(t_m) dP(w) \]

by Tonelli's theorem

\[ = \int_{\Omega} \int_{-\infty}^{+\infty} G_w(\min(x_1 - t_1, y_1)) dF(t_1) \ldots \]

\[ \int_{-\infty}^{+\infty} G_w(\min(x_m - t_m, y_m)) dF(t_m) dP(w) \]

\[ = \int_{\Omega} H_w(x_1, y_1) \ldots H_w(x_m, y_m) dP(w) \]

where
Remark 7.1. If \( W \) and \( Y \) are two random variables with distribution functions \( G_w(w) \) and \( F(y) \), respectively, then the joint distribution of \( W + Y \) and \( W \) is

\[
P(W + Y \leq x, W \leq y) = \int_{-\infty}^{+\infty} P(W + t \leq x, W \leq y) \, dF(t) - \int_{-\infty}^{+\infty} P(W < \min(x - t, y)) \, dF(t) = H_w(x, y).
\]

In what follows, if \( Y \) and \( Z \) are two random variables, then we will denote the joint distribution function of \( Z + Y \) and \( Z \) by \( F_{Z+Y,Z}(x, y) \).

As an application of Theorem 7.1, the following corollary presents the solution for a general problem which was approached by David (1973) under the assumption of normality of the distributions involved.

Corollary 7.1. Let \( \{X_i\} \) be a sequence of exchangeable random variables on \((\Omega, G, P)\) such that the joint distributions have the representation (7.5), let \( \{Y_i\} \) be a sequence of independent identically distributed random variables with distribution \( F_Y(y) \) and let
\[ M_n = \max(X_1, X_2, \ldots, X_n) \quad \text{and} \quad M_n^* = \max(X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n). \]

Suppose that there exist real sequences \( \{a_n > 0\} \), \( \{c_n > 0\} \), \( \{b_n\} \) and \( \{d_n\} \), and a random variable \( Z \) such that

\[
P^n_{Z+Y,Z}(a_n x + b_n, c_n y + d_n) \xrightarrow{w} G(x, y)
\]

where \( G(x, y) \) is a bivariate extreme distribution. If there exists a non-degenerate distribution \( A(t) \) concentrated on \((0, \infty)\) and such that for all \( t \) in its continuity set

\[
\lim_{u \to x_0^- \atop v \to y_0^-} \frac{\log \int_{-\infty}^{+\infty} G_w(\min(u - y, v)) \, dF_Z(y)}{\log F_{Z+Y,Z}(u, v)} \leq t
\]

(7.6)

\( (x_0, y_0 : F_{Z+Y,Z}(x_0, y_0) = 1; F_{Z+Y,Z}(u, v) < 1 \) for \( u < x_0, v < y_0; \)

\( x_0 \leq \infty, \ y_0 \leq \infty) \), then

\[
\lim_{n \to \infty} P(M_n^* \leq a_n x + b_n, M_n \leq c_n y + d_n) = \int_0^{\infty} [G(x, y)]^t \, dA(t).
\]

Proof: By Lemma 7.1, the sequence \( \{(X_1 + Y_1, X_1)\} \) is exchangeable and such that the joint distributions \( H_n(x_1, y_1, \ldots, x_m, y_m) \)

for any number \( m \) of these bivariate random variables have the
representation

\[ H_m(x_1, y_1, \ldots, x_m, y_m) = \int_{\Omega} H(x_1, y_1) \ldots H(x_m, y_m) \, dP(\omega) \]

where

\[ H(x, y) = \int_{-\infty}^{+\infty} G_{\omega}(\min(x - t, y)) \, dF_{\omega}(t) \]

Therefore, since all conditions of Theorem 7.1 are satisfied for the sequence \{(X_1 + Y_1, X_1)\}, the desired result follows. \[ \]

The following examples give an idea of the applications of the above results.

**Example 7.1.** Let \(\{Y_1\}\) and \(\{Z_1\}\) be two independent sequences of independent identically distributed random variables with distributions normal(\(\mu\), \(\sigma^2\)) and normal(\(\nu\), \(\tau^2\)), respectively. Then, by Corollary 5.5, \(M_n = \max(Z_1, Z_2, \ldots, Z_n)\) and \(M_n^* = \max(Z_1 + Y_1, Z_2 + Y_2, \ldots, Z_n + Y_n)\) are asymptotically independent. So there exist real sequences \(\{a_n > 0\}\), \(\{c_n > 0\}\), \(\{b_n\}\) and \(\{d_n\}\) such that

\[ F_{Z+Y_n}(a_nx + b_n, c_ny + d_n) \xrightarrow{w} \Lambda(x) \Lambda(y) \]

since the normal distributions are attracted to the extreme distribution \(\Lambda(x)\) (see Example 2.1). Therefore, if we have a sequence \(\{X_i\}\) of exchangeable random variables on \((\Omega, G, P)\) such that the joint
distributions have the representation (7.5), and condition (7.6) holds for a non-degenerate distribution function $A(t)$ concentrated on $(0, \infty)$, then, by Corollary 7.1,

$$\lim_{n \to \infty} P(M_n^* \leq a_n X_n + b_n, M_n \leq c_n Y_n + d_n) = \int_0^\infty [A(x) A(y)]^t \, dA(t)$$

where $M_n^* = \max(X_1 + Y_1, \ldots, X_n + Y_n)$ and $M_n = \max(X_1, \ldots, X_n)$.

This result was obtained in David (1973) under a condition similar to (7.6).

Example 7.2. Let $\{Y_i\}$ and $\{Z_i\}$ be two independent sequences of independent identically distributed random variables with a gamma $(\nu, \beta)$-distribution with $\nu$ an integer. Then, by Corollary 5.6, $M_n^* = \max(Z_1, Z_2, \ldots, Z_n)$ and $M_n^* = \max(Z_1 + Y_1, Z_2 + Y_2, \ldots, Z_n + Y_n)$ are asymptotically independent. Then, since the gamma($k, \beta$)-distributions with $k$ an integer are attracted to the extreme distribution $A(X)$ (see Example 2.2), there exist real sequences $\{a_n > 0\}$, $\{b_n > 0\}$, $\{c_n\}$ and $\{d_n\}$ such that

$$\frac{F^n_{Z+Y,Z}(a_n X_n + b_n, c_n Y_n + d_n)}{A(X) A(Y)} \xrightarrow{\text{w}} A(X) A(Y).$$

Therefore, if we have a sequence $\{X_i\}$ of exchangeable random variables on $(\Omega, \mathcal{G}, P)$ such that the joint distributions admit the representation (7.5), and condition (7.6) holds for a non-degenerate
distribution function $A(t)$ concentrated on $(0, \infty)$, then, by Corollary 7.1,

$$
\lim_{n \to \infty} P(M^*_n \leq a_n x + b_n, M_n \leq c_n y + d_n) = \int_0^\infty \{A(x)A(y)\}^t \, dA(t)
$$

where $M^*_n = \max(X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n)$ and $M_n = \max(X_1, X_2, \ldots, X_n)$.

Some more examples similar to Examples 7.1 and 7.2 can be obtained by means of Remark 5.1 and Remark 5.2.


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X. APPENDIX

A. Proof of Theorem 2.15

(a) Suppose that (2.10) holds. Let

\[ \gamma_n = \left( \log n \cdot c \right)^{1/\alpha} \quad \text{and} \quad \delta_n = -\frac{\beta \log \log n \cdot c}{\alpha^2 \left( \log n \cdot c \right)^{1-1/\alpha}}. \]

Then for all real \( x \)

\[ F^n(a_n x + b_n) = \left( 1 - \frac{n \left[ 1 - F(a_n x + \gamma_n + \delta_n) \right]}{n} \right). \quad (A.1) \]

Note that

\[ n \left[ 1 - F(a_n x + \gamma_n + \delta_n) \right] \]

\[ = n(a_n x + \gamma_n + \delta_n)^\beta \exp \{(a_n x + \gamma_n + \delta_n)^\alpha \} \left[ 1 - F(a_n x + \gamma_n + \delta_n) \right] \]

\[ \times \frac{1}{(a_n x + \gamma_n + \delta_n)^\beta \exp \{(a_n x + \gamma_n + \delta_n)^\alpha \}}. \quad (A.2) \]

Now we have that
\[(a_n x + \gamma_n + \delta_n)^\beta = \gamma_n^\beta \left(1 + \frac{a_n x + \delta_n}{\gamma_n}\right)^\beta\]

\[{\log n c}^{\beta/\alpha} \left(1 + \frac{x - (\beta/\alpha) \log \log n c}{\alpha \log n c}\right)^\beta\].

On the other hand, since
\[\frac{a_n x + \delta_n}{\gamma_n} = \frac{x - (\beta/\alpha) \log \log n c}{\alpha \log n c}\]

as \(n \to \infty\), for \(n\) sufficiently large we can apply the Binomial theorem, so

\[(a_n x + \gamma_n + \delta_n)^\alpha = \gamma_n^\alpha \left(1 + \frac{a_n x + \delta_n}{\gamma_n}\right)^\alpha\]

\[= \gamma_n^\alpha \left[1 + \left(\frac{\alpha}{1}\right) \frac{a_n x + \delta_n}{\gamma_n} + \sum_{k=2}^{\infty} \left(\frac{\alpha}{1}\right) \left(\frac{a_n x + \delta_n}{\gamma_n}\right)^k\right]\]

\[= \gamma_n^\alpha \gamma_n^{\alpha - 1} (a_n x + \delta_n) + \sum_{k=2}^{\infty} \left(\frac{\alpha}{1}\right) \left(\frac{a_n x + \delta_n}{\gamma_n}\right)^k\]

\[= \log n c + x - \frac{\beta}{\alpha} \log \log n c + s_n(x)\]

where

\[s_n(x) = \sum_{k=2}^{\infty} \left(\frac{\alpha}{k}\right) \left(\frac{a_n x + \delta_n}{\gamma_n}\right)^k\]
\[
= \sum_{k=2}^{\infty} \binom{\alpha}{k} \left( \frac{x - (\beta/\alpha) \log \log n c}{\alpha^{k-1}} \right)^{k-1} \alpha^k (\log n c)^{k-1}.
\]  \hspace{1cm} (A.3)

Hence

\[
\exp \{(a_n x + \gamma_n + \delta_n)^\alpha\} = n c^{1/\log n c} e^{\beta/\alpha} e^{x + s_n(x)}.
\]

Therefore, (A.2) is equal to

\[
(a_n x + \gamma_n + \delta_n)^\beta \exp \{(a_n x + \gamma_n + \delta_n)^\alpha\} \left[1 - F(a_n x + \gamma_n + \delta_n)\right]
\]

\[
	imes \frac{1}{(1 - \frac{x - (\beta/\alpha) \log \log n c}{\alpha \log n c})^\beta e^{x + s_n(x)}}. \hspace{1cm} (A.4)
\]

Assuming that \( s_n(x) \to 0 \) as \( n \to \infty \) for all real \( x \), and since

\[
a_n x + \gamma_n + \delta_n \to \infty \quad \text{as} \quad n \to \infty,
\]

we have that as \( n \) tends to infinity, expression (A.4) tends to \( e^{-x} \). That is,

\[
n \{1 - F(a_n x + \gamma_n + \delta_n)\} \to e^{-x} \quad \text{for all real} \quad x.
\]

Thus, using equality (A.1), we have that

\[
F^{n}(a_n x + b_n) \to \exp \{-e^{-x}\} = \Lambda(x) \quad \text{for all real} \quad x.
\]
That is, (2.11) holds with (2.12). In order to complete this part of the proof, we still have to show that

\[ \lim_{n \to \infty} s_n(x) = 0 \quad \text{for all real } x. \]

For any real \( x \), notice that if we let \( k = \ell + 2 \) in (A.3), then

\[
s_n(x) = \sum_{\ell=0}^{\infty} \binom{\alpha}{\ell+2} \frac{(x - (\beta/\alpha) \log \log n c)^{\ell+2}}{\alpha^{\ell+2}(\log n c)^{\ell+1}} = k_n(x) \sum_{\ell=0}^{\infty} c_{\ell}(g_n(x))^{\ell} \]

where

\[
k_n(x) = \frac{(x - (\beta/\alpha) \log \log n c)^2}{\alpha^2 \log n c},
\]

\[
c_{\ell} = \binom{\alpha}{\ell+2},
\]

\[
g_n(x) = \frac{x - (\beta/\alpha) \log \log n c}{\alpha \log n c}.
\]

Consider the power series \( \sum_{\ell=0}^{\infty} c_{\ell} y^{\ell} \). Note that
Then, by the ratio test, the series converges for $|y| < 1$. Since $g_n(x) \to 0$ as $n \to \infty$, there exists $N = N(x)$ such that for all $n > N$ we have $|g_n(x)| < 1$. Hence, for each $x$ the series

$$\sum_{\ell=0}^{\infty} c_\ell (g_n(x))^\ell$$

converges for all $n > N$. Therefore, since $k_n(x) \to 0$ as $n \to \infty$ for all $x$, 

$$s_n(x) = k_n(x) \sum_{\ell=0}^{\infty} c_\ell (g_n(x))^\ell \to 0 \text{ for all real } x.$$  

(b) Suppose that (2.11) holds with (2.12). Denote $a_m x + b_m$ by $x_m$. Consider the function $U: (x_2(\beta), \infty) \to \mathbb{R}^+$ defined by

$$U(x) = x^{-\beta} e^{-x^{\alpha}}$$

where $x_2(\beta) = (|\beta|/\alpha)^{1/\alpha}$. Notice that

$$U'(x) = (-\beta - \alpha x^{\alpha}) x^{\beta - 1} e^{-x^{\alpha}} \leq 0 \text{ for all } x \in (x_2(\beta), \infty).$$

Since $U(x)$ is a continuous function, we have then that $U(x)$ is non-increasing on $(x_2(\beta), \infty)$. 

\[ \left| \frac{c_{\ell+1} y^{\ell+1}}{c_\ell y^\ell} \right| = |y| \left| \frac{\frac{\alpha}{\ell+3}}{\frac{\alpha}{\ell+2}} \right| = |y| \left| \frac{\alpha - \ell - 2}{\ell + 3} \right| \xrightarrow{\ell \to \infty} |y|. \]
Now, define for \( t > 0 \) the integer \( n = n(t) \) by

\[
n = \min \{ m : x_{m+1} > t \} .
\]

Then

\[
x_n \leq t < x_{n+1} .
\]

Since \( U(x) \) and \( 1 - F(x) \) are positive, non-increasing function on \( (x_0(\beta), \infty) \)

\[
\frac{1 - F(x_{n+1})}{U(x_n)} \leq \frac{1 - F(t)}{U(t)} \leq \frac{1 - F(x_n)}{U(x_{n+1})} .
\]

Therefore

\[
\frac{1}{1 + 1/n} \cdot \frac{(n+1) \{1 - F(x_{n+1})\}}{n U(x_n)} \leq t^\beta e^{t^\alpha} \{1 - F(t)\}
\]

\[
\leq \frac{n \{1 - F(x_n)\}}{(n+1) U(x_{n+1})} \cdot (1 + 1/n) . \quad (A.5)
\]

Now, since (2.11) holds with (2.12), by Lemma 2.1, we have that

\[
m \{1 - F(x_m)\} \quad \xrightarrow{m \to \infty} \quad \log \Lambda(x) = e^{-x} .
\]
On the other hand, as we saw in part (a),

\[ m U(x_m) = \frac{m}{(a_m x + \gamma_m + \delta_m)^B \exp \left\{ (a_m x + \gamma_m + \delta_m)^B \right\}} \]

\[ = \frac{1}{(1 + \frac{x - (\beta/\alpha) \log \log m \log c}{\alpha \log m c})^B} \frac{x + s_m(x)}{c e^{x}} \]

\[ \xrightarrow{m \to \infty} c^{-1} e^{-x} \cdot \]

Therefore, since \( x_m \to \infty \) as \( m \to \infty \), if \( t \) tends to infinity then \( n = n(t) \) tends to infinity, and the extreme sides of the inequalities (A.5) tend to \( c \). Thus

\[ t^\beta e^{t^\alpha} \left\{ 1 - F(t) \right\} \xrightarrow{t \to \infty} c \cdot \]

E. Proof of Lemma 5.1

Suppose that

\[ \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \cdot \]
Without loss of generality, assume that $\sigma_1 \leq \sigma_2$. Consider the trivariate random variable $S' = (U, V, W)$ with normal($\mu$, $\Sigma^*$)-distribution where

$$
\begin{align*}
\nu &= \begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}
\quad \text{and} 
\Sigma^* &= \begin{pmatrix}
t_1^2 & 0 & t_{13} \\
0 & t_2^2 & 0 \\
t_{13} & 0 & t_3^2
\end{pmatrix} .
\end{align*}
$$

Take the $2 \times 3$ matrix

$$
D = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & \beta
\end{pmatrix}
$$

which has rank 2. Then (see Anderson, 1958, page 25) $Z = DS$ has a normal($D\nu$, $D \Sigma^* D'$)-distribution. Hence, $Z$ has a normal($\mu$, $\Sigma$)-distribution if and only if $D\mu = \mu$ and $D \Sigma^* D' = \Sigma$, that is, if and only if

$$
\begin{align*}
\nu_1 + \nu_3 &= \mu_1 , \\
\nu_2 + \nu_3 &= \mu_2 
\end{align*}
$$

(B.1)

and

$$
\begin{align*}
\sigma_1^2 &= t_1^2 + t_3^2 + 2t_{13} \\
\sigma_2^2 &= t_2^2 + t_3^2 \\
\sigma_{12} &= t_3^2 + t_{13} .
\end{align*}
$$

(B.2)
On one hand, if we let $\nu_1 = \mu_1$, $\nu_2 = \mu_2$ and $\nu_3 = 0$, Equations (B.1) are satisfied. On the other hand, since $|\rho| < 1$, we have that

$$-\sigma_1 \sigma_2 < \sigma_{12} < \sigma_1 \sigma_2.$$  \hspace{1cm} (B.3)

We consider two cases:

First suppose that $\sigma_{12} > 0$. Then, from Inequalities (B.3) we have that

$$\frac{\sigma_{12}}{\sigma_1^2} < \frac{\sigma_2^2}{\sigma_{12}}.$$  \hspace{1cm}

Therefore, there exists $\beta \in \mathbb{R}$ such that

$$0 < \frac{\sigma_{12}}{\sigma_1^2} < \beta < \frac{\sigma_2^2}{\sigma_{12}}.$$  \hspace{1cm}

Hence, we can see that

$$\tau_1^2 = \sigma_1^2 - \frac{1}{\beta} \sigma_{12} > 0$$

$$\tau_2^2 = \sigma_2^2 - \beta \sigma_{12} > 0$$

$$\tau_3^2 = \frac{1}{\beta} \sigma_{12} > 0$$

satisfy Equations (B.2).
Second suppose that $\sigma_{12} < 0$. Then from Inequalities (B.3) we have that

$$\frac{\sigma_2^2}{\sigma_{12}^2} < \frac{\sigma_{12}}{\sigma_1^2}.$$ 

Therefore there exists $\beta \in \mathbb{R}$ such that

$$\frac{\sigma_2^2}{\sigma_{12}^2} < \beta < \frac{\sigma_{12}}{\sigma_1^2} < 0.$$ 

Hence we can see that

$$\begin{align*}
\sigma_1^2 &= \sigma_1^2 - \frac{1}{\beta} \sigma_{12} > 0 \\
\sigma_2^2 &= \sigma_2^2 - \beta \sigma_{12} > 0 \\
\sigma_3^2 &= \frac{1}{\beta} \sigma_{12} > 0
\end{align*}$$

satisfy Equations (B.2). Now, since $\text{cov}(U, V) = 0$, $\text{cov}(U, W) = 0$, $\text{cov}(V, W) = 0$ and $(U, V, W)'$ is normally distributed, we have that, $U$, $V$ and $W$ are independent. Therefore since $Z' = (U + W, V + \beta W)$, the proof is complete.