Experimental investigation of flow between rotating spheres

Adnan Mohammad Waked

Iowa State University

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by

Adnan Mohammad Waked

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# TABLE OF CONTENTS

## LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>xi</td>
</tr>
</tbody>
</table>

## CHAPTER I. INTRODUCTION

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Purpose of This Research</td>
<td>1</td>
</tr>
<tr>
<td>Statement of the Problem</td>
<td>2</td>
</tr>
<tr>
<td>The Governing Equations</td>
<td>3</td>
</tr>
<tr>
<td>Review of Literature</td>
<td>7</td>
</tr>
<tr>
<td>Theoretical research</td>
<td>7</td>
</tr>
<tr>
<td>Experimental research</td>
<td>13</td>
</tr>
<tr>
<td>Description of the Flow</td>
<td>17</td>
</tr>
</tbody>
</table>

## CHAPTER II. THE EXPERIMENT

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Apparatus</td>
<td>25</td>
</tr>
<tr>
<td>The Experiments</td>
<td>28</td>
</tr>
<tr>
<td>Introduction</td>
<td>28</td>
</tr>
<tr>
<td>Flow visualization</td>
<td>29</td>
</tr>
<tr>
<td>Dye injection</td>
<td>29</td>
</tr>
<tr>
<td>The use of aluminum powder</td>
<td>30</td>
</tr>
<tr>
<td>Mutually buoyant particles</td>
<td>31</td>
</tr>
<tr>
<td>Torque measurements</td>
<td>32</td>
</tr>
</tbody>
</table>

## CHAPTER III. LOW REYNOLDS NUMBERS

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>39</td>
</tr>
<tr>
<td>Character of the Flow Field</td>
<td>40</td>
</tr>
<tr>
<td>The case when one of the spheres is rotating</td>
<td>40</td>
</tr>
<tr>
<td>The case when both spheres are rotating</td>
<td>56</td>
</tr>
<tr>
<td>Torque Measurements Results</td>
<td>61</td>
</tr>
</tbody>
</table>

## CHAPTER IV. INTERMEDIATE REYNOLDS NUMBERS

<table>
<thead>
<tr>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>72</td>
</tr>
<tr>
<td>Character of the Flow Field</td>
<td>74</td>
</tr>
<tr>
<td>The case of a stationary inner sphere ($\mu = \infty$)</td>
<td>74</td>
</tr>
<tr>
<td>The case of a stationary outer sphere ($\mu = 0.0$)</td>
<td>85</td>
</tr>
<tr>
<td>With the two spheres rotating together</td>
<td>86</td>
</tr>
<tr>
<td>Torque Measurements Results</td>
<td>98</td>
</tr>
<tr>
<td>One sphere rotates while the other is stationary</td>
<td>99</td>
</tr>
<tr>
<td>Both spheres are rotating</td>
<td>102</td>
</tr>
</tbody>
</table>
# CHAPTER V. LARGE REYNOLDS NUMBERS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>114</td>
</tr>
<tr>
<td>The Case of a Stationary Inner Sphere ($\mu = \infty$)</td>
<td>116</td>
</tr>
<tr>
<td>The Case of a Stationary Outer Sphere ($\mu = 0.0$)</td>
<td>121</td>
</tr>
<tr>
<td>The Case When Both Spheres are Rotating Together</td>
<td>129</td>
</tr>
</tbody>
</table>

# CHAPTER VI. CONCLUSION

- REFERENCES: 148a

# ACKNOWLEDGMENTS

- ACKNOWLEDGMENTS: 153

# APPENDIX A. THE EXPERIMENTS

- The Experimental Apparatus: 154
- Fluids Used and Viscosity Measurements: 161
- Wires and their Spring Constants: 163
- Derivation of the Equations Related to Experiments: 164
- Error Discussion: 166
- Uncertainty Estimate: 168a

# APPENDIX B. PERTURBATION SOLUTION TORQUE RESULTS

- APPENDIX B. PERTURBATION SOLUTION TORQUE RESULTS: 169
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>The geometry of the flow between rotating concentric spheres</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>The stream function, $\psi$, and the angular velocity, $\omega$, with stationary outer sphere, for different Reynolds numbers, $\eta = 0.5$ and $\mu = 0.0$</td>
<td>19</td>
</tr>
<tr>
<td>3</td>
<td>The stream function, $\psi$, and the angular velocity, $\omega$, with stationary inner sphere, for different Reynolds numbers, $\eta = 0.5$ and $\mu = \infty$</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>The stream function, $\psi$, and the angular velocity, $\omega$, for both spheres rotating with the same angular velocity but in opposite directions for $Re = 100$ and $Re = 500$, $\mu = -0.5$, $\eta = 0.5$</td>
<td>23</td>
</tr>
<tr>
<td>5</td>
<td>The stream function, $\psi$, and the angular velocity, $\omega$, when the outside sphere is rotating with half the angular velocity of the inside one and in the opposite direction for two different Reynolds numbers, $\mu = -0.5$ and $\eta = 0.5$</td>
<td>24</td>
</tr>
<tr>
<td>6</td>
<td>The experimental apparatus</td>
<td>26a</td>
</tr>
<tr>
<td>7</td>
<td>The torque as a function of Reynolds number when the inside sphere is rotating alone, for $\eta = 0.304$</td>
<td>34</td>
</tr>
<tr>
<td>8</td>
<td>The torque as a function of Reynolds number when the inside sphere is rotating alone, for $\eta = 0.44$</td>
<td>35</td>
</tr>
<tr>
<td>9</td>
<td>The torque as a function of Reynolds number when the outside sphere is rotating alone, for $\eta = 0.304$</td>
<td>36</td>
</tr>
<tr>
<td>10</td>
<td>The torque as a function of Reynolds number when the outside sphere is rotating alone, for $\eta = 0.44$</td>
<td>37</td>
</tr>
</tbody>
</table>
11 Development of the basic laminar flow with a stationary outer sphere for $\eta = 0.44^a$, $\mu = 0.0, \text{Re} = 33$

12 Development of the basic laminar flow with a stationary outer sphere for $\eta = 0.304^a$, $\mu = 0.0, \text{Re} = 30$

13 The sequence of photographs (1-13) shows the positions which sphere B takes when crossing the meridian plane, with a stationary outer sphere for $\eta = 0.44^a, \text{Re} = 30, \mu = 0$. Sphere A was placed approximately in the center of the secondary flow swirl

14 Photographs (1-5) at times 0.0, 0.21, 0.29, 0.42 and 0.79 sec respectively, show one revolution sphere B makes with a stationary outer sphere, $\eta = 0.44^a, \mu = 0.0, \text{Re} = 30$

15 Photographs (1-5) at times 0.0, 0.21, 0.38, 0.46 and 0.54 sec respectively, show one revolution sphere B makes with a stationary outer sphere for $\eta = 0.44^a, \mu = 0.0, \text{Re} = 100$

16 Development of the basic laminar flow with a stationary inner sphere for $\eta = 0.44^a$, $\mu = \infty, \text{Re} = 20$

17 Photographs (1-19) show the position which sphere B takes when crossing the meridian plane with a stationary inner sphere for $\eta = 0.44^a, \mu = \infty, \text{Re} = 30$. Sphere A was placed approximately in the center of the stationary flow swirl

18 Comparison between the experimental results and the theoretical solution, with a stationary inner sphere, $\mu = \infty$

19 Comparison of the flow pattern for two different radius ratios, $\eta = 0.44$ and $\eta = 0.304$, with a stationary inner sphere $\mu = \infty, \text{Re} = 50$

20 General nature of the meridional flow and the region of the effect of each sphere depending on $\mu$ and $\eta$ for small Re
21 The secondary flow swirl reverses its direction when $\mu$ changes from $\mu > 1$ to $\mu < 1$ 58
22 Development of the basic laminar flow with the two spheres rotating together and in opposite directions for $\eta = 0.44$, $\mu = -0.29$, $Re = 30$ 60
23 The size of the two swirls as a function of the angular velocity ratio, when the two spheres rotate in opposite directions for small $Re$ 62
24 The flow pattern as a function of the angular velocity ratio with the two spheres rotating in opposite directions for $\eta = 0.44$ 63
25 The torque curve up to $Re = 300$ with a stationary outer sphere for $\eta = 0.44$ 66
26 The torque curve up to $Re = 300$, with a stationary inner sphere, for $\eta = 0.44$ 68
27 The torque curve up to $Re = 300$ when one sphere is rotating and the other is stationary, for $\eta = 0.304$ 70
28 The flow pattern with a stationary inner sphere for different Reynolds numbers, $\eta = 0.44$, $\mu = \infty$ 75
29 Comparison between the experimental results and the theoretical solution with a stationary inner sphere $\mu = \infty$ 78
30 Comparison between the experimental results and the theoretical solution with a stationary inner sphere $\mu = \infty$ 79
31 Comparison between the experimental results and the theoretical solutions with a stationary inner sphere $\mu = \infty$ 81
32 The flow pattern with a stationary inner sphere for different Reynolds numbers, $\eta = 0.304$, $\mu = \infty$ 82
33 Comparison of the flow pattern for two different radius ratios: \( \eta = 0.44 \) and \( \eta = 0.304 \), with the inner sphere stationary for \( Re = 500 \), \( \mu = \infty \)

34 The flow pattern with a stationary outer sphere for different Reynolds numbers, \( \eta = 0.44, \mu = 0 \)

35 The center of the swirl approaches the equator as Reynolds number increases when the inside sphere is rotating alone

36 Comparison between the experimental results and the theoretical solution with a stationary outer sphere, \( \mu = 0 \)

37 Comparison between the experimental results and the theoretical solution with a stationary outer sphere, \( \mu = 0.0 \)

38 Comparison between the experimental results and the theoretical solution with a stationary outer sphere, \( \mu = 0.0, Re = 1000 \)

39 The flow pattern with a stationary outer sphere for low and high Reynolds numbers, \( \eta = 0.304, \mu = 0.0 \)

40 The flow when the two spheres are rotating in opposite directions with angular velocity ratio \( \mu = -0.3 \), but for different Reynolds numbers, \( \eta = 0.44 \)

41 Rotating the inside sphere affects the flow pattern resulting from the rotation of the outside sphere, \( Re = 100, \eta = 0.44 \)

42 The flow pattern when the two spheres are rotating in the same direction, and for different angular velocity ratios, \( \eta = 0.44, Re = 125 \)

43 The torque curve between \( Re = 400 \) and \( Re = 1000 \), with a stationary inner sphere for \( \eta = 0.44 \)
The torque curve between Re = 300 and Re = 950, with a stationary outer sphere for \( \eta = 0.44 \)

The torque curve between Re = 500 and Re = 2000, when one sphere is rotating and the other is stationary, for \( \eta = 0.304 \)

The torque as a function of the angular velocity ratio, for \( \eta = 0.44, \) Re = 100

The torque curve as a function of the angular velocity ratio, for different Reynolds numbers, for \( \eta = 0.44 \)

The torque curve as a function of the angular velocity ratio for different Reynolds numbers, for \( \eta = 0.44 \)

The torque curve as a function of the angular velocity ratio for different Reynolds numbers. It shows the change in shape of the torque curve at Re = 900

The torque curve as a function of the angular velocity ratio for \( \eta = 0.304, \) Re = 100

Comparison between the experimental results and the theoretical solution with a stationary inner sphere, \( \mu = \infty \)

The torque curve between Re = 1000 and Re = 2200 with a stationary inner sphere, for \( \eta = 0.44 \)

The torque curve between Re = 2500 and Re = 10000 with a stationary inner sphere, for \( \eta = 0.44 \)

The torque curve between Re = 5000 and Re = 11000 when one sphere is rotating and the other is stationary, for \( \eta = 0.304 \)
56 The torque curve between Re = 800 and Re = 2000, with a stationary outer sphere, for \( \eta = 0.44 \) 124

57 The torque curve between Re = 2000 and Re = 10000 with a stationary outer sphere, for \( \eta = 0.44 \) 125

58 Critical Reynolds number as a function of the radius ratio \( \eta \), for \( \mu = 0.0 \) 128

59 The flow when the two spheres are rotating in the same direction with almost the same angular velocity, for \( \eta = 0.44 \) 131

60 The general character of Proudman's solution for the case of almost rigid rotation, and the experimental results for this case 132

61 The torque curve as a function of the angular velocity ratio for different Reynolds numbers, for \( \eta = 0.44 \) 134

62 The torque as a function of the angular velocity ratio for different Reynolds numbers, for \( \eta = 0.44 \) 135

63 The turbulent region as a function of the angular velocity ratio when the two spheres are rotating in opposite directions, \( \eta = 0.304 \) and \( Re_1 = 7800 \) 137

64 The turbulent region also will be a function of the angular velocity if the two spheres rotate in the same direction, \( Re_1 = 7800 \) 139

65 The size of the turbulence region as a function of Reynolds number based on the angular velocity of the outside sphere, while the inside sphere rotates at constant angular velocity 140

66 The size of the turbulence region as a function of Reynolds number based on the angular velocity of the outside sphere for different inside sphere's angular velocities, for \( \eta = 0.44 \) 143
67 The region of turbulent and laminar flows for a radius ratio $\eta = 0.44$

68 Regions of laminar and turbulent flows for different radius ratios

69 The regions of laminar and turbulent flows for different radius ratios

70 The region of stability and instability for two concentric rotating cylinders

71 Plexiglass assembly

72 The mercury switch

73 The air bearings
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Radii of the inner spheres and the corresponding radius ratios</td>
<td>27</td>
</tr>
<tr>
<td>2</td>
<td>Stokes regions, for the flow cases being considered</td>
<td>38</td>
</tr>
<tr>
<td>3</td>
<td>The critical Reynolds numbers for different gap sizes</td>
<td>127</td>
</tr>
<tr>
<td>4</td>
<td>The wires used and their spring constants</td>
<td>164</td>
</tr>
<tr>
<td>5</td>
<td>The constants a and b when both spheres rotate in the same direction, for $\eta = 0.304$</td>
<td>172</td>
</tr>
<tr>
<td>6</td>
<td>The constants a and b when both spheres rotate in opposite directions, for $\eta = 0.304$</td>
<td>173</td>
</tr>
<tr>
<td>7</td>
<td>The constants a and b when both spheres rotate in the same direction, for $\eta = 0.44$</td>
<td>174</td>
</tr>
<tr>
<td>8</td>
<td>The constants a and b when both spheres rotate in opposite directions, for $\eta = 0.44$</td>
<td>175</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>a, b</td>
<td>Constants used in torque calculations.</td>
<td></td>
</tr>
<tr>
<td>$f_\xi(r)$</td>
<td>Basic flow angular velocity component.</td>
<td></td>
</tr>
<tr>
<td>$g_\xi(r)$</td>
<td>Basic flow stream function component.</td>
<td></td>
</tr>
<tr>
<td>J</td>
<td>Polar moment of inertia.</td>
<td></td>
</tr>
<tr>
<td>K</td>
<td>The spring constant of the torsion wire.</td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>Dimensional torque required to rotate spheres.</td>
<td></td>
</tr>
<tr>
<td>$\bar{M}$</td>
<td>Dimensionless torque required to rotate spheres.</td>
<td></td>
</tr>
<tr>
<td>$N_t$</td>
<td>Order of Legendre polynomial truncation.</td>
<td></td>
</tr>
<tr>
<td>$P_\ell(\Theta)$</td>
<td>$\ell$th order Legendre polynomial.</td>
<td></td>
</tr>
<tr>
<td>$R_1$</td>
<td>Radius of inner sphere.</td>
<td></td>
</tr>
<tr>
<td>$R_2$</td>
<td>Radius of outer sphere.</td>
<td></td>
</tr>
<tr>
<td>Re</td>
<td>Reynolds number.</td>
<td></td>
</tr>
<tr>
<td>$Re_1$</td>
<td>Reynolds number based on inner sphere angular velocity.</td>
<td></td>
</tr>
<tr>
<td>$Re_2$</td>
<td>Reynolds number based on outer sphere angular velocity.</td>
<td></td>
</tr>
<tr>
<td>$Re_{cr}$</td>
<td>Critical Reynolds number.</td>
<td></td>
</tr>
<tr>
<td>$Re_E$</td>
<td>Critical Reynolds number for stability (Energy Theory).</td>
<td></td>
</tr>
<tr>
<td>$Re_L$</td>
<td>Critical Reynolds number for instability (Linear Theory).</td>
<td></td>
</tr>
<tr>
<td>r</td>
<td>Nondimensional radial coordinate.</td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>Specific gravity.</td>
<td></td>
</tr>
<tr>
<td>U</td>
<td>The basic flow.</td>
<td></td>
</tr>
<tr>
<td>u</td>
<td>Dynamic viscosity.</td>
<td></td>
</tr>
</tbody>
</table>
\( u_r, u_\theta, u_\phi \) \quad \text{Nondimensional basic flow velocity components.}

\( \omega_n \) \quad \text{Natural frequency}

\( \alpha \) \quad \text{The twist angle.}

\( \eta \) \quad \text{The radius ratio.}

\( \theta \) \quad \text{Latitude coordinate}

\( \mu \) \quad \text{Angular velocity ratio.}

\( \tau \) \quad \text{Period of oscillation.}

\( \tau_{\theta r} \) \quad \text{The component of the rate of stress.}

\( \phi \) \quad \text{Longitude coordinate.}

\( \psi(r, \theta) \) \quad \text{Nondimensional stream function in the meridian plane.}

\( \omega \) \quad \text{Angular velocity about axis of rotation.}

\( \Omega(r, \theta) \) \quad \text{Nondimensional angular velocity function.}

\( \Omega_1 \) \quad \text{Angular velocity of inner sphere.}

\( \Omega_2 \) \quad \text{Angular velocity of outer sphere.}

\( \Theta_0 \) \quad \text{Characteristic angular velocity.}

\( \nu \) \quad \text{Kinematic viscosity.}

\( \mathcal{D}^2 \) \quad \text{Differential operator.}

\( \frac{d}{dr} \theta \) \quad = \frac{\partial}{\partial r}

\( \frac{d}{d\theta} \) \quad = \frac{\partial}{\partial \theta}

\( \frac{d}{dr} \) \quad = \frac{d}{dr} \)
CHAPTER I. INTRODUCTION

Purpose of This Research

This research is a continuation of previous research (under NSF Grant 35513) which has resulted in a better understanding (theoretical and experimental) of the flow in a spherical annulus when the outer sphere is stationary. The work was carried out by Munson and Menguturk (1) and Menguturk (2). The present research deals with situations in which both spheres may rotate.

Flow visualization and torque measurements techniques were used to study the steady flow of an incompressible viscous fluid contained in a spherical annulus. Dye injection was used to study the primary and secondary flows at low and moderate Reynolds numbers, and aluminum particles suspended in the fluid were used to study the stability and turbulence at larger Reynolds numbers. The torque measurements technique was used to verify the flow visualization results and to evaluate the torque needed to rotate the spheres at different rates. Other results concerning various aspects of spherical annulus flow (almost rigid rotation, free shear layers, contrarotating secondary flow swirls, etc.) are also presented.

The experiments were carried out with a wide range of angular velocity ratios with one of the two spheres rotating
while the other is stationary and with both spheres rotating together.

Statement of the Problem

This thesis deals with the steady flow of an incompressible viscous fluid contained between two concentric spheres which rotate at constant angular velocities about a common axis. The understanding of the behavior of this flow is of interest in the practical applications dealing with viscometry, design of spherical hydrodynamic bearings, oceanic and atmospheric circulations, and problems of astrophysics and geophysics. Aside from such applications, the flow geometry is of interest because it provides a relatively simple geometry (allowing theoretical solutions) in which most of the important fluid mechanics phenomena occur (creeping flow, Stokes flow, boundary layers, free shear layers, secondary flow, rotating flow effects, turbulence, etc.).

A considerable amount of theoretical work has been done previously. This investigation deals with experimental results for medium and wide gap situations, that is, when the two spheres are not nearly the same size. The experiments were performed for one of the spheres rotating while the other is stationary or for both spheres rotating together.

The inner and outer spheres have radii $R_1$ and $R_2$ and rotate with constant angular velocities $\Omega_1$ and $\Omega_2$, respec-
tively. We denote the radius ratio $\eta = R_1/R_2$ and the angular velocity ratio as $\mu = \Omega_2/\Omega_1$. The geometry of the flow is illustrated in Figure 1.

The Governing Equations

We consider flows which are symmetric with respect to the equator and are axisymmetric, that is independent of the longitude, $\theta$. Because of the incompressibility and symmetry, the Navier-Stokes equations can be written in terms of two functions: $\psi(r, \theta)$, the stream function in the meridian plane ($\phi = \text{constant}$), and $\Omega(r, \theta)$, the angular velocity function. The $\theta$-component of the momentum equation and the $\phi$-component of the vorticity equation thus become the governing system (3) and can be written as follows:

\[
\frac{\partial \Omega}{\partial t} - \frac{\psi \Omega_r}{r^2 \sin \theta} \frac{\partial}{\partial r} \frac{\partial}{\partial \theta} = \frac{1}{\text{Re}} \frac{\partial^2 \Omega}{\partial \theta^2} \tag{1}
\]

\[
\frac{\partial}{\partial t}(\nabla^2 \psi) + \frac{2\Omega}{r^2 \sin^2 \theta} (\Omega_r \cos \theta - \Omega_\theta \sin \theta)
- \frac{1}{r^2 \sin \theta} [\psi_r (\nabla^2 \psi)_\theta - \psi_\theta (\nabla^2 \psi)_r]
+ \frac{2 \nabla^2 \psi}{r^2 \sin^2 \theta} [\psi_r \cos \theta - \psi_\theta \sin \theta] = \frac{1}{\text{Re}} \frac{\partial \psi}{\partial \theta} \tag{2}
\]
Figure 1. The geometry of the flow between rotating concentric spheres
Here \( ( )_r = \frac{\partial}{\partial r} \) and \( ( )_\theta = \frac{\partial}{\partial \theta} \) and the operator \( \overline{D}^2 \) is defined by

\[
\overline{D}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{1}{r^2} \cot \theta \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta}.
\]

These equations are written in nondimensional form by using the radius of the outer sphere as a characteristic length, \( R_2 \), and an angular velocity, \( \Omega_0 \), as the characteristic speed parameter. Thus the Reynolds number is

\[
Re = \frac{\Omega_0 R_2^2}{v},
\]

where \( v \) is the kinematic viscosity. In general, \( \Omega_0 \) will be either \( \Omega_1 \) or \( \Omega_2 \), the angular velocity of the inner or outer sphere, respectively, depending on the particular situation being considered. The independent variables \( r \) and \( \theta \) range as follows:

\[
\eta \leq r \leq 1, \quad 0 \leq \theta \leq \pi/2.
\]

This range of \( \theta \) (rather than \( 0 \leq \theta \leq \pi \)) is due to the assumed symmetry about the equator.

The nondimensional velocities \( (u_r, u_\theta, u_\phi) \), can be obtained from \( \psi \) and \( \Omega \) as follows:

\[
u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}.
\]
\[
\begin{align*}
    u_\theta &= -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \\
    u_\phi &= \frac{\Omega}{r \sin \theta}.
\end{align*}
\]

Change to the physical variables, denoted by ( )*, is accomplished according to

\[
\begin{align*}
    r^* &= R_2 r, \quad \psi^* = R_2^3 \Omega \psi, \quad \Omega^* = R_2^2 \Omega \Omega
\end{align*}
\]

\[
\begin{align*}
    u_r &= \frac{1}{r^*^2 \sin \theta} \frac{\partial \psi^*}{\partial \theta} \\
    u_\theta^* &= \frac{-1}{r^* \sin \theta} \frac{\partial \psi^*}{\partial r^*} \quad \text{and} \\
    u_\phi^* &= \frac{\Omega^*}{r^* \sin \theta}.
\end{align*}
\]

The boundary conditions are the nonslip conditions giving \( \psi = \frac{\partial \psi}{\partial n} = 0 \) on the boundaries, with \( n \) denoting the unit normal, and that \( \Omega \) is prescribed on the boundaries since the angular velocities are given.

Solutions of Equations 1 and 2 with the prescribed boundary conditions gives the desired flow between rotating spheres. These equations are nonlinear partial differential equations; one is the fourth order and the other is second order, in terms of a meridian plane stream function \( \psi \), and an angular velocity function \( \Omega \). Since the equations are nonlinear it does not appear likely that an exact solution
is obtainable, and for this reason the only available solutions are either numerical solutions or special perturbation solutions valid in the limit of large or small parameters (such as Reynolds number).

The basic motion consists of a "primary motion" about the common axis, combined with the "secondary motion" in the meridian plane. Although the secondary motion is small relative to the motion about the axis of rotation when the Reynolds number is small, it can become comparable to the primary motion for larger Reynolds numbers.

Review of Literature

Previous works regarding the flow between concentric rotation spheres can be divided into two main approaches: theoretical and experimental.

Theoretical research

Most theoretical studies concerning this problem can be classified into one of the following categories:

a) For low Reynolds numbers the problem can be solved by the method of regular perturbation, which permits representation of the solution as a series in integer powers of Reynolds numbers.

b) Numerical integration of the nonlinear partial differential equations governing the flow, either by direct methods of Galerkin type or by the use of finite difference
methods for intermediate Reynolds numbers.

c) A boundary layer or single perturbation technique for large Reynolds numbers and special geometries.

d) Stability of the steady laminar motion.

Early studies tend to neglect the secondary flow and assume the motion consists entirely of the primary flow about the axis of rotation (4). It is not difficult to show, however, that except for the trivial case of rigid rotation a secondary flow must be present (5).

Ovseenko (6) proved that an analytical perturbation solution in terms of a power series in the Reynolds number will be appropriate for low or moderate Re values. Due to the complexities of the governing equations, he limited his consideration to terms no greater than Re^3. Munson (3) considered the results through Re^7 and gave the perturbation solution of the governing Equations 1 and 2 in the form

\[
\psi(r,\theta) = \sum_{l=1}^{\infty} \sum_{j=0}^{l} \text{Re}[\sin^2 \theta P_j(\theta) g_j^l(r)]^\text{odd even}
\]

\[
\Omega(r,\theta) = \sum_{l=0}^{\infty} \sum_{j=0}^{l} \text{Re}[\sin^2 \theta P_j(\theta) f_j^l(r)]^\text{even even}
\]

where \(P_j(\theta)\) is the \(j\)th-order Legendre polynomial. Thus the \(\theta\) dependence of \(\psi\) and \(\Omega\) separates from the \(r\) dependence for
the perturbation solution and allows the governing system to be written as an ordinary differential equation problem for the component functions $f_{j\ell}(r)$, $g_{j\ell}(r)$. This set of equations is obtained by substitution of the form given by Equation 3 into the governing Equations 1 and 2, and equating the like powers of Re. After lengthy calculations, the problem reduces to the evaluation of a large number of coefficients insuring that the perturbation equations and the boundary conditions are satisfied. The large number of coefficients involved in the higher-order solution (through $Re^7$) dictates the use of computer.

For higher Reynolds numbers Munson (3) and Munson and Joseph (7) expand the functions $\psi$ and $\Omega$ in terms of a series in Legendre polynomials:

$$
\psi(r,\theta) = \sum_{\ell=0}^{\infty} \sin^2 \theta P_\ell(\theta) g_{j\ell}(r)
$$

$$
\Omega(r,\theta) = \sum_{\ell=0}^{\infty} \sin^2 \theta P_\ell(\theta) f_{j\ell}(r)
$$

where $P_\ell(\theta)$ is the Legendre polynomial of the angle $\theta$ of order $\ell$.

The series in Equation 4 can be truncated at some appropriate number of terms, $N_t$, and substituted into the governing Equations 1 and 2. After some algebra the system reduces to a set of coupled nonlinear ordinary differential
equations. The lowest order truncation, $N_t = 1$, will give

$$\psi(r, \theta) = \sin^2 \theta \cos \theta g_1(r)$$

$$\Omega(r, \theta) = \sin^2 \theta f_0(r)$$

(5)

with the corresponding governing equations, results from substituting Equation 5 into 1 and 2. The accuracy of this representation (the lowest-order Galerkin approximation) is not expected to be good for any but small Reynolds numbers. High-order truncations, $N_t > 1$ allowing consideration of larger Reynolds numbers, provide the same general form for the governing system, but the number of terms increases very rapidly with $N_t$. The governing nonlinear boundary value problem was integrated numerically using a Runge-Kutta-Gill technique for various combinations of Reynolds number, radius ratio, angular velocity ratio, at various truncation orders, $N_t$.

For large Reynolds numbers the problem can be solved numerically. Pearson (8) has numerically integrated the time dependent Navier-Stokes equations for several ratios of angular velocities of the two spheres and for Reynolds number range 10-1500. Greenspan (9) adapted the numerical method developed for the study of secondary flow in a curved tube to study the viscous incompressible flow between two rotating spheres. The Navier-Stokes equations are approxi-
mated by difference equations which are associated with diagonally dominant linear algebraic systems.

For high Reynolds numbers the theoretical work is primarily of a boundary layer or singular-perturbation character (10-16). Proudman (11) obtained the almost rigid rotation solution in terms of a boundary layer inviscid core solution for large Reynolds numbers, when the two spheres rotate at nearly the same rate (the almost rigid rotation). He found that the cylindrical surface that touches the inner sphere (its axis is the axis of rotation) is a singular surface (shear layer) in which velocity gradients are very large. Outside this cylinder the fluid rotates as a rigid body with the same angular velocity as the outer sphere. Other linear theory boundary layer analysis have been conducted by Carrier (15) and Pedlosky (16) for stratified fluids in a spherical annulus.

The stability of the basic laminar flow was also studied either by applying the energy theory, or the linear stability theory. The energy theory, which has its beginnings with Reynolds (17) and Orr (18), has in recent times been greatly expanded by Serrin (19) and Joseph (20). The method consists of considering the time rate of change of the kinetic energy of an arbitrary disturbance in the flow field. If the kinetic energy decreases as a function of time, the flow is stable. Munson (3) and Munson and Joseph (21) obtained the critical
Reynolds number, $R_{e_B}$, such that the flow is definitely stable for $Re < R_{e_B}$, for various cases of flow in a spherical annulus. Works concerning this method can be found also in references 17-20, 22-24.

The linear theory of hydrodynamic stability considers only infinitesimal disturbances and provides a critical Reynolds number, $R_{e_L}$, such that the basic flow is definitely unstable for $Re > R_{e_L}$. It can say nothing concerning the possibility of instabilities caused by disturbances of finite size. Bratukhin (25) obtained an approximate linear stability result for flow between spheres with the outer sphere stationary and radius ratio $\eta = 0.5$ by using the Stokes-flow approximation as the basic laminar flow. Sorokin, Khlebutin, and Shaidurov (26) attempted to verify experimentally Bratukhin's (25) results but were not able to observe transitions indicating instability. A similar lack of any evidence of instability in the flow in a wide-gap spherical annulus was obtained by Khlebutin (27) for $\eta < 0.7$. Yakushin (28, 29) and Zierep and Sawatzki (30) have carried out approximate linear stability calculations for the narrow-gap case and have done experiments verifying the theory. Menguturk (2) carried out experimental and theoretical work for a stationary outer sphere, and found that the observed instability occurs at a Reynolds number close to the critical value of the energy stability theory.
Experimental research

Though the geometry of the spherical annulus flow is relatively simple it seems that the implementation of this problem in the experimental sense is a rather difficult one, especially when both spheres are rotating. This could be the reason why there are fewer reports concerning experimental work.

Charron (31) studied the problem for a stationary outer sphere. To visualize the flow he used a tracer composed of a mixture of alcohol, dye and water of the same density as that of the oil used. He described the flow as if every spherical layer of a thickness, dr, is moving as a rigid body. The measured torque was reported as being proportional to the angular speed.

Sorokin, Khlebutin, and Shaidurov (26) performed their experiments to verify Bratukhin's (25) analytical predictions. They used water-glycerin to fill the gap between an outer stationary sphere and a rotating inner sphere with a radius ratio $\eta = 0.5$. They measured the torque for the Reynolds number range: $0.7-2800$. Their results pointed out that in the range where departure from stability was expected, a crises-free transition from slow flow to boundary layer type of flow was observed.

Khlebutin (27) conducted a series of experiments considering narrow gaps ($\eta = 0.85-0.98$), and for stationary
outer sphere. The spheres were made of plexiglass. Aluminum powder was used for the flow visualization purposes. His results show that at low Reynolds numbers the torque is proportional to the angular velocity. The critical Reynolds number depends on the radius ratio $\eta$, and for $\eta = 1$ there is a vortex formation similar to that of Taylor vortices between rotating cylinders. For $\eta \ll 1$ the results show no sudden break in the torque curve as would be expected for a transition from the basic laminar flow.

Zierep and Sawatzki (30) used a rotating inner sphere made of aluminum and a stationary outer sphere made of plexiglass. The gap was filled with silicone oil. Aluminum powder was used as a flow tracer. Stokes flow was observed up to $Re = 3.3 \times 10^3$ for $\eta = 0.95$ and $Re = 600$ for $\eta = 0.847$. Next followed a regime of boundary layer type and finally the flow became unstable, passing various unstable flow configurations. Flow visualization and torque measurements were employed to detect these configurations. For small gap and for low Reynolds numbers the streamlines are concentric circles around the axis of rotation (primary flow only). At higher Reynolds numbers, the presence of the secondary motion was observed as spirals moving from the equator to the poles in the neighborhood of the outer sphere, and from the poles to the equator in the neighborhood of the inner sphere.

Menguturk (2) used a stationary outer sphere made of
casting resin and three different inner spheres with radius ratios 0.304, 0.44, and 0.881. Flow visualization and torque measurements were used to study the laminar flow, flow stability and turbulent flow in concentric and eccentric annuli. The results indicate that the basic laminar flow (primary and secondary flow) in concentric and eccentric spherical annuli is as previously predicted. The character of the flow is dependent upon the Reynolds number, and the radius ratio. For the narrow gap annulus the instability takes place as Taylor type vortices near the equator, with other transitions at larger Reynolds numbers. For a wide gap, the flow is found to be sublinearly unstable, with the observed instability occurring at a Reynolds number close to the value of energy stability theory. Mengeturk's (2) results are of interest and will be referred to in the following chapters.

Morales-Gomez (32) used three different inner spheres manufactured by spinning thin aluminum sheets to form hemispheres and then welded to form complete spheres. Two outer spheres were formed by plexiglass sheets. With the outside sphere stationary the experiment was performed for \( \eta = 0.512, 0.587, 0.647, 0.742 \) and 0.889. Chevron machine oil with some red transmission oil or Chevron machine oil with kerosene and red transmission oil were used to fill the gaps between the two concentric spheres. Aluminum powder was used as a tracer. With the outside sphere stationary he studied the
laminar flow and stability of the flow.

Nakabayashi's (33, 34) work was devoted to the flow in a narrow gap, $0.829 \leq \eta \leq 0.989$, and with a stationary outside sphere, $\mu = 0$, or with a stationary inner sphere, $\mu = \infty$. The outer sphere was made of acrylate and the inner sphere was made of aluminum alloy. Water, glycerine-water solution, spindle oil, refrigerator oil and turbine oil were used with aluminum powder as a tracer. He verified the existence of four basic flow regimes for both $\mu = 0$ and $\mu = \infty$. An empirical formula for the coefficient of friction resistance is obtained for the four mentioned regimes.

One of the important physical properties obtained from the basic flow between rotating spheres is the torque required to rotate the spheres at a given rate. Walters and Waters (35) have considered the torque on a sphere which rotates in a bath of visco-elastic material contained between concentric spheres by using a first-order perturbation theory for the slow flow. Ovseenko (6) has considered the torque in terms of a second-order perturbation solution for a Newtonian fluid. Munson (3) put the torque in dimensionless form as follows:

$$ M = \frac{8}{3} \pi \mu \Omega_0 R^2 \bar{M} . $$

(6)

Here $\bar{M}$ is a nondimensional torque which is a function of $\mu$ and $\eta$, and $Re. \; u$ is the viscosity of the fluid. With the
outside sphere stationary Menguturk (2) measured the torque experimentally for $\eta = 0.304$, 0.44 and 0.881, and for a wide range of Reynolds number. He detected four breaks in the torque curve indicating change in the flow character for a medium-sized gap, $\eta = 0.44$, while Sorokin, Khlebutin, and Shaidurov (26) were unable to detect any transition for $\eta = 0.5$. Zierep and Sawatzki (30) present data indicating various flow transitions for the narrow gap, $\eta \approx 1$. Morales-Gomez (32) found for $\mu = 0$ that a curve of the form, $\bar{M} = k \text{Re}^{3/2}$, will fit the experimental data for the torque for moderate and large Reynolds numbers. Here the coefficient, $k$, depends on the radius ratio $\eta$. Nakabayashi (34) experimentally investigated the frictional resistance for the flow between two spheres one of which rotates. He verified the existence of four basic flow regimes for both cases. Kendall (36) gave a correction to the retarding torque experienced by two rotating concentric spheres.

Additional information can be found in references (37-43).

Description of the Flow

The character of the flow is dependent upon the Reynolds number, $\text{Re}$, the ratio of the radii of the two spheres, $\eta$, and the angular velocity ratio, $\mu$. It can be characterized roughly as follows. For low Reynolds number, the flow field
is spherical in character. That is, the lines of constant angular velocity are circles on the $\omega$ maps (where $\omega$ is the angular velocity about the axis of rotation) corresponding to spherical shells of constant angular velocities as shown in Figure 2a. At these low Reynolds numbers the flow in the meridian plane, as indicated by the $\psi$ maps consists of fairly regular toroidal swirls, the direction of flow dependent on the relative rates of rotation of the two spheres. For example, with the outer sphere stationary the centrifugal force of the fluid dragged along near the rotating inner sphere increases as the equator is approached. The net result is that the fluid is thrown from the inner sphere and produces a counterclockwise swirl as shown in Figure 2. A similar but opposite result is obtained when the inner sphere is stationary. The resulting clockwise swirl is shown in Figure 3. If both spheres are rotating, these two opposing forces may combine to produce contrarotating swirls as shown in Figure 5.

As the Reynolds number is increased, various changes in the basic flow field may develop depending on the relative rates of rotation of the two spheres. If the outer sphere is the dominant one at low $\text{Re}$ (that is the clockwise swirl is dominant), the flow field then tends toward a typical two-dimensional (Taylor-Proudman) character (11) as $\text{Re}$ increases. This is clearly seen in Figure 3 for the case of $\mu = \infty$, that is, the inner sphere is stationary. Even at $\text{Re} = 100$ the
Figure 2. The stream function, \( \psi \), and the angular velocity, \( \omega \), with stationary outer sphere, for different Reynolds numbers, \( \eta = 0.5 \) and \( \mu = 0.0 \) (3).
Figure 3. The stream function, $\psi$, and the angular velocity, $\omega$, with stationary inner sphere, for different Reynolds numbers, $\eta = 0.5$ and $\mu = \infty$ (3)
lines of constant angular velocity have changed radically from the spherical character of \( \text{Re} = 10 \) into a more nearly cylindrical character \((3)\). At \( \text{Re} = 500 \) this character is more pronounced, including the beginning of a small reverse flow region near the equator as indicated by \( \psi \) map.

As for the flow when the inner sphere is dominant, no boundary layer or Taylor-Proudman effects are obtained as \( \text{Re} \) increases. The counterclockwise swirl remains, increases in strength, and tends to be centered nearer the equator as shown in Figure 2. At the same time the angular velocity contours lose their spherical character, but do not take on a cylindrical character as when the rotation of the outside sphere dominates. In fact, the angular velocity contours are rather nondescript \((3)\).

The character of the flow field when both spheres are rotating changes also as \( \text{Re} \) is increased. For example, as shown in Figure 4, if both spheres rotate with the same angular velocity but in opposite directions, that is \( \mu = -1 \), the outer sphere dominates at \( \text{Re} = 100 \) and causes one clockwise swirl. For \( \text{Re} = 500 \) the cylindrical sheath and reverse flow near the equator (characteristic of \( \mu = \infty \)) begin to appear. At the same time, however, a small counterclockwise swirl (characteristic of \( \mu = 0 \)) develops near the polar region of the inner sphere. With the outer sphere rotating at half the rate of the inner one and in the opposite
direction ($\mu = -0.5$), the two contrarotating swirls are present at any Re as shown in Figure 5. The counterclockwise swirl near the inner sphere exerts more influence as Re increases, and the tendency toward a cylindrical sheath character is not so great as in other cases.
Figure 4. The stream function, $\psi$, and the angular velocity, $\omega$, for both spheres rotating with the same angular velocity but in opposite directions for $Re = 100$ and $Re = 500$, $\mu = -1$, $\eta = 0.5$ (3)
Figure 5. The stream function, $\psi$, and the angular velocity, $\omega$, when the outside sphere is rotating with half the angular velocity of the inside one and in the opposite direction for two different Reynolds numbers, $\mu = -0.5$ and $\eta = 0.5$ (3)
CHAPTER II. THE EXPERIMENT

In this chapter we will discuss the experimental apparatus and techniques used to study the viscous incompressible flow between two rotating spheres, when one of them rotates or when both are rotating together.

The Apparatus

The apparatus used in the experiments is shown in Figure 6. It consists of two spherical surfaces between which the flow takes place. The outer shell is made of two identical hemispheres built separately of a clear casting resin, 0.5 in (1.27 cm) thickness and 5.1 in (12.95 cm) diameter. They are fastened together by means of a plexiglass bracket to complete the outer sphere. The bottom of the outer sphere is screwed to a telescopic shaft which leads to a variable speed motor through a belt-gear mechanism.

Six different inner spheres, made of a variety of materials, were used throughout the experiment. In each case the sphere is attached to a variable speed motor by means of a shaft. The assembly is suspended from the ceiling by a torsion wire of a known spring constant. Two air bearings are used to prevent the assembly from vibrating or slanting.

The two spheres are contained within a plexiglass box which is filled with the same fluid contained between the
The ceiling

Belt

Angular deflection indicator & scale

Torsion wire

Mercury switch
Mercury switch

Air bearings

Motor-shaft assembly for inner sphere

Plexi glass assembly

Motor controller
Figure 6. The experimental apparatus
Figure 6 (Continued)
two spheres. This helps to make the visualization of the flow more reliable by reducing optical refractions. A mercury switch was installed to eliminate the unwanted torque generated from twisting the electrical cord of the inner sphere motor. For more details on the components of the apparatus, the wires, and the fluids used in the experiments, refer to Appendix A.

Six different inner spheres were used in conducting the experiments. The diameters of these spheres and the corresponding gaps are shown in Table 1.

Spheres A, C, E and F were used for studying the stability of the laminar flow only. Spheres B (\( \eta = 0.304 \)) and D (\( \eta = 0.44 \)) were used for flow visualization and torque

<table>
<thead>
<tr>
<th>Sphere</th>
<th>Radius of inner sphere, ( R_1 )</th>
<th>Radius ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.510 in, 1.295 cm</td>
<td>0.200</td>
</tr>
<tr>
<td>B</td>
<td>0.775 in, 1.969 cm</td>
<td>0.304</td>
</tr>
<tr>
<td>C</td>
<td>1.000 in, 2.540 cm</td>
<td>0.390</td>
</tr>
<tr>
<td>D</td>
<td>1.122 in, 2.850 cm</td>
<td>0.440</td>
</tr>
<tr>
<td>E</td>
<td>1.500 in, 3.810 cm</td>
<td>0.590</td>
</tr>
<tr>
<td>F</td>
<td>2.247 in, 5.707 cm</td>
<td>0.881</td>
</tr>
</tbody>
</table>
measurements over a wide range of Reynolds numbers. In all cases the radius of the outside sphere was 2.55 in (6.475 cm).

Dow Corning 200 silicone fluid with viscosities ranging from 5 up to 1330 centistokes, and four different torsion wires with diameters 0.024, 0.020, 0.012 and 0.010 in (0.061, 0.051, 0.030 and 0.025 cm) were used in the experiments.

The Experiments

Introduction

In this investigation we focus our attention on the flow in the medium-sized gap spherical annulus ($\eta \neq 1$, $\eta \neq 0$), considering the following situations.

i) The laminar flow between concentric spheres for $\eta = 0.304$ and 0.44 for the following cases:
   - The outside sphere stationary, $\mu = 0.0$;
   - The inside sphere stationary, $\mu = \infty$;
   - Both spheres rotating with an angular velocity ratio range of $-2 < \mu < 2$, and for different Reynolds numbers.

ii) Stability considerations of the basic laminar flow.

iii) Almost rigid rotation.

The experiments were performed for a wide range of Reynolds numbers ($10 < Re < 10^4$), where Reynolds number was defined as
\[ \text{Re} = \frac{\Omega_0 R_2^2}{v} \]

with

\( R_2 \) = radius of the outside sphere
\( \Omega_0 \) = characteristic angular velocity
\( v \) = kinematic viscosity of the oil used.

The choice of \( \Omega_0 \) depends on the particular situation being considered. However, it will be \( \Omega_1 \) with a stationary outer sphere, \( \Omega_2 \) with a stationary inner sphere.

**Flow visualization**

Flow visualization techniques were applied with the use of three kinds of tracers.

**Dye injection** A small amount of the fluid used in the experiment was colored with a blue dye (CALCO Oil Blue ZV) and injected into the flow field when needed. This technique was used in studying the development of the basic laminar flow, and to reveal the laminar flow pattern. We were able to photograph the flow pattern for a variety of flow situations presented in the following chapters. Dye was injected through the shaft hole at the top of the outside sphere with the use of a suitable-sized needle and syringe. The location where the dye was injected differs from one case to the other with an attempt to get the best possible
picture for the flow pattern. For example, if the inside sphere were stationary, the dye was injected near the outside sphere. It was easy to observe the travel of the dye in spirals down to the equator, where it leaves the outside rotating spheres and moves towards the inside stationary sphere. Close to the inside sphere the dye moves up in spirals around the axis of rotation towards the pole, and so on. This process was repeated for different Reynolds numbers.

At low Reynolds number the fluid was noted to make many revolutions about the axis of rotation before completing one secondary flow swirl. This is due to the weakness of the secondary flow in the meridian plane compared with the primary flow around the axis of rotation. This made it possible to get a good sequence of photographs to illustrate the development of the basic laminar flow.

The experiment was repeated with the outside sphere stationary, and with both spheres rotating at the same time. For the later case the dye was injected into two different locations in the flow field; one close to the inside sphere and the other close to the outside one.

The use of aluminum powder Due to the fact that the dye blends thoroughly with the fluid over a considerable amount of time, aluminum powder was used at higher Reynolds numbers. The use of the aluminum powder was very convenient
in studying the transitions in the flow field. The aluminum flakes align themselves relative to the fluid shear, allowing the flow visualization even at very high Reynolds numbers, and giving an opportunity to observe transition stages and the final breakdown to turbulence. This technique was also used to evaluate the critical Reynolds number, $Re_{cr}$ (above which the flow is fully turbulent), for all the spheres listed in Table 1. In all cases the fluid used was silicone oil with a kinematic viscosity, $\nu = 5$ centistokes. One good advantage of using aluminum powder was being able to repeat the experiment as many times as desired, without having to replace the fluid each time as was necessary with the dye experiments. It should be noted here that it would be rather difficult to observe the development of the basic laminar flow by using the aluminum powder. Hence the use of dye and aluminum flakes techniques.

**Mutually buoyant particles** A spherical particle of about $3/32$ in diameter, made of a material which has almost the same density as that of the fluid used, was inserted in the annulus between the two spheres. A moving picture was taken to show the movement of this particle corresponding to the particular case being examined. The path of the particle will describe a streamline in the flow field (steady-state situation). This technique is useful in studying flow cases where streamlines change their directions due to a slight
change in the parameters.

**Torque measurements**

The determination of the torque necessary to rotate the spheres can provide considerable insight into understanding the flow. To apply this method, the torque is determined in nondimensional form and plotted against other dimensionless parameters, such as the Reynolds number. As long as the flow is stable, the torque curve is smooth and continuous. However, any transition in the flow will produce a change in the torque with the resulting break or discontinuity in the torque curve. For stability considerations, this method is very effective in determining virtually all the transitions that the flow may undergo.

The dimensionless torque for the geometry under consideration is given by

\[
\bar{M} = 8266.58 \frac{K\alpha}{vS\Omega_0}
\]  

(7)

where

- \(\bar{M}\) = dimensionless torque
- \(K\) = the torsion wire's spring constant, in lb/rad
- \(\alpha\) = the angular deflection of the torsion wire, degree
- \(v\) = kinematic viscosity of the fluid, centistokes (cs)
- \(S\) = specific gravity of the fluid
- \(\Omega_0\) = characteristic angular velocity, rpm.
The Reynolds number is obtained from the equation

\[ \text{Re} = 439.48 \frac{\Omega \text{(rpm)} \cdot \nu}{v \text{(cs)}} \]  

(8)

The derivation of these two equations is illustrated in Appendix A. Also for more information about determining the wire's spring constant, the viscosity of the fluids, and other parameters related to the experiments, refer to Appendix A.

By measuring the angular deflection, \( \alpha \), and knowing the spring constant, \( K \), it was possible to calculate the dimensionless torque for all the flow situations mentioned, covering a wide range of Reynolds numbers \( (10 < \text{Re} < 10^4) \).

Figures 7 and 8 present the torque as a function of Reynolds number with a stationary outside sphere, \( \mu = 0.0 \), for a radius ratio \( \eta = 0.304 \) and 0.44 respectively. Figures 9 and 10 present the torque curve with a stationary inner sphere for a radius ratio \( \mu = 0.304 \) and 0.44 respectively. In all four cases the torque agrees with the Stokes-flow solution for low Reynolds numbers, that is, when secondary flows are negligible compared to the primary flow. Table 2 shows the Stokes-flow region where the dimensionless torque remains constant in each of the above mentioned cases.
Figure 7. The torque as a function of Reynolds number when the inside sphere is rotating alone, for $\eta = 0.304$. 

- $\Delta \nu = 256.56 \text{ cs}$
- $\circ \nu = 110 \text{ cs}$
- $\diamond \nu = 35 \text{ cs}$
- $\cdot \nu = 9.27 \text{ cs}$
- $\ast \nu = 5.43 \text{ cs}$
- $\eta = 0.304$
- $\mu = 0.0$
- $\Omega_0 = \Omega_1$
Figure 8. The torque as a function of Reynolds number when the inside sphere is rotating alone, for $\eta = 0.44$. 

The figure shows the relationship between the non-dimensional torque $\bar{M}$ and the Reynolds number $Re$, with various data points representing different velocities $\nu$ for $\eta = 0.44$. The theoretical curve is also plotted for comparison.
Figure 9. The torque as a function of Reynolds number when the outside sphere is rotating alone, for $\eta = 0.304$.
Figure 10. The torque as a function of Reynolds number when the outside sphere is rotating alone, for $\eta = 0.44$.
Table 2. Stokes regions, for the flow cases being considered

<table>
<thead>
<tr>
<th>Radius ratio ( \eta )</th>
<th>Angular velocity ratio ( \mu )</th>
<th>Stokes region</th>
<th>Constant torque ( \text{\textit{M}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.304</td>
<td>0.0</td>
<td>up to Re = 80</td>
<td>0.0867</td>
</tr>
<tr>
<td></td>
<td>( \infty )</td>
<td>up to Re = 10</td>
<td>0.0867</td>
</tr>
<tr>
<td>0.44</td>
<td>0.0</td>
<td>up to Re = 45</td>
<td>0.279</td>
</tr>
<tr>
<td></td>
<td>( \infty )</td>
<td>up to Re = 25</td>
<td>0.279</td>
</tr>
</tbody>
</table>

In the succeeding chapters we will present the torque curves when both spheres are rotating over a wide range of angular velocity ratios. Also we will take a close look at the torque results for all the flow cases under consideration.
CHAPTER III. LOW REYNOLDS NUMBERS

In this chapter we discuss cases where the Reynolds number is relatively small.

Introduction

For small Reynolds numbers, the solution of the boundary value problem governing the flow can be obtained analytically by expanding the basic flow, \( U \), in the form of a series in positive integer powers of Reynolds number

\[
U = \sum_{n=0}^{\infty} U_n \text{Re}^n
\]

The zeroth term of the expansion Equation 3 describes the flow when the inertial forces are negligibly small compared to the viscous forces \( (\text{Re} \ll 1) \). In this classical case the flow can be described as circles around the axis \((4a)\)-primary flow only.

Ovseenko (6) included inertia effects but limited his consideration to terms no greater than \( \text{Re}^3 \), because of the complexities of the governing equations. He also proved that the radius of convergence of such a representation is nonzero. The calculation of its specific value is another problem.

Munson (3) considered the results through \( \text{Re}^7 \), the
equations governing his perturbation solution are given by Equation 3. The solution was calculated for various cases; that is, for the various combinations of radius ratio \( \eta \) and angular velocity ratio \( \mu \). Comparing the perturbation solution with the true solution, that is, a Galerkin-type procedure, for \( \eta = 0.5 \), one finds that for \( \text{Re} < 10 \) the low order solution and the true solution are essentially the same. For \( \text{Re} = 50 \) the lowest-order solution is in error by about 20%. However, the fifth-order solution (through \( \text{Re}^5 \)) agrees very well with the actual solution for \( \text{Re} = 50 \) (about 1% difference). For \( \text{Re} = 80 \) even the seventh-order solution is in error by about 15%, but as the order increases the perturbation solution appears to be converging to the true solution (7). He also stated that the inclusion of terms of order greater than \( \text{Re}^3 \) is seen to provide a fairly accurate description of the flow field for Reynolds numbers considerably above the \( \text{Re} = 10 \) limit of the low-order consideration.

Character of the Flow Field

The case when one of the spheres is rotating

For low Reynolds numbers the flow field is spherical in character. That is, the lines of constant angular velocity are circles on \( \omega \) maps, corresponding to spherical shells of constant angular velocity. At these low Reynolds numbers the flow in the meridian plane consists of fairly regular toroidal
swirls; the direction of this secondary flow is dependent on the relative rates of rotation of the two spheres. If one of the spheres is rotating and the other is stationary, the fluid close to the rotating sphere moves from the poles to the equator, and the fluid close to the stationary sphere moves from the equator to the poles. The two sides will join to form a closed swirl. An explanatory example is with the outer sphere stationary, for which the centrifugal forces of the fluid dragged along near the rotating inner sphere increase as the equator is approached. The net result is that fluid is thrown from the inner sphere and produces a counterclockwise swirl (looking at the top right hand side quarter of the meridian plane).

Flow visualization studies show that the basic flow is as indicated by the above mentioned theoretical calculations. The primary flow is around the axis of rotation with the secondary flow in the meridian plane. The sequence of photographs shown in Figures 11 and 12 illustrate the development of the secondary flow streamlines for Re \(\approx 30\) and \(\mu = 0\). Figure 11 is for a medium gap situation, \(\eta = 0.44\), while Figure 12 is for a larger gap situation, \(\eta = 0.304\). In each case we note that the fluid makes many revolutions about the axis of rotation before completing one secondary flow swirl. This shows that the secondary flow is weak compared with the primary flow. The character of the flow is basically the
Figure 11. Development of the basic laminar flow with a stationary outer sphere for $\eta = 0.44, \mu = 0.0, \text{ Re} = 33$. 
Figure 12. Development of the basic laminar flow with a stationary outer sphere for $\eta = 0.304, \mu = 0.0, Re = 30$
Figure 12 (Continued)
same for the two different radius ratios at this low Reynolds number. However, due to the larger gap size in the case of $\eta = 0.304$, the development of the secondary flow streamlines takes more revolutions to complete the swirl than the case of $\eta = 0.44$.

Two mutually buoyant small spheres were placed in the fluid. One of them, A, was approximately in the center of the secondary flow swirl. A moving picture was taken to record the flow with the inner sphere rotating and for $Re = 30$, $\eta = 0.44$. The sequence of photographs (1-13) in Figure 13 shows the position which sphere B takes when crossing the meridian plane. Sphere A is noticed to move in circles about the axis of rotation. Photograph 14 was produced by superposing the previous 13 photographs to show the shape of the secondary flow swirl.

Comparing Figures 14 and 15 which show the secondary flow swirls for $Re = 30$ and 100 respectively, one finds that at low Reynolds number ($Re = 30$) the fluid makes more revolutions to complete the swirl than the case of larger Reynolds number ($Re = 100$). For $Re = 100$, sphere B makes more than half of a secondary flow swirl with one revolution in 0.54 sec. For $Re = 30$, sphere B makes a small fraction of a swirl in one revolution in 0.79 sec.

A similar but opposite result is obtained when the inner sphere is stationary. The fluid pushed by centrifugal forces
Figure 13. The sequence of photographs (1-13) shows the positions which sphere B takes when crossing the meridian plane, with a stationary outer sphere for \( \eta = 0.44, \) Re = 30, \( \mu = 0. \) Sphere A was placed approximately in the center of the secondary flow swirl.
Figure 14. Photographs (1-5) at times 0.0, 0.21, 0.29, 0.42 and 0.70 sec. respectively, show one revolution sphere B makes with a stationary outer sphere, \( n = 0.44 \), \( \mu = 0.0 \), \( Re = 30 \).
Figure 15. Photographs (1-5) at times 0.0, 0.21, 0.38, 0.46 and 0.54 sec. respectively, show one revolution sphere B makes with a stationary outer sphere for $n = 0.44$, $\mu = 0.0$, $Re = 100$. 
generated by the outer rotating sphere will produce a clockwise swirl. The sequence of photographs shown in Figure 16 illustrates the development of the basic laminar flow for Re = 20. This flow consists of a primary flow around the axis of rotation and a secondary flow in the meridian plane in the form of a clockwise swirl. The sequence of photographs (1-19) in Figure 17 shows the positions which the mutually buoyant sphere B takes when crossing the meridian plane. Sphere A is placed approximately in the center of the secondary swirl and it noticeably moves in circles about the axis of rotation. Photograph 20 is produced by superposing the sequence (1-19) to show the shape of the clockwise swirl produced by the rotation of the outside sphere.

Figure 18 shows the agreement of the flow visualization results with a previous theoretical solution for the flow at low Reynolds numbers (3). The photograph was taken for conditions of Re = 20 and for a radius ratio $\eta = 0.44$, while the theoretical solution was for Re = 20 and a radius ratio $\eta = 0.5$. Figure 19 shows the flow visualization results for two different radius ratios, $\eta = 0.44$ and $\eta = 0.304$, for Re = 50. From the photographs it can be seen that for $\eta = 0.304$ the secondary flow streamlines have started to change from their spherical character into a more cylindrical character, while for the case when $\eta = 0.44$ such a change is not apparent. This means that the larger the gap the lower will be the
Figure 16. Development of the basic laminar flow with a stationary inner sphere for $\eta = 0.44$, $\mu = \infty$, $Re = 20$. 
Figure 16 (Continued)
Figure 17. Photographs (1-19) show the position which sphere B takes when crossing the meridian plane with a stationary inner sphere for $\eta = 0.44$, $\mu = \infty$, $Re = 30$. Sphere A was placed approximately in the center of the secondary flow swir
Figure 18. Comparison between the experimental results and the theoretical solution, with a stationary inner sphere, $\mu = \infty$ (3)
\( \eta = 0.44 \quad \eta = 0.304 \)

Figure 19. Comparison of the flow pattern for two different radius ratios, \( \eta = 0.44 \) and \( \eta = 0.304 \), with a stationary inner sphere \( \mu = \infty \), \( Re = 50 \).
values of Reynolds numbers at which this change occurs.

The flow visualization results compared very well with Menguturk's (2) results for stationary outer sphere for both medium and wide gap (\(\eta = 0.44\) and 0.304 respectively).

The case when both spheres are rotating

When both spheres are rotating, the two centrifugal force fields resulting from the rotation of the spheres may combine to produce one swirl or two contrarotating swirls. For example, if both spheres rotate at the same rate but in opposite directions (\(\mu = -1\)), the outer sphere dominates and causes a clockwise swirl. For other values of \(\mu\) there may be two contrarotating swirls. Figure 20 shows the possible types of meridional flow and the region of the effect of each sphere, as a function of the values of the radius ratio \(\eta\) and the angular velocity ratio \(\mu\). It shows that only one swirl is possible when the two spheres rotate in the same direction, that is positive \(\mu\). For \(\mu > 1\) the outer sphere dominates producing a clockwise swirl. However, when \(0 < \mu < 1\), the inner sphere dominates producing a counterclockwise swirl. When both spheres rotate with the same angular velocity, \(\mu = 1\), the fluid will rotate as a solid body, and no secondary flow is expected. This is clear in Figure 21 where the photographs (1-5) and (7-11) show that the secondary flow spirals reverse their directions when \(\mu\) changes from \(\mu > 1\)
Figure 20. General nature of the meridional flow and the region of the effect of each sphere depending on $\mu$ and $\eta$ for small $\text{Re}$ (44)
Photographs (1-5) show the direction of the secondary flow spirals (clockwise) when 
\[ \Omega_1 = (1-\epsilon)\Omega_2, \epsilon \ll 1 \]
\[ \mu > 1 \]

Photographs (7-11) show the direction of the secondary flow spirals (counterclockwise) when 
\[ \Omega_1 = (1+\epsilon)\Omega_2 \]
\[ \mu < 1 \]

Figure 21. The secondary flow swirl reverses its direction when \( \mu \) changes from \( \mu > 1 \) to \( \mu < 1 \).
(photographs 1-5) to \( \mu < 1 \) (photographs 7-11). When the two spheres rotate in opposite directions, two swirls, one clockwise and the other counterclockwise, may occur depending on the two parameters \( \eta \) and \( \mu \). From Figure 20 can be seen the regions of all types of flow: Region I, where the inner sphere dominates, one counterclockwise swirl is possible; Region III, where the outer sphere dominates, one clockwise swirl is possible; and Region II, where two contrarotating swirls are possible. This region is bounded by the two curves \( \mu_1 \) and \( \mu_2 \) (44), where

\[
\mu_1 = -\frac{14\eta^6 + 53\eta^5 + 77\eta^4 + 51\eta^3 + 15\eta^2}{2\eta^6 + 14\eta^5 + 102\eta^4 + 93\eta^2 + 42\eta + 6}
\]  \hspace{1cm} (10)

\[
\mu_2 = -\frac{6\eta^6 + 42\eta^5 + 93\eta^4 + 102\eta^3 + 56\eta^2 + 14\eta + 2}{15\eta^4 + 51\eta^3 + 77\eta^2 + 53\eta + 14}
\]  \hspace{1cm} (11)

Figure 22 shows the development of the two secondary flow swirls in the meridian plane for an angular velocity ratio \( \mu = -0.29 \) and Re = 30. We note from the sequence of photographs that it takes more revolutions to complete the outside swirl than to complete the inside one. This is because the inside swirl results from the rotation of the inner sphere which has higher angular velocity than the outer sphere. Besides that, the fluid has to travel a longer path to complete the outside swirl.
Figure 22. Development of the basic laminar flow with the two spheres rotating together and in opposite directions for $\eta = 0.44$, $\mu = -0.29$, $Re = 30$
The size of each swirl for a certain Reynolds number depends on the angular velocity ratio $\mu$, and the radius ratio $\eta$. Figure 2j shows the dependence of the size of the swirls on the angular velocity ratio $\mu$, for a radius ratio $\eta = 0.44$. From the photographs we see that the inside swirl shrinks with the increase of the angular velocity ratio.

Figure 2k shows the theoretical (36) (see Appendix B) and experimental results of the dependence of the size of the two swirls on $\mu$ for two different radius ratios, $\eta = 0.44$, and 0.304. We see that for the larger gap ($\eta = 0.304$) the experimental and theoretical results agree very well, while for the smaller gap ($\eta = 0.44$) there is a noticeable difference between experiment and theory.

Torque Measurements Results

The determination of the torque necessary to rotate the spheres is of interest for several reasons. Hence the torque is determined in nondimensional form and plotted as a function of the important dimensionless parameters—Reynolds number, radius ratio and angular velocity ratio. This method is very helpful in studying the stability properties of the basic flow between spheres, by determining the transitions that the flow undergoes. At the low Reynolds numbers discussed in this section, the stability of the flow cannot be questioned. Thus the importance of this method is to
Figure 23. The size of the two swirls as a function of the angular velocity ratio, when the two spheres rotate in opposite directions for small $Re$. 

\[ \eta = 0.304, \eta = 0.44 \]

Experiment, $\eta = 0.44$

Experiment, $\eta = 0.304$

\[ \zeta = \frac{r}{R_2} \]
Figure 24. The flow pattern as a function of the angular velocity ratio with the two spheres rotating in opposite directions for \( \eta = 0.44 \).
determine the torque required to rotate the spheres at a given rate. The actual torque, $M$, needed to rotate the spheres may be obtained as follows:

$$M = 2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} ur^3 \sin^2 \theta \frac{\tau_{\theta r}}{r=r^*} \, d\theta d\phi$$  \hspace{1cm} (13)

where $u$ is the viscosity, $r^*$ the dimensional radius, and $\tau_{\theta r}$ is the component of the rate of stress. This torque may be written in dimensionless form as

$$M = \frac{8\pi}{3} u \Omega R^3 \bar{M}$$  \hspace{1cm} (14)

where $\bar{M}$ is a nondimensional torque which is a function of the radius ratio $\eta$, the angular velocity ratio $\mu$, and the Reynolds number $Re$. At low Reynolds numbers (negligible inertia effects, lowest order perturbation solution) $\bar{M}$ is independent of $Re$ and is given by (3)

$$\bar{M} = \frac{3\eta^3(1-\mu)}{1 - \eta^3}$$  \hspace{1cm} (15)

With the outside sphere stationary Menguturk (2) experimentally determined the torque needed to rotate the inside sphere. His results for a medium gap, $\eta = 0.44$, agreed very well with Munson's theoretical solution (3). The Stokes flow region (where $\bar{M}$ is more or less constant) is
up to \( \text{Re} \approx 100 \). For a narrow gap, \( \eta = 0.881 \), the Stokes flow region would extend to \( \text{Re} \approx 1000 \). In both cases the Reynolds number was defined as

\[
\text{Re} = \frac{\Omega R_2^2}{v}.
\]

He added that a more descriptive definition for the narrow gap will be

\[
\text{Re}_{\text{eff}} = \frac{R_2(R_2 - R_1)\Omega}{v} = (1 - \eta)\text{Re}.
\]

With this definition for Reynolds number, the Stokes flow region for a narrow gap would extend to \( \text{Re}_{\text{eff}} \approx 470 \).

Menguturk (2) also studied the case of a wide gap, \( \eta = 0.304 \), where the torque \( \bar{M} \) remains constant up to \( \text{Re} = 50 \). Comparing the results for the three above mentioned gaps, one finds that the Stokes flow region decreases as \( \eta \) decreases.

Morales-Gomez (32) studied the cases when \( \eta = 0.512, 0.587, 0.647, 0.742 \) and \( 0.889 \), with stationary outer sphere in all cases. His results for low Reynolds numbers do not corroborate the analytical slow flow solution (Stokes-flow). He believed that this was the result of the torque produced by the Coulomb friction of the shaft and bearing being a comparable order of magnitude as that of the torque required to rotate the sphere in the fluid.

Figure 25 shows the experimental torque results for a
Figure 25. The torque curve up to $Re = 300$ with a stationary outer sphere for $\eta = 0.44$
stationary outer sphere with \( \eta = 0.44 \) for Reynolds numbers up to 300. The second order perturbation solution (3) is also shown. It is noted that the Stokes-flow region is up to \( \text{Re} \approx 45 \), where the dimensionless torque \( \bar{M} \) remains constant at a value of 0.279. The dimensionless torque was calculated from the experimental data as

\[
\bar{M} = \frac{3K\alpha}{8\mu\Omega_0 R_2^3}
\]  

(16)

where

- \( K \) = the torsion wire constant, in lb/rad
- \( \alpha \) = the angular deflection of the wire, rad
- \( \mu \) = viscosity of the fluid, lb sec/in²
- \( \Omega_0 = \Omega_1 \).

The extent of the Stokes-flow region was found to be different from Menguturk's results (up to \( \text{Re} = 100 \)). However, the results agree well with the second-order perturbation solution up to \( \text{Re} \approx 70 \).

Figure 26 presents the torque when the inside sphere is stationary, for a radius ratio \( \eta = 0.44 \). The Stokes-flow region is up to \( \text{Re} \approx 25 \) in this case. Comparing the experimental data with the theoretical Galerkin-type solution (45),

\[1\]The complete torque curve covering the entire Reynolds number range investigated is given in Figure 8.
Figure 26. The torque curve up to $Re = 300$, with a stationary inner sphere, for $\eta = 0.44$
we find a difference in the torque value ranging between 0-6% for $\text{Re} < 100$ while the perturbation solution gives an error of 15%, for $\text{Re} = 100$. This error grows rapidly with the increase of Reynolds number. The torque was calculated using the angular velocity of the outer sphere as the characteristic angular velocity, $\Omega_0 = \Omega_2$. It is interesting to note here that for Stokes-flow, the torque needed to rotate either one of the spheres would be the same $M = 0.279$, that is, independent of $\mu$. This, of course, is due to the absence of inertia effects in the Stokes-flow region—only the shear effects are of importance.

The torque was also measured in the case of the larger gap, $\eta = 0.304$, with each one of the spheres rotating alone. This is shown in Figure 27. With the inside sphere stationary ($\mu = \infty$) the Stokes-flow region is up to $\text{Re} = 10$. On the other hand, the Stokes-flow region is up to $\text{Re} = 80$ in the case of a stationary outer sphere ($\mu = 0.0$). The perturbation solution is also presented for both cases. For $\mu = 0.0$ the experimental results agree well with the mentioned theoretical solution. However, this is not true in the case of $\mu = \infty$ where the perturbation solution diverges from the experimental results at a smaller Reynolds number.

It is clear from all the results we have so far that the dimensionless torque is independent of $\text{Re}$ at low Reynolds numbers due to weakness of the secondary flow. For larger
Figure 27. The torque curve up to $Re = 300$ when one sphere is rotating and the other is stationary, for $\eta = 0.304$.
Reynolds numbers the secondary flow becomes more important, and this leads to an increase in the torque.

Menguturk's (2) results for the three different gaps ($\eta = 0.304$, $0.44$ and $0.881$) show that the Stokes-flow region decreases as $\eta$ decreases. This is in agreement with the Stokes-flow approximation. However, in our experiment, we were not able to reach this conclusion, due to the difficulties that arose in measuring the torque at low Reynolds number; in the case of the larger gap, see Appendix A.
CHAPTER IV. INTERMEDIATE REYNOLDS NUMBERS

Introduction

In this chapter we discuss the flow between concentric rotating spheres at moderate Reynolds numbers. Most of the available work concerning flow between concentric spheres is concentrated on cases of low Reynolds numbers (low and high-order perturbation solutions) or of large Reynolds numbers (boundary layer or singular perturbation techniques). The solution of the nonlinear partial differential Equations 1 and 2 in the case of intermediate Reynolds numbers is the most complex. The only available solutions are numerical solutions obtained by difference methods (8, 9, 46) or direct (Galerkin-type procedure) methods (7, 47-49).

In the difference methods, the region between the two spheres is covered by a mesh with spacing $\Delta \theta$ and $\Delta r$ in directions of increasing $\theta$ and $r$ respectively. The governing equations are then approximated by difference equations. The solution for these difference equations is carried out numerically. For more details refer to reference (8).

The other method is to expand the functions $\psi$ and $\Omega$ in a Legendre polynomial series representation. The result is an infinite set of coupled equations. An appropriate truncation of the series results in a finite number of coupled ordinary differential equations which can be integrated.
numerically. Essentially the method is an appropriate Galerkin-type procedure used to reduce the original partial differential equations to ordinary differential equations. The method works well with the series truncated at a reasonable value, $N_t$. Two advantages of solving the original problem in this fashion (as opposed to a numerical integration of the original partial differential equations) are the following. For a given physical system (given $Re$, $\eta$, $\mu$) the computer time necessary to generate a solution in terms of the Legendre polynomial expansion is less than that required to integrate the original partial differential equations. Secondly, the results in terms of numerical values for the $r$ dependence and Legendre polynomial representation for the $\theta$ dependence are more convenient for theoretical stability considerations than a set of values of $\psi$, $\Omega$ at various mesh points throughout the $r$ and $\theta$ range (7).

Both methods (difference and direct) are rather cumbersome. The application of the difference method is related to a significant expenditure of computer time, while the application of the direct method requires a large number, $N_t$, of basis functions. This leads to algebraic systems of high orders ($\Psi$).

A third method intermediate between the difference and the direct methods is to expand the solution in terms of series in powers of the angle $\theta$. The coefficients in these
series are functions of the distance from the center, \( r \). An infinite number of coupled systems of ordinary differential equations of fourth-order with boundary conditions specified at the boundaries are generated. As is usual in the direct method, the system is cut off at some \( N \) systems of fourth-order and is solved by the adjustment method (50).

Character of the Flow Field

The case of a stationary inner sphere (\( \mu = \infty \))

As discussed in the previous chapter, for low Reynolds numbers the flow field is spherical in character. The flow in the meridian plane consists of a clockwise swirl. As the Reynolds number is increased, the flow field tends toward a typical two dimensionality (Taylor-Proudman) character (3). The sequence of photographs a-g in Figure 28 shows the flow pattern with a stationary inner sphere and a radius ratio of 0.44 for \( \text{Re} = 20, 50, 110, 150, 500, 900 \) and 2000 respectively. It is clear that the character of the flow field changes from spherical at low Reynolds numbers (\( \text{Re} = 20, 50 \)) to more cylindrical at higher Reynolds numbers (\( \text{Re} = 110 \) and up). For \( \text{Re} = 110 \) we notice the change not only from spherical to more cylindrical but also the formation of a recirculation zone (vortex in opposite direction to the main circulation) near the equator and in the neighborhood of the outside rotating sphere. As the Reynolds number increases this zone
Figure 78. The flow pattern with a stationary inner sphere for different Reynolds numbers, $\eta = 0.04, \mu = \infty$
Figure 28 (Continued)
expands towards the poles pushing the cylindrical surfaces (surfaces of equal angular velocities) towards the stationary inner sphere. The radius of the cylindrical surfaces, $r_{c_y}$, approaches the value of the radius of the inner sphere as Reynolds number increases as shown in the photographs (c-g). The rotation outside the cylinder is almost of solid body character (8, 44), much affected by the outer sphere. Regions of high velocity gradients are formed close to the cylinder, $r_{c_y} = R_1$, while boundary layers are formed on the sphere inside the cylinder (44).

Figure 29 shows the experimental results for $Re = 110$, $\eta = 0.44$ as compared with the third order truncation solution, $N_t = 3$ (3) for $Re = 100$, $\eta = 0.5$. This solution, which is indistinguishable from that of Pearson (8), still does not indicate the formation of a recirculation zone as is clear in the photograph. We note here the difference in both $\eta$ and $Re$ in favor of delaying the formation of the recirculation zone. That is, a smaller $\eta$ and a larger $Re$ for the experimental situation.

At $Re = 500$ the beginning of a small reverse flow region near the equator is indicated by the seventh-order truncation solution (3) for $\eta = 0.5$ as shown in Figure 30. This solution is compared with the experimental results for $\eta = 0.44$ which show a larger recirculation zone near the equator. Both results, experimental and theoretical, show the change to more
Re = 110, $\eta = 0.44$

Re = 100, $\eta = 0.5$

Figure 29. Comparison between the experimental results and the theoretical solution (3) with a stationary inner sphere $\mu = \infty$. 
Re = 500, \( \eta = 0.44 \)

Re = 500, \( \eta = 0.5 \)

Figure 30. Comparison between the experimental results and the theoretical solution (3) with a stationary inner sphere \( \mu = \infty \)
cylindrical character.

Figure 31 shows the experimental results for \( \eta = 0.44 \) together with the seventh-order truncation solution (3) and the difference solutions (8, 9), for \( \eta = 0.5 \). For Re = 1000 the three theoretical solutions show the formation of a small counterclockwise swirl near the equator. However, Greenspan (9) indicated that the streamline \( \psi = (20) \times 10^{-4} \) is, in fact, two disjoint closed curves as shown in Figure 31a. The seventh-order solution compared favorably with Pearson's (8) solution in many ways although the stream function was not described well by the seventh-order solution in some regions of the flow field. For this large Reynolds number it seems that higher order truncation is needed (3).

The sequence of photographs (a-g) in Figure 32 shows the flow pattern for a different radius ratio, \( \eta = 0.304 \). The photographs were taken with a stationary inner sphere and for Re = 20, 70, 100, 250, 400, 500 and 850. The flow in this case is similar to the previous case, \( \eta = 0.44 \), except that the change from spherical to more cylindrical character starts at lower Reynolds numbers in the case of \( \eta = 0.304 \) than that of \( \eta = 0.44 \). We note that the Reynolds number is based on the radius of the outer sphere \( R_2 \), and not the gap size \( R_2 - R_1 \). This means a larger gap for smaller radius ratio \( \eta \); thus "more freedom" for secondary flow.

Figure 33 (a, b) shows the flow pattern for \( \eta = 0.44 \) and
Figure 31. Comparison between the experimental results and the theoretical solutions (3, 8, 9) with a stationary inner sphere $\mu = \infty$. 

Re = 900, $\eta = 0.44$
Figure 32. The flow pattern with a stationary inner sphere for different Reynolds numbers, $\eta = 0.304, \mu = \infty$
d) \( \text{Re} = 250 \)

e) \( \text{Re} = 400 \)

f) \( \text{Re} = 500 \)

g) \( \text{Re} = 850 \)

Figure 32 (Continued)
Figure 33. Comparison of the flow pattern for two different radius ratios: $\eta = 0.44$ and $\eta = 0.304$, with the inner sphere stationary for $Re = 500$, $\mu = \infty$. 

\[ \text{a) } \eta = 0.44 \quad \text{b) } \eta = 0.304 \]
and $\eta = 0.304$ respectively. At $Re = 500$ the recirculation zone is larger in the case of $\eta = 0.304$ and the flow is more nearly cylindrical. The radius of the cylinder is smaller for $\eta = 0.304$, which indicates that the radius is approaching the inside sphere radius in both cases.

**The case of a stationary outer sphere ($\mu = 0.0$)**

The character of the basic flow field is much different from the previous case, $\mu = \infty$. No boundary layer or Taylor-Proudman effects are obtained as the Reynolds number is increased. The photographs a-d in Figure 34 show the flow pattern for a radius ratio $\eta = 0.44$ and for $Re = 125, 500, 750$ and 1000 respectively. The flow in all cases consists of one counterclockwise swirl. The swirl increases in strength and tends to be centered nearer the equator as Reynolds number is increased. At the same time the angular velocity contours lose their spherical character, but do not take on a cylindrical character as in the case of the rotating outer sphere (3).

Figure 35 shows that the center of the swirl approaches the equator as the Reynolds number increases. The figure shows the experimental results and other available theoretical results (3, 8). For low Reynolds numbers the secondary flow is very weak as compared with the primary flow. Thus, no change is expected in the shape of the counterclockwise
Figure 34. The flow pattern with a stationary outer sphere for different Reynolds numbers, \( \eta = 0.44, \mu = 0 \)
Figure 35. The center of the swirl approaches the equator as Reynolds number increases when the inside sphere is rotating alone.
swirl, as indicated by the perturbation solution. With the increase of Reynolds number the secondary flow is of more importance, and the inertia forces of the fluid close to the rotating sphere are greater. These forces increase as the fluid moves from the pole to the equator where the fluid is thrown from the inner sphere in a jet-like flow. The velocity of the fluid increases as the Reynolds number is increased. The fluid moves from a high velocity region to lower velocities as it continues the swirl. The continuity of the flow will produce a thinner layer nearer the equator as the Reynolds number is increased.

The experimental results are compared to the theoretical solutions (3, 8) in Figures 36, 37 and 38 for Re = 100, 400 and 1000 respectively. In all cases the experimental results are for \( \eta = 0.44 \) while the theoretical solutions were for \( \eta = 0.5 \).

Figure 39 shows the experimental results for low and high Reynolds number for a radius ratio \( \eta = 0.304 \). The flow pattern in this case is similar to the previous case (\( \eta = 0.44 \)).

With the two spheres rotating together

As was discussed in Chapter III, for low Reynolds numbers the rotation of the two spheres may produce two contrarotating secondary flow swirls depending on the parameters \( \mu \) and \( \eta \) (see Figure 20). The regions shown in Figure 20 apply only
Re = 125, \eta = 0.44

Re = 100, \eta = 0.5

Figure 36. Comparison between the experimental results and the theoretical solution (3) with a stationary outer sphere, \mu = 0.
Figure 37. Comparison between the experimental results and the theoretical solution (3) with a stationary outer sphere, $\mu = 0.0$.
Figure 38. Comparison between the experimental results and the theoretical solution (3,8) with a stationary outer sphere, $\mu = 0.0$, $Re = 1000$.
Figure 39. The flow pattern with a stationary outer sphere for low and high Reynolds numbers, \( \eta = 0.304 \), \( \mu = 0.0 \).
for cases of low Reynolds numbers. However, for higher Reynolds numbers the flow field will behave differently. For example, if at low Reynolds number two contrarotating swirls exist ($\eta = 0.5$, $\mu = -0.5$), then the increase of Reynolds number leads to a predominant increase in intensity of the counterclockwise swirl (dominant of the inner sphere). Figure 40 shows the two contrarotating swirls for the case of $\eta = 0.44$ and $\mu = -0.3$. On the other hand, if for small Reynolds numbers a single clockwise swirl exists ($\eta = 0.5$, $\mu = -1$), the increase of Reynolds number may lead to the formation of a small counterclockwise swirl (dominant of the inner sphere) near the polar region of the inner sphere. It also leads to the formation of a cylindrical sheath and a recirculation zone near the equator (characteristics of $\mu = \infty$).

If the two spheres rotate in the same direction, a single clockwise swirl is produced in the cases of $\mu > 1$. The swirl is of spherical character at low Reynolds numbers. As Reynolds number is increased the flow field tends toward a cylindrical character. The rotation of the inner sphere delays this tendency. That is, it will start at higher Reynolds number than that corresponding to a stationary inner sphere ($\mu = \infty$). As shown in Figure 41, for $Re = 100$, photograph a ($\mu = \infty$) shows the flow of more cylindrical character than that of photograph b ($\mu = 1.2$). We also notice
a) $Re = 30$

b) $Re = 100$

Figure 40. The flow when the two spheres are rotating in opposite directions with angular velocity ratio $\mu = -0.3$, but for different Reynolds numbers, $\eta = 0.4\mu$. 
Figure 41. Rotating the inside sphere affects the flow pattern resulting from the rotation of the outside sphere, $Re = 100$, $\eta = 0.44$.
Figure 42. The flow pattern when the two spheres are rotating in the same direction, and for different angular velocity ratios, $\eta = 0.4\mu$, $Re = 125$.
Figure 42 (Continued)
a formation of a recirculation zone near the equator in the case of \( \mu = \infty \).

The sequence of photographs (a-f) in Figure 4.2 shows the flow for different combinations of angular velocities \( \mu = 0.0, 0.3, 0.63, 0.8, 1.2 \) and 1.5. For photographs (a-d) where \( 0 < \mu < 1 \) the secondary flow consists of a single counterclockwise swirl of a spherical character. Photographs e and f where \( \mu > 1 \), the secondary flow consists of a single clockwise swirl.

**Torque Measurements Results**

As discussed in Chapter III, for low Reynolds numbers (negligible inertia effects) the torque is independent of Re. For larger Reynolds number, where the inertia effects and the secondary flow are of more importance, the torque increases as Reynolds number is increased as shown in Figures 4.3-4.5. The dimensionless torque is calculated from the experimental data using the form given in Equation 17. This is done for the cases of stationary inner sphere, stationary outer sphere and with both spheres rotating together. We will discuss the results for each case separately and compare them with other reported results (experimental and theoretical) when possible.
One sphere rotates while the other is stationary

Figure 43 shows the dimensionless torque, $\bar{M}$, vs. Reynolds number (Re = 300-1000) for a stationary inner sphere and $\eta = 0.44$ ($\Omega_0 = \Omega_2$). We note that this curve is a continuation of the torque curve shown in Figure 26. In both figures the curve increases continuously up to $Re \approx 575$. At this Reynolds number the curve changes its slope and the dimensionless torque remains constant (independent of Reynolds number) between $Re = 575$ and $Re = 775$. Flow visualization either by dye injection or aluminum flakes suspended in the fluid did not indicate any instability at this range of Reynolds number. However, the photograph taken for the flow at $Re = 500$ (Figure 28e) shows that in the recirculation zone near the equator there is a possibility that a small counterclockwise swirl is starting to build up in that region. The existence of this swirl will reduce the effect of the main secondary flow swirl. Since it is primarily the secondary flow that causes the torque to be greater than the low Re values, it is not surprising that such a change in the torque curve occurs near the Re for which the secondary flow pattern changes considerably.

Figure 44 shows the torque curve ($\Omega_0 = \Omega_1$) between $Re = 300$ and $Re = 950$ with a stationary outer sphere and for $\eta = 0.44$. The dimensionless torque in Figures 25 and 44 is continuously increasing up to $Re = 950$. Flow visualization did
Figure 43. The torque curve between Re = 400 and Re = 1000, with a stationary inner sphere for $\eta = 0.44$. 

\[ \nu = 30.74 \text{ cs} \]
\[ \eta = 0.44 \]
\[ \mu = \infty \]
\[ \Omega_0 = \Omega_2 \]
Figure 44. The torque curve between $Re = 300$ and $Re = 950$, with a stationary outer sphere for $\eta = 0.44$
not show any change in the behavior of the flow in this range. However, Menguturk's (2) results for the same case ($\eta = 0.44$, $\mu = 0.0$) show a break in the torque curve at $Re = 540$. At this break an unstable mode appears in the form of small spots or puffs of turbulence at the center of the secondary flow swirl. Also presented in Figure 44 is the torque calculated by the Legendre polynomial series solution (3). This solution shows an error of up to 16% at some points.

The torque was also measured for a different sized gap, $\eta = 0.304$. The results shown in Figure 45 are for a Reynolds number range 500-2000. For both cases, when the inner sphere is stationary ($\mu = \infty$) or when the outer sphere is stationary ($\mu = 0$), the torque curves are very similar in character. The dimensionless torque is independent of Reynolds number, where it stays constant between $Re \approx 800$ and $Re \approx 1050$. It is believed that this case is similar to the previous one ($\eta = 0.44$). Menguturk's (2) results for $\mu = 0$ do not indicate any break in the torque curve for $Re < 4500$.

Both spheres are rotating

The dimensionless torque in this case was determined also by using Equation 17 with the characteristic angular velocity $\Omega_\circ = \Omega_1$ (the angular velocity of the inner sphere), Figures 46, 47 and 50; or $\Omega_\circ = \Omega_1 - \Omega_2$ (difference between
Figure 45. The torque curve between Re = 500 and Re = 2000, when one sphere is rotating and the other is stationary, for η = 0.304
the angular velocities of the two rotating spheres, Figures 48, 49 and 51. During the experiment the angular velocity of the inner sphere was kept constant and the data points were taken for different outer sphere angular velocities with an angular velocity range; $-2.25 < \mu < 2.25$. The Reynolds number is defined as

$$\text{Re} = \frac{\Omega_1 R_2^2}{v}.$$ 

The experiment was repeated for different Reynolds numbers.

Figure 46 shows the dimensionless torque vs. the angular velocity ratio $\mu$ for $\text{Re} = 100$. The angular velocity of the inner sphere is taken as the characteristic angular velocity and held constant during the experiment. This will make the dimensionless torque proportional (to some degree) to the difference in the angular velocities $\Omega_1 - \Omega_2$. For $0 < \mu < 1$ this difference decreases with the increase of $\mu$ producing a decrease in the measured dimensionless torque. For $\mu = 1$ (case of rigid rotation) the torque measured is zero. Increasing the speed of the outer sphere (that is for $\mu > 1$) the difference in the angular velocity $\Omega_1 - \Omega_2$ is increasing producing a rise in the measured torque. For $\mu = 0$ the value of the torque is the same as given in Figure 25 for a stationary outer sphere $\mu = 0.0$. The figure also shows the second-order perturbation solution which shows an agreement.
Figure 46. The torque as a function of the angular velocity ratio, for $\eta = 0.44$, $Re = 100$

\[ Re = \frac{\Omega_1 R^2}{\nu} \]
\[ \Omega_0 = \Omega_1 \]
\[ \eta = 0.44 \]
with the experimental data for $-1.5 < \mu < 1.5$. The Stokes-flow solution is also shown in the same figure. For $\mu < 0$ we expect that the fact of having two contrarotating swirls (in the range of $-0.53 < \mu < -0.23$ in this specific case) will reduce the torque in that range. This is shown theoretically by the perturbation solution. That is, the difference between the $Re = 100$ and $Re = 0$ curves is due to secondary flow effects. In the range $-0.53 < \mu < -0.23$ the two curves nearly coincide. This is more clearly shown in Figure 47 where the torque drops slightly in the range where the two contrarotating swirls exist. At a Reynolds number of 100 the secondary flow is still of little importance so the drop is very small compared with larger Reynolds numbers cases. In Figure 48 where $\Omega_0 = \Omega_1 - \Omega_2$ (that is, the dimensionless torque is based on shear effects only), the dimensionless torque is expected to remain constant at low Reynolds number. However, the rising of the torque curve in Figure 48 ($Re = 100$) when $\mu > 0$ is felt to be due to the secondary flow effects caused by the rotation of both spheres. For $\mu < 0$ the difference in the angular velocities is increasing with the decreasing of $\mu$. If the effective Reynolds number is thought to be based on this difference, then the decreasing of $\mu$ means an increase in the effective Reynolds number. Consequently, the secondary flow is of more importance and the torque increases.
Figure 47. The torque curve as a function of the angular velocity ratio, for different Reynolds numbers, for $\eta = 0.44$.

\[ \text{Re} = \frac{\Omega_1 R_2^2}{\nu}, \quad \eta = 0.44 \]

$\Omega_0 = \Omega_1$

- $\circ$ Re = 900
- $\triangle$ Re = 600
- $\square$ Re = 250

Equation for Reynolds number:

\[ \text{Re} = \frac{\Omega_1 R_2^2}{\nu} \]
Figure 48. The torque curve as a function of the angular velocity ratio for different Reynolds numbers, for $\eta = 0.44$.
For the cases of \( Re = 250 \) and \( Re = 600 \), the behavior of the torque curves in Figures 47 and 48 can be explained in a similar way. It is shown in both figures that when the spheres rotate in opposite directions the dimensionless torque decreases with the increase of \( \Omega_2 \), that is, the decreasing of \( \mu \) (in the range \(-0.36 < \mu < 0\) for \( Re = 250 \) and \(-0.46 < \mu < 0\) for \( Re = 600 \)). The region close to the outside sphere is affected by its rotation and in this region a clockwise swirl is produced which will reduce the effect of the main counterclockwise swirl (dominant of the inner sphere) on the torque measured. As the clockwise swirl grows in size (with the increase of \( \Omega_2 \)) the dimensionless torque will drop until it reaches its minimum where the two contrarotating swirls are of approximately the same strength. Increasing \( \Omega_2 \) will strengthen the clockwise swirl and finally diminish the counterclockwise one. This of course will mean an increase in the dimensionless torque as \( \Omega_2 \) is increased. After that the behavior of the torque curve will be similar to that of a stationary inner sphere, \( \mu = \infty \).

An interesting case occurs when \( Re = 900 \), for which the torque curve shows a change in the behavior from that for lower Reynolds number, especially for \( 0 < \mu < 1 \). This is shown in Figure 49, where for \( Re < 900 \) the dimensionless torque increases with the increase of \( \mu \) for \( 0 < \mu < 1 \). Here the characteristic angular velocity is \( \Omega_0 = \Omega_1 - \Omega_2 \). For \( Re = 900 \)
the torque curve drops up to $0 < \mu < 0.3$ and starts to rise again (similar to the cases of lower Reynolds numbers). We note here that for higher Reynolds number, $Re > 1000$, the torque curve did not show any rise in that region (see Figures 49 and 62).

In Figures 50 and 51 the results for larger sized gap $\eta = 0.304$ are presented. In Figure 50 we see that the perturbation solution and the experimental results agree well in the range $-1.5 < \mu < 1.5$. From Figure 51 it can be seen that the effect of the two contrarotating swirls ($-0.38 < \mu < -0.12$) was not clear at this low Reynolds number.
Figure 49. The torque curve as a function of the angular velocity ratio for different Reynolds numbers. It shows the change in the shape of the torque curve at Re = 900.

\[ \eta = \frac{0.44}{\text{Re}} \]

\[ \text{Re} = \frac{\Omega_1 R_2^2}{\nu} \]

\[ \Omega_0 = \Omega_1 - \Omega_2 \]
Figure 50. The torque curve as a function of the angular velocity ratio for $\gamma = 0.304$, $Re = 100$
Figure 51. The torque curve as a function of the angular velocity ratio for \( \eta = 0.304 \), \( \text{Re} = 100 \).
CHAPTER V. LARGE REYNOLDS NUMBERS

Introduction

This chapter deals with the flow between concentric spheres at large Reynolds numbers. It also includes the stability of the basic laminar flow and consideration of turbulent flows. For this range of Reynolds numbers the theoretical work is primarily of a boundary-layer or singular-perturbation character (10-16).

It is known that for large Reynolds numbers the established laminar flow becomes unstable. This problem also arises with the study of flow in a spherical annulus. Besides the natural necessity of identifying the critical Reynolds numbers, the study of stability of the flow in a spherical annulus also has purely hydrodynamic interest for two reasons (44):

1) The flow occurs in a closed region, in contrast to a flow in a region of infinite extent, such as plane and cylindrical Couette and Poiseuille flow.

2) The nature of the basic flow depends on Reynolds number, which is also unusual for this type of problem.

The stability of the basic laminar flow in a spherical annulus has been studied either by applying the energy theory (17-20, 22-24), or the linear stability theory (2, 25-30).
The energy theory considers the time rate of change of the kinetic energy of an arbitrary disturbance in the flow field. If the kinetic energy decreases as a function of time, the flow is stable. The theory provides a critical Reynolds number, $Re_E$, such that the flow is definitely stable to all disturbances if $Re < Re_E$.

The linear theory of hydrodynamic stability considers only infinitesimal disturbances and provides a critical Reynolds number, $Re_L$, such that the basic flow is definitely unstable for $Re > Re_L$. It can say nothing concerning the possibility of instability caused by disturbances of finite size. On the other hand, the energy theory considers any size disturbance (large or small). The stability of the basic flow in a spherical annulus is strongly dependent on the radius ratio, $\eta$, and the angular velocity ratio, $\mu$. The dependence on the radius ratio is not only quantitative, but qualitative as well. In particular, the transitions involved in the wide-gap situation do not involve the Taylor-type vortices of the narrow-gap case and are much less drastic and more difficult to detect \( (44) \). When both spheres rotate, the flow field may consist of a turbulent circular cylinder region about the axis of rotation surrounded by a laminar flow region. For a given radius ratio, the sizes of the two regions depend on the angular velocity ratio $\mu$, and the Reynolds number $Re$. 
The experimental results will be presented under three categories:

1) Stationary inner sphere (\( \mu = \infty \))
2) Stationary outer sphere (\( \mu = 0.0 \))
3) Both spheres are rotating.

The results are compared with other available results; theoretical and experimental, when it is possible.

The Case of a Stationary Inner Sphere (\( \mu = \infty \))

As indicated before for a stationary inner sphere the secondary flow circulation tends towards a cylindrical sheath of radius approximately equal to \( R_1 \), and a recirculation zone develops near the equator, as the Reynolds number is increased. In Figure 31a it is seen that for \( Re = 1000 \) the streamline \( \psi = (20) \times 10^{\frac{1}{4}} \) is two disjoint closed curves. For \( Re > 1000 \) not only does this phenomenon continue to develop in the central portion of the region, but it also appears in the equatorial region as shown in Figure 52b. For \( Re = 3000, \eta = .5 \) the streamline \( \psi = 1 \times 10^{\frac{1}{4}} \) is, in fact, two disjoint closed curves. The size of the counterclockwise swirl near the equator is larger than for cases of smaller Reynolds numbers (see Figure 31a). The photograph (a) in Figure 52 is for a radius ratio \( \eta = 0.44 \) and \( Re = 2000 \). It shows the cylindrical character of the flow.

In Figures 53 and 54 the dimensionless torque is plotted
Figure 52. Comparison between the experimental results and the theoretical solution (9) with a stationary inner sphere, $\mu = \infty$
Figure 53. The torque curve between Re=1000 and Re=2200 with a stationary inner sphere, for $\eta = 0.44$.

- $\nu = 30.74$ cs
- $\nu = 18.44$ cs
- $\nu = 9.34$ cs

$\eta = 0.44$, $\mu = \infty$, $\Omega_0 = \Omega_2$
Figure 54. The torque curve between Re=2500 and Re=10000 with a stationary inner sphere, for η = 0.44
as a function of Reynolds number. Figure 53 shows that the
dimensionless torque increases with the increase of Reynolds
number between Re = 1000 and Re = 2200. For 2300 < Re < 3750
this increase continues as shown in Figure 54. The dimen­sionless torque then remains constant for 3750 < Re < 4800.
As shown there is a break in the torque curve at Re = 4800,
where the dimensionless torque starts to increase with the
increase of Reynolds number. Flow visualization using
aluminum flakes did not show any transition in the flow
field at this point. In fact the flow visualization did not
show any transition or instability in the flow at all in the
Reynolds number range considered (up to 104). The break in
the curve is felt to be due to the growing importance of the
recirculation zone and the forming of a counterclockwise
swirl near the equator. Munson and Joseph (21) found in the
case of η = 0.5 the critical Reynolds number Re_E = 440 and
Re_E = 750 for η = 0.75. The only available experimental
work for the case of a stationary inner sphere to the
author's knowledge is that of Nakabayashi (33, 34), in which
he calculated the coefficient of viscous frictioned moment,
C_M, for a narrow gap situation (0.829 ≤ η ≤ 0.989). The relation between C_M and the Reynolds number for those narrow gap
situations shows the existence of four possible modes
(regimes) of flow. As noted previously, the narrow gap and
wide gap situations are qualitatively and quantitatively
different.

In Figure 55 the torque results for a radius ratio $\eta$ of 0.304 are presented. The dimensionless torque is shown as a function of Reynolds number between $Re = 5000$ and $Re = 11000$. The dimensionless torque increases with the increase of Reynolds number. The flow did not show any transition or instability, either by a break in the torque curve, or during flow visualization using aluminum flakes.

Other cases with different radius ratios 0.2, 0.39, 0.59 and 0.881 were examined visually by using aluminum flakes. For a stationary inner sphere the flow did not show an instability up to $Re = 10000$, the maximum investigated.

In their review for the flow in spherical layers, Yavorskaya and Astaf'yeva (44) concluded that the flow generated by rotation of the outer sphere only ($\mu = \infty$) is always stable. The experimental results presented here tend to support this conclusion.

The Case of a Stationary Outer Sphere ($\mu = 0.0$)

Most of the experimental investigations regarding the flow between two concentric spheres have concentrated on the case of a stationary outer sphere. Zierep and Sawatzki (30) studied the case of a narrow gap. They found that the flow becomes turbulent at $Re = 53,500$ and $158,000$ for $\eta = 0.956$ and 0.975 respectively. For a narrow gap it was noticed that
Figure 55. The torque curve between Re=5000 and Re=11000 when one sphere is rotating and the other is stationary, for $\eta = 0.304$
after passing the critical Taylor number, $Ta = 41.3$, Taylor vortices develop close to the equator. For those narrow gap situations the instability occurs very similarly to that in the case of rotating cylinders. In fact the critical Taylor number for the spherical annulus and the cylindrical annulus is essentially the same. The Taylor number is defined here as:

$$\sqrt{Ta} = \frac{R^2 \Omega_1 (1 - \eta)}{v \sqrt{\eta (1 - \eta)}}$$

The axis of these vortices have a spiral form and end free in the flow field. From the end of the vortices up to the poles the laminar flow remains stable. With the increase of the Reynolds number the axes of the vortices become wavy and the flow becomes turbulent after passing some intermediate states. Similar results were obtained by Menguturk (2) for a radius ratio $\eta = 0.881$, where the flow becomes turbulent between $Re = 10,000$ and $Re = 14,000$.

For a medium sized gap situation the dimensionless torque is shown in Figures 56 and 57. The results are for $\eta = 0.44$ and a stationary outer sphere. Figure 56 shows that the torque increases with an increase of Reynolds number between $Re = 800$ and $Re = 1850$. Figure 57 shows the torque curve between $Re = 2000$ and $Re = 10000$. The dimensionless torque is constant between $Re = 2750$ and $Re = 4000$. Flow visualization showed that the flow becomes turbulent at
Figure 56. The torque curve between Re = 800 and Re = 2000, with a stationary outer sphere, for \( \eta = 0.44 \).
Figure 57. The torque curve between $Re = 2000$ and $Re = 10000$ with a stationary outer sphere, for $\eta = 0.44$. 

$v = 9.34$ cs
$v = 5.59$ cs
$\eta = 0.44$
$\mu = 0.0$
$\Omega_o = \Omega_1$
Re \approx 3800. In this situation the transition to turbulent occurs directly from the basic laminar flow without an intermediate Taylor vortex type laminar flow involved. For 4000 < Re < 6000 the dimensionless torque increases as the Reynolds number is increased, while it remains constant between Re = 6000 and Re = 8500. At Re \approx 8000 the turbulent flow breaks down into a finer structure. It is felt that the break in the torque curve at Re \approx 8500 is due to this change in the flow structure.

The above results for the case of \( \eta = 0.44 \) disagrees with Menguturk's (2) results for the same case. Menguturk found that the flow suddenly becomes turbulent at Re = 2300.

Morales-Gomez (32) found the critical Reynolds number for the cases when \( \eta = 0.512, 0.587, 0.647, 0.742 \) and 0.889 to be \( Re_{cr}^* = 1953, 2577, 2527, 3070 \) and 1253 respectively. We note here that \( Re_{cr}^* \) is the value of Reynolds number at which the laminar flow loses its stability.

The torque curve in Figure 55 shows the dimensionless torque as a function of Reynolds number for the case of \( \eta = 0.304 \) and \( \mu = 0 \). For 5000 < Re < 7600 the dimensionless torque increases with the increasing of Reynolds number. The dimensionless torque remains constant between Re = 7600 and Re = 10200, where it starts to increase again. Flow visualization with the use of aluminum flakes shows the flow becomes suddenly turbulent at Re \approx 7800.
Table 3. The critical Reynolds numbers for different gap sizes

<table>
<thead>
<tr>
<th>Sphere</th>
<th>$\eta$</th>
<th>$Re_{cr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.200</td>
<td>44000</td>
</tr>
<tr>
<td>B</td>
<td>0.304</td>
<td>7800</td>
</tr>
<tr>
<td>C</td>
<td>0.390</td>
<td>4290</td>
</tr>
<tr>
<td>D</td>
<td>0.440</td>
<td>3800</td>
</tr>
<tr>
<td>E</td>
<td>0.590</td>
<td>4590</td>
</tr>
<tr>
<td>F</td>
<td>0.881</td>
<td>4900</td>
</tr>
</tbody>
</table>

Figure 58 shows the critical Reynolds number, $Re_{cr}$, as a function of the radius ratio $\eta$. Here $Re_{cr}$ is the value of Reynolds number at which the laminar flow changes to turbulent. Six spheres of different sizes were used to determine this relation. The critical Reynolds numbers for each case is shown in Table 3.

It is clear from Figure 58 that for $\eta = 0.46$ the flow becomes turbulent at Reynolds number lower than any other radius ratio situation. We denote the radius ratio at this point $\eta^*$. In thin layers, $\eta > \eta^*$, flow becomes turbulent at sufficiently large Reynolds number. The thinner the layer the larger the critical number, and the flow is more stable. On the other hand, for thicker layer, $\eta < \eta^*$, the thicker
Figure 58. Critical Reynolds number as a function of the radius ratio $\eta$, for $\mu = 0.0$
the layer the larger the critical Reynolds number and the flow is more stable.

The Case When Both Spheres are Rotating Together

In this section we discuss the case of almost rigid rotation as well as the torque measurements results for other combinations of angular velocities (-2 < \( \mu < 2 \)). Also we determine the criteria governing whether the flow is laminar or turbulent.

In the case of almost rigid rotation, the difference in the angular velocity of the two spheres is small
\[ \Omega_1 = (1 - \varepsilon) \Omega_2, \text{ and } |\varepsilon| << 1 \]. In such a case the governing equations can be appropriately linearized in a manner that is independent of the Reynolds number. Proudman (11) has considered this case and has obtained the solution in terms of an Ekman-layer on each sphere, a cylindrical sheath layer with a radius equal to the radius of the inner sphere, and an essentially inviscid core region elsewhere. Outside the cylinder with radius of that of the inner sphere, the fluid rotates as a rigid body with the same angular velocity of the outer sphere. Inside the cylinder the velocity distribution in the inviscid core is determined by the velocity distribution in the Ekman-layers on the spheres. The cylindrical surface is a singular surface in which velocity gradients are very large. The structure of the cylindrical
shear layer is very complex and not fully understood at present (3). Thus, except for the Ekman-layer on each sphere, the flow is essentially vertical, that is, in the direction of the axis of rotation. Therefore, except for these viscous layers, a superposition of a solid body rotation \( \bar{\Omega}_o \), such that the inner sphere rotates at \((1+\varepsilon)\Omega_2 + \bar{\Omega}_o\) and the outer at \(\Omega_2 + \bar{\Omega}_o\), should have little effect on the flow field. Thus, perhaps except for secondary effects in the boundary layer the general flow field should have the same character whether the spheres are rotating at \((1-\varepsilon)\Omega_2 + \bar{\Omega}_o\) and \(\Omega_2 + \bar{\Omega}_o\), provided that the Reynolds number is large enough to obtain the "cylindrical sheath" character (3).

Figure 59 shows the flow field when both spheres rotate with slightly different angular velocities (\(\mu \approx 0.9\)), and for a radius ratio \(\eta\) of 0.44. For \(Re = 50\) the character of the flow is spherical and no shear layer is formed. For \(Re = 335\) the flow is of a more cylindrical character. The formation of the cylindrical shear layer is obvious at \(Re = 3000\) as shown in Figure 59c for an angular velocity ratio \(\mu\) of 0.94.

Figure 60b shows the general character of Proudman's (11) solution, while the photograph a shows the experimental results for the case of almost rigid rotation at \(Re = 3000\).

The dimensionless torque as a function of the angular
Figure 59. The flow when the two spheres are rotating in the same direction with almost the same angular velocity, for \( \eta = 0.44 \)
Figure 60. The general character of Proudman's solution (11) for the case of almost rigid rotation, and the experimental results for this case.
velocity ratio \( \mu \) is shown in Figures 61 and 62, for \( \text{Re} = 1000, 2000 \) and 5000. In Figure 61 the angular velocity of the inner sphere is taken as the characteristic angular velocity \( (\Omega_0 = \Omega_1) \). While in Figure 62 the difference in the angular velocities of the two spheres is taken to be the characteristic angular velocity. The curves for the three Reynolds numbers are similar in character. However, as the Reynolds number is increased the effect of the contrarotating secondary flow swirls is greater, and for a larger range of velocity ratio. This is clear in Figure 62. For \( \text{Re} = 1000 \) the dimensionless torque decreases with the decrease of \( \mu \), when both spheres are rotating in opposite directions, and for \( -0.5 \leq \mu < 0 \). The dimensionless curve reaches its minimum at \( \mu = -0.5 \). In the other cases (\( \text{Re} = 2000 \) and 5000) the minimum is at \( \mu = -0.6 \) and \(-0.85\) respectively. We recall this case at intermediate Reynolds numbers discussed in the preceding chapter. The minimum was at \( \mu = -0.36, -0.46 \) and \(-0.48 \) for \( \text{Re} = 250, 600 \) and 900 respectively (see Figures 48 and 49). In all cases presented in Figures 61 and 62 the dimensionless torque increases with the decreasing of \( \mu \) (for negative \( \mu \)) after reaching its minimum as mentioned above.

When both spheres rotate in the same direction the dimensionless torque decreases with the increasing of \( \mu \) for \( 0 \leq \mu \leq 1 \). For \( \mu > 1 \) the dimensionless torque increases with
Figure 61. The torque curve as a function of the angular velocity ratio for different Reynolds numbers, for $\eta = 0.44$

Reynolds number equation:

$$Re = \frac{\Omega R^2}{\nu}, \quad \eta = 0.44$$

- $\Omega_0 = \Omega_1$
- $\bullet$ Re = 5000
- $\triangle$ Re = 2000
- $\bullet$ Re = 1000

Parameter $\mu$ is also plotted on the x-axis.
Figure 62. The torque as a function of the angular velocity ratio for different Reynolds numbers, for $\eta = 0.44$. 

The diagram shows the relationship between the torque and the angular velocity ratio for different Reynolds numbers ($Re = 5000$, $Re = 2000$, $Re = 1000$), with $\eta = 0.44$. The equation $Re = \frac{\Omega_1 R_2^2}{\nu}$ is also depicted, where $\Omega_0 = \Omega_1 - \Omega_2$. The diagram includes a variety of data points and curves illustrating the torque values for each Reynolds number and angular velocity ratio. The vertical axis represents the torque values, and the horizontal axis represents the angular velocity ratio ($\mu$).
the increase of $\mu$, as was explained in the previous chapter for $Re = 600$ and $Re = 900$.

As was discussed before, with the inner sphere rotating alone the flow becomes turbulent at sufficiently high Reynolds number, depending on the radius ratio $\eta$ as shown in Figure 58. However, it is possible to transform some or all of the turbulent flow region into laminar flow by rotating the outer sphere. We define two Reynolds numbers for this flow:

$$Re_1 = \frac{\Omega_1 R_2^2}{v}$$

based on the inner sphere angular velocity and

$$Re_2 = \frac{\Omega_2 R_2^2}{v}$$

based on the outer sphere angular velocity. The nature of the flow (laminar or turbulent) was obtained by flow visualization techniques using aluminum flakes suspended in the fluid. These techniques show that the flow may be either entirely laminar, entirely turbulent or some combination of laminar and turbulent flow existing side by side. The photographs (a-c) in Figure 63 show the regions of turbulent and laminar flows and the cylindrical boundary between them for a radius ratio $\eta = 0.304$. For $Re_1 = 7800$ the flow is entirely turbulent at $Re_2 = 0$. In photograph b, where the
a) $Re_2 = 0.0$, $\mu = 0.0$, $L/R_2 = 1$

b) $Re_2 = -3420$, $\mu = -.45$, $L/R_2 = 0.54$

c) $Re_2 = -4290$, $\mu = -.55$, $L/R_2 = 0.48$

Figure 63. The turbulent region as a function of the angular velocity ratio when the two spheres are rotating in opposite directions, $\eta = 0.3:4$ and $Re_1 = 7800$
outer sphere was rotating in an opposite direction with $Re_2 = -3420$, the flow field consists of a turbulent circular cylinder region of radius $\ell$ about the axis of rotation surrounded by a laminar flow region. Here $\frac{\ell}{R_2} = 0.54$. Photograph c is for $Re_2 = -4290$, it shows that the turbulent region is smaller than in photograph b. That is, $\ell/R_2$ is smaller than for the situation in photograph b. Similarly if both spheres rotate in the same direction, the turbulent and laminar flow regions may exist together separated by a cylindrical boundary as shown in Figure 64. Inside the cylindrical boundary the flow is turbulent and outside it the flow is laminar. Figure 65 shows that for a given value of $Re_1$, an increase in $Re_2$ causes the size of the turbulent region to shrink; that is, a decrease in $\ell/R_2$.

Figure 66 shows similar results for a radius ratio $\eta = 0.44$. With the outer sphere stationary the flow becomes turbulent at $Re = 3800$ where $\ell/R_2 = 1$. Rotating the outer sphere while holding $Re$, at 3800 results in transforming some of the turbulent flow region into laminar flow. Increasing $Re_2$ causes the size of the laminar region to grow and finally the flow becomes completely laminar at certain $Re_2$. The experiment was repeated for different values of $Re_1$ ($Re = 5300, 6700$ and 8000). The results are shown in Figure 66.

The reduction in the size of the turbulent flow region
\[ \mu = 0.65, \quad \frac{L}{R_2} = 0.51 \]

Figure 64. The turbulent region also will be a function of the angular velocity if the two spheres rotate in the same direction, \( Re_1 = 7500 \)
Figure 65. The size of the turbulence region as a function of Reynolds number based on the angular velocity of the outside sphere, while the inside sphere rotates at constant angular velocity.
caused by rotation of the outer sphere can be explained by consideration of the distribution of the angular momentum of the fluid within the annulus. A somewhat similar situation occurs in the Taylor vortex type instability found in the rotating cylinder situation. The inviscid Rayleigh stability criteria states that the flow between rotating cylinders is stable if and only if the angular momentum increases monotonically outwards \((41)\). Hence the flow is expected to be inviscidly unstable if only the inner cylinder rotates and inviscidly stable if only the outer cylinder rotates. Such reasoning carries over to the spherical annulus situation. This is shown by the fact that for large enough Reynolds numbers the flow becomes turbulent if \(\mu = 0\), but the flow remains laminar for \(\mu = \infty\).

For the rotating cylinder geometry, if the cylinders rotate in opposite directions it is possible to have a portion of the gap between the cylinders "stable" and a portion "unstable". In such a situation the angular momentum increases outward over a portion of the gap, while it decreases outward over the remainder of the gap. Hence a region of instability (Taylor vortices) occurs next to a region of stability (no Taylor vortices).

It is felt that this same mechanism works to produce the stabilization of the turbulent flow in a spherical annulus when the angular velocity of the outer sphere is increased.
Figure 67 shows the different regions of flow for a radius ratio of 0.44. In region I the flow is completely laminar, in region II the flow is completely turbulent, and region III the flow is partly turbulent and partly laminar. The curve for 50% turbulent is also shown in the figure. This curve was produced from Figure 66 for $l/R_2 = 0.5$ and for the different values of $Re_1$ considered.

The results shown in Figures 68 and 69 are for $\eta = 0.304$, 0.39, 0.44 and 0.59. For each case the region above the curve is that for which the flow is completely turbulent, and the region below it is either partially or fully laminar. The general shape of the curve dividing these regions is very similar to the stability boundary obtained for the rotating cylinders (41), Figure 70.

All the cases considered are for a relatively wide gap situation, where the transition from the basic laminar flow to turbulent flow may occur directly without an intermediate Taylor vortex type laminar flow involved as in the case of small gap width ($\eta \approx 1$). For a small gap width, the transition from basic laminar flow to turbulent flow is very similar to that obtained in the flow between rotating cylinders. That is, the original instability occurs in the form of Taylor vortices which at large enough Reynolds numbers break down into turbulent (2).
Figure 66. The size of the turbulence region as a function of Reynolds number based on the angular velocity of the outside sphere for different inside sphere's angular velocities, for $\eta = 0.44$. 

$$
\text{Re}_1 = \frac{\Omega_1 R_2}{\nu} \\
\text{Re}_2 = \frac{\Omega_2 R_2}{\nu}
$$
Figure 67. The region of turbulent and laminar flows for a radius ratio $\eta = 0.44$. 

- **I**: Laminar
- **II**: Partially Turbulent
- **III**: Completely Turbulent

Re$_1$ and Re$_2$ are the Reynolds numbers for regions 1 and 2, respectively.
Figure 68. Regions of laminar and turbulent flows for different radius ratios

\[ \text{Re}_1 = \frac{\Omega_1 R_2^2}{\nu} \]

\[ \text{Re}_2 = \frac{\Omega_2 R_2^2}{\nu} \]
Figure 69. The regions of laminar and turbulent flows for different radius ratios

\[ \Re_1 = \frac{\Omega_1 R_2^2}{\nu}, \quad \Re_2 = \frac{\Omega_2 R_2^2}{\nu} \]

- $\Re = 0.59$
- $\Re = 0.39$
Two concentric rotating cylinders, the outside rotates with an angular velocity $\Omega_2$ and the inside rotates with $\Omega_1$.

Figure 70. The region of stability and instability for two concentric rotating cylinders (41)
CHAPTER VI. CONCLUSION

During this investigation we were able to study the viscous incompressible flow between rotating spheres for a medium-sized gap, when one of the spheres is rotating or when both are rotating together. The previous experimental works concerning this kind of flow are concentrated on the case of a stationary outer sphere, except that of Nakabayashi (33, 34) where he measured the frictional torque for a thin gap situation and with a stationary inner sphere.

With the inside sphere stationary ($\mu = \infty$) the flow visualization results show the character of the flow pattern is as previously predicted theoretically. The photographs taken for the flow at different Reynolds numbers show that the flow is very spherical in character at low Reynolds numbers, while for large Reynolds numbers the flow becomes very two-dimensional in character. In this case the flow did not show any instability or transition to turbulence over the range of Reynolds number under consideration (up to $10^5$). The torque needed to rotate the outer sphere is presented.

In the case of a stationary outer sphere ($\mu = 0$) the transition to turbulent may occur directly from the basic laminar flow. The critical Reynolds number $Re_{cr}$, for which the flow is fully turbulent if $Re > Re_{cr}$, is presented. The results show that for $\eta > 0.46$ the smaller the gap the higher
Re_{cr} is, while for \( \eta < 0.46 \) the larger the gap the higher Re_{cr} will be. The torque needed to rotate the inside sphere is presented in a nondimensional form and plotted versus the Reynolds number. The transition to turbulent from laminar in this case was also shown by a break in the torque curve.

When both spheres rotate, the results show that the flow may be either entirely laminar, entirely turbulent or some combination of laminar and turbulent flow existing side by side. In this case the flow field consists of a turbulent circular cylinder about the axis of rotation surrounded by a laminar region. The size of these regions depends on the values of the angular velocities of the spheres. The torque needed to rotate the spheres in this case is presented for different values of Reynolds number and angular velocity rates. The presence of the two contrarotating secondary flow swirls at a certain range of angular velocity ratio affects the value of the measured torque in that range.

Results concerning the development of the basic laminar flow in the mentioned cases and the almost rigid rotation are also presented.
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APPENDIX A. THE EXPERIMENTS

In this appendix we present the details of the experimental apparatus, and other related topics such as the fluids used and their viscosities, determination of the spring constants of the torsional wires, and the derivation of the equations used in connection with the experiments.

The Experimental Apparatus

1. The spheres and the plexiglass assembly

The apparatus used in the experiments is illustrated in Figures 6, 71, 72 and 73. It consists of two spherical surfaces between which the flow takes place. The outer shell is made of two identical hemispheres built separately of a clear casting resin, 0.5 inch (1.27 cm) thickness, 5.1 inch (12.95 cm) diameter, and fastened together by means of a plexiglass bracket to complete the outer sphere. A small hole was drilled through the top of the outer sphere for entry of the shaft carrying the inner sphere. A hub was glued to the bottom of the outer sphere where a telescopic shaft was screwed to it. The shaft leads to a variable speed motor through a belt-gear mechanism.

The inner sphere is attached to a variable speed motor by means of a shaft. The motor is mounted on \( \Gamma \)-shaped bracket which is connected to the inside sleeve of an air
bearing designed to keep the inner sphere and its motor lined with the axis of rotation of both spheres. The assembly of the inner sphere, the motor and its bracket, and the inside sleeve is suspended from the ceiling by a torsion wire of known spring constant.

The two spheres are contained within a plexiglass box which is filled with the same fluid used between the two spheres. This helps make the visualization of the flow more reliable by reducing optical refractions. A hole was drilled in the bottom of the box to install a bearing through which the telescopic shaft passes, and an oil seal to prevent leakage as shown in Figure 71.

2. The mercury switch

To eliminate the unwanted torque generated from the twisting of the electrical cord for the power input of the inner sphere motor, a mercury switch was installed as shown in Figure 72. The power supply cord consists of five separate wires. These five wires are placed into five separate small mercury pools lined up with the axis of rotation. The other end of the wire is glued to the bottom of the mercury pool and is in contact with the mercury. This end leads to the motor controller. The use of the mercury switch allows the inner sphere assembly to swing freely without any restriction. Consequently the torque transmitted to the torsional wire will be purely due to the rotation of the
Figure 71. Plexiglass assembly
Figure 72. The mercury switch
spheres. Each of the mercury pools, 0.75 in (1.91 cm) diameter and 0.75 in (1.91 cm) depth, is contained in a plexiglass holder. The holders are glued to a plexiglass bracket, which is attached to the main frame and can be adjusted so that the pools are lined up with the wires. The wires are supported by five holders, carried by a rectangular aluminum frame. This frame is placed between the air bearing and the torsional wire.

3. The air bearings

The air bearings were designed to keep the inner sphere assembly lined up with the axis of rotation and prevent it from vibrating or slanting. Figure 73 shows the two air bearings used in the experiment. Each bearing consists of an inner and outer sleeve with a clearance of 0.020 inch (0.05 cm). The outer sleeve is surrounded by an air chamber where the pressure was kept at about 30 psi. The air is injected into the clearance between the two sleeves through a set of four nozzles, 0.02 inch (0.05 cm) diameter, placed on the circumference of the outer sleeve. All parts of the bearing were made out of aluminum.

We note here that during some experiments we did not use the air bearing at all, and the assembly was suspended freely from the ceiling. Also, in some experiments the upper air bearing was used alone.
Figure 73. The air bearings
4. Measurements of the twist angle

The reflection of the air discharged from the air bearings on the nearby objects could generate additional torque transmitted to the torsion wire. The value of this torque changes when the inner sphere-assembly changes its location. To eliminate the effect of this torque the assembly is kept in the same location at all times. Thus the additional torque (if there is any) will remain the same during the experiment and it will be taken as the zero reading.

The torque generated by the rotation of the spheres is absorbed by twisting the top end of the wire until the assembly goes back to its original location. An indicator is attached to the assembly to locate the origin. Another indicator is attached to the top end of the wire to indicate the angular deflection, \( \alpha \), with the use of a large angular scale.

The top end of the wire is connected to a pulley which is driven by a small variable speed motor through a belt. The motor is connected with a clockwise, counterclockwise switch so the wire can be twisted either way depending on the direction of rotation of the spheres.

5. Motors and motor controllers

To rotate the inner and outer spheres, two identical motors with two motor controllers were used. The motors have the following specifications:
Motomatic-Model E-650 M
Electrocraft Corporation
Input power: 115V - 50/60 Hz
Speed range: 0-3000 rpm
Torque: 5 in lb.

The motor controllers' specifications are as follows:

E - 650M Motor Controller
Input power: 105-125 VAC 50/60 Hz
Line regulation: 1%
Output current: 2.2A.

A Dayton gearmotor was used to twist the top end of the torsion wire with specifications:

Model: 2Z803
Speed: 0-100 rpm
Torque: 27 in lb
Input power: 115V AC/DC
Horse power: 1/15
Full load amps = 1.3A.

All three motors have reverse switches.

Fluids Used and Viscosity Measurements

1. Silicone oil

Dow Corning 200 silicone fluid with viscosities ranging from 5 to 1330 centistokes was used. The general specification of this fluid and its special features are as follows:
Type: Dimethyl Siloxane Polymer

Physical Form: Fluids with viscosities ranging from 0.65 to 100,000 centistokes

Special Properties: Thermal stability, water repellency, high dielectric strength, low surface tension, nontoxic and transparent.

Fluids with viscosities 5, 10, 100, 1330 centistokes were available, but we were able to construct any desired viscosity simply by mixing different viscosities together according to a chart supplied by the manufacturer.

2. Syrup

Karo Light Corn Syrup was used during some flow visualization experiments, where small change in Reynolds number during the experiment was insignificant. To get a wide range of viscosities the syrup was diluted with water at different ratios. The syrup was used because it costs less than the silicone oil and is easier to clean from the apparatus.

Silicone oil was used in all the experiments dealing with torque measurements.

The viscosities were determined by means of Calibrated Cannon-Fenske Routine Viscometers.
Wires and their Spring Constants

Steel spring wires (Precision Steel Warehouse, Inc.) with four different diameters, 0.024, 0.020, 0.012 and 0.010 in (0.061, 0.051, 0.030 and 0.025 cm), were used throughout the experiment. In all cases the length of the wire was 52.5 in (133.35 cm).

To determine the spring constant of each wire, a disc with a known moment of inertia was hung on the wire to determine its natural frequency. The weight of the disc is close to the weight of the inner sphere-motor assembly. After determining the natural frequency, the wire's spring constant was then calculated as follows:

\[ w_n = \sqrt{\frac{K}{J}} \quad \text{and} \quad w_n = \frac{2\pi}{T} \]

where

- \( w_n \) = natural frequency of the disc, sec\(^{-1}\)
- \( K \) = wire's spring constant, in lb/rad
- \( J \) = polar moment of inertia of the disc, slug in\(^2\)
- \( T \) = period of oscillation, sec

Eliminating \( w_n \) between the two equations gives:

\[ K = J\left(\frac{2\pi}{T}\right)^2 \]

The spring constant was then calculated for each wire before using it. This procedure was repeated using a
Table 4. The wires used and their spring constants

<table>
<thead>
<tr>
<th>Diameter of the wire, in</th>
<th>K (spring constant), in lb/rad</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.024</td>
<td>$5.704 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.020</td>
<td>$3.321 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.012</td>
<td>$3.85 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.010</td>
<td>$2.0662 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

different disc. The spring constants for the wires used are as shown in Table 4.

Derivation of the Equations Related to Experiments

The torque needed to rotate the spheres may be written as:

$$M = \frac{8}{3} \pi u \Omega_o R_2^3 \bar{M}$$

where

- $M$ = torque, in lb
- $u$ = viscosity of the fluid, lb sec/in$^2$
- $\Omega_o$ = characteristic angular velocity, rad/sec
- $R_2$ = diameter of the outer sphere, in
- $\bar{M}$ = dimensionless torque

at low Reynolds numbers $\bar{M}$ is independent of $Re$ and given by:
\[
\bar{M} = \frac{3\eta^3(1 - \mu)}{1 - \eta^3}
\]

where

\[
\mu = \text{the angular velocity ratio} = \frac{\Omega_2}{\Omega_1}
\]

\[
\eta = \text{the radius ratio} = \frac{R_1}{R_2}
\]

The experimental value of the torque is simply

\[
M = K\alpha
\]

where

\[
K = \text{wire spring constant, in lb/rad}
\]
\[
\alpha = \text{the angular deflection, rad.}
\]

We eliminate the dimensional torque, \(M\), between these two equations to give

\[
K\alpha = \frac{8\pi u\Omega_0 R_2^3}{3} \bar{M}
\]

or

\[
\bar{M} = \frac{3K\alpha}{8\pi R_2^3 \Omega_0 u}.
\]

For the geometry we have, that is \(R_2 = 2.55\) in, the dimensionless torque will be

\[
\bar{M} = \frac{3}{8\pi} \frac{K(\text{in \text{lb}})}{\Omega (\text{rad})} \alpha (\text{deg}) \frac{2\pi}{360} \left(\frac{0.00209}{144} \frac{\Omega_0 (\text{rpm})}{60} \left(2.55\right)^3\right)
\]
The Reynolds number is defined as

\[
Re = \frac{\Omega (\text{rpm})}{\nu} \frac{2\pi}{250} \frac{(2.55)^2}{(0.001076) \times 144} 
\]

\[
Re = 439.48 \frac{\Omega (\text{rpm})}{\nu (\text{CS})} .
\]

Error Discussion

Several factors were felt to have contributed to experimental errors. These are classified in two categories, direct and indirect errors.

1. Direct experimental errors

These are errors directly involved in the torque measurements. Under this category we mention mechanical errors, manufacturing defects, the effect of the earth magnetic field, etc.
The mechanical errors result from several factors. For example, due to the presence of the air bearing, it was felt that additional torque could be transmitted to the torsion wire if the location of the inner sphere-assembly is not constant. To eliminate this torque the assembly is kept in the same location at all times. The torque generated by the rotation of the spheres is absorbed by twisting the top end of the wire until the assembly goes back to its original location. Other additional torque would have been generated from the twisting of the electrical cord of the inner sphere motor. This torque was also eliminated with the use of the mercury switch as was explained before.

The manufacturing defects include those of manufacturing the spheres, shafts and the air bearings. The air bearings were checked and cleaned frequently to minimize any possible error.

The compass effects (the interaction between the earth magnetic field and the permanent magnet within the inner sphere-motor) are felt to have significant results on the torque measurements in the case of \( \eta = 0.304 \) since the torque is small compared with the case of \( \eta = 0.44 \). To eliminate this compass effects the inner sphere-motor was rapped with thin sheet of Netic and Conetic metal (Profection Mica) to isolate the motor's permanent magnet from the earth magnetic field. In this case a viscous damper was used to
replace the air bearings.

In addition, the portion of the inner sphere shaft which was immersed in the fluid created extra torque, introducing another error. The magnitude of this error was calculated by Menguturk (2) to be of about 1%, in the case of $\eta = 0.44$. For the larger gap situation, $\eta = 0.30$, a thinner shaft was used in an attempt to decrease the error. A large angular scale was used in order to have a sufficiently accurate reading for the angular deflection of the torsion wire.

2. Indirect experimental errors

We mention here those agents which did not have any direct bearing on the execution of the experiments. The errors involved in the determination of the spring constant and the viscosities of the fluids used fall in this category. These errors affected the torque results through the calculations indicated by Equation 17.

Overall, where it is possible to compare the experimental results with exact theoretical results, the comparison is extremely good. It is felt, due to these comparisons and the repeatability of the data, that the results presented are quite accurate.

Uncertainty Estimate

In this section we compute the uncertainty in the data using the method described by Kline and McClintock (51). If
R is a linear function of \( n \) independent variables \( v_1 \), each of which is normally distributed, then the relation between the interval for the variables \( w_1 \), and the interval for the result \( w_R \), which gives the same odds for each of the variables and for the result is

\[
w_R = \left[ \left( \frac{\partial R}{\partial v_1} w_1 \right)^2 + \left( \frac{\partial R}{\partial v_2} w_2 \right)^2 + \ldots + \left( \frac{\partial R}{\partial v_n} w_n \right)^2 \right]^{1/2} \quad (19)
\]

Equation 19 might be used directly as an approximation for calculating the uncertainty in the results.

This method will be used to calculate the uncertainty in the torque results. Equation 17 is used to calculate the torque throughout the experiment. Thus the torque uncertainty interval \( w_M \) can be calculated using Equation 19:

\[
w_M = (8266.58) \left[ \frac{a}{vS_\Omega R_3} w_K \right]^2 + \left( \frac{K}{vS_\Omega R_3} w_a \right)^2 + \left( \frac{-3K\alpha}{vS_\Omega R_2} w_R \right)^2
\]

\[+ \left( \frac{-\alpha K}{v^2 S_\Omega R_2} w_v \right)^2 + \left( \frac{-\alpha K}{vS_\Omega R_2} w_s \right)^2 + \left( \frac{-\alpha K}{vS_\Omega R_2} w_\Omega \right)^2 \right]^{1/2}
\]

this equation can be simplified when divided by Equation 17 to nondimensionalize it

\[
\frac{w_M}{M} = \left[ \left( \frac{w_K}{K} \right)^2 + \left( \frac{w_a}{a} \right)^2 + \left( \frac{3w_R}{R_2} \right)^2 \right]^{1/2} + \left( \frac{w_v}{v} \right)^2 + \left( \frac{w_s}{s} \right)^2 + \left( \frac{w_\Omega}{\Omega} \right)^2 \right]^{1/2}
\]

(20)
Equation 20 is used to calculate the uncertainty in the torque data. As a sample of this calculation, we show the results of three data points with different values of the main variables.

\[ \alpha = 325 \pm 0.2500 \text{ deg} \]
\[ \Omega_0 = 42.86 \pm 0.0250 \text{ rpm} \]
\[ K = (5.70 \pm 0.0112) \times 10^{-3} \text{ in lb/rad} \]
\[ V = 1335 \pm 0.1 \text{ CS} \]
\[ S = 0.971 \pm 0.0005 \]
\[ R_2 = 2.55 \pm 0.0050 \text{ in} \]
\[ \frac{\tilde{W}}{\tilde{M}} = 0.0063 \]

\[ \alpha = 145 \pm 0.2500 \text{ deg} \]
\[ \Omega_0 = 90.68 \pm 0.0125 \text{ rpm} \]
\[ K = (2.1069 \pm 0.0004) \times 10^{-4} \text{ in lb/rad} \]
\[ V = 9.27 \pm 0.0013 \text{ CS} \]
\[ S = 0.9325 \pm 0.0005 \]
\[ R_2 = 2.55 \pm 0.0050 \text{ in} \]
\[ \frac{\tilde{W}}{\tilde{M}} = 0.00616 \]

\[ \alpha = 97 \pm 0.2500 \text{ deg} \]
\[ \Omega_0 = 80.54 \pm 0.0141 \text{ rpm} \]
\[ K = (2.1069 \pm 0.0004) \times 10^{-4} \text{ in lb/rad} \]
\[ V = 5.27 \pm 0.0007 \text{ CS} \]
\[ S = 0.916 \pm 0.0005 \]
\[ R_2 = 2.55 \pm 0.005 \text{ in} \]
\[ \frac{\bar{W}_M}{M} = 0.0064 \]

All the uncertainty intervals for the variables are based on the same odds, twenty to one.

The calculations show that the uncertainty interval in the torque values is about 0.6%.
APPENDIX B. PERTURBATION SOLUTION TORQUE RESULTS

For sufficiently small Reynolds numbers the flow between concentric rotating spheres can be written in terms of a power series as

\[ \psi(r, \theta) = \sum_{l=1}^{\infty} \sum_{j=1}^{l} \text{Re}[\sin^{2} \theta P_{j}(\theta)g_{j,l}(r)] \]

where \( P_{j}(\theta) \) is the jth-order Legendre polynomial.

The torque, \( M \), required to rotate the spheres at a given rate is given by

\[ M = 2 \pi \int_{0}^{\pi} \int_{0}^{2\pi} ur^{3} \sin^{2} \theta \tau_{\theta r} \sin \theta d\phi d\theta \]

where \( u \) is the fluid viscosity and \( \tau_{\theta r} \) is the component of the stress tensor. This torque can be put into dimensionless form by writing

\[ M = \frac{8}{3} \pi u \Omega R^{3} \bar{M} \]

where \( \bar{M} = \bar{M}(\eta, \mu) \) is the dimensionless torque.

Through terms of order \( \text{Re}^{2} \), the above perturbation
solution can be written as

\[ \psi = \text{Re} \sin^2 \theta \, P_1(\theta) g_{11}(r) \]

\[ \Omega = \sin^2 \theta \, f_{oo}(r) + \text{Re}^2 \sin^2 \theta \left[ P_2(\theta) f_{o2}(r) + f_{oo}(r) \right] \]

and

\[ \tilde{M} = |(f'_{oo} - 2f_{oo}) + \text{Re}^2(f'_{o2} - \frac{1}{2} f'_{22})|_{r=1} \]

where

\[ (\cdot)' = \frac{d}{dr}. \]

Use of the solution of the governing equations for the component functions \( f_{oo}, f_{o2}, f_{22} \) and \( g_{11} \) allows the dimensionless torque to be written as:

\[ \tilde{M} = a + b\text{Re}^2 \]

where

\[ a = 3|a_1| \quad \text{and} \]

\[ b = \left| \frac{4}{15} a_1 A_1 + \frac{3}{50} a_1^2 + \frac{1}{5} a_1 A_2 + C_1 + \frac{2}{15} a_2 A_1 \right. \]

\[ - \left. \frac{2}{5} a_2 A_2 - \frac{2}{5} a_1 A_3 - 2C_2 - \frac{3}{10} a_2^2 + \frac{6}{5} a_1 A_4 \right. \]

\[ + \left. \frac{4}{5} a_2 A_3 + \frac{3}{5} a_2 A_4 \right| . \]

Here
\[ a_1 = \frac{(1 - \mu)\eta}{1 - \eta^3} \]
\[ a_2 = \frac{(\mu - \eta^3)}{(1 - \eta^3)} \]
\[ C_1 = -\frac{a_1 a_2}{4} \]

and

\[ C_2 = \frac{a_1^2}{4}. \]

The remaining constants, \( A_1, A_2, A_3, A_4 \) occur in the function \( g_{11} \) as follows

\[ g_{11} = A_1 r^{-2} + C_2 r^{-1} + A_2 + C_1 r^2 + A_3 r^3 + A_4 r^5. \]

For given value of \( \eta \) and \( \mu \) these constants are determined by use of the boundary conditions \( g_{11} = g_{11}' = 0 \) at \( r = \eta \) and \( r = 1 \).

The values of \( a \) and \( b \) for \( \eta = 0.304 \) and \( \eta = 0.44 \) are given as a function of \( \mu \) in Tables 5-8.

The conditions for which two swirls exist and the size of the two swirls can, for low Reynolds numbers, be obtained from the above equations by setting \( g_{11} = 0 \). That is, the boundary conditions specify that \( \psi \) (hence \( g_{11} \)) vanish on the boundaries \( (r = \eta, r = 1) \). Likewise another root for \( g_{11} = 0 \) gives the boundary separating the two secondary flow swirls which exist for a certain range of the angular velocity ratio.
Table 5. The constants $a$ and $b$ when both spheres rotate in the same direction, for $\eta = 0.304$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$a$</th>
<th>$b$</th>
<th>$\mu$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.07805</td>
<td>$42563 \times 10^{-6}$</td>
<td>1.4</td>
<td>0.03469</td>
<td>$50189 \times 10^{-5}$</td>
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<tr>
<td>0.2</td>
<td>0.06938</td>
<td>$64969 \times 10^{-6}$</td>
<td>1.5</td>
<td>0.04336</td>
<td>$70821 \times 10^{-5}$</td>
</tr>
<tr>
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<td>$19454 \times 10^{-4}$</td>
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<td>0.08672</td>
<td>$23724 \times 10^{-4}$</td>
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<tr>
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<td>2.1</td>
<td>0.09539</td>
<td>$58524 \times 10^{-4}$</td>
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<td>0.10406</td>
<td>$33883 \times 10^{-4}$</td>
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<td>$53612 \times 10^{-4}$</td>
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Table 6. The constants $a$ and $b$ when both spheres rotate in opposite directions, for $\eta = 0.304$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$a$</th>
<th>$b$</th>
<th>$\mu$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
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<td>0.0</td>
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<td>$0.17247 \times 10^{-4}$</td>
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<td>$0.21051 \times 10^{-4}$</td>
</tr>
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<td>$0.35594 \times 10^{-4}$</td>
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<td>0.13875</td>
<td>$0.13649 \times 10^{-5}$</td>
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<td>0.25149</td>
<td>$0.41585 \times 10^{-4}$</td>
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<tr>
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<td>$0.22111 \times 10^{-5}$</td>
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<tr>
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<td>$0.55456 \times 10^{-4}$</td>
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<td>0.28617</td>
<td>$0.72045 \times 10^{-4}$</td>
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<td>0.18211</td>
<td>$0.85553 \times 10^{-5}$</td>
<td>-2.4</td>
<td>0.29485</td>
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<td>-2.5</td>
<td>0.30352</td>
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### Table 7. The constants $a$ and $b$ when both spheres rotate in the same direction, for $\eta = 0.44$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$a$</th>
<th>$b$</th>
<th>$\mu$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.4</td>
<td>.11174</td>
<td>$1.1669 \times 10^{-4}$</td>
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<td>1.5</td>
<td>.13967</td>
<td>$1.6336 \times 10^{-4}$</td>
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<td>.19554</td>
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<td>.16761</td>
<td>$2.1798 \times 10^{-4}$</td>
</tr>
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<td>.19554</td>
<td>$2.8143 \times 10^{-4}$</td>
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<td>.13967</td>
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<td>1.8</td>
<td>.22348</td>
<td>$3.5420 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.6</td>
<td>.11174</td>
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<td>1.9</td>
<td>.25141</td>
<td>$4.3690 \times 10^{-4}$</td>
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<tr>
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<td>.08380</td>
<td>$3.0993 \times 10^{-5}$</td>
<td>2.0</td>
<td>.27935</td>
<td>$5.3010 \times 10^{-4}$</td>
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<td>.30728</td>
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<tr>
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<td>.33522</td>
<td>$7.5040 \times 10^{-4}$</td>
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<td>.36315</td>
<td>$8.7868 \times 10^{-4}$</td>
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<td>.02793</td>
<td>$1.9914 \times 10^{-5}$</td>
<td>2.4</td>
<td>.39109</td>
<td>$1.0198 \times 10^{-3}$</td>
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<td>.08380</td>
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</table>
**Table 8. The constants a and b when both spheres rotate in opposite directions, for η = 0.44**

<table>
<thead>
<tr>
<th>μ</th>
<th>a</th>
<th>b</th>
<th>μ</th>
<th>a</th>
<th>b</th>
</tr>
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<tbody>
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<td>0.64250</td>
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<td>0.35010 x 10^{-4}</td>
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<tr>
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<td>0.72630</td>
<td>0.42739 x 10^{-4}</td>
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<td>0.75424</td>
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<td>0.81011</td>
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<td>0.97853 x 10^{-4}</td>
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