Regression estimation for multivariate normal distributions

Grace Ogechukwukanma Esimai

Iowa State University
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for multivariate normal distributions

by

Grace Ogechukwuakanma Esimai

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For the Major Department

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I. INTRODUCTION

A. Introduction

Although statisticians and research workers have, for a long time, appealed to preliminary tests of significance as a technique in their investigations, it was only in the last three decades that proper evaluation of their effects on subsequent inferences is being made. These inference procedures incorporating preliminary tests usually occur in incompletely specified models. Recently Bancroft and Han (1977) has given a more appropriate designation of such inference procedures and termed them as inference based on conditional specification.

The term, conditional specification as opposed to unconditional specification, is used to describe the situation when the research worker is uncertain regarding the initial specification of a model for his investigation. He may wish to determine the final specification based on available data, usually by using preliminary tests. However, the research worker, either from experience or some knowledge about the investigation, may be able to choose a complete model for his study. In such a case, the research worker has an unconditional specification.

A bibliography on inference based on conditional specifications was recently compiled by Bancroft and Han (1977). These include estimation, prediction, hypothesis testing and
others. In this dissertation, we shall consider the regression estimation of a population mean under conditional specification.

B. Literature Review

The earliest paper on the effect of preliminary tests was due to Bancroft (1944). He discussed the bias in the case of estimation of variances on the basis of a preliminary F-test. Since then, many statisticians and research workers have worked on inference procedures based on conditional specification. Most of these studies used the terminologies "inference procedures incorporating preliminary test(s)," "pooling data," or "inference for incompletely specified models." Recently, in a note by Bancroft and Han (1977), the terminology "inference based on conditional specification" was suggested as a broader representation of the phrases used in the past. In this section, we shall review briefly the estimation of the mean.

In 1948, Mosteller studied the estimation of a population mean by pooling independent samples on the basis of a significance test. He investigated what he called the 'Disadvantage Coefficient' which is the efficiency of the never pool estimator relative to the preliminary test estimator. Bennett (1952) evaluated the bias and distribution of estimates of means based on one or more preliminary tests of significance. He extended the work of Mosteller to the cases where the two
population variances may be known but unequal or equal but unknown. Preliminary tests were also used by Bennett (1956) to provide interval estimates for the mean and variance of a normal population.

Kitagawa (1963) continued the investigations of Bennett (1952) on the distribution of the preliminary test estimator for the mean of a normal distribution when the variance is unknown. He derived the bias and m.s.e. and expressed these as infinite sums which are very difficult to compute. However, Han and Bancroft (1968) worked on the same problem and were able to express the bias and m.s.e. as finite sums which are much easier to evaluate. They also recommended a procedure for determining a proper choice of the significance level of the preliminary test to ensure a relative efficiency to be larger than some preassigned value.

A little before this time, Kale and Bancroft (1967) had considered the problem of pooling means of two independent random samples from discrete distributions (particularly the Poisson and binomial) which can be approximated by normal distributions after appropriate transformations. They studied two samples from $N(\mu_1, \sigma^2)$ $i = 1, 2$ assuming the parameter of interest was $\mu_1$ and $\sigma^2$ was known. An estimator $\bar{x}^*$ was proposed both for the estimation of $\mu_1$ and for the test of $H_0: \mu_1 = \mu_0$. The bias and m.s.e. of $\bar{x}^*$ and the size and power of the overall hypothesis testing procedure were studied. They
recommended the preliminary test should be at the .25 level for the control of the m.s.e. and size of the test procedure based on $\bar{x}^*$. In 1971, Brogan used a preliminary test of significance and two-stage sampling to derive an estimator for the mean of a normal distribution. He derived the bias and m.s.e. and compared the latter, for a fixed total sampling cost, to the m.s.e. of some other estimation procedures. Ahsanullah (1971) studied the problem of estimation of the mean $\mu_1$ of one of the components of a bivariate normal distribution with equal marginal variances from a sample of size n. The result of a preliminary test of $H_0: \mu_1 = \mu_2$ was used to define an estimator for $\mu_1$ where $\mu_2$ is the mean of the other component of the distribution. He studied the m.s.e. of the preliminary test estimator and tabulated its efficiency relative to the usual estimator. He also used the selection procedure recommended by Han and Bancroft (1968) to compute tables which can be used to determine a proper choice of the significance level of the preliminary test.

Bancroft (1972) gave a summary of some recent advances in inference procedures using preliminary tests of significance. He briefly outlined the theory behind the use of preliminary tests in estimation, tests of hypothesis and prediction. This is based primarily on the desire to make inferences for incompletely specified models. Useful applications of preliminary
tests of significance based on results obtained in earlier papers were given in the text by Bancroft (1968).

In 1973, Han (1973a) introduced the use of preliminary tests into regression estimation for bivariate normal distributions. In estimating the mean $\mu_y$ of one of the components of a bivariate normal distribution and the mean $\mu_x$ of the other variable is known, the investigator can use $X$ in a regression estimation to increase precision. When $\mu_x$ is unknown, Han proposed the use of a regression estimator which depends on the outcome of the preliminary test of $H_0: \mu_x = \mu_0$. He studied the bias and m.s.e. of the preliminary test estimator and discussed the relative efficiency. Later, the same year, Han (1973b) extended his study to the case when the mean of $X$ is unknown and double sampling can be employed. If in addition, the investigator has partial information about $\mu_x$, then Han proposed to perform a preliminary test and use the preliminary test estimator. He derived the bias, m.s.e. and relative efficiency of the preliminary test estimator and gave recommendations of the levels of the preliminary test and optimum allocation of sample sizes.

At the same time, many other statisticians and research workers have shown concern about estimation with high precision. Consequently, many workers in the field were also carrying out investigations and proposing new estimators based on certain criteria. One such investigation was given by
Stein (1955) who discussed the inadmissibility of the usual estimator for the mean of a multivariate normal distribution for \( p \geq 3 \). He proposed a spherically symmetric estimator which is essentially a shrunken estimator. James and Stein (1960) continued with the same studies and gave more precise forms and merits of the shrunken estimator for the cases when the covariance matrix is either known or unknown. In 1960, Stein investigated the improvement in m.s.e. by a transformation, on the regression coefficient \( \hat{\beta} \), of the form \( C \hat{\beta} : 0 < C < 1 \) which is a shortening of the vector \( \hat{\beta} \).

In 1968, Thompson (1968a) studied various ways of shrinking the minimum variance unbiased estimator of a population mean towards some known origin, thereby reducing its m.s.e. He employed a preliminary test of significance as a shrinking procedure. Later in the same year, Thompson (1968b) extended his work to shrinkage towards an interval centered at some origin.

C. An Overview of the Present Research and Summary of Results

The present thesis is divided into three main parts. The first part is an effort to extend the studies of Han (1973a) for bivariate normal distributions to \((p+1)\) variate normal distributions \((p+1 > 2)\). The second part attempts to extend the method of double sampling with partial information on auxiliary variables first studied by Han (1973b) for one
auxiliary variable to the case where the auxiliary variable is a px1 vector. The last part considers regression estimators with certain shrunken estimators for the mean of the auxiliary variable and compares them with the preliminary test estimators.

In Chapter II, Section B, we define the preliminary test estimator, \( \hat{\mu} \), for the general p+1 variate normal distribution and study its bias when the covariance matrix, \( \Sigma \), is known. In Chapter II, Section C, we derive and discuss the m.s.e. of \( \hat{\mu} \) for \( \Sigma \) known. In Chapter II, Section D, the relative efficiency \( e \), of \( \hat{\mu} \) is considered while Chapter II, Sections E and F, respectively, deal with the derivation and discussion of the properties of the Bias and m.s.e. of \( \hat{\mu} \) when \( \Sigma \) is unknown. Chapter II, Section G, gives the expression for and some computed values of the relative efficiency \( e' \).

In Chapter III, we consider double sampling with partial information on auxiliary variables and for \( \Sigma \) known, we define the preliminary test estimator \( \hat{\mu}_{lr} \) and exhibit its properties in Section B. The m.s.e. and the relative efficiency of \( \hat{\mu}_{lr} \) are given and studied in Chapter III, Sections C and D, respectively. Chapter III, Section E, furnishes a discussion of the optimal sample design and some comparisons. When \( \Sigma \) is unknown, the bias, m.s.e. and relative efficiency \( e_2 \) of \( \hat{\mu}_{lr} \) are derived and investigated in Chapter III, Sections F, G and H, respectively.
In Chapter IV, we consider regression estimators with certain shrunken estimators for the mean of the auxiliary variable. A shrunken regression estimator \( \hat{\mu}^* \) is given in Chapter IV, Section B, following Thompson (1968a). The relative efficiency \( e^* \) of \( \hat{\mu}^* \) is also discussed. In Chapter IV, Section C, a shrunken regression estimator \( \hat{\mu}_1 \) is constructed following James and Stein (1960). We also give an expression for its relative efficiency.

In general, the bias and m.s.e. of the preliminary test estimators are found to be functions of \( n, \mu_x, \Sigma_{12} \) and \( \alpha \) where \( \Sigma_{12} \) is the covariance between \( Y \) and \( X \). When \( \Sigma \) is known, the bias and m.s.e. are found in terms of the cumulative distribution of the noncentral Chi-squared distribution. For \( p = 1 \), Han (1973a, 1973b) expressed the bias and m.s.e. in terms of the cumulative distribution function of the standard normal distribution. Thus the computations in this dissertation afford a further empirical verification of the results of Han (1975) on some relationships between noncentral Chi-squared and normal distributions. The properties of the bias and m.s.e. for \( p > 1 \) are found to be identical with those recorded for \( p = 1 \).

When \( \Sigma \) is unknown, the bias and m.s.e. of the preliminary test estimators are also found to be functions of \( n, \mu_x, \Sigma_{12} \) and \( \alpha \), but in terms of the cumulative distribution of the noncentral F-distribution. For \( p = 1 \), Han (1973a, 1973b)
expressed these in terms of the moments of normal distribution. The properties of the bias and m.s.e. for $p > 1$ are in general found to be identical with those recorded for $p = 1$.

The m.s.e. of the shrunken regression estimator is found to be a function of $n$, $\Sigma$ and $\mu_x$. The efficiency of the preliminary test estimator relative to the shrunken regression estimator is generally found to be greater than unity when $\mu_x = 0$, or when the null hypothesis of the preliminary test is true. The value of the relative efficiency then decreases to a value smaller than unity, increases to above unity and finally decreases to unity as components of $\mu_x$ increase.
II. REGRESSION ESTIMATION
FOR MULTIVARIATE NORMAL DISTRIBUTIONS

A. Introduction

Consider that we have a multivariate normal population, that is, consider the case:

\[ \mathbf{Y} \sim \mathcal{N}(\mathbf{\mu}, \Sigma) \]

where

\[
\begin{pmatrix}
\mathbf{Y} \\
\mathbf{X}_1 \\
\vdots \\
\mathbf{X}_p
\end{pmatrix}
= \begin{pmatrix}
\mathbf{Y} \\
\mathbf{X}_1 \\
\vdots \\
\mathbf{X}_p
\end{pmatrix}
= \begin{pmatrix}
\mathbf{Y} \\
\mathbf{X}_1 \\
\vdots \\
\mathbf{X}_p
\end{pmatrix}
= \begin{pmatrix}
\mathbf{Y} \\
\mathbf{X}_1 \\
\vdots \\
\mathbf{X}_p
\end{pmatrix}
\]

(2.1)

Suppose we are interested in estimating the mean \( \mu \). This happens in an investigation that the investigator is interested in primarily one variable while he uses other variables as auxiliary information. Following Han (1973a), we may use the remaining \( p \) variables as ancillary variables to increase precision. If \( \boldsymbol{\mu}_x \) and \( \Sigma \) are known and we have a random sample of
size n, we can use the regression estimator defined as

\[ \hat{\mu} = \bar{y} + \Sigma_{12} \Sigma_{22}^{-1} (\mu_X - \bar{X}) \]

where

\[ \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i; \bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i. \]

In this case we know the variance of the regression estimator is

\[ \frac{1}{n} \sigma^2 - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \] and if \( \frac{1}{n} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \)

is considerably large, we have an appreciable gain in precision. If \( \mu_X \) is unknown, one may use \( \bar{y} \) to estimate \( \mu \). However, it may happen that from certain sources, the experimenter may expect that \( \mu_X = \mu_0 \) but not sure for certainty. In this case, a preliminary test of \( H_0: \mu_X = \mu_0 \) can be performed and the estimator is made to depend on the result of the preliminary test. In this chapter we shall consider the properties of this preliminary test estimator.

B. The Preliminary Test Estimator and its Bias when \( \Sigma \) is Known

Assume \( (y_1, X_{11}, \ldots, X_{p1}) i = 1, \ldots, n \) is a random sample from the \((p+1)\)-variate normal distribution \( N(\mu, \Sigma) \). Suppose \( \Sigma \) is known and \( \mu \) unknown. Consider the hypotheses:
\( H_0: \mu_x = \mu_0 \)

\( H_1: \mu_x \neq \mu_0 \)

Wlog (without loss of generality) we take \( \mu_0 \) to be the null vector 0. The test statistic for \( H_0 \) versus \( H_1 \) is \( n(\bar{X}' \Sigma_{22}^{-1} \bar{X}) \) which has a Chi-squared distribution, \( \chi_p^2 \), with \( p \) degrees of freedom. A size \( \alpha \) test is to reject \( H_0 \) if \( n(\bar{X}' \Sigma_{22}^{-1} \bar{X}) > \chi_p^{2, \alpha} \) where \( \chi_p^{2, \alpha} \) is the 100(1-\( \alpha \)) percentage point of \( \chi_p^2 \). Therefore if we let \( \chi_p^{2, \alpha} = c \) and denote the acceptance region \([n \bar{X}' \Sigma_{22}^{-1} \bar{X}: n \bar{X}' \Sigma_{22}^{-1} \bar{X} < c]\) by \( A \) then the preliminary test estimator can be written as

\[
\hat{\mu} = \begin{cases} 
\bar{y} - \Sigma_{12} \Sigma_{22}^{-1} \bar{X} & \text{given } A \\
\bar{y} & \text{given } \bar{A}
\end{cases} \tag{2.2}
\]

The expected value of \( \hat{\mu} \) is

\[
E(\hat{\mu}) = E((\bar{y} - \Sigma_{12} \Sigma_{22}^{-1} \bar{X})|A) \ P(A) + E(\bar{y}|\bar{A}) \ P(\bar{A}) \tag{2.3}
\]

\[
= E(\bar{y}) - \Sigma_{12} \Sigma_{22}^{-1} E(\bar{X}|A) P(A)
\]

But \( E(\bar{y}) = \mu \). Hence the bias of \( \hat{\mu} \) is the second term and if we denote this by \( B \), we can write

\[
B = -\Sigma_{12} \Sigma_{22}^{-1} E(\bar{X}|A) P(A) \tag{2.4}
\]

Now we know \( \bar{X} \sim N(\mu_x, \Sigma_{22}^{-1}) \) and since \( \Sigma_{22} \) is positive definite, \( T \) a nonsingular \( T \in \mathbb{T}'T = \Sigma_{22}^{-1} \). Let \( \bar{Z} = T \bar{X} \). Therefore
\[ Z \sim N(TH_x, \frac{1}{n}I) \]

\[ \sim N(\nu_x, \frac{1}{n}I) \] say

and we can write

\[ B = -\Sigma_{12} \Sigma_{22}^{-1} T^{-1} E[Z|n(Z'Z) \leq c] \cdot P[n(Z'Z) \leq c] \]

where

\[ \{n(Z'Z):n(Z'Z) \leq c\} = A. \]

Hence

\[ B = -\Sigma_{12} \Sigma_{22}^{-1} T^{-1} E[Z|A]P(A) \quad (2.5) \]

In general \( n(Z'Z) \) has a noncentral Chi-squared distribution with \( p \) degrees of freedom and noncentrality parameter \( \lambda = n(\mu_x'T'T\mu_x) = \nu_x'\nu_x \). We shall denote the \( i \)-th component of the pxl vector \( \nu_x \) by \( \nu_x^{(1)} \).

Now consider

\[ R = P(A) = \int_0^c e^{-\frac{1}{2}t} \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{1}{2} \right)^j h_{p+2j}(t) dt \quad (2.6) \]

where \( h_{p+2j}(\cdot) \) is the probability density function of \( \chi^2_{p+2j} \).

Differentiating (2.6) with respect to (w.r.t.) \( \nu_x^{(1)} \) following the method of justification in the Appendix, we obtain
\[ \frac{\partial R}{\partial \nu_1} = c \int_0^\infty e^{-\frac{1}{2} \lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n}{\nu_1} \nu_1(1) j(\lambda)^{j-1} h_{p+2j}(t) dt \]

\[ - \int_0^\infty \frac{1}{2} 2n\nu_1(1) e^{-\frac{1}{2} \lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{\nu_1} (\lambda)^j h_{p+2j}(t) dt \] (2.7)

or

\[ \frac{\partial R}{\partial \nu_1} = n[H_{p+2}(c;\lambda) - P(A)] \nu_1(1) \] (2.8)

where \( H_{p+2}(c;\lambda) \) is the cumulative distribution function of the noncentral Chi-squared distribution with \( p+2 \) degrees of freedom and noncentrality parameter \( \lambda \).

Alternatively, we can evaluate \( P(A) \) by the use of the distribution of \( Z \) and write

\[ R = P(A) = \prod_{j=1}^{P} \frac{1}{\sqrt{2\pi}} \frac{n}{\sqrt{\nu_1}} e^{\frac{-n}{2}(Z(j) - \nu_1(1))^2} \] (2.9)

since components of \( Z \) are independent. If we differentiate (2.9) w.r.t. \( \nu_1(1) \) using the method of justification in the Appendix, we obtain

\[ \frac{\partial R}{\partial \nu_1(1)} = \prod_{j=1}^{P} \frac{\sqrt{n}}{\sqrt{2\pi}} n\frac{1}{2} 2(Z(i) - \nu_1(1)) e^{\frac{-n}{2}(Z(j) - \nu_1(1))^2} dZ(j) \] (2.10)
\[
\frac{\partial R}{\partial \nu_x^{(1)}} = n E[Z^{(1)}|A] P(A) - n \nu_x^{(1)} P(A) \tag{2.11}
\]

To obtain \(E(Z^{(1)}|A)P(A)\) we equate (2.8) and (2.11),

\[
n[H_{p+2}(c;\lambda) - P(A)]\nu_x^{(1)} = nE[Z^{(1)}|A]P(A) - n \nu_x^{(1)} P(A)
\]

which gives

\[
E(Z^{(1)}|A)P(A) = H_{p+2}(c;\lambda)\nu_x^{(1)} \tag{2.12}
\]

Substituting (2.12) in (2.5) and noting that the conditional expectation of a vector is defined as the vector of the conditional expectation of its components, we have

\[
B = -\Sigma_{12} \Sigma_{22}^{-1} T^{-1} H_{p+2}(c;\lambda)\nu_x = -\Sigma_{12} \Sigma_{22}^{-1} \mu_x H_{p+2}(c;\lambda)
\]

As a partial check, when \(c=0\), the estimator reduces to the usual estimator \(\tilde{y}\) which is the case when we always reject the null hypothesis. In this case, \(B=0\). When \(c=\infty\), the null hypothesis is always accepted and the regression estimator \(\bar{y} - \Sigma_{12} \Sigma_{22}^{-1} \bar{x}\) is used. The bias in this case reduces to the bias for the regression estimator, i.e.,

\[
B = -\Sigma_{12} \Sigma_{22}^{-1} T^{-1} \nu_x = -\Sigma_{12} \Sigma_{22}^{-1} \mu_x.
\]
We now check the result with that of Han (1973a) when p = 1. Without loss of generality we let $\Sigma_{22} = I$ and $\sigma^2 = 1$. Therefore

$$\sqrt{n} B = -K \Sigma_{p+2}(c; \lambda) \text{ where } K = \Sigma_{12} \mu_x \sqrt{n}.$$  

For p = 1, $\Sigma_{12} = \rho$ and we let $\mu_x \sqrt{n} = a$.

Hence $K = \rho a$ and we observe that $\sqrt{n} B$ changes sign with $\rho$ or $a$. Therefore for p = 1, we may only study the bias for positive values of $\rho$ and $a$. It is obvious to see that $\sqrt{n} B$ is a function of $\rho$, $a$ and $\alpha$. The values of $-\sqrt{n} B$ for certain values of $\rho$, $a$ and $\alpha$ were computed and examined and only very few of these are given in Table 2.1.

**Table 2.1. Values of $-\sqrt{n} B$ for p = 1.**

<table>
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<th>$\alpha = .05$</th>
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<tr>
<td>.1</td>
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<td>.5</td>
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</tr>
<tr>
<td>3.0</td>
<td>0.0215</td>
<td>0.1076</td>
</tr>
</tbody>
</table>
From the computed values we note the following properties of the Bias.

1. The bias is zero when \( \mu_x = 0 \). This corresponds to the case when the hypothesis is true.
2. The value of the bias generally increases with \( \rho \) and decreases as \( \alpha \) increases.
3. For fixed \( n \), \( \alpha \) and \( \rho \), the bias first increases then decreases to zero as \( \mu_x \) increases.

We also note that the values of \( -\sqrt{nB} \) given in Table 2.1 are identical with the values obtained by Han (1973a). The only difference is that while Han's results were given to three decimal places, the values here are computed to four decimal places. The above properties of the Bias were also recorded by Han. Furthermore, we note that Han expressed the bias in terms of functions of the distribution function and probability density function of the standard normal distribution while in this paper, the bias is expressed in terms of the cumulative distribution function of the noncentral Chi-squared distribution with an odd degree of freedom. The above results thus provide an empirical verification of the theoretical results obtained by Han (1975) on some relationships between noncentral Chi-squared and normal distributions.

For \( p=2 \), the values of \( -\sqrt{nB} \) for some values of \( \Sigma_{12} \), \( \mu_x \sqrt{n} \) and \( \alpha \) are given in Table 2.2. Since the bias changes sign with \( \mu_x \), the values were computed for only positive values of \( \mu_x \).
Table 2.2. Values of $-\sqrt{n}B$ for $p=2$.

<table>
<thead>
<tr>
<th>$\Sigma_{12}$</th>
<th>$\alpha = .05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>(.5, 0)</td>
<td>-.1931</td>
</tr>
<tr>
<td>(.5, .5)</td>
<td>0</td>
</tr>
<tr>
<td>(1.0, 0)</td>
<td>-.3447</td>
</tr>
<tr>
<td>(1.0, .5)</td>
<td>-.1656</td>
</tr>
<tr>
<td>(1.5, 0)</td>
<td>-.4191</td>
</tr>
<tr>
<td>(1.5, .5)</td>
<td>-.2672</td>
</tr>
<tr>
<td>(1.5, 1.0)</td>
<td>-.1162</td>
</tr>
<tr>
<td>(1.5, 1.5)</td>
<td>0</td>
</tr>
<tr>
<td>(2.0, 0)</td>
<td>-.4018</td>
</tr>
<tr>
<td>(2.0, .5)</td>
<td>-.2866</td>
</tr>
<tr>
<td>(2.0, 1.0)</td>
<td>-.1637</td>
</tr>
<tr>
<td>(2.0, 1.5)</td>
<td>-.0625</td>
</tr>
<tr>
<td>(2.0, 2.0)</td>
<td>0</td>
</tr>
<tr>
<td>(2.5, 0)</td>
<td>-.3126</td>
</tr>
<tr>
<td>(2.5, .5)</td>
<td>-.2365</td>
</tr>
<tr>
<td>(2.5, 1.0)</td>
<td>-.1496</td>
</tr>
<tr>
<td>(2.5, 1.5)</td>
<td>-.0744</td>
</tr>
<tr>
<td>(2.5, 2.0)</td>
<td>-.0242</td>
</tr>
<tr>
<td>(2.5, 2.5)</td>
<td>0</td>
</tr>
<tr>
<td>(3.0, 0)</td>
<td>-.1978</td>
</tr>
<tr>
<td>(3.0, .5)</td>
<td>-.1551</td>
</tr>
<tr>
<td>(3.0, 1.0)</td>
<td>-.1031</td>
</tr>
<tr>
<td>(3.0, 1.5)</td>
<td>-.0563</td>
</tr>
<tr>
<td>(3.0, 2.0)</td>
<td>-.0237</td>
</tr>
<tr>
<td>(3.0, 2.5)</td>
<td>-.0064</td>
</tr>
<tr>
<td>(3.0, 3.0)</td>
<td>0</td>
</tr>
</tbody>
</table>

These values are obtained for different combinations of $\mu_x/\sqrt{n}$ and $\Sigma_{12}$, with $\alpha = .05$. The table provides a comprehensive view of the relationship between $\Sigma_{12}$ and $\sqrt{n}B$ for various values of $\alpha$. The values are calculated by considering the effect of different mean values on the distribution of $B$.
Table 2.2. (continued)

<table>
<thead>
<tr>
<th>Σ^n_{12} (μ_x/√n)^t</th>
<th>α = .20</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.5</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>(.5, 0)</td>
<td>-1.117</td>
</tr>
<tr>
<td>(1.0, 0)</td>
<td>-1.117</td>
</tr>
<tr>
<td>(.5, .5)</td>
<td>0</td>
</tr>
<tr>
<td>(1.0, .5)</td>
<td>-1.845</td>
</tr>
<tr>
<td>(1.0, 1.0)</td>
<td>-1.903</td>
</tr>
<tr>
<td>(1.5, .5)</td>
<td>-1.179</td>
</tr>
<tr>
<td>(1.5, 1.0)</td>
<td>-0.0472</td>
</tr>
<tr>
<td>(1.5, 1.5)</td>
<td>0.0187</td>
</tr>
<tr>
<td>(2.0, 0)</td>
<td>-1.1507</td>
</tr>
<tr>
<td>(2.0, .5)</td>
<td>-1.047</td>
</tr>
<tr>
<td>(2.0, 1.0)</td>
<td>-0.0554</td>
</tr>
<tr>
<td>(2.0, 1.5)</td>
<td>0.0157</td>
</tr>
<tr>
<td>(2.0, 2.0)</td>
<td>0</td>
</tr>
<tr>
<td>(2.0, 2.5)</td>
<td>0</td>
</tr>
<tr>
<td>(2.5, .5)</td>
<td>-0.0692</td>
</tr>
<tr>
<td>(2.5, 1.0)</td>
<td>-0.0408</td>
</tr>
<tr>
<td>(2.5, 1.5)</td>
<td>-0.0182</td>
</tr>
<tr>
<td>(2.5, 2.0)</td>
<td>-0.0051</td>
</tr>
<tr>
<td>(2.5, 2.5)</td>
<td>0</td>
</tr>
<tr>
<td>(3.0, 0)</td>
<td>-0.0462</td>
</tr>
<tr>
<td>(3.0, .5)</td>
<td>-0.0355</td>
</tr>
<tr>
<td>(3.0, 1.0)</td>
<td>-0.0221</td>
</tr>
<tr>
<td>(3.0, 1.5)</td>
<td>-0.0109</td>
</tr>
<tr>
<td>(3.0, 2.0)</td>
<td>-0.0040</td>
</tr>
<tr>
<td>(3.0, 2.5)</td>
<td>-0.0009</td>
</tr>
<tr>
<td>(3.0, 3.0)</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 2.2. (continued)

<table>
<thead>
<tr>
<th>Σ_{12} (μ_x√n)'</th>
<th>α = .50</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(-.5)</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>0</td>
</tr>
<tr>
<td>(.5, 0)</td>
<td>-.0348</td>
</tr>
<tr>
<td>(.5, .5)</td>
<td>0</td>
</tr>
<tr>
<td>(1.0, 0)</td>
<td>-.0578</td>
</tr>
<tr>
<td>(1.0, .5)</td>
<td>-.0235</td>
</tr>
<tr>
<td>(1.0, 1.0)</td>
<td>0</td>
</tr>
<tr>
<td>(1.5, 0)</td>
<td>-.0474</td>
</tr>
<tr>
<td>(1.5, .5)</td>
<td>-.0286</td>
</tr>
<tr>
<td>(1.5, 1.0)</td>
<td>-.0106</td>
</tr>
<tr>
<td>(1.5, 1.5)</td>
<td>0</td>
</tr>
<tr>
<td>(2.0, 0)</td>
<td>-.0314</td>
</tr>
<tr>
<td>(2.0, .5)</td>
<td>-.0213</td>
</tr>
<tr>
<td>(2.0, 1.0)</td>
<td>-.0105</td>
</tr>
<tr>
<td>(2.0, 1.5)</td>
<td>-.0032</td>
</tr>
<tr>
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<td>0</td>
</tr>
<tr>
<td>(2.5, 0)</td>
<td>-.0159</td>
</tr>
<tr>
<td>(2.5, .5)</td>
<td>-.0115</td>
</tr>
<tr>
<td>(2.5, 1.0)</td>
<td>-.0063</td>
</tr>
<tr>
<td>(2.5, 1.5)</td>
<td>-.0025</td>
</tr>
<tr>
<td>(2.5, 2.0)</td>
<td>.0006</td>
</tr>
<tr>
<td>(2.5, 2.5)</td>
<td>0</td>
</tr>
<tr>
<td>(3.0, 0)</td>
<td>-.0062</td>
</tr>
<tr>
<td>(3.0, .5)</td>
<td>-.0047</td>
</tr>
<tr>
<td>(3.0, 1.0)</td>
<td>-.0027</td>
</tr>
<tr>
<td>(3.0, 1.5)</td>
<td>-.0012</td>
</tr>
<tr>
<td>(3.0, 2.0)</td>
<td>-.0004</td>
</tr>
<tr>
<td>(3.0, 2.5)</td>
<td>.0001</td>
</tr>
<tr>
<td>(3.0, 3.0)</td>
<td>0</td>
</tr>
</tbody>
</table>
From Table 2.2 we observe the following properties of the Bias for \( p=2 \).

1. The bias is zero when \( \mu_x = 0 \). Again this corresponds to the case when the hypothesis is true.

2. The value of the bias generally increases with \( \Sigma_{12} \) but decreases as \( \alpha \) increases.

3. The bias is zero if \( \mu_x \) has identical components and \( \Sigma_{12} \) has components which differ only in sign.

4. The bias is negative if either \( \mu_x \) or \( \Sigma_{12} \) has non-identical but positive components and the other has components which differ only in sign.

5. If \( n, \alpha, \Sigma_{12} \) and a component of \( \mu_x \) are fixed, the bias first increases then decreases to zero as the other component of \( \mu_x \) increases.

C. The M.S.E. of \( \hat{\mu} \) when \( \Sigma \) is Known

In order to find the M.S.E. of \( \hat{\mu} \), we first consider

\[
V(\hat{\mu}) = E(\hat{\mu}^2) - [E(\hat{\mu})]^2 .
\] (2.13)

Also we can write

\[
\hat{\mu} = \begin{cases} 
\bar{y} - \Sigma_{12} \Sigma^{-1} 22 T^{-1} Z & \text{given } A \\
\bar{y} & \text{given } \bar{A}
\end{cases}
\]
Therefore
\[
E(\hat{\mu}^2) = E[(\bar{y} - \Sigma_{12} \Sigma_{22}^{-1} \mathbf{T}^{-1} \mathbf{Z})^2 | A] \, P(A) \\
+ E(\bar{y}^2 | \bar{A}) \, P(\bar{A})
\]

\[
= E(\bar{y}^2) - 2 \Sigma_{12} \Sigma_{22}^{-1} \mathbf{T}^{-1} E(\bar{y} \mathbf{Z} | A) \, P(A) \tag{2.14} \\
+ \Sigma_{12} \Sigma_{22}^{-1} \mathbf{T}^{-1} E(\mathbf{ZZ}' | A) \, P(A) \mathbf{T}^{-1} \Sigma_{22}^{-1} \mathbf{Z}_2
\]

Therefore to evaluate \( E(\hat{\mu}^2) \), we need to find \( E(\mathbf{ZZ}' | A)P(A) \) and \( E(\bar{y} \mathbf{Z} | A)P(A) \). Let us consider \( E(\mathbf{ZZ}' | A)P(A) \) and denote the \( i \)-th component of \( \mathbf{Z} \) by \( \mathbf{Z}^{(1)} \). We need actually consider \( E[(\mathbf{Z}^{(1)})^2 | A] \) \( P(A) \) and \( E(\mathbf{Z}^{(1)} \mathbf{Z}^{(k)} | A)P(A) \) for \( i \neq k \). These can be evaluated by using the second derivatives of \( R \).

Differentiating (2.7) w.r.t. \( v_x^{(1)} \), we have

\[
\frac{\partial^2 R}{\partial v_x^{(1)}^2} = \int_0^\infty \frac{c}{2} e^{-\frac{1}{2} \lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{2n}{\lambda} x \right)^2 j (j-1) \left( \frac{\lambda}{2} x \right)^{j-2} h_{p+2j} (t) dt
\]

\[
+ \int_0^\infty \frac{c}{2} e^{-\frac{1}{2} \lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n}{\lambda} j \left( \frac{\lambda}{2} \right)^{j-1} h_{p+2j} (t) dt
\]

\[
- \int_0^\infty \frac{1}{2} 2n e^{-\frac{1}{2} \lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n v_x^{(1)}}{\lambda} j (j-1) h_{p+2j} (t) dt
\]

\[
- \int_0^\infty \frac{1}{2} 2n e^{-\frac{1}{2} \lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\lambda}{2} h_{p+2j} (t) dt
\]
\[ + \int_0^1 \left( \frac{1}{2} \right) 2n\nu_j \left( \begin{array}{c} \frac{1}{2} \\ \nu_j(t) \end{array} \right) e^{-\frac{1}{2} \lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{\lambda}{2} \right)^j \mathrm{h}_{p+2j}(t) \, dt \]

\[ - \int_0^1 \frac{1}{2} 2n\nu_x \left( \begin{array}{c} \frac{1}{2} \\ \nu_x(t) \end{array} \right) e^{-\frac{1}{2} \lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{2n\nu_x(t)}{2} \right)^j \left( \frac{\lambda}{2} \right)^{j-1} \mathrm{h}_{p+2j}(t) \, dt \]

\[ = n^2(\nu_x(1))^2 \mathrm{H}_{p+4}(c; \lambda) + n \mathrm{H}_{p+2}(c; \lambda) - n^2(\nu_x(1))^2 \mathrm{H}_{p+2}(c; \lambda) - n \mathrm{P}(A) + n^2(\nu_x(1))^2 \mathrm{P}(A) - n^2(\nu_x(1))^2 \mathrm{H}_{p+2}(c; \lambda) \]

\[ = n^2(\nu_x(1))^2 \mathrm{H}_{p+4}(c; \lambda) + n(1 - 2n(\nu_x(1))^2) \mathrm{H}_{p+2}(c; \lambda) \]

\[ + n[n(\nu_x(1))^2 - 1] \mathrm{P}(A) \quad (2.15) \]

Similarly differentiating (2.9) twice w.r.t. \( \nu_x(1) \), we obtain

\[ \frac{\partial^2 R}{\partial \nu_x(1)^2} = \int_A \prod_{j=1}^P \sqrt{n} \frac{1}{\sqrt{2\pi}} (\frac{n}{2})^2 z(j)(z(j) - \nu_x(1)) e^{-\frac{n}{2}(z(j) - \nu_x(1))^2} \, d\nu_x \]

\[ - \int_A \prod_{j=1}^P \sqrt{n} \frac{1}{\sqrt{2\pi}} (\frac{n}{2})^2 \mathrm{e}^{\frac{n}{2}(z(j) - \nu_x(1))^2} \, d\nu_x \]

\[ - \int_A \prod_{j=1}^P \sqrt{n} \frac{1}{\sqrt{2\pi}} (\frac{n}{2})^2 \nu_x(1)(z(j) - \nu_x(1)) e^{-\frac{n}{2}(z(j) - \nu_x(1))^2} \, d\nu_x \]
\[ \frac{\partial^2 R}{\partial \nu_x(1)^2} = n^2 E[(Z^{(1)})^2 | A] P(A) - 2n^2 \nu_x(1)^2 H_{p+2}(c; \lambda) \] (2.17)

\[ + n^2(\nu_x(1))^2 P(A) - n P(A) \]

Equating (2.15) and (2.17), we have

\[ n^2 E[(Z^{(1)})^2 | A] P(A) + n^2 \nu_x(1)^2 [P(A) - 2H_{p+2}(c; \lambda)] \]

\[ - n P(A) = n^2(\nu_x(1))^2 H_{p+4}(c; \lambda) + n[1-2n(\nu_x(1)^2)H_{p+2}(c; \lambda)] \]

\[ + n [n(\nu_x(1))^2 - 1] P(A) \]

or

\[ E(Z^{(1)}^2 | A) P(A) = (\nu_x(1))^2 H_{p+4}(c; \lambda) + \frac{1}{n} H_{p+2}(c; \lambda) \]

Next we find \( E(Z^{(1)} Z^{(k)} | A) P(A) \), first by differentiating (2.7) and (2.10) w.r.t. \( \nu_x^{(k)} \), then equating the results.
From (2.7),

\[
\frac{\partial R}{\partial \nu_x} = \frac{\partial \nu_x}{\partial \nu_x} = \int_0^\infty \sum_{J=0}^\infty \frac{1}{\sqrt{2\pi J!}} \frac{2\nu_x(1) \cdot 2\nu_x(k)}{J \lambda} \frac{1}{2} \int_0^\infty \frac{J-J-1(\lambda)J-1h_p+2J(t)dt}{\lambda} J(\lambda)J-1h_p+2J(t)dt
\]

Similarly from (2.10),

\[
\frac{\partial R}{\partial \nu_x} = \int_0^\infty \sum_{J=1}^\infty \frac{1}{\sqrt{2\pi J!}} \frac{2\nu_x(1) \cdot 2\nu_x(k)}{J \lambda} \frac{1}{2} \int_0^\infty \frac{J-J-1(\lambda)J-1h_p+2J(t)dt}{\lambda} J(\lambda)J-1h_p+2J(t)dt
\]

\[
= n^2 \nu_x \cdot \nu_x \cdot \lambda_p+4(c;\lambda) - 2n^2 \nu_x \cdot \nu_x \cdot \lambda_p+2(c;\lambda)
\]

\[
+ n^2 \nu_x \cdot \nu_x \cdot P(A).
\]
Hence by (2.12), (2.19) becomes

\[
\frac{\partial R}{\partial \nu_x(k) \partial \nu_x(i)} = n^2 E[(Z(1)Z(k))|A] P(A) - n^2 \nu_x(1) H_{p+2}(c;\lambda) \nu_x(k)
\]

\[
- n^2 \nu_x(k) H_{p+2}(c;\lambda) \nu_x(1) + n^2 \nu_x(1) \nu_x(k) P(A)
\]

Therefore, equating (2.18) and (2.20 we have

\[
E[Z(1)Z(k)|A] P(A) = H_{p+4}(c;\lambda) \nu_x(1) \nu_x(k)
\]

(2.21)

We may now let \( E[ZZ'|A]P(A) = D \) where \( D \) is a \( p \times p \) matrix with

i-th diagonal element = \( (\nu_x(1))^2 H_{p+4}(c;\lambda) + \frac{1}{n} H_{p+2}(c;\lambda) \) and

the \((i,k)\)-th off diagonal element = \( H_{p+4}(c;\lambda) \nu_x(i) \nu_x(k) \).

Finally we note that

\[
E(\bar{Y}Z|A)P(A) = E(E(\bar{Y}Z|Z,A)|A)P(A)
\]

\[
= E[\bar{Z} E[\bar{Y}|Z]|A]P(A)
\]

\[
= E[\bar{Z} [\mu + \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{22}^{T-1} (Z - \nu_x)]|A]P(A)
\]

\[
= \mu E(Z|A)P(A) + E(Z\Sigma_{12} \Sigma_{22}^{-1} \Sigma_{22}^{T-1} Z|A)P(A)
\]

\[
- E(Z\Sigma_{12} \Sigma_{22}^{-1} \nu_x|A)P(A)
\]

But since \( \Sigma_{12} \Sigma_{22}^{-1} T^{-1} Z \) is a scalar,

\[
\Sigma_{12} \Sigma_{22}^{-1} T^{-1} Z = Z' T^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{T-1} Z
\]

Similarly \( \Sigma_{12} \Sigma_{22}^{-1} \nu_x = \nu_x' T^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{T-1} \nu_x \).
Hence

\[ E(\tilde{Y}Z|A)P(A) = \mu E(Z|A)P(A) + E[ZZ'|A]P(A)T^{-1}_{22}E_{22} \]

\[- \ E(Z|A)P(A)\gamma_x'T^{-1}_{22}E_{22} \]

or

\[ E(\tilde{Y}Z|A)P(A) = \mu H_{p+2}(c;\lambda)\gamma_x + DT^{-1}_{22}E_{22} \]

\[- \ H_{p+2}(c;\lambda)\gamma_x\gamma_x'T^{-1}_{22}E_{22} \]

(2.22)

We also note \( E(\tilde{Y}^2) = \frac{1}{n} \sigma^2 + \mu^2 \). Substituting these into (2.14) we obtain

\[ E(\hat{\mu}^2) = \frac{1}{n} \sigma^2 + \mu^2 - 2\mu E_{12}E_{22}H_{p+2}(c;\lambda)\gamma_x \]

\[- \ E_{12}E_{22}DT^{-1}_{22}E_{22} + 2E_{12}E_{22}\gamma_x\gamma_x'T^{-1}_{22}E_{22}H_{p+2}(c;\lambda) \]

From the last section we have

\[ [E(\hat{\mu})]^2 = [\mu - E_{12}E_{22}H_{p+2}(c;\lambda)\gamma_x]^2 \]

\[ = \mu^2 - 2\mu E_{12}E_{22}H_{p+2}(c;\lambda)\gamma_x \]

\[ + E_{12}E_{22}[H_{p+2}(c;\lambda)] \gamma_x\gamma_x'T^{-1}_{22}E_{22} \]

Substitution in (2.13) yields
\( V(\hat{u}) = \frac{1}{n} \sigma^2 + \mu^2 - 2 \mu \Sigma_{12} \Sigma_{22}^{-1} T^{-1} H_{p+2}(c; \lambda) \nu_x \)

\[- \Sigma_{12} \Sigma_{22}^{-1} T^{-1} D T^{-1} \Sigma_{22}^{-1} \Sigma_{21} + 2 \Sigma_{12} \Sigma_{22}^{-1} T^{-1} \nu_{x-x} T^{-1} \Sigma_{22}^{-1} \Sigma_{21} H_{p+2}(c; \lambda) \]

\[- \mu^2 + 2 \mu \Sigma_{12} \Sigma_{22}^{-1} H_{p+2}(c; \lambda) \nu_x \]

\[- \Sigma_{12} \Sigma_{22}^{-1} T^{-1} [H_{p+2}(c; \lambda)]^2 \nu_{x-x} T^{-1} \Sigma_{22}^{-1} \Sigma_{21} \]

\[= \frac{1}{n} \sigma^2 - \Sigma_{12} \Sigma_{22}^{-1} T^{-1} D T^{-1} \Sigma_{22}^{-1} \Sigma_{21} \quad (2.23) \]

\[- \Sigma_{12} \Sigma_{22}^{-1} T^{-1} [H_{p+2}(c; \lambda)]^2 \nu_{x-x} T^{-1} \Sigma_{22}^{-1} \Sigma_{21} \]

\[+ 2 \Sigma_{12} \Sigma_{22}^{-1} T^{-1} \nu_{x-x} T^{-1} \Sigma_{22}^{-1} H_{p+2}(c; \lambda) \]

Again as partial checks, when \( c=0 \), we always use \( \hat{u} = \bar{y} \) and \( V(\hat{u}) = \frac{1}{n} \sigma^2 \). When \( c=\infty \), we always use \( \hat{u} = \bar{y} - \Sigma_{12} \Sigma_{22}^{-1} x \) and \( V(\hat{u}) = \frac{1}{n} (\sigma^2 - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}) = V(\hat{u}) \).

Now the M.S.E. is defined as M.S.E. = Variance + (Bias)^2.

Hence

\[ \text{M.S.E.}(\hat{u}) = \frac{1}{n} \sigma^2 - \Sigma_{12} \Sigma_{22}^{-1} T^{-1} D T^{-1} \Sigma_{22}^{-1} \Sigma_{21} \]

\[- \Sigma_{12} \Sigma_{22}^{-1} T^{-1} [H_{p+2}(c; \lambda)]^2 \nu_{x-x} T^{-1} \Sigma_{22}^{-1} \Sigma_{21} \]

\[+ \Sigma_{12} \Sigma_{22}^{-1} T^{-1} [H_{p+2}(c; \lambda)]^2 \nu_{x-x} T^{-1} \Sigma_{22}^{-1} \Sigma_{21} \]

\[+ 2 \Sigma_{12} \Sigma_{22}^{-1} T^{-1} \nu_{x-x} T^{-1} \Sigma_{22}^{-1} \Sigma_{21} H_{p+2}(c; \lambda) \]
M.S.E.($\hat{\mu}$) = $\frac{1}{n} \sigma^2 - \Sigma_{12}^{\Sigma^{-1} \mathbf{T}^{-1} \mathbf{D} \mathbf{T}^T - \Sigma^{-1} \Sigma_2 21}$

$$+ 2 \Sigma_{12}^{\Sigma^{-1} \mathbf{T}^{-1} \mathbf{y} \mathbf{y}' \mathbf{y}'^T - \Sigma^{-1} \Sigma_2 21 \mathbf{H}_p + 4 (c; \lambda)$$

$$= \frac{1}{n} \sigma^2 - \Sigma_{12}^{\Sigma^{-1} \mathbf{H}_x \mathbf{H}_x' \mathbf{H}_x' \mathbf{H}_x \Sigma^{-1} \Sigma_2 21 \mathbf{H}_p + 4 (c; \lambda)$$

$$- \frac{1}{n} \Sigma_{12}^{\Sigma^{-1} \Sigma_2 21 \mathbf{H}_p + 2 (c; \lambda)}$$

$$+ 2 \Sigma_{12}^{\Sigma^{-1} \mathbf{H}_x \mathbf{H}_x' \mathbf{H}_x' \mathbf{H}_x \Sigma^{-1} \Sigma_2 21 \mathbf{H}_p + 2 (c; \lambda)}$$

(2.24)

D. Relative Efficiency (e)

In practice, we may want to select an estimator for $\mu$ with the smallest bias and M.S.E. Since bias is a part of M.S.E., it is reasonable to consider only the M.S.E. Using (2.24) we may compare the performance of the preliminary test estimator, $\hat{\mu}$, with the usual estimator $\bar{y}$. The relative efficiency of $\hat{\mu}$ to $\bar{y}$ is defined as

$$e = \frac{1}{\text{M.S.E.}(\hat{\mu})} \bigg/ \frac{1}{\text{M.S.E.}(\bar{y})}$$

(2.25)

Now using (2.24) and since $\bar{y}$ is unbiased, $\text{M.S.E.}(\bar{y}) = \mathbb{V}(\bar{y}) = \frac{1}{n} \sigma^2$. Hence

$$e = \frac{\frac{1}{n} \sigma^2}{\frac{1}{n} \sigma^2 - \Sigma_{12}^{\Sigma^{-1} \mathbf{H}_x \mathbf{H}_x' \mathbf{H}_x' \mathbf{H}_x \Sigma^{-1} \Sigma_2 21 \mathbf{H}_p + 4 (c; \lambda) - \frac{1}{n} \Sigma_{12}^{\Sigma^{-1} \Sigma_2 21 \mathbf{H}_p + 2 (c; \lambda)}$$

$$+ 2 \Sigma_{12}^{\Sigma^{-1} \mathbf{H}_x \mathbf{H}_x' \mathbf{H}_x' \mathbf{H}_x \Sigma^{-1} \Sigma_2 21 \mathbf{H}_p + 2 (c; \lambda)}$$

$$= \frac{1}{1+k(a)}$$
where

$$k(a) = \frac{n}{\sigma^2} \left\{ -\Sigma_{12} \Sigma_{22}^{-1} x_x' \Sigma_{22}^{-1} H_p + 4 (c; \lambda) - \frac{1}{n} \Sigma_{12} \Sigma_{22}^{-1} H_p + 2 (c; \lambda) \right\} + 2 \Sigma_{12} \Sigma_{22}^{-1} x_x' \Sigma_{22}^{-1} H_p + 2 (c; \lambda) \} \quad (2.26)$$

Wlog we let $\Sigma_{22} = I$ and $\sigma^2 = 1$. Therefore for $p=1$, $\Sigma_{12} = \rho$. Table 2.3 gives the values of $e$ for $p=1$ and some choices of $\rho, a$ and $\alpha$.

Table 2.3. Values of $e$ for $p=1$.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\alpha = .05$</th>
<th>$\alpha = .5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\cdot .5$</td>
<td>$\cdot .9$</td>
</tr>
<tr>
<td>0</td>
<td>1.0073</td>
<td>1.2199</td>
</tr>
<tr>
<td>.5</td>
<td>1.0044</td>
<td>1.1244</td>
</tr>
<tr>
<td>1.0</td>
<td>.9974</td>
<td>.9398</td>
</tr>
<tr>
<td>1.5</td>
<td>.9900</td>
<td>.7976</td>
</tr>
<tr>
<td>2.0</td>
<td>.9858</td>
<td>.7357</td>
</tr>
<tr>
<td>2.5</td>
<td>.9866</td>
<td>.7462</td>
</tr>
<tr>
<td>3.0</td>
<td>.9906</td>
<td>.8078</td>
</tr>
</tbody>
</table>

From Table 2.3 we observe that for fixed $n$, $\rho$ and $\alpha$, the relative efficiency of $\hat{\mu}$ assumes its maximum value when $\mu_x = 0$, it then decreases to a minimum and then increases as $\mu_x$ increases. For fixed $n$, $\mu_x$ and $\alpha$, $e$ is an increasing function.
of $\rho$ while for fixed $n$, $\mu_x$ and $\rho$, $e$ is a decreasing function of $\alpha$.

The selection procedure for an estimator or the level of the preliminary test such that the relative efficiency is the largest when $\mu_x$ equals the origin, say 0, suggested by the experimenter's prior knowledge, and is at least as large as some $e_{\min}$ when $\mu_x \neq 0$ was first recommended by Han and Bancroft (1968) and was later used by Han (1973a) for the case $p=1$ for the present problem. The values of $e_{\min}$ and $e_{\max}$ at certain values of $\rho$ and $\alpha$ are given in Table 2.4 where $e_{\max}$ is the value of $e$ at $\mu_x = 0$.

Table 2.4. Values of $e_{\min}$ and $e_{\max}$ for $p=1$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\rho$</th>
<th>.1</th>
<th>.5</th>
<th>.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>.50</td>
<td>$e_{\max}$</td>
<td>1.0007</td>
<td>1.0182</td>
<td>1.0613</td>
</tr>
<tr>
<td></td>
<td>$e_{\min}$</td>
<td>0.9992</td>
<td>0.9793</td>
<td>0.9340</td>
</tr>
<tr>
<td>.05</td>
<td>$e_{\max}$</td>
<td>1.0073</td>
<td>1.2199</td>
<td>2.4036</td>
</tr>
<tr>
<td></td>
<td>$e_{\min}$</td>
<td>0.9858</td>
<td>0.7337</td>
<td>0.4621</td>
</tr>
</tbody>
</table>

Thus for $\rho = .9$, a preliminary test at $\alpha = .05$ ensures the relative efficiency of the preliminary test estimator will be at least 0.4621 and may be as large as 2.4036 when the null hypothesis of the preliminary test is true or $\mu_x = 0$. For a more detailed table and full discussion on the properties and uses of the above table, one is referred to Han (1973a).
Table 2.5. Values of \( e \) for \( p=2 \).

<table>
<thead>
<tr>
<th>( \Sigma_{12} ) ((u_x/n)^{1/2})</th>
<th>( \alpha = .05 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(-.5)</td>
</tr>
<tr>
<td>((0, 0))</td>
<td>1.6673</td>
</tr>
<tr>
<td>((.5, 0))</td>
<td>1.4792</td>
</tr>
<tr>
<td>((.5, .5))</td>
<td>1.5930</td>
</tr>
<tr>
<td>((1.0, 0))</td>
<td>1.3336</td>
</tr>
<tr>
<td>((1.0, .5))</td>
<td>1.3819</td>
</tr>
<tr>
<td>((1.0, 1.0))</td>
<td>1.4124</td>
</tr>
<tr>
<td>((1.5, 0))</td>
<td>.8723</td>
</tr>
<tr>
<td>((1.5, .5))</td>
<td>1.0933</td>
</tr>
<tr>
<td>((1.5, 1.0))</td>
<td>1.2382</td>
</tr>
<tr>
<td>((1.5, 1.5))</td>
<td>1.2219</td>
</tr>
<tr>
<td>((2.0, 0))</td>
<td>.7348</td>
</tr>
<tr>
<td>((2.0, .5))</td>
<td>.9004</td>
</tr>
<tr>
<td>((2.0, 1.0))</td>
<td>1.0499</td>
</tr>
<tr>
<td>((2.0, 1.5))</td>
<td>1.1142</td>
</tr>
<tr>
<td>((2.0, 2.0))</td>
<td>1.0914</td>
</tr>
<tr>
<td>((2.5, 0))</td>
<td>.6950</td>
</tr>
<tr>
<td>((2.5, .5))</td>
<td>.8169</td>
</tr>
<tr>
<td>((2.5, 1.0))</td>
<td>.9398</td>
</tr>
<tr>
<td>((2.5, 1.5))</td>
<td>1.0198</td>
</tr>
<tr>
<td>((2.5, 2.0))</td>
<td>1.0409</td>
</tr>
<tr>
<td>((2.5, 2.5))</td>
<td>1.0279</td>
</tr>
<tr>
<td>((3.0, 0))</td>
<td>.7273</td>
</tr>
<tr>
<td>((3.0, .5))</td>
<td>.8152</td>
</tr>
<tr>
<td>((3.0, 1.0))</td>
<td>.9066</td>
</tr>
<tr>
<td>((3.0, 1.5))</td>
<td>.9743</td>
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<tr>
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</tr>
<tr>
<td>((3.0, 2.5))</td>
<td>1.0105</td>
</tr>
<tr>
<td>((3.0, 3.0))</td>
<td>1.0059</td>
</tr>
<tr>
<td>$\Sigma_{12}^{*}$</td>
<td>$\alpha = .20$</td>
</tr>
<tr>
<td>-----------------</td>
<td>-----------------</td>
</tr>
<tr>
<td></td>
<td>(.5)</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>1.3141</td>
</tr>
<tr>
<td>(.5, 0)</td>
<td>1.2197</td>
</tr>
<tr>
<td>(.5, .5)</td>
<td>1.2634</td>
</tr>
<tr>
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<td>1.0414</td>
</tr>
<tr>
<td>(1.0, .5)</td>
<td>1.1574</td>
</tr>
<tr>
<td>(1.0, 1.0)</td>
<td>1.1579</td>
</tr>
<tr>
<td>(1.5, 0)</td>
<td>.9099</td>
</tr>
<tr>
<td>(1.5, .5)</td>
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</tr>
<tr>
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<td>1.0691</td>
</tr>
<tr>
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<td>.8570</td>
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<td>.9748</td>
</tr>
<tr>
<td>(3.0, 1.5)</td>
<td>.9936</td>
</tr>
<tr>
<td>(3.0, 2.0)</td>
<td>1.0006</td>
</tr>
<tr>
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<td>1.0014</td>
</tr>
<tr>
<td>(3.0, 3.0)</td>
<td>1.0007</td>
</tr>
<tr>
<td>$\mu_n^*/\sqrt{n}$</td>
<td>$\Sigma_{12}$</td>
</tr>
<tr>
<td>-----------------</td>
<td>---------------</td>
</tr>
<tr>
<td></td>
<td>$(-.5)$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td></td>
</tr>
<tr>
<td>$(0, 0)$</td>
<td>1.0831</td>
</tr>
<tr>
<td>$(.5, 0)$</td>
<td>1.0571</td>
</tr>
<tr>
<td>$(1.0, .5)$</td>
<td>1.0673</td>
</tr>
<tr>
<td>$(1.0, 0)$</td>
<td>1.0055</td>
</tr>
<tr>
<td>$(1.0, .5)$</td>
<td>1.0378</td>
</tr>
<tr>
<td>$(1.0, 1.0)$</td>
<td>1.0361</td>
</tr>
<tr>
<td>$(1.5, 0)$</td>
<td>.9688</td>
</tr>
<tr>
<td>$(1.5, .5)$</td>
<td>1.0029</td>
</tr>
<tr>
<td>$(1.5, 1.0)$</td>
<td>1.0167</td>
</tr>
<tr>
<td>$(1.5, 1.5)$</td>
<td>1.0130</td>
</tr>
<tr>
<td>$(2.0, 0)$</td>
<td>.9607</td>
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<tr>
<td>$(2.0, .5)$</td>
<td>.9856</td>
</tr>
<tr>
<td>$(2.0, 1.0)$</td>
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</tr>
<tr>
<td>$(2.0, 1.5)$</td>
<td>1.0049</td>
</tr>
<tr>
<td>$(2.0, 2.0)$</td>
<td>1.0031</td>
</tr>
<tr>
<td>$(2.5, 0)$</td>
<td>.9713</td>
</tr>
<tr>
<td>$(2.5, .5)$</td>
<td>.9852</td>
</tr>
<tr>
<td>$(2.5, 1.0)$</td>
<td>.9956</td>
</tr>
<tr>
<td>$(2.5, 1.5)$</td>
<td>1.0002</td>
</tr>
<tr>
<td>$(2.5, 2.0)$</td>
<td>1.0010</td>
</tr>
<tr>
<td>$(2.5, 2.5)$</td>
<td>1.0005</td>
</tr>
<tr>
<td>$(3.0, 0)$</td>
<td>.9853</td>
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<tr>
<td>$(3.0, .5)$</td>
<td>.9913</td>
</tr>
<tr>
<td>$(3.0, 1.0)$</td>
<td>.9964</td>
</tr>
<tr>
<td>$(3.0, 1.5)$</td>
<td>.9991</td>
</tr>
<tr>
<td>$(3.0, 2.0)$</td>
<td>1.0000</td>
</tr>
<tr>
<td>$(3.0, 2.5)$</td>
<td>1.0001</td>
</tr>
<tr>
<td>$(3.0, 3.0)$</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
few values for \( p=1 \) given here in Table 2.4 are only computed as a partial check of the general results obtained in this paper. The values agree with the results of Han (1973a).

For \( p=2 \), the values of \( e \) are given in Table 2.5 for some choices of \( \Sigma_{12}, \mu_x \sqrt{n} \) and \( \alpha \). Since \( e \) is a symmetric function of \( \Sigma_{12} \) and \( \mu_x \), values are computed for only positive values of \( \Sigma_{12} \) and \( \mu_x \) when the components are identical.

From Table 2.5 and (2.26) we note the following properties of \( e \) for \( p=2 \).

1. The relative efficiency is maximum when \( \mu_x = 0 \) for fixed \( n, \alpha \) and \( \Sigma_{12} \). This corresponds to the case when the null hypothesis is true.

2. The maximum value of \( e \) increases with \( \Sigma_{12} \) for any given \( \alpha \) but decreases as \( \alpha \) increases for a given \( \Sigma_{12} \).

3. For fixed \( n, \mu_x, \alpha \) and \( \Sigma_{12} \), the relative efficiency is generally larger when the components of \( \Sigma_{12} \) have different signs than when the signs are identical.

4. The relative efficiency remains the same for values of \( \Sigma_{12} \) which differ only in sign.

5. For fixed \( \alpha, n, \Sigma_{12} \) and some component of \( \mu_x \), the relative efficiency decreases to a minimum and then increases as the other component increases.

We also observe that since \( \Sigma \) is positive definite, its determinant is greater than zero. Consequently for identical components of \( \Sigma_{12} \), say \( \Sigma_{12} = (a, a) \), then
\[
\Sigma = \begin{pmatrix}
1 & a & a \\
a & 1 & 0 \\
a & 0 & 1 \\
\end{pmatrix}
\]

\[|\Sigma| = 1 - a^2 - a^2 = 1 - 2a^2 > 0\]

\[a^2 < \frac{1}{2} \text{ or } |a| < \frac{1}{\sqrt{2}} = .70\]

Thus the relative efficiency and Bias of \(\hat{\mu}\) do not exist for values such as \(\Sigma_{12} = (.9, .9), (.8, .8)\). Similarly for non-identical components of \(\Sigma_{12}\), say \(\Sigma_{12} = (a, b)\), then

\[
\Sigma = \begin{pmatrix}
1 & a & b \\
a & 1 & 0 \\
b & 0 & 1 \\
\end{pmatrix}
\]

\[|\Sigma| = 1 - a^2 - b^2 > 0\]

\[a^2 + b^2 < 1\]

Hence the relative efficiency and bias of \((\hat{\mu})\) do not exist for such values of \(\Sigma_{12}\) as \((.9, .7)\), etc.

Following Han (1973a), it is possible to extend the computation of \(e_{\max}\) and \(e_{\min}\) to any value of \(p\) so that an investigator can select an estimator or \(\alpha\) such that \(e_{\max}\) occurs when \(\mu_x = 0\) and \(e\) is at least as large as \(e_{\min}\) when \(\mu_x \neq 0\). Table 2.6 gives the values of \(e_{\max}\) and \(e_{\min}\) for some choices of \(\alpha\),
\[ \Sigma_{12} = (.5, .5) \] and \( p = 2 \). It also gives \( \mu_x^* \) which is the value of \( \mu_x \) about which \( e_{\text{min}} \) occurs for a search at .05 intervals.

Table 2.6. Values of \( e_{\text{min}} \) and \( e_{\text{max}} \) for \( p = 2 \).

<table>
<thead>
<tr>
<th>( \Sigma_{12} )</th>
<th>( .05 )</th>
<th>( .10 )</th>
<th>( .20 )</th>
<th>( .30 )</th>
<th>( .40 )</th>
<th>( .50 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_{\text{max}} )</td>
<td>1.6673</td>
<td>1.5032</td>
<td>1.3141</td>
<td>1.2040</td>
<td>1.1322</td>
<td>1.0831</td>
</tr>
<tr>
<td>( e_{\text{min}} ) (( .5, .5 ))</td>
<td>.4976</td>
<td>.5904</td>
<td>.7098</td>
<td>.7941</td>
<td>.8587</td>
<td>.9085</td>
</tr>
<tr>
<td>( \mu_x^* ) (( 1.65, 1.60, 1.40, 1.35, 1.35, 1.35 ))</td>
<td>(( 1.65, 1.60, 1.40, 1.35, 1.35, 1.35 ))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus for a relative efficiency of at least .75, with the above selection procedure, the investigator would use \( \alpha = .30 \) for the preliminary test when \( \Sigma_{12} = (.5, .5) \). This choice guarantees a relative efficiency of at least .79. The relative efficiency in this case can be as large as 1.2040. Also from Table 2.6, we observe as before that for fixed \( \Sigma_{12} \),

1. \( e_{\text{max}} \) is a decreasing function of \( \alpha \),
2. \( e_{\text{min}} \) is an increasing function of \( \alpha \), and
3. \( \mu_x^* \) has identical components and is a decreasing function of \( \alpha \). We note that the negative values of \( \mu_x^* \) also give the same minimum values.

E. Bias of \( \hat{\mu} \) when \( \Sigma \) is Unknown

When \( \Sigma \) is unknown and assume that \( \mu_0 = 0 \), the preliminary test estimator is defined as
\[ \hat{\mu} = \begin{cases} \bar{y} - S_{12}S_{22}^{-1}\bar{x} & \text{if } nm(\bar{x}'S_{22}^{-1}\bar{x}) \leq T_0^2 \\ \bar{y} & \text{if } nm(\bar{x}'S_{22}^{-1}\bar{x}) > T_0^2 \end{cases} \]  

(2.27)

where \( m = n-1 \), \( T_0^2 \) is the 100(1-\( \alpha \))th percentile of a central Hotelling's \( T^2 \) distribution with \( m \) degrees of freedom, and

\[ \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \]

\[ S_{22} = \sum_{i=1}^{n} (X_i - \bar{X})(X_i - \bar{X})' \]

\[ S_{12} = \sum_{i=1}^{n} (y_i - \bar{y})(X_i - \bar{X})' \]

\[ S_{11} = \sum_{i=1}^{n} (y_i - \bar{y})^2 \]

In this section, we shall obtain the bias of \( \hat{\mu} \). If we denote the acceptance region for the preliminary test \{nm(\bar{x}'S_{22}^{-1}\bar{x}):nm(\bar{x}'S_{22}^{-1}\bar{x}) \leq T_0^2 \} by \( G \), then

\[
E(\hat{\mu}) = E((\bar{y} - S_{12}S_{22}^{-1}\bar{x})|G) P(G) + E(\bar{y}|\bar{G}) P(\bar{G})
\]

\[= E(\bar{y}) - \frac{S_{12}S_{22}^{-1}E(\bar{x})}{P(G)} \]

(2.29)

Since \( E(\bar{y}) = \mu \), the second term is the bias and we write

\[ b = - \frac{E(S_{12}S_{22}^{-1}\bar{x})}{P(G)} \]

(2.30a)
Let $f(\boldsymbol{X})$ be the multivariate normal density of $\boldsymbol{X}$ and 
$g(S_{22}, S_{12}, S_{11})$ be the density of $S_{22}$, $S_{11}$ and $S_{12}$ which have 
a Wishart distribution. Then

$$E[S_{12}S_{22}^{-1} \mid G] \quad P(G)$$ (2.30b) 

$$= \int_{G} S_{12}S_{22}^{-1}f(\boldsymbol{X})g(S_{22}, S_{12}, S_{11})d\boldsymbol{X}dS_{22}dS_{12}dS_{11}$$

Following Han (1973a) and Rao (1965), we make the following transformations. Since $S_{22}$ is positive definite, there exists 
a nonsingular matrix $B \in B'\in B = S_{22}$. Also $\exists$ a nonsingular 
matrix $T \in T'\in T = E_{22}^{-1}$. Let

$$W_1 = TB'B^t$$

$$W_2 = [S_{12} - E_{12}T'TB]E_{11.2}^{1/2}B^{-1}.$$ 

But since $E_{11.2}$ is a constant scalar, we let

$$E_{11.2} = k^2$$ (2.31)

Therefore

$$W_2 = \frac{1}{k}[S_{12}B^{-1} - E_{12}T'TB']$$

$$W_3 = k^2(S_{11} - S_{12}B^{-1}B'^{-1}S_{21})$$

From (2.31),
\[ B'B = T^{-1}W_1T^{-1} = T^{-1}W_1W_1^T T^{-1} \]

\[ S_{12} = ([KW_2 + \Sigma_{12} T'TB'] B \]

Therefore,
\[ S_{12}S_{22}^{-1} = S_{12}B^{-1}B^{-1} = \frac{1}{2}KW_2W_1^T + \Sigma_{12}T'T \]

Substituting in (2.30b), we have
\[ E\{S_{12}S_{22}^{-1}/X\} \cdot P(G) \]

\[ = \int_{G} \int_{x} (KW_2W_1^T + \Sigma_{12}T'T) f(x)g(w_1, w_2, w_3) dw_3 dw_2 dw_1 dx \]

We claim
\[ W_1 \sim W(I,n-1) \]
\[ W_2 \sim N(0,I) \]
\[ W_3 \sim W(1,n-p-1) \]

and the three are mutually independent. To prove the above claim, we note

(a) \[ S_{22} = B'B \sim W(T^{-1}T^{-1}, n-1) \]

\[ W_1 = TB'B'T \sim W(TT^{-1}T^{-1}T', n-1) \]

\[ = W(I,n-1) . \]
(b) Given the \( X_i \)'s, the conditional distribution of \( W_2' \) is normal since \( W_2'|X_i \)'s is a linear combination of the \( y_i \)'s which are normally distributed. If we denote this conditional distribution by \( g(W_2'|X) \), and write

\[
W_2' = \frac{1}{K} B'^{-1}(S_{21} - B'B'T\Sigma_{21})
\]

then we only need find the mean and variance of \( W_2'|X \).

\[
S_{21} = \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}) = \sum_{i=1}^{n} (X_i - \bar{X})Y_i.
\]

Now there exists an \( n \)-th order Helmert matrix \( C = (c_{ij}) \) such that \( U_1 = (U_{11} \ldots U_{p1}) \), \( U_{j1} = i \)-th element of \( CX_j \), \( \omega_i = i \)-th element of \( CY \) so that making an orthogonal transformation,

\[
S_{21} = \sum_{i=1}^{m} U_{i1} \omega_i \quad \text{where} \ m = n-1.
\]

\[
\omega_i | U_1 \sim N(\Sigma_1 T^{-1} U_1, \Sigma_{1i.2})
\]

\[
= N(U_1' T^{-1} \Sigma_{21}, \Sigma_{1i.2}).
\]

Therefore

\[
E(S_{21}|U) = \sum_{i=1}^{m} U_{i1} U_{i1}' T^{-1} \Sigma_{21}
\]

\[
= B'B'T\Sigma_{21}.
\]
Hence

\[ \text{E}(W_2|U) = \frac{1}{K} B^{-1}(B'B'T\Sigma_{21} - B'B'T\Sigma_{21}) \]

\[ = 0 . \]

\[ \text{Var}(W_2|U) = \frac{1}{K^2} B^{-1} V(S_{21}|U) B^{-1} \]

\[ = \frac{1}{K^2} B^{-1} \left( \sum_{i=1}^{m} U_i \Sigma_{11.2} U_i' \right) B^{-1} \]

where recall \( K^2 = \Sigma_{11.2} \) is a scalar.

\[ = \frac{1}{K^2} \Sigma_{11.2} B^{-1}B'B'B^{-1} = I' \]

Therefore \( (W_2|X) \sim N(0,I) \) and this does not depend on \( X \).

Hence \( W_2 \sim N(0,I) \).

(c) \( W_3 = \frac{1}{K^2}(S_{11} - S_{12}B^{-1}B'^{-1}S_{21}) \) and from Anderson (1958), Theorems 4.3.2 and 4.3.3, we know \( K^2 W_3 = S_{11} - S_{12}B^{-1}B'^{-1} \sim \tilde{W}(\Sigma_{11.2}, n-p-1) \) and hence \( \tilde{W}_3 \sim \tilde{W}(1, n-p-1) \). Finally, to establish the mutual independence of \( W_1, W_2 \) and \( W_3 \) we note that by fixing the \( X_1 \)'s, we also fix \( W_1 \) and since the conditional distribution of either \( W_2 \) or \( W_3 \) with \( X_1 \) fixed does not depend on \( X \), then either conditional distribution is equivalent to the actual unconditional distribution and each is independent of \( W_1 \). Thus \( W_1 \) is independent of \( W_2 \) and \( W_3 \). To show \( W_2 \) and \( W_3 \) independent we employ Cochran's theorem as follows:
Let
\[ u_1 = y_1 - \mu - \Sigma_{12} T' T (x_1 - \mu_{x_1}) \sim \text{NID}(0, \Sigma_{11,2}) \]
and
\[ V = \bar{x} - \mu_x \sim \text{N}(0, T^{-1} T^{-1}) \]
be fixed. Then
\begin{align*}
\bar{u} &= \bar{y} - \mu - \Sigma_{12} T' T (\bar{x}_1 - \mu_x) \sim \text{N}(0, \frac{1}{n} \Sigma_{11,2}) \\
\end{align*}

\[ \frac{nu^2}{\Sigma_{11,2}} = \frac{n[\bar{y} - \mu - \Sigma_{12} T' T (\bar{x}_1 - \mu_x)]^2}{\Sigma_{11,2}} \sim \chi^2(1) \]

\[ S_{uV} = \sum_{i=1}^{n} u_1 (V_i - \bar{V})' = S_{12} - \Sigma_{12} T' T B' B \sim \text{N}(0, \Sigma_{11,2} B' B) \]

\[ W_2 = \frac{1}{\Sigma_{11,2}^{1.2}} (S_{12} T^{-1} T^{-1} B' B) B^{-1} \sim \text{N}(0, I) \]

\[ \Sigma_{11,2} (S_{uV} S_{VV}^{-1} S_{V'u}) = \bar{W}_2 W_2' \sim \chi^2(p) \]

where
\[ S_{VV} = \sum_{i=1}^{n} (V_i - \bar{V})(V_i - \bar{V})' = B' B. \]

Similarly defining
\[ S_{uu} = \sum_{i=1}^{n} (u_i - \bar{u})^2, \]
then finally
\[ \Sigma_{11}^{-1} \cdot 2 (S_{uu} - S_{uv} S_{vv}^{-1} S_{vu}) \]

\[ = W_3 = \Sigma_{11}^{-1} (S_{11} - S_{12} B^{-1} B^{-1} S_{21}) \sim \chi^2(n-p-1). \]

But

\[ S_{uu} = \sum_{i=1}^{n} u_i^2 - n \bar{u}^2 = S_{uv} S_{vv}^{-1} S_{vu} + (S_{uu} - S_{uv} S_{vv}^{-1} S_{vu}) \]

and hence

\[ \Sigma_{11}^{-1} \cdot 2 \sum_{i=1}^{n} u_i^2 = \Sigma_{11}^{-1} \cdot 2 n \bar{u}^2 + \Sigma_{11}^{-1} (S_{uv} S_{vv}^{-1} S_{vu}) \]

\[ + \Sigma_{11}^{-1} (S_{uu} - S_{uv} S_{vv}^{-1} S_{vu}) \]

or

\[ \Sigma_{11}^{-1} \cdot 2 \sum_{i=1}^{n} (y_i - \mu)^2 = 2(\bar{y} - \mu) \Sigma_{12} T' T(\Sigma_{11}^{-1} u_{x_1}) \]

\[ + (X_1 - \mu_{X_1})' T' \Sigma_{21} \Sigma_{12} T' T(\Sigma_{11}^{-1} u_{x_1}) \]

\[ = \Sigma_{11}^{-1} \cdot 2 n [\bar{y} - \mu_{X_1} - \Sigma_{12} T' T(\Sigma_{11}^{-1} u_{x_1})]^2 + W_2 W_2' + W_3 \]

or

\[ \chi^2(n) = \chi^2(1) + \chi^2(p) + \chi^2(n-p-1). \]

Thus \( W_2 \) and \( W_3 \) are independent.

Therefore \( W_1, W_2 \) and \( W_3 \) have a joint distribution given by

\[ g(W_1, W_2, W_3) = C_0 e^{-\frac{1}{2} \text{tr}(W_2 W_2' + W_3 W_3') } \frac{1}{2!(n-2p-2)} |W_3|^{\frac{1}{2}} |W_1|^{\frac{1}{2}} \]

\[ (2.33) \]
The region of integration is given by
\[ G = \{ \mathbf{w} : \mathbf{w}'\mathbf{X}'\mathbf{W}_1^{-1}\mathbf{X} \leq T_0^2 \} \]

The integral in (2.32) becomes
\[
\begin{align*}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (K\mathbf{W}_2\mathbf{W}_1 + \Sigma_{12}\mathbf{T}'\mathbf{T}) \bar{f}(\mathbf{x}) C_0 e^{-\frac{1}{2}\text{tr}(\mathbf{W}_1\mathbf{W}_2 + \mathbf{W}_3 + \mathbf{W}_1)} \prod_{i=1}^{n-p-2} |W_i|^{\frac{1}{2}(n-p-2)} dW_3 dW_2 dW_1 d\mathbf{x}.
\end{align*}
\]

But from independence and the fact \( E(W_2) = 0 \), we know the first term is zero. The second integral is equivalent to
\[
\begin{align*}
\frac{1}{2^{p(n-1)} \pi^{p(p-1)/2}} \Gamma\left[\frac{1}{2}(n-1)\right] \prod_{i=1}^{n-p-2} |W_i|^{\frac{1}{2}(n-p-2)} \bar{f}(\mathbf{x}) d\mathbf{x} dW_1.
\end{align*}
\]

Now \( \mathbf{x} \sim N(\mathbf{u}_x, \frac{1}{n}T^{-1}T^{-1}) \). Let \( \mathbf{z} = T\mathbf{x} \). Therefore
\[ Z \sim N(\mathbf{T}_x, \frac{1}{n} I) \]

\[ \mathbf{X} = N(\mathbf{V}_x, \frac{1}{n} I) \text{ say.} \]

Now \( \mathbf{X} = \mathbf{T}^{-1}Z \) and \( \mathbf{X}' \mathbf{W}_1^{-1} \mathbf{X} = \mathbf{Z}' \mathbf{T}^{-1} \mathbf{W}_1^{-1} \mathbf{T}^{-1} \mathbf{Z} = \mathbf{Z}' \mathbf{W}_1^{-1} \mathbf{Z} \), we have \( \mathcal{G} = \{ \text{nm}(\mathbf{Z}' \mathbf{W}_1^{-1} \mathbf{Z}) : \text{nm}(\mathbf{Z}' \mathbf{W}_1^{-1} \mathbf{Z}) < T^2 \} \). Hence we wish to evaluate

\[
\sum_{1}^{T^2} \prod_{i=1}^{1/2(n-1)} G \frac{1/2 \text{tr} \mathbf{W}_1}{\Gamma(1/2(n-1))} \frac{1/2(n-p-2)}{2^{1/2(2p(n-1))} \pi^{1/2(p-1)} \Gamma(1/2(n-1))} \int \mathbf{Z}(\mathbf{Z}) d\mathbf{Z} d\mathbf{W}_1
\]

(2.34)

where \( \text{nm}(\mathbf{Z}' \mathbf{W}_1^{-1} \mathbf{Z}) = T^2 \) has the Hotelling's \( T^2 \) distribution with \( n-1 \) degrees of freedom and \( \frac{T^2(n-p)}{p(n-1)} = \mathcal{F}^{*} \) has the noncentral \( F \)-distribution with \( p \) and \( n-p \) degrees of freedom and non-centrality parameter \( \lambda = \frac{\mathbf{V}_x' \mathbf{V}_x}{\mathbf{T}_x} \).

Following Alam and Risvi (1967), we define a random variable \( \mathcal{G} \) given by

\[ \mathcal{G} = \frac{p \mathcal{F}^{*}}{n-p} = \frac{T^2}{n-1} \]

\( \mathcal{G} \) has the density function

\[
f(g) = f_{p, n-p}(g, \lambda) = \frac{1}{\Gamma(n-p/2)} \sum_{j=0}^{\infty} \frac{\lambda^{j}}{2^{j} \Gamma\left(\frac{j}{2}\right)} \int_{g}^{(p+1, n-p)}(g) \, dg
\]

(2.35)

where
\[ G_{\left(\frac{p+j}{2}, \frac{n-p}{2}\right)}(g) = \frac{g^{p+j-1} \Gamma\left(\frac{p+j+n-p}{2}\right)}{(1+g)^{\frac{p+j+n-p}{2}} \Gamma\left(\frac{p+j}{2}\right)} \]

Therefore

\[ P(G) = P(T^2 \leq T_0^2) \]

\[ = P(T^2 \leq \frac{p(n-1)}{n-p} F_{p,n-p}(\alpha)) \]

\[ = P\left(\frac{T^2}{n-1} \leq \frac{p}{n-p} F_{p,n-p}(\alpha)\right) \]

\[ = P(G \leq c) \]

where \( G \) has the density function given in (2.35) and \( c = \frac{p}{n-p} F_{p,n-p}(\alpha) \) where \( F_{p,n-p}(\alpha) \) is the 100(1-\( \alpha \)) percent point of the central \( F \)-distribution with \( p \) and \( n-p \) degrees of freedom.

Therefore

\[ R = P(G) = \int_0^c \frac{e^{-\frac{1}{2} \lambda}}{\Gamma\left(\frac{n-p}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\lambda}{2}\right)^j G_{\left(\frac{p+j}{2}, \frac{n-p}{2}\right)}(g) dg \]

Differentiating (2.37) w.r.t. \( v_x^{(1)} \), we have

\[ \frac{\partial R}{\partial v_x^{(1)}} = \int_0^c \frac{e^{-\frac{1}{2} \lambda}}{\Gamma\left(\frac{n-p}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\lambda}{2}\right)^j \frac{2n v_x^{(1)}}{2} g_{\left(\frac{p+j}{2}, \frac{n-p}{2}\right)}(g) dg \]
For the first term if we let \( j-1 = j' \), then \( j = j'+1 \) and consequently we may write

\[
\frac{\partial R}{\partial \nu_x} = \int_0^\infty \frac{c}{2\Gamma(n-p/2)} e^{-\frac{1}{2}\lambda} \sum_{j'=0}^\infty \frac{2\nu_x(1)}{2} \left( \frac{\lambda}{2} \right)^j j! G \left( \frac{p+2+j', n-p}{2} \right) dg
\]

so that

\[
\frac{\partial R}{\partial \nu_x} = \nu_x \left( G^*_{p+2, n-p}(c; \lambda) - \nu_x P(G) \right) \quad (2.38b)
\]

where \( G^*_{p+2, n-p}(c; \lambda) \) is the cumulative distribution of the non-central \( G \) random variable with \( p+2, n-p \) degrees of freedom and noncentrality parameter \( \lambda \). Also making use of the separate distributions of \( Z \) and \( W_1 \) and noting that these are independent, we may write

\[
R = P(G) = \prod_{j=1}^p \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{n}{2}(Z(j)-\nu_x(j))^2}
\]

\[
= \frac{1}{2^{p(n-1)} \pi^{(p-1)/2} \prod_{i=1}^p \Gamma\left[ \frac{1}{2}(n-1) \right]} \frac{1}{n^{p-2}} \frac{1}{\frac{1}{2}(n-p-2)} \frac{1}{\frac{1}{2}(n-1)} \frac{1}{\frac{1}{2}(n-p)} |W_1|^\frac{1}{2}(n-p-2) e^{-\frac{1}{2}trW_1} dZ(j) dW_1
\]

(2.39)
Differentiating (2.39) w.r.t. \( v_{x}^{(i)} \), we obtain

\[
\frac{\partial R}{\partial v_{x}^{(i)}} = \int_{\mathbb{G}} \prod_{j=1}^{p} \frac{\sqrt{\pi}}{\sqrt{2\pi}} \cdot \frac{n}{2} \cdot 2(z(j) - v_{x}^{(i)})e^{-\frac{n}{2}(z(j) - v_{x}^{(i)}))^2}
\]

\[
\cdot \frac{1}{2(n-p-2)} \cdot \frac{1}{2} \text{tr} W_{1} \cdot \frac{1}{4} \text{tr} (n-1) \cdot p \cdot \prod_{i=1}^{n} \Gamma \left[ \frac{1}{2}(n-i) \right]
\]

Hence

\[
\frac{\partial R}{\partial v_{x}^{(i)}} = \frac{1}{2} \text{tr} W_{1} \cdot \frac{1}{4} \text{tr} (n-1) \cdot p \cdot \prod_{i=1}^{n} \Gamma \left[ \frac{1}{2}(n-i) \right]
\]

(2.40)

\[
Z^{(i)} g(Z) dZ dW_{1} - n v_{x}^{(i)} P(G)
\]

Equating (2.38b) and (2.40) we have

\[
\frac{1}{2} \text{tr} W_{1} \cdot \frac{1}{4} \text{tr} (n-1) \cdot p \cdot \prod_{i=1}^{n} \Gamma \left[ \frac{1}{2}(n-i) \right]
\]

(2.41)

\[
Z^{(i)} g(Z) dZ dW_{1} = v_{x}^{(i)} G_{p+2, n-p}(\cdot; \lambda)
\]

Finally we let
\[
\begin{align*}
\sum_{i=1}^{p\pi(n-1)} \prod_{i=1}^{p\pi(p-1)} \prod_{i=1}^{p\pi} \left[ \frac{1}{\sqrt{2(n-1)}} \right] \int_{\mathbb{R}^d} e^{-\frac{1}{2} \text{tr} W_1^T W_1} \frac{1}{2(n-p-2)} G(\mathbf{z}) \, d\mathbf{z} \, dW_1 = I(\mathbf{z}(1))
\end{align*}
\]

and note (2.34) is \( \Sigma_{12} T^T \Sigma_{12}^{-1} I(\mathbf{z}) \) where \( I(\mathbf{z}) \) is a pxl vector with \( i \)-th component \( I(\mathbf{z}(i)) \). From (2.41), \( I(\mathbf{z}) = \sum_{x} G^*_{p+2,n-p}(c; \lambda) \). Hence (2.34) becomes

\[
\begin{align*}
\sum_{12} T^T \Sigma_{12}^{-1} \sum_{x} G^*_{p+2,n-p}(c; \lambda) \\
= \sum_{12} \sum_{22} \sum_{x} G^*_{p+2,n-p}(c; \lambda)
\end{align*}
\]

where

\[
G^*_{p+2,n-p}(c; \lambda) = P(G \leq c)
\]

and \( G \) has \( p+2 \) and \( n-p \) degrees of freedom and noncentrality parameter \( \lambda \). Therefore

\[
G^*_{p+2,n-p}(c; \lambda) = P\left( \frac{n-p}{p+2} G \leq \frac{n-p}{p+2} c \right)
\]

\[
= P\left( F^*_{p+2,n-p} \leq \frac{n-p}{p+2} \cdot \frac{p}{n-p} F_{p,n-p}(\alpha) \right)
\]

\[
= P\left( F^*_{p+2,n-p} \leq c_2 \right)
\]

where

\[
c_2 = \frac{p}{p+2} F_{p,n-p}(\alpha)
\]
Therefore Bias = \(-\sum_{12} \Sigma_{p+2, n-p}(c_2; \lambda)\) where \(F_{p+2, n-p}(c_2; \lambda)\) is the cumulative distribution function of the noncentral F distribution with \(p+2\) and \(n-p\) degrees of freedom and non-centrality parameter \(\lambda\).

As a partial check, when \(c_2=0\), the estimator reduces to the usual estimator \(\bar{y}\) which is the case when we always reject the null hypothesis. In this case Bias = 0. When \(c_2=\infty\), the null hypothesis is always accepted and the regression estimator \(\bar{y} - S_{12} S_{22}^{-1} \mu_x\) is always used. The bias in this case is the usual bias for the regression estimator since \(F_{p+2, n-p}(c_2; \lambda) = 1\) and Bias = \(-\sum_{12} \Sigma_{p+2, n-p}(c_2; \lambda)\).

Now for the purpose of comparison with the results of Han (1973a), we let \(p = 1\) and

\[
\rho = \frac{1}{\sigma} \Sigma_{12} T',
\]

\[
\Sigma_{12} T' T = \sigma \rho T
\]

so

\[
-\frac{\text{Bias}}{\sigma} = \rho T' \mu_x F_{p+2, n-p}(c_2; \lambda).
\]  \hspace{1cm} (2.42)

For \(p = 1\) we have

\[
-\frac{\text{Bias}}{\sigma} = \rho \frac{\mu_x}{\sigma_x} F_{3, n-1} \left(\frac{\mu_x}{\sigma_x}, \frac{n}{n-1}(\alpha), \frac{n}{n-1} \right) \frac{\mu_x^2}{\sigma_x^2}.
\]
Wlog we set $\Sigma_{22} = I$ and $\sigma^2 = 1$. Therefore for $p = 1$, $\Sigma_{12} = \rho$ and we study the bias for positive values of $\rho$ and $\mu_x$ since Bias changes sign with either parameter. Table 2.7 gives the values of $-\text{Bias}$ for $p = 1$, $n = 9$ and some choices of $\mu_x$, $\rho$ and $\alpha$.

Table 2.7. Values of $-\text{Bias}$ for $p = 1$ and $n = 9$.

<table>
<thead>
<tr>
<th>$\mu_x$</th>
<th>$\alpha = .05$</th>
<th>$\alpha = .10$</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$\rho$</td>
<td>$\rho$</td>
</tr>
<tr>
<td>$0$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$0.3$</td>
<td>.020</td>
<td>.102</td>
</tr>
<tr>
<td>$0.6$</td>
<td>.028</td>
<td>.139</td>
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<tr>
<td>$0.9$</td>
<td>.020</td>
<td>.101</td>
</tr>
<tr>
<td>$1.2$</td>
<td>.008</td>
<td>.042</td>
</tr>
<tr>
<td>$1.5$</td>
<td>.002</td>
<td>.011</td>
</tr>
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<td></td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The above values are essentially the same as those obtained by Han (1973a) although differences are observed. The differences occur because the expression for the bias given here is in terms of noncentral $F$ distribution, while that of Han is given in terms of moments of normal distributions. Therefore there may be rounding off errors in the computation. We can observe that $\text{Bias} = 0$ when $\mu_x = 0$ or when the null
hypothesis of the preliminary test is true. Also it can be seen that \( -Bias \) is an increasing function of \( \rho \) for fixed \( n, \alpha \) and \( \mu_x \) but a decreasing function of \( \alpha \) for fixed \( n, \rho \) and \( \mu_x \). However, \(-Bias\) increases and then decreases to zero as \( \mu_x \) increases whenever \( \alpha, n \) and \( \rho \) are fixed.

For \( p = 2 \) and \( n = 9 \), the values of \(-Bias\) are given in Table 2.8 for some values of \( \Sigma_{12} \), \( \mu_x \) and \( \alpha \).

Table 2.8. Values of \(-Bias\) for \( p = 2 \) and \( n = 9 \).

<table>
<thead>
<tr>
<th>( \Sigma'_{12} )</th>
<th>( \mu'_x )</th>
<th>( \alpha = .05 )</th>
</tr>
</thead>
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<tr>
<td>( \Sigma'_{12} )</td>
<td>( \mu'_x )</td>
<td>( .5 )</td>
</tr>
<tr>
<td>( 0, 0)</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( .3, 0)</td>
<td>-0.1193</td>
<td>-0.1193</td>
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<tr>
<td>( .3, .3)</td>
<td>-0.1112</td>
<td>0.0</td>
</tr>
<tr>
<td>( .6, 0)</td>
<td>-0.1911</td>
<td>-0.1911</td>
</tr>
<tr>
<td>( .6, .3)</td>
<td>-0.1761</td>
<td>-0.0880</td>
</tr>
<tr>
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</tr>
<tr>
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From Table 2.8, the following properties of the Bias of $\hat{\mu}$ are apparent.

1. The bias is zero when the null hypothesis of the preliminary test of significance is true, that is, when $\mu_x = 0$.

2. For fixed $n$, $\alpha$ and $\mu_x$, the value of the bias generally increases with $\Sigma_{12}$.

3. For fixed $n$, $\alpha$ and $\Sigma_{12}$, the bias generally decreases as $\alpha$ increases.
4. The bias is zero when either \( \mu_x \) or \( \Sigma_{12} \) has identical components and the other has components which differ only in sign.

5. For fixed \( n \), \( \Sigma_{12} \) and \( \alpha \) and some component of \( \mu_x \), the value of the bias first increases, then decreases to zero as the other component increases.

F. The M.S.E. of \( \hat{\mu} \) when \( \Sigma \) is Unknown

The M.S.E. of \( \hat{\mu} \) is

\[
\text{M.S.E.}(\hat{\mu}) = V(\hat{\mu}) + B^2 \tag{2.43}
\]

where

\[
B = \text{Bias}(\hat{\mu}),
\]

and

\[
V(\hat{\mu}) = E(\hat{\mu}^2) - [E(\hat{\mu})]^2.
\]

When \( \Sigma \) is unknown, the preliminary test estimator is given in (2.27). Hence, making use of the notations of Section E,

\[
E(\hat{\mu}^2) = E\{ (\bar{y} - S_{12}s_{22}^{-1}\bar{x})^2 | \Gamma \} P(\Gamma)
\]

\[
+ E(\bar{y}^2 | \Gamma) P(\Gamma)
\]

\[
= E(\bar{y}^2) - 2E[S_{12}s_{22}^{-1}\bar{y}\bar{x} | \Gamma] P(\Gamma)
\]

\[
+ E[(S_{12}s_{22}^{-1}\bar{y}, s_{22}^{-1}s_{21}) | \Gamma] P(\Gamma) \tag{2.44}
\]
Thus we need to evaluate
\[ E[S_{12} S_{22}^{-1} y x | G] P(G) \] and \[ E[(S_{12} S_{22}^{-1} y x') S_{22}^{-1} x_1] | G] P(G) \]

Since \( S_{12} S_{22}^{-1} y x \) is a scalar,
\[ E[(S_{12} S_{22}^{-1} y x') S_{22}^{-1} x_1] | G] P(G) \]

Now using the transformation of (2.31), we have
\[
\text{tr} \cdots \int \left\{ (K T' W_1^2 W_2 + T' T x_2) (K W_2 W_1^{-T} + \Sigma_{12} T' T) y x' f(x) \right\} g(W_1, W_2, W_3) dW_3 dW_2 dW_1 d\bar{x}
\]
\[
= K^2 \text{tr} \cdots \int_{0}^{\infty} T' W_1^2 W_2 W_1^{-T} y x' f(x) c.e \left[ \frac{1}{2} \text{tr}(W_2 W_3 W_1) \right] \left| W_3 \right|^{1/2}(n-p-3) dW_3 dW_2 dW_1 d\bar{x}
\]
\[
+ K^2 \text{tr} \cdots \int_{-\infty}^{0} T' W_1^2 W_2 W_1^{-T} y x' f(x) c.e \left[ \frac{1}{2} \text{tr}(W_2 W_3 W_1) \right] \left| W_3 \right|^{1/2}(n-p-3) dW_3 dW_2 dW_1 d\bar{x}
\]
\[ + \text{tr} \ldots \int_{-\infty}^{\infty} \int_{G} \Sigma_{12} T' \Sigma_{12} T' f(\mathbf{X}) c e^{-\frac{1}{2} \text{tr}(W'W_2 + W_3 + W_1)} \]

\[ \cdot \left| W_3 \right|^{\frac{1}{2}(n-p-3)} \left| W_1 \right|^{\frac{1}{2}(n-p-2)} dW_3 dW_2 dW_1 d\mathbf{X} \]

\[ + \text{tr} \ldots \int_{G} \Sigma_{12} T' T' \Sigma_{12} T' f(\mathbf{X}) c e^{-\frac{1}{2} \text{tr}(W_2 + W_3 + W_1)} \]

\[ \cdot \left| W_3 \right|^{\frac{1}{2}(n-p-3)} dW_3 dW_2 dW_1 d\mathbf{X} \]

Now from independence and the fact

\[ c \int_{-\infty}^{\infty} W_2 W_2 e^{-\frac{1}{2} \text{tr} W_1 W_2 W_2} dW_2 = I, \]

the first term equals

\[ \frac{k^2}{2^{\frac{1}{2}(n-1)} \pi^{\frac{1}{2}p(p-1)}} \left| \prod_{i=1}^{\frac{1}{2}(n-1)} \Gamma \left[ \frac{1}{2}(n-i) \right] \right| \left| W_1 \right|^{\frac{1}{2}(n-p-2)} \int_{G} \int_{\mathbf{X}}\int_{T^2} f(\mathbf{X}) e^{-\frac{1}{2} \text{tr} W_1} dW_1 d\mathbf{X} \]

\[ = \frac{k^2}{2^{\frac{1}{2}(p(n-1))} \pi^{\frac{1}{2}p(p-1)}} \left| \prod_{i=1}^{\frac{1}{2}(n-1)} \Gamma \left[ \frac{1}{2}(n-i) \right] \right| \left| W_1 \right|^{\frac{1}{2}(n-p-2)} \int_{G} \int_{\mathbf{X}}\int_{T^2} f(\mathbf{X}) e^{-\frac{1}{2} \text{tr} W_1} dW_1 d\mathbf{X} \]

(where we recall \( G = \{ \mathbf{n}\mathbf{X}'T'W_1^{-1}\mathbf{T}\mathbf{X} : \mathbf{n}\mathbf{X}'T'W_1^{-1}\mathbf{T}\mathbf{X} < T_0^2 \} \) and \( T'W_1^{-1}T = S_2^{-1} \))
\[ = k^2 E\left[ \bar{X}'S^{-1}\bar{X} \mid nm\bar{X}'S_2^{-1}\bar{X} \leq T_0^2 \right] \]
\[ = k^2 E\left[ \frac{p_t}{n(n-p)} \mid nm\frac{p_t}{n(n-p)} \leq T_0^2 \right] \]
\[ = \frac{k^2 p}{n(n-p)} E[t \mid t \leq \frac{n(n-p)}{nm} \cdot \frac{p(n-1)}{n-p} F_{p,n-p}] \]
\[ = \frac{k^2 p}{n(n-p)} E[t \mid t \leq F_{p,n-p}(\alpha)] \]
\[ = \frac{k^2 p}{n(n-p)} \int_0^d t f(t) = Q \quad (2.45) \]

where \( d = F_{p,n-p}(\alpha) \) and \( t \) has the noncentral \( F \) distribution with \( p \) and \( n-p \) degrees of freedom and noncentrality parameter \( \lambda = \frac{n\mu'\Sigma_{22}^{-1}\Sigma_{21}}{2} \).

Also from independence and the fact \( E(W_2) = 0 \), the second and third terms are zero and the fourth term is equivalent to

\[ \text{tr} \int_{\mathbf{G}} T'T \Sigma_{21} \Sigma_{12} T'T^{-1} \frac{ZZ'}{T'} T^{-1} g(Z) c_1 e^{-\frac{1}{2} tr W_1} |W_1|^\frac{1}{2}(n-p-2) \frac{1}{2} \frac{1}{n-1} dZ dW_1 \]
\[ = \frac{1}{2^p(n-1)} \frac{1}{P} g(Z) c_1 e^{-\frac{1}{2} tr W_1} |W_1|^\frac{1}{2}(n-p-2) \frac{1}{2} \frac{1}{n-1} dZ dW_1 \quad (2.46) \]

We evaluate the diagonal elements of the above integral by differentiating each of the two representations of \( P(\mathbf{G}) \) twice w.r.t. \( v_x^{(1)} \) and equating the results. The off-diagonal elements can similarly be evaluated by differentiating the two
representations first w.r.t. $\nu_x^{(1)}$ and then w.r.t. $\nu_x^{(I)}$ and equating the results.

Differentiating (2.38a) w.r.t. $\nu_x^{(1)}$, we have

$$
\frac{aR}{\nu_x^{(1)}^2} = \int_0^{\infty} \frac{1}{\Gamma\left(\frac{n-p}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{j!} j^{-(\lambda_j/2)-1} \left(\frac{2\nu_x^{(1)}}{2}\right) g_{\frac{p+j}{2}, \frac{n-p}{2}}(g) dg 
$$

$$
+ \int_0^{\infty} \frac{1}{\Gamma\left(\frac{n-p}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{j!} j^{-(\lambda_j/2)-1} \left(\frac{2n}{2}\right) g_{\frac{p+j}{2}, \frac{n-p}{2}}(g) dg 
$$

$$
- \int_0^{\infty} \frac{2n}{\nu_x^{(1)}} \frac{1}{\Gamma\left(\frac{n-p}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{j!} j^{-(\lambda_j/2)-1} \left(\frac{2
u_x^{(1)}}{2}\right) g_{\frac{p+j}{2}, \frac{n-p}{2}}(g) dg 
$$

$$
- \int_0^{\infty} \frac{2n}{\nu_x^{(1)}} \frac{1}{\Gamma\left(\frac{n-p}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{j!} j^{-(\lambda_j/2)-1} \left(\frac{2
u_x^{(1)}}{2}\right) g_{\frac{p+j}{2}, \frac{n-p}{2}}(g) dg 
$$

$$
+ \int_0^{\infty} \frac{1}{\nu_x^{(1)}} \frac{1}{\Gamma\left(\frac{n-p}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{j!} j^{-(\lambda_j/2)-1} \left(\frac{2
u_x^{(1)}}{2}\right) g_{\frac{p+j}{2}, \frac{n-p}{2}}(g) dg 
$$

$$
- \int_0^{\infty} \frac{1}{\nu_x^{(1)}} \frac{1}{\Gamma\left(\frac{n-p}{2}\right)} \sum_{j=0}^{\infty} \frac{1}{j!} j^{-(\lambda_j/2)-1} \left(\frac{2
u_x^{(1)}}{2}\right) g_{\frac{p+j}{2}, \frac{n-p}{2}}(g) dg 
$$
Therefore

\[ \frac{\partial R}{\partial \psi_x(1)} = n^2(\psi_x(1))^2 G_{p+4,n-p}(c;\lambda) + nG_{p+2,n-p}(c;\lambda) \]

\[ - n^2(\psi_x(1))^2 G_{p+2,n-p}(c;\lambda) - nP(\lambda) + n^2(\psi_x(1))^2 P(\lambda) \]

Similarly differentiating (2.40) w.r.t. \( \psi_x(1) \), we obtain

\[ \frac{\partial^2 R}{\partial \psi_x(1)^2} = \int \cdots \int \prod_{j=1}^{P} \frac{\sqrt{n}}{\sqrt{2\pi}} \left( \frac{n}{2} \right)^2 (\psi_{1}(1) - \psi_{x}(1))^2 e^{-\frac{n}{2}(\psi_{1}(j) - \psi_{x}(j))^2} \]

\[ \cdot \frac{|W_{1}|^{\frac{1}{2}(n-p-2)} e^{-\frac{1}{2}trW_{1}}}{2^{P(n-1)} \pi^{P(p-1)} \prod_{j=1}^{P} \Gamma \left[ \frac{1}{2}(n-1) \right]} \]

\[ - \int \cdots \int \prod_{j=1}^{P} \frac{\sqrt{n}}{\sqrt{2\pi}} \left( \frac{n}{2} \right)^2 e^{-\frac{n}{2}(\psi_{1}(j) - \psi_{x}(j))^2} \]

\[ \cdot \frac{W_{1}^{\frac{1}{2}(n-p-3)} e^{-\frac{1}{2}trW_{1}}}{2^{P(n-1)} \pi^{P(p-1)} \prod_{j=1}^{P} \Gamma \left[ \frac{1}{2}(n-1) \right]} \]
Therefore

$$\frac{\partial^2 R}{\partial v_x(1)^2} = n^2 c_1 \int_G z(1)^2 g(z) e^{-\frac{1}{2} \text{tr} W_1} \frac{1}{2} |W_1| |z|^{\frac{1}{2}(n-p-2)} \, dz(j) \, dw_1$$

$$- 2n^2 c_1 v_x(z(1)^2 g(z) e^{-\frac{1}{2} \text{tr} W_1} \frac{1}{2} |W_1| |z|^{\frac{1}{2}(n-p-2)} \, dz(j) \, dw_1$$

$$+ n^2 v_x^2 P(G) - nP(G)$$

But from equating (2.38b) and (2.40), we know the middle term

$$= -2n^2 v_x^2 G^*_{p+2,n-p(c;\lambda)}. \text{ Hence}$$

$$\frac{\partial^2 R}{\partial v_x(1)^2} = n^2 c_1 \int_G z(1)^2 g(z) e^{-\frac{1}{2} \text{tr} W_1} \frac{1}{2} |W_1| |z|^{\frac{1}{2}(n-p-2)} \, dz(j) \, dw_1$$

$$- 2n^2 v_x^2 G^*_{p+2,n-p(c;\lambda)} + n^2 v_x^2 P(G) - nP(G)$$

Equating (2.47) and (2.48) yields

$$\frac{1}{2^p(p-1) \prod_{i=1}^p r[i^{\frac{1}{2}}(n-1)]} \int_G z(1)^2 g(z) e^{-\frac{1}{2} \text{tr} W_1} \frac{1}{2} |W_1| |z|^{\frac{1}{2}(n-p-2)} \, dz(j) \, dw_1$$

$$dz(j) \, dw_1 = (v_x(z(1)^2 g(z) e^{-\frac{1}{2} \text{tr} W_1} \frac{1}{2} |W_1| |z|^{\frac{1}{2}(n-p-2)} \, dz(j) \, dw_1$$

$$+ G^*_{p+4,n-p(c;\lambda)} + \frac{1}{n} G^*_{p+2,n-p(c;\lambda)}$$

(2.49)
Next we differentiate (2.38a) w.r.t. \( v_x^{(K)} \) to get

\[
\frac{3R}{\partial v_x^{(K)} / \partial v_x^{(1)}} = c \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n v_x^{(1)}}{\Gamma\left(\frac{n-p}{2}\right)} \cdot \frac{2n v_x^{(K)}}{\Gamma\left(\frac{n-p}{2}\right)} \cdot \frac{1}{2^{j-1}} \left(\frac{\lambda}{2}\right)^{j-1}
\]

\[
G_{p+j, \frac{n-p}{2}}^{(g)}(g) dg = \int_0^1 \left( \frac{1}{2} \cdot 2n v_x^{(1)} \right) e^{-\frac{1}{2} \lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n v_x^{(1)}}{\Gamma\left(\frac{n-p}{2}\right)} \cdot \frac{1}{2^{j-1}} \left(\frac{\lambda}{2}\right)^{j-1}
\]

\[
G_{p+j, \frac{n-p}{2}}^{(g)}(g) dg = \int_0^1 \left( \frac{1}{2} \cdot 2n v_x^{(1)} \right) v_x^{(1)} e^{-\frac{1}{2} \lambda} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n v_x^{(K)}}{\Gamma\left(\frac{n-p}{2}\right)} \cdot \frac{1}{2^{j-1}} \left(\frac{\lambda}{2}\right)^{j-1}
\]

\[
G_{p+j, \frac{n-p}{2}}^{(g)}(g) dg = n^2 v_x^{(1)} v_x^{(K)} G_{p+4, \frac{n-p}{2}}^{(g)}(c; \lambda) - 2n^2 v_x^{(1)} v_x^{(K)}
\]

\[
G_{p+2, \frac{n-p}{2}}^{(g)}(c; \lambda) + n^2 v_x^{(1)} v_x^{(K)} P(G)
\]

(2.50)
Similarly from (2.40),

\[
\frac{\partial R}{\partial \nu_x(K) \partial \nu_x(1)} = \mathcal{G} \left[ \frac{\sqrt{n}}{\sqrt{2\pi}} \left( \frac{2n}{2} z(1) - \nu_x(1) \right) \left( z(K) - \nu_x(K) \right) \right] \]

\[
= \frac{n^2}{\frac{2}{4} (n-1)^{1/4} p(p-1) \prod_{i=1}^{p} \Gamma \left[ \frac{1}{2}(n-1) \right]} \mathcal{G} z(1) z(K) e^{-\frac{n}{2} (z(j) - \nu_x(j))^2} \]

\[
\cdot \left| W_1 \right|^2 e^{-\frac{1}{2} \text{tr} W_1} g(Z) dZ(j) dW_1 \]

\[
= \frac{n^2 \nu_x(K)}{\frac{2}{4} (n-1)^{1/4} p(p-1) \prod_{i=1}^{p} \Gamma \left[ \frac{1}{2}(n-1) \right]} \mathcal{G} z(1) e^{-\frac{n}{2} (z(j) - \nu_x(j))^2} \]

\[
\cdot \left| W_1 \right|^2 e^{-\frac{1}{2} \text{tr} W_1} g(Z) dZ(j) dW_1 \]

\[
= \frac{n^2 \nu_x(1)}{\frac{2}{4} (n-1)^{1/4} p(p-1) \prod_{i=1}^{p} \Gamma \left[ \frac{1}{2}(n-1) \right]} \mathcal{G} z(K) e^{-\frac{n}{2} (z(j) - \nu_x(j))^2} \]

\[
\cdot \left| W_1 \right|^2 e^{-\frac{1}{2} \text{tr} W_1} g(Z) dZ(j) dW_1 \]

\[
+ n^2 \nu_x(1) \nu_x(K) \mathcal{G} \]
Again we know each of the two middle terms equals
\[-n^2v_{x}(1)v_{x}(K)_{p+2,n-p}(c;\lambda)\] from (2.38) and (2.40). Therefore

\[\frac{3R}{\partial v_{x}(K)\partial v_{x}(1)} = \frac{n^2}{2^\frac{n}{2}(n-1)\frac{1}{p}(p-1)\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n-1)]}\int_{G} z_{(1)}z_{(K)}\]

\[= \frac{n}{2}(z(j)-v_{x}(j))^2 |w_1|^2 e^{-\frac{1}{2}trw_1g(z)dz(j)dw_1} e^{-\frac{1}{2}trw_1g(z)dz(j)dw_1}\]

\[-2n^2v_{x}(1)v_{x}(K)_{p+2,n-p}(c;\lambda) + n^2v_{x}(1)v_{x}(K)_{p+4,n-p}(c;\lambda)\]

Equating (2.50) and (2.51) we have

\[\frac{n^2}{2^\frac{n}{2}(n-1)\frac{1}{p}(p-1)\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n-1)]}\int_{G} z_{(1)}z_{(K)} e^{-\frac{1}{2}(z(j)-v_{x}(j))^2}\]

\[= \frac{1}{2}(n-p-2) e^{-\frac{1}{2}trw_1g(z)dz(j)dw_1} = v_{x}(1)v_{x}(K)_{p+4,n-p}(c;\lambda)\]

We may now let

\[\frac{1}{2^\frac{n}{2}(n-1)\frac{1}{p}(p-1)\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n-1)]}\int_{G} z_{z}z_{g(z)} e^{-\frac{1}{2}trw_1z_{1}z_{w_1}} |w_1|^2 e^{-\frac{1}{2}trw_1g(z)dz(j)dw_1}\]
= M where M is a pxp matrix with i-th diagonal element
= \((v_x(1))^2_{p+4,n-p}(c;\lambda) + \frac{1}{n} g_{p+2,n-p}(c;\lambda)\) and the (i,k)th off-diagonal element = \(v_x(v_x(1),g)_{p+4,n-p}(c;\lambda)\). Hence

\[
E[(S_{12}^{-1}\overline{X}_1 S_{22}^{-1} S_{22}^{1}) | G] P(G)
\]

= \(Q + \text{tr} T^{-1} T'E_{22}^{-1} E_{12}^{-1} T^{-1} M\)

= \(Q + \text{tr} T^{-1} E_{22}^{-1} E_{12}^{-1} T^{-1} M\)  \( (2.54) \)

= \(Q + \text{tr} E_{12}^{-1} T^{-1} M T^{-1} E_{22}^{-1} E_{22}^{-1} T^{-1} M\)

since the second term is a scalar.

Next we note that

\[
E[S_{12}^{-1} \overline{X}_1 S_{22}^{-1} \overline{X} | G] P(G)
\]

= \(E(E(S_{12}^{-1} \overline{Y} | S, G) | G) P(G)\)

= \(E(E(S_{12}^{-1} \overline{Y} | S, G) | G) P(G)\)

= \(E(S_{12}^{-1} \overline{X} E[\overline{Y} | \overline{X}, G]) P(G)\)

= \(E(S_{12}^{-1} \overline{X} \mu + E(S_{12}^{-1} \overline{X} E[S_{12}^{-1} \overline{X} | G] P(G)\)

= \(E(S_{12}^{-1} \overline{X} E[S_{12}^{-1} \overline{X} | G] P(G)\) + \(E(S_{12}^{-1} \overline{X} E[S_{12}^{-1} \overline{X} | G] P(G)\)

= \(\mu E[S_{12}^{-1} \overline{X} | G] P(G) + E[S_{12}^{-1} \overline{X} E[S_{12}^{-1} \overline{X} | G] P(G)\)
But since $\Sigma_{12}^{-1} x_2$ and $\Sigma_{12}^{-1} x_2$ are scalars, they are equal to $x_2 \Sigma_{22}^{-1} x_2$ and $\mu x_2 \Sigma_{22}^{-1} x_2$, respectively. Hence

$$E[S_{12} x_2 XX | G] P(G) = \mu E[S_{12} x_2 XX | G] P(G) + E[(S_{12} x_2 XX \Sigma_{22}^{-1} x_2 | G] P(G) - E(S_{12} x_2 XX \Sigma_{22}^{-1} x_2 | G] P(G)$$

To evaluate the middle term,

$$E[S_{12} x_2 XX \Sigma_{22}^{-1} x_2 | G] P(G) = E\{tr(S_{12} x_2 XX \Sigma_{22}^{-1} x_2 | G)\} P(G) = tr \Sigma_{22}^{-1} x_2 E[S_{12} x_2 XX | G] P(G)$$

and with the transformation of (2.31)

$$= tr T^T \Sigma_{21} g(t) KW_2 W_1 \Sigma_{12} \Sigma_{12}^T XX f(x) g(W_1, W_2, W_3) dW_3 dW_2 dW_1 d\bar{x}$$

$$= tr T^T \Sigma_{21} g(t) KW_2 W_1 \Sigma_{12} \Sigma_{12}^T XX f(x) g(W_1, W_2, W_3) dW_3 dW_2 dW_1 d\bar{x}$$
But from independence and the fact $E(W_2) = 0$, the first term $= 0$ and using (2.46) and (2.53)

$$E[S_{12} S_{22}^{-1} \bar{X} X^{-1} \Sigma_{22}^{2} 21 | G] P(G)$$

$$= \text{tr} T^{-1} T' \Sigma_{21} E_{12} T' T^{-1} M$$

$$= E_{12} \Sigma_{22}^{-1} T^{-1} \Sigma_{22}^{-1} E_{22}^{*} 21$$

Therefore

$$E(S_{12} S_{22}^{-1} \bar{X} X | G) P(G)$$

$$= w \Sigma_{12} E_{22}^{-1} \bar{X} X^{-1} \mu_{x} G_{x}^{*} p+2, n-p (c; \lambda)$$

$$+ \Sigma_{12} E_{22}^{-1} T^{-1} \Sigma_{22}^{-1} E_{22}^{*} 21$$

$$- E_{12} E_{22}^{-1} T^{-1} \Sigma_{22}^{-1} E_{22}^{*} 21 G_{x}^{*} p+2, n-p (c; \lambda)$$

Finally $E(\bar{y}^2) = \frac{1}{n} \sigma^2 + \mu^2$. Substituting into (2.44), we have

$$E(\bar{y}^2) = \frac{1}{n} \sigma^2 + \mu^2 - 2 \mu \Sigma_{12} E_{22}^{-1} \bar{X} X^{-1} \mu_{x} G_{x}^{*} p+2, n-p (c; \lambda)$$

$$- \Sigma_{12} E_{22}^{-1} T^{-1} \Sigma_{22}^{-1} E_{22}^{*} 21$$

$$+ 2 \Sigma_{12} E_{22}^{-1} \bar{X} X^{-1} \mu_{x} G_{x}^{*} p+2, n-p (c; \lambda)$$

$$+ Q$$
\[ [E(\hat{\mu})]^2 = [\mu - \Sigma_{12} \Sigma_{22}^{-1} \nu \Sigma \Sigma_{p+2, n-p}^2(c; \lambda)]^2 \]
\[ = \mu^2 - 2\mu \Sigma_{12} \Sigma_{22}^{-1} \nu \Sigma \Sigma_{p+2, n-p}^2(c; \lambda) \]
\[ + \Sigma_{12} \Sigma_{22}^{-1} \nu \Sigma \Sigma_{p+2, n-p}^2(c; \lambda) \]
\[ = \nu - 2\mu \Sigma_{12} \Sigma_{22}^{-1} \nu \Sigma \Sigma_{p+2, n-p}^2(c; \lambda) \]
\[ + \Sigma_{12} \Sigma_{22}^{-1} \nu \Sigma \Sigma_{p+2, n-p}^2(c; \lambda) \]

Therefore

\[ V(\hat{\mu}) = E(\hat{\mu}^2) - [E(\hat{\mu})]^2 \]
\[ = \frac{1}{n} \sigma^2 - \Sigma_{12} \Sigma_{22}^{-1} \nu \Sigma \Sigma_{p+2, n-p}^2(c; \lambda) \]
\[ + \Sigma_{12} \Sigma_{22}^{-1} \nu \Sigma \Sigma_{p+2, n-p}^2(c; \lambda) \]
\[ - \Sigma_{12} \Sigma_{22}^{-1} \nu \Sigma \Sigma_{p+2, n-p}^2(c; \lambda) \]
\[ = \frac{1}{n} \sigma^2 - \Sigma_{12} \Sigma_{22}^{-1} \nu \Sigma \Sigma_{p+2, n-p}^2(c; \lambda) \]
\[ + \Sigma_{12} \Sigma_{22}^{-1} \nu \Sigma \Sigma_{p+2, n-p}^2(c; \lambda) \]
\[ - \Sigma_{12} \Sigma_{22}^{-1} \nu \Sigma \Sigma_{p+2, n-p}^2(c; \lambda) \]

To be able to make any partial checks, we need to compute the variance of \( \hat{\mu} \) for the cases when \( c = 0 \) and when \( c = \infty \).

When \( c = 0 \), we always reject the null hypothesis and so the estimator reduces to \( \hat{\mu} = \bar{y} \) with variance

\[ V(\hat{\mu}) = V(\bar{y}) = \frac{1}{n} \sigma^2 \]

For \( c = \infty \), we always accept and so use the estimator

\[ \hat{\mu} = \bar{y} - \Sigma_{12} \Sigma_{22}^{-1} \Delta \]
Now

\[
V(\bar{y} - S_{12}S_{22}^{-1}\bar{x})
\]

\[
= E V(\bar{y} - S_{12}S_{22}^{-1}\bar{x}|x_1',s) + V E(\bar{y} - S_{12}S_{22}^{-1}\bar{x}|x_1',s)
\]

\[
= E \left[ \frac{1}{n} \sum_{i=1}^{n} (\bar{y} - \bar{y}) (\bar{y} - \bar{x}) S_{22}^{-1} x_i's \right]
\]

\[
+ \sum_{i=1}^{n} \bar{x} S_{22}^{-1} x_i'
\]

\[
+ V[\mu + \sum_{i=1}^{n} (\bar{x} - \bar{y}) S_{22}^{-1} x_i's)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} + \sum_{i=1}^{n} E(\bar{y} - S_{12}S_{22}^{-1}\bar{x})
\]

But \(nm(\bar{x} S_{22}^{-1}) = T^2 \sim \text{noncentral } T^2 \text{ distribution with } n-1 \text{ degree of freedom where recall } m = n-1 \text{ and } \frac{T^2(n-p)}{p(n-1)} \sim \text{noncentral } F \text{ distribution with } p \text{ and } n-p \text{ degrees of freedom and noncentrality parameter } \frac{\mu^T S_{22}^{-1} \mu}{n} \text{. Hence}

\[
V(\bar{y} - S_{12}S_{22}^{-1}\bar{x})
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} + \sum_{i=1}^{n} E(\bar{y} - S_{12}S_{22}^{-1}\bar{x})
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} + \sum_{i=1}^{n} \frac{p}{n(n-p)} E(t)
\]

where \(t \sim F_{p-n-p}(\lambda) \). But

\[
E(t) = \frac{n-p}{n-p-2} \left[ 1 + \frac{2\mu^T S_{22}^{-1} \mu}{p} \right]
\]
Hence

\[ V(\tilde{y} - S_{12} S_{22}^{-1} \tilde{x}) = \frac{1}{n} \Sigma_{11} \cdot 2 + \Sigma_{11} \cdot 2 \frac{p}{n(n-p-2)} [1 + \frac{2\nu_x^{1} \Sigma_{22}^{1} \mu_{x}}{p}] \]

(2.58)

So now for partial checks, when \( c = 0 \), (2.56) is \( V(\hat{u}) = \frac{1}{n} \sigma^2 \) which is the variance of the estimator when we always reject. For \( c = \infty \), we note that

\[ Q = K^2 \frac{p}{n(n-p)} \int_0^\infty f(t) dt \]

\[ = \Sigma_{11} \cdot 2 \frac{p}{n(n-p)} E(t) \]

\[ = \Sigma_{11} \cdot 2 \frac{p}{n(n-p-2)} [1 + \frac{2\nu_x^{1} \Sigma_{22}^{1} \mu_{x}}{p}] \]

and (2.56) reduces to

\[ \frac{1}{n} \sigma^2 - \Sigma_{12} \Sigma_{22}^{1} \mu_{x} \Sigma_{22}^{1} \Sigma_{21} - \Sigma_{12} \Sigma_{22}^{1} \Sigma_{21} \]

\[ + \Sigma_{11} \cdot 2 \frac{p}{n(n-p-2)} [1 + \frac{2\nu_x^{1} \Sigma_{22}^{1} \mu_{x}}{p}] \]

\[ + 2\Sigma_{12} \Sigma_{22}^{1} \mu_{x} \Sigma_{22}^{1} \Sigma_{21} \]

\[ - \Sigma_{12} \Sigma_{22}^{1} \mu_{x} \Sigma_{22}^{1} \Sigma_{21} \]

\[ = \frac{1}{n} \Sigma_{11} \cdot 2 + \Sigma_{11} \cdot 2 \frac{p}{n(n-p-2)} [1 + \frac{2\nu_x^{1} \Sigma_{22}^{1} \mu_{x}}{p}] \]
which by (2.58) is the variance of the regression estimator when we always accept $H_0$.

Now we obtain the M.S.E. of $\hat{\mu}$.

$$\text{M.S.E.}(\hat{\mu}) = \frac{1}{n} \sigma^2 - \Sigma_{12}^1 \Sigma_{22}^1 T_{-1}^{1} M_{0}^{1} - \Sigma_{12}^1 \Sigma_{22}^1 E_{21} + Q$$

$$+ 2\Sigma_{12}^1 \Sigma_{22}^1 \mu_1 \mu_1^T \Sigma_{22}^1 G^* \Sigma_{p+2, n-p}(c; \lambda)$$

$$- \Sigma_{12}^1 \Sigma_{22}^1 T_{-1}^{1} \Sigma_{22}^1 E_{21} \left[ G^* \Sigma_{p+2, n-p}(c; \lambda) \right]^2$$

$$+ \Sigma_{12}^1 \Sigma_{22}^1 T_{-1}^{1} \Sigma_{22}^1 E_{21} \left[ G^* \Sigma_{p+2, n-p}(c; \lambda) \right]^2$$

or

$$\text{M.S.E.}(\hat{\mu}) = \frac{1}{n} \sigma^2 - \Sigma_{12}^1 \Sigma_{22}^1 T_{-1}^{1} M_{0}^{1} - \Sigma_{12}^1 \Sigma_{22}^1 E_{21} + Q$$

$$+ 2\Sigma_{12}^1 \Sigma_{22}^1 \mu_1 \mu_1^T \Sigma_{22}^1 G^* \Sigma_{p+2, n-p}(c; \lambda)$$

$$= \frac{1}{n} \sigma^2 - \Sigma_{12}^1 \Sigma_{22}^1 \mu_1 \mu_1^T \Sigma_{22}^1 G^* \Sigma_{p+4, n-p}(c; \lambda)$$

$$- \frac{1}{n} \Sigma_{12}^1 \Sigma_{22}^1 G^* \Sigma_{p+2, n-p}(c; \lambda) + Q$$

$$+ 2\Sigma_{12}^1 \Sigma_{22}^1 \mu_1 \mu_1^T \Sigma_{22}^1 G^* \Sigma_{p+2, n-p}(c; \lambda)$$

$$(2.59)$$

G. Relative Efficiency ($e'$)

To evaluate the gain and loss of precision of the preliminary test estimator, we consider the relative efficiency of $\hat{\mu}$ to the usual estimator $\bar{y}$. This is defined as

$$e' = \frac{1}{\text{M.S.E.}(\hat{\mu})} \frac{1}{\text{M.S.E.}(\bar{y})}$$
so that using (2.59),

\[ e' = \frac{1}{1 + h} \quad (2.60) \]

where

\[
h = \frac{n}{\sigma^2} \left\{ 2 \Sigma_{12} \Sigma_{22}^{-1} \mu_x \mu_x' \Sigma_{22}^{-1} \Sigma_{21} G^*_{p+2, n-p}(c; \lambda) \right. \\
- \left. \Sigma_{12} \Sigma_{22}^{-1} \mu_x \mu_x' \Sigma_{22}^{-1} \Sigma_{21} G^*_{p+4, n-p}(c; \lambda) \right. \\
- \frac{1}{n} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} G^*_{p+2, n-p}(c; \lambda) + Q \}
\]

Wlog we let \( \Sigma_{22} = I \), \( \sigma^2 = 1 \). Therefore

\[
h = 2n \Sigma_{12} \mu_x \mu_x' \Sigma_{21} F^*_{p+2, n-p}(c_2; \lambda) - n \Sigma_{12} \mu_x \mu_x' \Sigma_{21} F^*_{p+4, n-p}(c_4; \lambda)

- \Sigma_{12} \Sigma_{21} F^*_{p+2, n-p}(c_2; \lambda) + (1 - \Sigma_{12} \Sigma_{21}) \frac{d}{n-p} \int t f(t) dt
\]

If we let \( \Sigma_{12} \mu_x = K_1 \), \( \Sigma_{12} \Sigma_{21} = g_1 \),

\[
h = 2n K_1^2 F^*_{p+2, n-p}(c_2; \lambda) - n K_1^2 F^*_{p+4, n-p}(c_4; \lambda)

- g_1 F^*_{p+2, n-p}(c_2; \lambda) + (1 - g_1) \frac{d}{n-p} \int t f(t) dt
\]

We note that \( e' \) is a function of \( n, \Sigma_{12}, \mu_x \) and \( \lambda \) for any given \( p \). For the computation of \( e' \) for certain choices of \( n, \Sigma_{12}, \mu_x \) and \( \lambda \), we use the incomplete Beta approximation to the noncentral \( F \) distribution. We denote the cumulative
distribution function of the noncentral F random variable with \( v_1 \) and \( v_2 \) degrees of freedom by \( F^*_{v_1,v_2}(d;\lambda) \) where \( \lambda \) is the noncentrality parameter. That is, we let

\[
\int f(t|v_1,v_2,\lambda)dt = F^*_{v_1,v_2}(d,\lambda).
\]

Therefore

\[
F^*_{v_1,v_2}(d,\lambda) = \int \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}(\frac{d}{2})}}{j!} (\frac{v_1}{2})^{\frac{1}{2}v_1+j} \frac{t^{\frac{1}{2}v_1+j-1}}{(1+\frac{v_1}{2}t)^{\frac{1}{2}(v_1+v_2+2j)}} dt
\]

and since \( 1-I_x(a,b) = I_1-x(b,a) \) where \( I_x(a,b) \) is the incomplete \( \beta \) function given in Karl Pearson (1934), then from Tiku (1967),

\[
F^*_{v_1,v_2}(d,\lambda) = \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}(\frac{d}{2})}}{j!} I_x(\frac{v_1}{2}+j,\frac{v_2}{2});x = \frac{v_1d}{v_1d+v_2}. \]

To obtain an analogue of (2.62) for

\[
\int t f(t|v_1,v_2,\lambda)dt,
\]

we use (2.61) and note
\[
d = \int_t^s f(t|v_1,v_2,\lambda) \, dt
\]

\[
= \int_0^s \sum_{j=0} e^{\frac{-\lambda}{2}\left(\frac{v_1}{2}\right)} \left(\frac{v_1}{v_2}\right)^{\frac{1}{2}v_1+j} \frac{\frac{1}{2}v_1+j}{t^2} \left(1+\frac{v_1}{v_2}\right) \frac{\frac{1}{2}(v_1+v_2+2j)}{B\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \, dt
\]

Let

\[
x = \frac{v_1 t}{v_1 t + v_2}
\]

or

\[
t = \frac{xv_2}{v_1 (1-x)}.
\]

Therefore

\[
dt = \frac{v_2}{v_1 (1-x)^2} \, dx
\]

Further let

\[
d_1 = \frac{v_1 d}{v_1 d + v_2},
\]

we have

\[
\sum_{j=0} e^{\frac{-\lambda}{2}\left(\frac{v_1}{2}\right)} \frac{\frac{1}{2}v_1+j}{j!} \int_0^{x(1-x)} \left[1 + \left(\frac{x}{1-x}\right)\right] \frac{\frac{1}{2}(v_1+v_2+2j)}{B\left(\frac{v_1}{2}, \frac{v_2}{2}\right)} \frac{v_2}{v_1 (1-x)^2} \, dx
\]
\[
\begin{align*}
&= \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \int_0^1 \frac{v_2}{v_1} \frac{v_1}{2} x^{v_1+j} (1-x)^{-\frac{1}{2}(v_1+j)} (1-x)^{-2[1-x]\frac{1}{2}(v_1+v_2+2j)} \frac{1}{B(\frac{1}{2}v_1+j, \frac{v_2}{2})} dx \\
&= \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \int_0^1 \frac{v_2}{v_1} \frac{v_1}{2} x^{v_1+j} (1-x)^{-\frac{1}{2}(v_2)^2-2} \frac{1}{B(\frac{1}{2}v_1+j, \frac{v_2}{2})} dx \\
&= \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \int_0^1 \frac{v_2}{v_1} \frac{v_1}{2} x^{v_1+j-1} (1-x)^{-\frac{1}{2}(v_2)^2-1} \frac{1}{B(\frac{1}{2}v_1+j, \frac{v_2}{2})} dx \\
\text{But} \\
B(\frac{1}{2}v_1+j, \frac{v_2}{2}) &= \frac{v_2}{v_1+2j} B(\frac{v_1+2}{2}+j, \frac{v_2-2}{2}) \\
\text{Therefore} \\
\int_0^1 t f(t) dt &= \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \int_0^1 \frac{v_2}{v_1} \frac{v_1}{2} x^{v_1+j} (1-x)^{-\frac{1}{2}(v_2)^2} \frac{1}{v_1+2j} \frac{v_1+2}{2} + j-1 \frac{v_2-2}{2} -1 \frac{1}{B(\frac{1}{2}v_1+j, \frac{v_2}{2})} dx \\
&= \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \frac{v_1+2j}{v_1} \frac{1}{B(\frac{1}{2}v_1+j, \frac{v_2}{2})} \int_0^1 \frac{v_2}{v_1} \frac{v_1}{2} x^{v_1+j-1} (1-x)^{-\frac{1}{2}(v_2)^2-1} \frac{1}{B(\frac{1}{2}v_1+j, \frac{v_2}{2})} dx \\
&= \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \frac{v_1+2j}{v_1} I_{d_1} (\frac{v_1+2}{2} + j, \frac{v_2-2}{2}) \\
&= (2.63)
\end{align*}
\]
where

\[ d_1 = \frac{\frac{1}{d} \cdot \log \left( \frac{1}{\nu_1 d + \nu_2} \right)}{\log \left( \frac{1}{\nu_1 d + \nu_2} \right)}. \]

For the purpose of comparison with the results of Han (1973a), we compute the values of \( e' \) for \( p = 1 \) and certain values of \( n, \Sigma_{12} = \rho, \mu_X \) and \( \alpha \). These values are shown in Table 2.9 and reveal no significant difference from the values obtained by Han. Han's results were in terms of moments of normal densities while the present results are expressed as a function of the cumulative distribution and the expected values of the truncated noncentral \( F \) distribution. Subroutines using an incomplete Beta distribution to approximate the noncentral \( F \) distribution were used in the computation and the slight differences for small values of \( \alpha \) are due to these approximations and rounding off errors.

Table 2.9 shows \( e' \) assumes its maximum value when \( \mu_X = 0 \). It then decreases to some minimum before increasing to 1.0 as \( \mu_X \) increases. The value of 1.0 for large values of \( \mu_X \) corresponds to the fact that when \( \mu_X \) gets very large, then the difference from zero is significant and we always reject the null hypothesis, thus making the two estimators the same. For fixed \( n, \mu_X \) and \( \alpha \), \( e' \) increases with \( \rho \) while for fixed \( n, \mu_X \) and \( \rho \), \( e' \) is a decreasing function of \( \alpha \).
The values of $e'$ for $p = 2$ are given in Table 2.10 for some values of $\Sigma_{12}$, $\mathbb{H}_X$, $n$ and $a$. From the table, we note the following properties.

1. The relative efficiency $e'$ is maximum when $\mathbb{H}_X = 0$ for fixed $n$, $a$ and $\Sigma_{12}$. This corresponds to the case when the null hypothesis is true.

2. For a fixed sample size, the maximum value of $e'$ increases with $\Sigma_{12}$ for any given $a$ but decreases as $a$ increases for a given $\Sigma_{12}$.

3. $e'$ remains the same for values of $\Sigma_{12}$ which differ only in sign.

4. For fixed $a$, $n$, $\Sigma_{12}$ and some component of $\mathbb{H}_X$, the relative efficiency decreases to a minimum and then increases as the other component increases.

5. The value of $e'$ equals 1.0 for large values of $n$ or $\mathbb{H}_X$. This is because the two estimators tend to be the same as $n$ gets large; while for large values of $\mathbb{H}_X$, we would always reject the null hypothesis and use $\hat{\mu} = \bar{y}$.

6. For a fixed $a$ and small values of $\mathbb{H}_X$, the value of $e'$ increases with $\Sigma_{12}$, but $e'$ is a decreasing function of $\Sigma_{12}$ for moderately large values of $\mathbb{H}_X$. For large values of $\mathbb{H}_X$, $e'$ equals 1.0 as explained in 5 above.
Table 2.9. Values of $e'$ for $p = .1$.

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<tr>
<th>$n = 9$</th>
<th>$\alpha = .05$</th>
<th>$\alpha = .10$</th>
<th>$\alpha = .20$</th>
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<td>$\mu_x$</td>
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<td>.5</td>
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<td>(0, 0)</td>
<td>1.0219</td>
<td>1.0824</td>
<td>1.2214</td>
<td>1.1477</td>
</tr>
<tr>
<td>(0, .5)</td>
<td>1.0074</td>
<td>0.9541</td>
<td>0.9291</td>
<td>0.9081</td>
</tr>
<tr>
<td>(0, 1.5)</td>
<td>1.0</td>
<td>0.9989</td>
<td>0.9979</td>
<td>0.9978</td>
</tr>
<tr>
<td>(0, 3.0)</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>(.5, .5)</td>
<td>0.9663</td>
<td>0.8814</td>
<td>0.7973</td>
<td>1.0207</td>
</tr>
<tr>
<td>(.5, 1.5)</td>
<td>0.9999</td>
<td>0.9991</td>
<td>0.9983</td>
<td>0.9995</td>
</tr>
<tr>
<td>(.5, 3.0)</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>(1.5, 1.5)</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>(1.5, 3.0)</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>(3.0, 3.0)</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>
III. DOUBLE SAMPLING WITH PARTIAL INFORMATION ON AUXILIARY VARIABLES

A. Introduction

Consider a $p+1$ variate normal population

\[
\begin{pmatrix}
Y \\
X
\end{pmatrix} \sim N(\mu, \Sigma)
\]

where $Y$ is a univariate random variable and $X$ is a $p \times 1$ random vector with $p \geq 1$,

\[
\mu = \begin{pmatrix}
\mu \\
\mu_X
\end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix}
\sigma^2 & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix} \quad (3.1)
\]

Suppose we are interested in estimating the population mean $\mu$ of $Y$. It is well known that the precision of the estimator can be increased if auxiliary information is available. For example, if the relationship is linear, a linear regression estimator may be constructed. We shall consider here the regression estimator. In the given multivariate normal distribution, the vector $X$ is correlated with $Y$ and so can be used as an ancillary variable to increase precision in estimating $\mu$. To use the regression estimator we need to know the population mean $\mu_X$ of $X$. When $\mu_X$ is unknown, we may take a preliminary sample to estimate it. This sampling procedure is the double sampling technique. In certain situations, an
investigator may have partial information about \( \mu_x \). In order to make use of this partial information, the investigator can perform a preliminary test about the hypothesis \( H_0 : \mu_x = \mu_0 \) versus \( H_1 : \mu_x \neq \mu_0 \) where \( \mu_0 \) is some constant vector that he believes that the population mean \( \mu_x \) should be based on the partial information.

As an example, consider estimating the average growth of some rats. It is known that the growth is highly correlated with the amount of a certain vitamin in the feed. Hence the vitamin content of the feed can be used as an auxiliary variable. The investigator usually does not know the population mean value of the vitamin content but from the growth of Neurospora mycelium (or some other fungus) on agar plates and the comparison of this with the growth on some control plates with known concentration of the vitamin, the experimenter may believe that the population mean should be \( \mu_0 \). Once a preliminary sample is available, the investigator may test \( H_0 : \mu_x = \mu_0 \) against \( H_1 : \mu_x \neq \mu_0 \). He then will use \( \mu_0 \) in the regression estimator if \( H_0 \) is accepted, otherwise he uses the sample mean based on the preliminary sample. This estimator is usually known as the preliminary test estimator. If the investigator's prior information or experience is reliable, then the true mean \( \mu_x \) of \( X \) will be expected to be very close to \( \mu_0 \). In this situation, the efficiency of the preliminary test estimation is very high. Thus in practice, it is desirable to use the
preliminary test estimator when some partial information is available to the investigator.

B. The Preliminary Test Estimator and its Bias when $\Sigma$ is Known

Let $(\frac{Y}{X})$ have a multivariate normal distribution as given in section A. We assume $X$ is cheaply observed while the pair $(Y, X)$ is more expensive to observe. We wish to estimate $\mu$, the population mean of $Y$. Let $(y_1, X_{11}, X_{21}, \ldots, X_{p1})', i = 1, \ldots, n_2$ be a random sample from $N(\mu, \Sigma)$. This is supplemented by $m$ more independent observations on $X' = (X_1, \ldots, X_p)$'. In practice, the sample of $n_2$ observations is usually a subsample from the sample of $n_1 = n_2 + m$ observations. From all the observations, we define

$$\bar{X}_1 = (\frac{1}{n_1} \sum_{i=1}^{n_1} X_{1i}, \ldots, \frac{1}{n_1} \sum_{i=1}^{n_1} X_{pi})',$$

and from the subsample in which $X$ and $Y$ are observed, we define

$$\bar{X}_2 = (\frac{1}{n_2} \sum_{i=1}^{n_2} X_{1i}, \ldots, \frac{1}{n_2} \sum_{i=1}^{n_2} X_{pi}), \quad \bar{y} = \frac{1}{n_2} \sum_{i=1}^{n_2} y_i.$$

If the vector $\mu_X$ and $\Sigma$ are known, then the regression estimator of $\mu$ is

$$\hat{\mu} = \bar{y} + \Sigma_{12} \Sigma_{22}^{-1} (\mu_X - \bar{X}_2).$$
The regression estimator is unbiased with variance
\[ \frac{1}{n_2} \{ \sigma^2 - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \}. \]

If
\[ \frac{1}{n_2} \{ \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \} \]
is considerably large, we have an appreciable gain in precision. If \( \mu_x \) is unknown and it happens that from certain sources, the experimenter is pretty sure but not certain that \( \mu_x = \mu_0 \), then he may perform a preliminary test of \( H_0: \mu_x = \mu_0 \). In this case he can make the regression estimator depend on the result of the preliminary test. The new estimator is then the preliminary test estimator. Without loss of generality, we let \( \mu_0 = 0 \). Thus the preliminary test estimator is defined as
\[ \hat{\mu}_{lr} = \begin{cases} \bar{y} - \Sigma_{12} \Sigma_{22}^{-1} \bar{X}_2 & \text{if } n_1(\bar{X}_1 \Sigma_{22}^{-1} \bar{X}_1) \leq \chi^2_{p, \alpha} \quad (3.2) \\ \bar{y} + \Sigma_{12} \Sigma_{22}^{-1} (\bar{X}_1 - \bar{X}_2) & \text{if } n_1(\bar{X}_1 \Sigma_{22}^{-1} \bar{X}_1) > \chi^2_{p, \alpha} \end{cases} \]
where the subscript \( lr \) denotes linear regression and \( \chi^2_{p, \alpha} \) is the \( 100(1-\alpha) \) percent point of the Chi-squared distribution with \( p \) degrees of freedom. \( \alpha \) is the level of significance of the preliminary test.

The joint distribution of \( (\bar{X}_1, \bar{X}_2, \bar{Y})' \) is normal with mean \( (\mu_x, \mu_x, \mu)' \) and covariance matrix
Denote the acceptance region for the preliminary test by $A$ and its complement by $\bar{A}$ and let $\chi_p^2, \alpha = c$. The expected value of $\hat{\mu}_{lr}$ is

$$E(\hat{\mu}_{lr}) = E\{(\bar{Y} - \Sigma_{12}^{-1}\bar{X})|A\}P(A)$$

$$= E(\bar{Y} - \Sigma_{12}^{-1}\bar{X}) + \Sigma_{12}^{-1}E\{\bar{X}_1|A\}P(A)$$

$$= \mu - \Sigma_{12}^{-1}\Sigma_{22}\mu_x + \Sigma_{12}^{-1}E\{\bar{X}_1|A\}P(A) \quad (3.3)$$

Hence the bias of $\hat{\mu}_{lr}$ is given as

$$B_1 = \Sigma_{12}^{-1}E\{\bar{X}_1|A\}P(A) - \Sigma_{12}^{-1}\Sigma_{22}\mu_x \quad (3.4)$$

In order to evaluate the bias, we need to find the first term.

Now $\bar{X}_1 \sim N(\mu_x, \frac{1}{n_1} \Sigma_{22})$ and since $\Sigma_{22}$ is positive definite, a nonsingular matrix $D \in D'D = \Sigma_{22}^{-1}$. Let
\[ Z = D\bar{X} \text{, then} \]
\[ Z \sim N(Du'X/n_1', I) \]
\[ N(\bar{X}'X/n_1', I) \text{ say.} \]

Hence \( \{n_1(Z'Z) : n_1(Z'Z) > c\} = A. \)

\[ B_1 = \Sigma_1 \Sigma_2^{-1} D^{-1} E[Z|A]P(A) - \Sigma_1 \Sigma_2^{-1} \mu_x \quad (3.5) \]

It is known that \( n_1(Z'Z) \) has a noncentral Chi-squared distribution with \( p \) degrees of freedom and noncentrality parameter \( \delta = n_1(\mu_x'D'D\mu_x) = n_1(\gamma_1 \gamma_x) \). \( \gamma_x \) is a px1 vector and we denote the i-th component by \( \gamma_x^{(i)} \). Hence

\[ T = P(A) = \int e^{-\frac{1}{2}\delta} \sum \frac{1}{j!} (\delta^j) h_{p+2j}(t) dt \quad (3.6) \]

where \( h_{p+2j}(\cdot) \) is the probability density function of \( \chi^2_{p+2j} \). Differentiating (3.6) with respect to \( \gamma_x^{(i)} \), we obtain

\[ \frac{\partial T}{\partial \gamma_x^{(i)}} = \frac{\partial}{\partial \gamma_x^{(i)}} \int e^{-\frac{1}{2}\delta} \sum \frac{1}{j!} (\delta^j) h_{p+2j}(t) dt \]

and by the Lebesgue Dominated Convergence Theorem (LDCT) as justified in the Appendix, we can take the differentiation inside the integral and have
\[ \frac{\partial T}{\partial \gamma_x^{(1)}} = \int e^{-\frac{1}{2} \delta \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n_1 \gamma^{(1)}}{2^j}} j(\frac{\delta}{2})^{j-1} h_{p+2j}(t) dt \] (3.7)

\[ - \int e^{-\frac{1}{2} \delta \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n_1 \gamma^{(1)}}{2^j}} j(\frac{\delta}{2})^{j-1} h_{p+2j}(t) dt \]

\[ = n_1 [1 - H_{p+2}(c; \delta) - P(\overline{\alpha})] \gamma_x^{(1)} \]

\[ = n_1 [P(\alpha) - H_{p+2}(c; \delta)] \gamma_x^{(1)} \]

where

\[ P(\alpha) = \int e^{-\frac{1}{2} \delta \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n_1 \gamma^{(1)}}{2^j}} j(\frac{\delta}{2})^{j-1} h_{p+2j}(t) dt \]

and \( H_{p+2}(c; \delta) \) is the cumulative distribution function of the noncentral Chi-squared distribution with \( p+2 \) degrees of freedom and noncentrality parameter \( \delta \).

Alternatively, we can evaluate \( P(\overline{\alpha}) \) by the use of the distribution of \( Z \) and write

\[ T = P(\overline{\alpha}) = \int \cdots \int \frac{1}{\sqrt{2\pi}} \sqrt{n_1} e^{-\frac{1}{2} (z(j) - \gamma_x^{(1)})^2} dZ(j) \] (3.8)

If we now differentiate (3.8) w.r.t. \( \gamma_x^{(1)} \) by the LDCT as shown in the Appendix, we have

\[ \frac{\partial T}{\partial \gamma_x^{(1)}} = \int \cdots \int \frac{n_1}{\sqrt{2\pi}} \sqrt{n_1} e^{-\frac{1}{2} (z(j) - \gamma_x^{(1)})^2} \cdot \frac{n_1}{2} (z(j) - \gamma_x^{(1)}) dZ(j) \]

\[ = n_1 E[Z^{(1)} | \overline{\alpha}] P(\overline{\alpha}) - n_1 \gamma_x^{(1)} P(\overline{\alpha}) \] (3.9)
Hence we may obtain $E(Z^{(1)}|\bar{A})P(\bar{A})$ by equating (3.7) and (3.9).

That is

$$n_1[1-H_{p+2}(c;\delta) - P(\bar{A})]Y_x^{(1)} = n_1[E(Z^{(1)}|\bar{A})P(\bar{A}) - Y_x^{(1)}P(\bar{A})]$$

$$E(Z^{(1)}|\bar{A})P(\bar{A}) = [1-H_{p+2}(c;\delta)]Y_x^{(1)}$$  \hspace{1cm} (3.10)

Substituting (3.10) in (3.5), then

$$B_1 = \Sigma_1^2\Sigma_2^{(2)}D^{-1}[1-H_{p+2}(c;\delta)]Y_x - \Sigma_1^2\Sigma_2^{(2)}\mu_x$$

$$= -\Sigma_1^2\Sigma_2^{(2)}D^{-1}Y_xH_{p+2}(c;\delta) = -\Sigma_1^2\Sigma_2^{(2)}\mu_xH_{p+2}(c;\delta)$$  \hspace{1cm} (3.11)

As a partial check, when $c = 0$, the estimator reduces to $\bar{y} + \Sigma_1^2\Sigma_2^{(2)}(\bar{y} - \bar{x})$ with zero bias which is the case when we always reject the null hypothesis. In this case $H_{p+2}(c;\delta) = 0$ and $B_1 = 0$. When $c = \infty$, the null hypothesis is always accepted and the estimator reduces to $\hat{\mu}_x = \bar{y} - \Sigma_1^2\Sigma_2^{(2)}\bar{x}$. Here $H_{p+2}(c;\delta) = 1$ and $B_1 = -\Sigma_1^2\Sigma_2^{(2)}\mu_x$ which is the bias for the regression estimator, $\bar{y} - \Sigma_1^2\Sigma_2^{(2)}\bar{x}$.

Wlog we let $\Sigma_2^{(2)} = I$ and $\sigma^2 = 1$. Again for $p = 1$, we observe that $B_1$ changes sign with $\Sigma_1 = \rho$ or $\mu_x$ so we need only study the bias for $\mu_x > 0$ and $\rho > 0$. The values of $-B_1$ for $p = 1$ and $n_1 = 30$ and certain values of $\rho$, $\mu_x$ and $\alpha$ are given in Table 3.1 and are independent of $n_2$. 
### Table 3.1. Values of $-B_1$ for $n_1 = .30$ and $p = 1.$

<table>
<thead>
<tr>
<th>$\mu_x$</th>
<th>$\alpha = .05$</th>
<th>$\alpha = .10$</th>
<th>$\alpha = .25$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0362</td>
<td>0.0465</td>
<td>0.0173</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0781</td>
<td>0.0722</td>
<td>0.0245</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0827</td>
<td>0.0698</td>
<td>0.0207</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0833</td>
<td>0.0489</td>
<td>0.0121</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0497</td>
<td>0.0256</td>
<td>0.0052</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0227</td>
<td>0.0079</td>
<td>0.0017</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0079</td>
<td>0.0030</td>
<td>0.0004</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0021</td>
<td>0.0005</td>
<td>0.0001</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0010</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

From Table 3.1 the following properties of the bias are obvious.

1. The bias is zero when $\mu_x = 0$ which is when the null hypothesis is true.

2. The bias is an increasing function of $\rho$, but a decreasing function of $\alpha$.

3. For fixed $n$, $\alpha$ and $\rho$, the bias first increases from zero and then decreases to zero as $\mu_x$ increases from zero to one.

We observe that the values obtained here correspond with those of Han (1973b) which are used as a further check of the expression for the bias.
For \( p = 2 \), the values of \(-B_1\) for \( n_1 = 30\) and certain values of \( \xi_{12}, \mu_x \) and \( \alpha \) are given in Table 3.2.

Table 3.2. Values of \(-B_1\) for \( p = 2 \) and \( n_1 = 30\).

<table>
<thead>
<tr>
<th>( \xi_{12} )</th>
<th>( \alpha = .05 )</th>
<th>( \alpha = .10 )</th>
<th>( \alpha = .25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (0, 0) )</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( (0.5, 0) )</td>
<td>0.0659</td>
<td>0.0274</td>
<td>0.0126</td>
</tr>
<tr>
<td>( (0.5, 0.5) )</td>
<td>0.0622</td>
<td>0.0511</td>
<td>0.0116</td>
</tr>
<tr>
<td>( (1.0, 0) )</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>( (1.0, 0.5) )</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>( (1.0, 1.0) )</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( (0, 0) )</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( (0.5, 0) )</td>
<td>0.0383</td>
<td>0.0090</td>
<td>0.0126</td>
</tr>
<tr>
<td>( (0.5, 0.5) )</td>
<td>0.0358</td>
<td>0.0165</td>
<td>0.0232</td>
</tr>
<tr>
<td>( (1.0, 0) )</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>( (1.0, 0.5) )</td>
<td>0.0001</td>
<td>0.0001</td>
<td>0.0000</td>
</tr>
<tr>
<td>( (1.0, 1.0) )</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
From Table 3.2 we note that for \( p = 2 \) and \( n_1 = 30 \)

1. The bias is zero when \( \mu_x = 0 \). This once more corresponds to the case when the null hypothesis is true.
2. The bias is generally an increasing function of \( \Sigma_{12} \), but a decreasing function of \( \alpha \).
3. For fixed \( n, \alpha \) and \( \Sigma_{12} \), the bias first increases from zero and then decreases to zero as \( \mu_x' \) increases from \( (0,0) \) to \( (1.0,1.0) \).

C. The M.S.E. of \( \hat{\mu}_{lr} \) when \( \Sigma \) is Known

By definition, the M.S.E. of \( \hat{\mu}_{lr} \) is given by \( \text{M.S.E.}(\hat{\mu}_{lr}) = V(\hat{\mu}_{lr}) + \text{Bias}^2 \). Therefore to find \( \text{M.S.E.}(\hat{\mu}_{lr}) \), we may first find

\[
V(\hat{\mu}_{lr}) = \text{E}(\hat{\mu}_{lr})^2 - \left[\text{E}(\hat{\mu}_{lr})\right]^2
\]

(3.12)

From (3.2), we have

\[
\text{E}(\hat{\mu}_{lr}^2) = \text{E}\left[\left(\overline{y} - \Sigma_{12}\Sigma_{22}^{-1}\bar{x}_2\right)^2|\bar{A}\right]\text{P}(\bar{A})
\]

\[
+ \text{E}\left[\left(\overline{y} - \Sigma_{12}\Sigma_{22}^{-1}\bar{x}_2\right)^2|\bar{A}\right]\text{P}(\bar{A}) + \text{E}\left[\left(\Sigma_{12}\Sigma_{22}^{-1}\bar{D}\Sigma_{22}^{-1}\bar{Z}\right)^2|\bar{A}\right]\text{P}(\bar{A})
\]

\[
+ 2\Sigma_{12}\Sigma_{22}^{-1}\text{E}\left[(\overline{y}\bar{Z})|\bar{A}\right]\text{P}(\bar{A}) - 2\Sigma_{12}\Sigma_{22}^{-1}\text{E}\left[(\bar{X}_1\bar{X}_2')|\bar{A}\right]\text{P}(\bar{A})\Sigma_{22}^{-1}\Sigma_{21}
\]

Therefore

\[
\text{E}(\hat{\mu}_{lr}^2) = \text{E}\left(\overline{y} - \Sigma_{12}\Sigma_{22}^{-1}\bar{x}_2\right)^2 + 2\Sigma_{12}\Sigma_{22}^{-1}\text{E}\left[(\overline{y}\bar{Z})|\bar{A}\right]\text{P}(\bar{A})
\]

(3.13)

\[- 2\Sigma_{12}\Sigma_{22}^{-1}\text{E}\left[(\bar{X}_1\bar{X}_2')|\bar{A}\right]\text{P}(\bar{A})\Sigma_{22}^{-1}\Sigma_{21}
\]

\[+ \Sigma_{12}\Sigma_{22}^{-1}\text{E}\left[\bar{Z}\bar{Z}'|\bar{A}\right]\text{P}(\bar{A})\Sigma_{22}^{-1}\Sigma_{21}
\]
Therefore to evaluate $E(\hat{\mathbf{Z}}_k^2)$, we need to find $E[\mathbf{Z}_{1}^2 \mid \mathbf{A}] P(\mathbf{A})$, $E[\mathbf{Z}_{2}^2 \mid \mathbf{A}] P(\mathbf{A})$ and $E[\mathbf{Z}_{1} \mathbf{Z}_{2} \mid \mathbf{A}] P(\mathbf{A})$. As before, we denote the $i$-th component of $\mathbf{Z}$ by $Z^{(i)}$ and note that it is sufficient to consider only $E[(Z^{(i)})^2 \mid \mathbf{A}] P(\mathbf{A})$ and $E[Z^{(i)} Z^{(K)} \mid \mathbf{A}] P(\mathbf{A})$ for $i \neq K$. To evaluate these, we use the second derivatives of $T$ where $T$ is given in (3.6). Thus differentiating (3.7) w.r.t. $\gamma^{(1)}_x$, we have

$$\frac{\partial^2 T}{\partial \gamma^{(1)}_x} = \int e^{-\frac{1}{2} \delta} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\delta}{2}\right)^j h_{p+2j}^2 (t) dt$$

$$+ \int e^{-\frac{1}{2} \delta} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\delta}{2}\right)^j h_{p+2j}^2 (t) dt$$

$$- \int \frac{1}{2} 2n_1 \gamma^{(1)}_x e^{-\frac{1}{2} \delta} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\delta}{2}\right)^j h_{p+2j}^2 (t) dt$$

$$- \int \frac{1}{2} 2n_1 e^{-\frac{1}{2} \delta} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\delta}{2}\right)^j h_{p+2j}^2 (t) dt$$

$$+ \int \left(\frac{1}{2} 2n_1 \gamma^{(1)}_x \right)^2 e^{-\frac{1}{2} \delta} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\delta}{2}\right)^j h_{p+2j}^2 (t) dt$$

$$- \int \frac{1}{2} 2n_1 \gamma^{(1)}_x e^{-\frac{1}{2} \delta} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\delta}{2}\right)^j h_{p+2j}^2 (t) dt$$
Therefore
\[
\frac{\partial^2 T}{\partial \gamma_x(1)^2} = n_1^2 (\gamma_x(1))^2 [1-H_p+4(c;\delta)] + n_1 [1-H_p+2(c;\delta)]
- n_1^2 (\gamma_x(1))^2 [1-H_p+2(c;\delta)] - n_1 P(\bar{A}) + n_1^2 (\gamma_x(1))^2 P(\bar{A})
- n_1^2 (\gamma_x(1))^2 [1-H_p+2(c;\delta)]
\]

Hence
\[
\frac{\partial^2 T}{\partial \gamma_x(1)^2} = n_1^2 (\gamma_x(1))^2 [1-H_p+4(c;\delta)] + n_1 [1-2n_1 (\gamma_x(1))^2] [1-H_p+2(c;\delta)]
+ n_1 [n_1 (\gamma_x(1))^2 - 1] P(\bar{A})
\]

Similarly, differentiating $T$ twice w.r.t. $\gamma_x(1)$ where $T$ is given in (3.8), we obtain
\[
\frac{\partial^2 T}{\partial \gamma_x(1)^2} = \int \ldots \int \sum_{j=1}^{P} \frac{n_1}{\sqrt{2\pi}} (\frac{n_1}{2})^2 \gamma_x(1)(Z(j) - \gamma_x(1)) e^{-\frac{n_1}{2}(Z(j) - \gamma_x(1))^2} dZ(j)
- \int \ldots \int \sum_{j=1}^{P} \frac{n_1}{\sqrt{2\pi}} (\frac{n_1}{2})^2 \gamma_x(1)(Z(j) - \gamma_x(1)) e^{-\frac{n_1}{2}(Z(j) - \gamma_x(1))^2} dZ(j)
- \int \ldots \int \sum_{j=1}^{P} \frac{n_1}{\sqrt{2\pi}} (\frac{n_1}{2})^2 \gamma_x(1)(Z(j) - \gamma_x(1)) e^{-\frac{n_1}{2}(Z(j) - \gamma_x(1))^2} dZ(j)
\]
Therefore

\[
\frac{\partial^2 T}{\partial \gamma_x^2} = n_1^2 E[Z(1)^2 | \bar{A}] P(\bar{A}) - n_1^2 \gamma_x^{(1)} E(Z(1) | \bar{A}) P(\bar{A}) - n_1 P(\bar{A}) - n_1^2 \gamma_x^{(1)} E(Z(1) | \bar{A}) P(\bar{A}) + n_1^2 (\gamma_x^{(1)})^2 P(\bar{A})
\]  

(3.15)

Hence from (3.10),

\[
\frac{\partial^2 T}{\partial \gamma_x^2} = n_1^2 E[Z(1)^2 | \bar{A}] P(\bar{A}) - 2n_1^2 (\gamma_x^{(1)})^2 [1-H_p+2(c;\delta)] + n_1^2 (\gamma_x^{(1)})^2 P(\bar{A}) - n_1^2 (\gamma_x^{(1)})^2 P(\bar{A})
\]  

(3.16)

Equating (3.16) and (3.14), we have

\[
n_1^2 E[Z(1)^2 | \bar{A}] P(\bar{A}) - 2n_1^2 (\gamma_x^{(1)})^2 [1-H_p+2(c;\delta)] + n_1^2 (\gamma_x^{(1)})^2 P(\bar{A}) - n_1 P(\bar{A}) = n_1^2 (\gamma_x^{(1)})^2 [1-H_p+4(c;\delta)] + n_1^2 (\gamma_x^{(1)})^2 - 1] P(\bar{A})
\]

or

\[
E[Z(1)^2 | \bar{A}] P(\bar{A}) = (\gamma_x^{(1)})^2 [1-H_p+4(c;\delta)] + \frac{1}{n_1} [1-H_p+2(c;\delta)]
\]

Next we find \(E[Z^{(1)} Z^{(K)} | \bar{A}] P(\bar{A})\). Differentiating \(T\) w.r.t. \(\gamma_x^{(1)}\) and then w.r.t. \(\gamma_x^{(K)}\) where \(T\) is given in (3.6), we have
\[
\begin{align*}
\frac{\partial T}{\partial \gamma_x^{(K)}} &= \int_{c} e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n_1 \gamma_x^{(1)}}{2} \frac{2n_1 \gamma_x^{(K)}}{2} j j^{-\frac{\delta}{2}} J_{-\frac{2}{2} h_p + 2j}(t) dt \\
- \int_{c} \frac{1}{2} \frac{2n_1 \gamma_x^{(K)}}{2} e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n_1 \gamma_x^{(1)}}{2} j j^{-\frac{\delta}{2}} J_{-\frac{2}{2} h_p + 2j}(t) dt \\
- \int_{c} \frac{2n_1 \gamma_x^{(1)}}{2} e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{2n_1 \gamma_x^{(K)}}{2} j j^{-\frac{\delta}{2}} J_{-\frac{2}{2} h_p + 2j}(t) dt \\
+ \int_{c} \left( \frac{2n_1}{2} \right)^2 \gamma_x^{(1)} \gamma_x^{(K)} e^{-\frac{1}{2}\delta} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\delta}{2} J_{-\frac{2}{2} h_p + 2j}(t) dt \\
\end{align*}
\]

\[
\frac{\partial T}{\partial \gamma_x^{(1)}} = n_1 \frac{2n_1 \gamma_x^{(1)} \gamma_x^{(K)} [1 - h_p + 4(\gamma_x^{(1)} \gamma_x^{(K)}) - 2n_1 \gamma_x^{(1)} \gamma_x^{(K)} [1 - h_p + 2(\gamma_x^{(1)} \gamma_x^{(K))]}]
\]

\[
+ n_1 \frac{2n_1 \gamma_x^{(1)} \gamma_x^{(K)}}{P(\bar{A})} 
\]

Similarly using (3.8) we have

\[
\begin{align*}
\frac{\partial T}{\partial \gamma_x^{(K)}} &= \int_{\bar{A}} \ldots \int_{\bar{A}} \frac{\sqrt{n_1}}{\sqrt{2\pi}} \left( \frac{-2n_1}{2} \right)^2 (\gamma_x^{(1)})(\gamma_x^{(K)}) \left( \gamma_x^{(1)})(\gamma_x^{(K)}) \right) \\
&\quad \times \frac{n_1}{2} (\gamma_x^{(1)})(\gamma_x^{(1)})^2 e^{-\frac{1}{2}(\gamma_x^{(1)})(\gamma_x^{(1)})^2} dZ(1) \\
\end{align*}
\]

Therefore
\[
\frac{\partial^2}{\partial \gamma_x^T \partial \gamma_x} = n_1 \mathrm{E}(Z(1)Z(K)\mid \bar{A})P(\bar{A}) - n_1 \mathbf{1}^T \mathbf{E}(Z(K)\mid \bar{A})P(\bar{A})
\]

\[
- n_1 \mathbf{1}^T \mathbf{E}(Z(1)\mid \bar{A})P(\bar{A}) + n_1 \gamma_x^T \gamma_x P(\bar{A})
\]  \hspace{1cm} (3.18)

Therefore using (3.10)

\[
\frac{\partial^2}{\partial \gamma_x^T \partial \gamma_x} = n_1 \mathbf{E}(Z(1)Z(K)\mid \bar{A})P(\bar{A}) - n_1 \mathbf{1}^T \mathbf{E}(1-H_p+2(c;\delta)) \gamma_x^T \gamma_x
\]

\[
- n_1 \mathbf{1}^T \mathbf{E}(1-H_p+2(c;\delta)) \gamma_x^T \gamma_x + n_1 \gamma_x^T \gamma_x P(\bar{A})
\]  \hspace{1cm} (3.19)

Equating (3.17) and (3.19), we have

\[
\mathbf{E}(Z(1)Z(K)\mid \bar{A})P(\bar{A}) = [1-H_p+4(c;\delta)] \gamma_x^T \gamma_x
\]  \hspace{1cm} (3.20)

and so we have evaluated \( \mathbf{E}(ZZ'\mid \bar{A})P(\bar{A}) \) completely. For convenience, we let \( \mathbf{E}(ZZ'\mid \bar{A})P(\bar{A}) = W \) where \( W \) is a pxp matrix with the \( i \)-th diagonal element

\[
\gamma_x^T \gamma_x \left( [1-H_p+4(c;\delta)] + \frac{1}{n_1} [1-H_p+2(c;\delta)] \right)
\]

and the \((1,K)\)th off diagonal element \([1-H_p+4(c;\delta)] \gamma_x^T \gamma_x \).

Next we evaluate other terms in (3.13).

\[
\mathbf{E}(\bar{y} - \Sigma_{12} \mathbf{E}^{-1} \bar{X}_2)^2 = \mathbf{E}(\bar{y}^2) - 2 \Sigma_{12} \Sigma_{22}^{-1} \mathbf{E}(\bar{Y} \bar{X}_2) + \Sigma_{12} \Sigma_{22}^{-1} \mathbf{E}(\bar{X}_2 \bar{X}_2') \Sigma_{22}^{-1} \Sigma_{21}
\]

\[
= \frac{1}{n_2} \sigma^2 + \mu^2 - 2 \Sigma_{12} \Sigma_{22}^{-1} \{ \text{Cov}(\bar{Y} \bar{X}_2) + \mathbf{E}(\bar{Y}) \mathbf{E}(\bar{X}_2) \}
\]

\[
+ \Sigma_{12} \Sigma_{22}^{-1} \{ \mathbf{E}(\bar{X}_2) + [\mathbf{E}(\bar{X}_2)][\mathbf{E}(\bar{X}_2)'] \Sigma_{22}^{-1} \Sigma_{21}
\]
\[
\begin{align*}
&= \frac{1}{n_2} \sigma^2 + \mu^2 - \frac{2}{n_2} \Sigma_1 \Sigma_2^{12} \Sigma_2^{21} - 2 \Sigma_1 \Sigma_2^{12} \Sigma_2^{22} \\
&\quad + \frac{1}{n_2} \Sigma_2^{12} \Sigma_2^{22} \Sigma_2^{12} + \Sigma_1 \Sigma_2^{12} \Sigma_2^{22} \Sigma_2^{12} \\
&= \frac{1}{n_2} \sigma^2 + \mu^2 - \frac{1}{n_2} \Sigma_1 \Sigma_2^{12} \Sigma_2^{22} \Sigma_2^{12} - 2 \Sigma_1 \Sigma_2^{12} \Sigma_2^{22} \mu \Sigma_2^{12} + \Sigma_1 \Sigma_2^{12} \Sigma_2^{22} \Sigma_2^{12} \Sigma_2^{12} \\
\end{align*}
\]

Now

\[
E[(\tilde{Y} \tilde{Z}) | \bar{A}] P(\bar{A}) = E[E(\tilde{Z} | \tilde{Z}, \bar{A})] P(\bar{A})
\]

\[
= E[E[\tilde{Z} | \tilde{Y}, \tilde{Z}] | \bar{A}] P(\bar{A})
\]

\[
= E[\tilde{Z} | \mu + \Sigma_1 \Sigma_2^{12} \Sigma_2^{12} \Sigma_2^{12} - \Sigma_2^{12} \Sigma_2^{12} \tilde{Z} | \bar{A}] P(\bar{A})
\]

\[
= \mu E[\tilde{Z} | \bar{A}] P(\bar{A}) + E[\tilde{Z} \Sigma_1 \Sigma_2^{12} \Sigma_2^{12} \tilde{Z} | \bar{A}] P(\bar{A})
\]

\[
- E[\Sigma_1 \Sigma_2^{12} \Sigma_2^{12} \tilde{Z} | \bar{A}] P(\bar{A})
\]

But since \( \Sigma_1 \Sigma_2^{12} \Sigma_2^{12} \tilde{Z} \) is a scalar and \( \Sigma_1 \Sigma_2^{12} \Sigma_2^{12} \tilde{Y}_x \) is a constant,

\[
\Sigma_1 \Sigma_2^{12} \Sigma_2^{12} \tilde{Z} = \tilde{Z}' \Sigma_2^{12} \Sigma_2^{12} \]

and

\[
\Sigma_1 \Sigma_2^{12} \Sigma_2^{12} \tilde{Y}_x = \tilde{Y}_x \Sigma_2^{12} \Sigma_2^{12}
\]

Hence

\[
E(\tilde{Y} \tilde{Z} | \bar{A}) P(\bar{A}) = \mu E(\tilde{Z} | \bar{A}) P(\bar{A}) + E[\tilde{Z} \tilde{Z}' | \bar{A}] P(\bar{A}) \Sigma_2^{12} \Sigma_2^{12} \tilde{Z} + \tilde{Z} P(\bar{A}) \Sigma_2^{12} \Sigma_2^{12} \tilde{Y}_x
\]

\[
- E(\tilde{Z} | \bar{A}) P(\bar{A}) \tilde{Y}_x \Sigma_2^{12} \Sigma_2^{12}
\]
or

\[ E(\tilde{Y}_2|\tilde{A})P(\tilde{A}) = \mu[1-H_p+2(c;\delta)]y_x + WD'^{-1}D_{22}^{-1}E_{22}^{-1} \]

\[ [1-H_p+2(c;\delta)]y_x y'D'_{22}^{-1}D_{22}^{-1}E_{22}^{-1} \quad (3.21) \]

Finally,

\[ E[\tilde{X}_1\tilde{X}_2|\tilde{A}]P(\tilde{A}) = E\{E(E[\tilde{X}_1\tilde{X}_2|\tilde{X}_1,\tilde{A}]\}P(\tilde{A}) \]

\[ = E\{\tilde{X}_1E[\tilde{X}_2\tilde{X}_1]|\tilde{A}\}P(\tilde{A}) \]

\[ = E\{\tilde{X}_1[\mu'_x + (\tilde{X}_1'-\mu'_x)]|\tilde{A}\}P(\tilde{A}) \]

\[ = E[(\tilde{X}_1\tilde{X}_1')|\tilde{A}]P(\tilde{A}) . \]

Therefore

\[ E[\tilde{X}_1\tilde{X}_2|\tilde{A}]P(\tilde{A}) = D'^{-1}WD'^{-1} \quad (3.22) \]

and substituting these into (3.13) and then into (3.12), we have

\[ V(\hat{\mu}_x) = \frac{1}{n_2} \sigma^2 - \frac{1}{n_2} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} - 2\Sigma_{12} \Sigma_{22}^{-1} \mu_x + \Sigma_{12} \Sigma_{22}^{-1} \mu_x \Sigma_{22}^{-1} \Sigma_{21} \]

\[ + 2\Sigma_{12} \Sigma_{22}^{-1} \mu_x [1-H_p+2(c;\delta)] - 2\Sigma_{12} \Sigma_{22}^{-1} \mu_x \Sigma_{22}^{-1} \Sigma_{21} [1-H_p+2(c;\delta)] \]

\[ + \Sigma_{12} \Sigma_{22}^{-1} D^{-1}WD'^{-1}D_{22}^{-1}E_{22}^{-1} + 2\mu_x \Sigma_{12} \Sigma_{22}^{-1}H_p+2(c;\delta) \]

\[ - \Sigma_{12} \Sigma_{22}^{-1}H_p+2(c;\delta)^2 \]
Noting that
\[
\Sigma_{12}^{-1/2} D^{-1/2} \Sigma_{22}^{-1/2} = \Sigma_{12}^{-1/2} \Sigma_{22}^{-1/2} \Sigma_{22}^{-1/2} [1 - H_p + 4(c; \delta)]
\]
and that \(D^{-1} D'^{-1} = \Sigma_{22}^2\), then

\[
V(\hat{\mu}_{lr}) = \frac{1}{n_2} \sigma^2 - \frac{1}{n_2} \Sigma_{12}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1} + \Sigma_{12}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1} [1 - H_p + 4(c; \delta)]
\]

\[- 2 \Sigma_{12}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1} [1 - H_p + 4(c; \delta)]
\]

\[+ \Sigma_{12}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1} [1 - H_p + 4(c; \delta)]
\]

\[- \frac{1}{n_1} \Sigma_{12}^{-1} \Sigma_{22}^{-1} [1 - H_p + 4(c; \delta)]
\]

\[- \Sigma_{12}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1} [1 - H_p + 4(c; \delta)]^2
\]

(3.23)

As a partial check, when \(c = 0\) and we always reject the null hypothesis, then \(\hat{\mu}_{lr}\) reduces to \(\bar{y} + \Sigma_{12}^{-1} \Sigma_{22}^{-1} (\bar{x}_1 - \bar{x}_2)\). Now

\[V[\bar{y} + \Sigma_{12}^{-1} \Sigma_{22}^{-1} (\bar{x}_1 - \bar{x}_2)] = V(\bar{y}) + V(\Sigma_{12}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1})
\]

\[+ 2 \Sigma_{12}^{-1} \Sigma_{22}^{-1} \text{Cov}(\bar{y}, \bar{x}_1) - 2 \Sigma_{12}^{-1} \Sigma_{22}^{-1} \text{Cov}(\bar{x}_1, \bar{x}_2) \Sigma_{22}^{-1} \Sigma_{22}^{-1}
\]

\[- 2 \Sigma_{12}^{-1} \Sigma_{22}^{-1} \text{Cov}(\bar{y}, \bar{x}_2) = \frac{1}{n_2} \sigma^2 + \frac{1}{n_1} \Sigma_{12}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1} + \frac{1}{n_2} \Sigma_{12}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1}
\]

\[+ \frac{2}{n_1} \Sigma_{12}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1} - \frac{2}{n_1} \Sigma_{12}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1} - \frac{2}{n_2} \Sigma_{12}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1}
\]
Therefore

\[ V(\bar{y} + \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} (\bar{x}_{i} - \bar{x}_{2})) = \frac{1}{n_2} \sigma^2 + \frac{1}{n_1} \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} - \frac{1}{n_2} \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \Sigma_{21} \] (3.24)

Therefore putting \( c = 0 \) in (3.23), i.e. \( H_{p+2}(c;\delta) = H_{p+4}(c;\delta) = 0 \)

\[ V(\hat{\mu}_{lr}) = \frac{1}{n_2} \sigma^2 - \frac{1}{n_2} \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \Sigma_{21} + \frac{1}{n_1} \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \Sigma_{21} \]

which is the same as (3.24).

When \( c = \infty \), we always accept \( H_0 \) and the preliminary test estimator reduces to \( \bar{y} - \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \bar{x}_{2} \). Now

\[ V(\bar{y} - \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \bar{x}_{2}) = V(\bar{y}) + V(\Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \bar{x}_{2}) - 2 \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \text{Cov}(\bar{y}, \bar{x}_{2}) \] (3.25)

\[ = \frac{1}{n_2} \sigma^2 + \frac{1}{n_2} \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \Sigma_{21} - \frac{2}{n_2} \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \Sigma_{21} + \frac{1}{n_2} \sigma^2 - \frac{1}{n_2} \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \Sigma_{21} \]

putting \( c = \infty \) in (3.23), then \( H_{p+2}(c;\delta) = H_{p+4}(c;\delta) = 1 \) and

\[ V(\hat{\mu}_{lr}) = \frac{1}{n_2} \sigma^2 - \frac{1}{n_2} \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \Sigma_{21} \]

which is the same as (3.25). We now give the M.S.E. of \( \hat{\mu}_{lr} \).

\[
\text{M.S.E.}(\hat{\mu}_{lr}) = V(\hat{\mu}_{lr}) + \text{Bias}^2 = \frac{1}{n_2} \sigma^2 + \frac{1}{n_1} \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} [1 - H_{p+2}(c;\delta)] \\
- \frac{1}{n_2} \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} + \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \Sigma_{22}^{l} \Sigma_{21} - 2 \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \Sigma_{22}^{l} \Sigma_{21} [1 - H_{p+2}(c;\delta)] \\
+ \Sigma_{i=1}^{12} \Sigma_{j=22}^{l} \Sigma_{22}^{l} \Sigma_{21} [1 - H_{p+4}(c;\delta)]
\]
or

\[ \text{M.S.E.}(\hat{\mu}_{x^r}) = g_1 + h_1 \]  (3.26)

where

\[ g_1 = \frac{1}{n_2} \sigma^2 + \frac{1}{n_1} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} - \frac{1}{n_2} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \]

and

\[ h_1 = \text{M.S.E.}(\hat{\mu}_{x^r}) - g_1 . \]

We note that \( g_1 \) is the variance of \( \bar{y} + \Sigma_{12} \Sigma_{22}^{-1} (\bar{x}_1 - \bar{x}_2) \) which is the linear regression estimator ignoring the information of \( \mu_x \).

D. Relative Efficiency (\( e_1 \))

In practice, we would want to select an estimator for \( \mu \) with the smallest bias and M.S.E.. Again we consider only the M.S.E. of the preliminary test estimator since bias is a part of M.S.E. Using (3.26), we compare the performance of the preliminary test estimator \( \hat{\mu}_{x^r} \) with the usual linear regression estimator, \( \bar{y} + \Sigma_{12} \Sigma_{22}^{-1} (\bar{x}_1 - \bar{x}_2) \), when the information of \( \mu_x \) is ignored. The relative efficiency of \( \hat{\mu}_{x^r} \) to \( \bar{y} + \Sigma_{12} \Sigma_{22}^{-1} (\bar{x}_1 - \bar{x}_2) \) is defined as

\[ e_1 = \frac{\text{M.S.E.}(\bar{y} + \Sigma_{12} \Sigma_{22}^{-1} (\bar{x}_1 - \bar{x}_2))}{\text{M.S.E.}(\hat{\mu}_{x^r})} \]  (3.27)
and since \( \bar{y} + \Sigma_{12}^{z} \Sigma_{22}^{-1}(\bar{x}_1 - \bar{x}_2) \) is unbiased, its M.S.E. is equal to its variance. Therefore, using (3.24) and (3.26), we have

\[
e_1 = \frac{g_1}{g_1 + h_1}
\]  

(3.28)

Wlog we let \( \Sigma_{22} = I \) and \( \sigma^2 = 1 \). Hence for \( p = 1, \Sigma_{12} = \rho \). Table 3.3 gives the values of \( e_1 \) for \( n_1 = 30, n_2 = 10, p = 1 \) and certain values of \( \rho, \mu_x \) and \( \alpha \).

Table 3.3. Values of \( e_1 \) for \( p = 1, n_1 = 30, n_2 = 10 \).

<table>
<thead>
<tr>
<th>( \mu_x )</th>
<th>( \rho )</th>
<th>( \alpha = .05 )</th>
<th>( \alpha = .10 )</th>
<th>( \alpha = .25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.7</td>
<td>.9</td>
<td>.7</td>
<td>.9</td>
<td>.7</td>
</tr>
<tr>
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</tr>
<tr>
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<td>.9992</td>
<td>.9980</td>
<td>.9998</td>
<td>.9995</td>
</tr>
</tbody>
</table>

From Table 3.3 we can easily observe the following properties.

1. The relative efficiency of \( \hat{\mu}_{lr} \) assumes its maximum value when \( \mu_x = 0 \).
2. For \( \mu_x \leq 0.1 \) and fixed \( \alpha, n_1 \) and \( n_2, e_1 \) is larger for \( p = 0.9 \) than for \( p = 0.7 \) and is a decreasing function of \( \alpha \) within this range.

3. For \( 0.2 \leq \mu_x \leq 0.7 \), \( e_1 \) is larger for \( p = 0.7 \) than for \( p = 0.9 \) and is an increasing function of \( \alpha \) within this range.

4. For \( \mu_x \geq 0.8 \), there is no appreciable difference in the values of \( e_1 \) for either values of \( p \) or different values of \( \alpha \).

5. For fixed \( n_1, n_2, p \) and \( \alpha, e_1 \) first decreases from a value above unity to some minimum and then increases again to unity as \( \mu_x \) increases.

Table 3.4 gives the values of \( e_1 \) for \( p = 2, n_1 = 30, n_2 = 10 \) and certain values of \( \Sigma_{12}, \alpha \) and \( \mu_x \). From the table, the following properties of \( e_1 \) are apparent.

1. \( e_1 \) has its maximum when \( \mu_x = 0 \).

2. The maximum at \( \mu_x = 0 \) is an increasing function of \( p \) for fixed \( \alpha, n_1 \) and \( n_2 \).

3. For fixed \( \alpha, n_1, n_2 \) and \( \Sigma_{12}, e_1 \) decreases from the maximum value to a minimum and then increases to unity as \( \mu_x \) increases from \((0,0)\) to \((1.0,1.0)\).

E. The Optimal Sample Design and Comparisons

The problem here is to find the optimum allocation of the sample sizes \( n_1 \) and \( n_2 \) for some given cost function. Usually the cost function is of the form
Table 3.4. Values of $e_1$ for $p = 2$, $n_1 = 30$ and $n_2 = 10$.

<table>
<thead>
<tr>
<th>$\Sigma_{12}$</th>
<th>$\alpha = .05$</th>
<th>$\alpha = .10$</th>
<th>$\alpha = .25$</th>
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where $c_1$ is the cost of observing the vector $X$ and $c_2$ is the cost of observing $Y$. The optimum values of $n_1$ and $n_2$ are obtained by minimizing the m.s.e.($\hat{\mu}_r$) given in (3.26) subject to the constraint (3.29). We recall that in practice, under the supposition of a conditional specification, the experimenter has only partial information based on which he believes that $\mu_X$ is close to 0. The relative efficiency of $\hat{\mu}_r$ is largest at $\mu_X = 0$ and so it would be best to consider the problem of optimum allocation under the optimum situation by letting $\mu_X = 0$ in m.s.e.($\hat{\mu}_r$).

When $\mu_X = 0$, from (3.26)

$$m.s.e. (\hat{\mu}_r) = \frac{K_1}{n_1} + \frac{K_2}{n_2}$$

(3.30)

where

$$K_1 = \Sigma_{12} \Sigma_{22}^{-1} \{ 1 - \Sigma_{22} \Sigma_{12} \}$$

and

$$K_2 = \sigma^2 - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

Thus we wish to minimize (3.30) subject to (3.29). From (3.29),

$$n_1 = \frac{C - n_2 c_2}{c_1}.$$
Thus

\[
\text{m.s.e.}(\hat{\mu}_{kr}) = \frac{K_1 c_1}{c-n_2 c_2} + \frac{K_2}{n_2}
\]

and

\[
\frac{\partial \text{m.s.e.}(\hat{\mu}_{kr})}{\partial n_2} = \frac{K_1 c_1 c_2}{(c-n_2 c_2)^2} - \frac{K_2}{n_2^2} = 0
\]

or

\[
n_2 = \frac{c\sqrt{K_2}}{\sqrt{K_1 c_1 c_2} + c_2 \sqrt{K_2}}
\]

Substituting in (3.29) we have

\[
n_1 = \frac{c\sqrt{K_1}}{\sqrt{K_2 c_1 c_2} + c_1 \sqrt{K_1}}
\]

Substituting for \(n_1\) and \(n_2\) in (3.30), the optimum value of \(\text{m.s.e.}(\hat{\mu}_{kr})\) is

\[
\text{m.s.e.}(\hat{\mu}_{kr})_{\text{opt}} = \frac{K_1 \{c\sqrt{K_2 c_1 c_2} + c_1 \sqrt{K_1}\}}{c\sqrt{K_1}} + \frac{K_2 \{c\sqrt{K_1 c_1 c_2} + c_2 \sqrt{K_2}\}}{c\sqrt{K_2}}
\]

\[
= \frac{K_1 c_1 + 2\sqrt{K_1 K_2 c_1 c_2} + c_2 K_2}{c}
\]

\[
= \frac{(\sqrt{K_1 c_1} + \sqrt{K_2 c_2})^2}{c}
\]

(3.31)
The regression estimator under double sampling without using the preliminary test is \( \bar{y} + \Sigma_{12} \Sigma_{22}^{-1} (\bar{x}_1 - \bar{x}_2) \) with variance

\[
\frac{\Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}}{n_1} + \frac{\sigma^2 - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}}{n_2} = \frac{K_1'}{n_1} + \frac{K_2'}{n_2}
\]

where

\[ K_1' = \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \]

and

\[ K_2' = \sigma^2 - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \]

Next we note that since \( \bar{y} + \Sigma_{12} \Sigma_{22}^{-1} (\bar{x}_1 - \bar{x}_2) \) is unbiased, its variance equals its m.s.e. and so denoting this m.s.e. by \( M \) and following the above method of minimizing m.s.e.\((\hat{u}_{lr})\) we find that

\[
M_{\text{opt}} = \frac{(\sqrt{K_1'} c_1 + \sqrt{K_2'} c_2)^2}{c} \quad (3.32)
\]

Now to compare (3.31) and (3.32) we note from (3.31) that \((1-H_p+2(c;0))\) is a decreasing function of \(c\) with a maximum equal to unity at \(c = 0\). Hence the numerator of m.s.e.\((\hat{u}_{lr})\) is at most as large as that of \(M_{\text{opt}}\) and so we are led to conclude that m.s.e.\((\hat{u}_{lr})_{\text{opt}} \leq M_{\text{opt}}\) with equality holding for \(c = 0\) which is the point at which the two estimators coincide.
We shall now compare \( \hat{u}_{lr} \) with the preliminary test estimator, \( \hat{u} \), of Chapter II for a fixed total budget. For the double sampling scheme, the cost function in (3.29) remains unchanged and when \( u_x = 0 \), we are led to the optimum value of m.s.e. (\( \hat{u}_{lr} \)) in (3.31). Under the optimum situation, we shall find the optimum value of m.s.e. (\( \hat{u} \)) which we denote by m.s.e. (\( \hat{u} \))\textsubscript{opt}. When \( u_x = 0 \), from (2.24), the m.s.e. of the preliminary test estimator is \( \frac{V}{n} \) where

\[
V = \sigma^2 - \sum_{i=1}^{2} \sum_{j=1}^{22} \sum_{k=21}^{H_{p+2}} (c;0) .
\]

If the total budget is devoted to a single sample, this sample has size

\[
n = \frac{c}{c_2}
\]

and

\[
m.s.e.(\hat{u})_{opt} = \frac{c_2V}{c}.
\]

Hence under the optimum situation, i.e. \( u_x = 0 \), double sampling gives a smaller m.s.e. if

\[
c_2V > (\sqrt{K_1c_1} + \sqrt{K_2c_2})^2 .
\]

When \( u_x \neq 0 \), from (3.26)

\[
m.s.e.(\hat{u}_{lr}) = \frac{K^*}{n_1} + \frac{K_2}{n_2} + \theta_1
\]
where $K_2$ is as defined in (3.30) and

$$K_1 = \sum_{ij} \Sigma_{i=1}^{n-1} \Sigma_{j=1}^{2} \Sigma_{2}^{1} [\Sigma_{2}^{0} + \Sigma_{2}^{P+2} (c; \delta)]$$

$$\theta_1 = 2 \sum_{ij} \Sigma_{i=1}^{n-1} \Sigma_{j=1}^{2} \Sigma_{2}^{1} \Sigma_{2}^{1} \Sigma_{2}^{0} + \Sigma_{2}^{P+2} (c; \delta)$$

$$- \sum_{ij} \Sigma_{i=1}^{n-1} \Sigma_{j=1}^{2} \Sigma_{2}^{1} \Sigma_{2}^{1} \Sigma_{2}^{0} + \Sigma_{2}^{P+4} (c; \delta)$$

Similarly, when $\mu_0 \neq 0$, from (2.24), m.s.e.($\hat{\mu}$) = $\frac{V^*}{n} + \theta_2$

where

$$V^* = \sigma^2 - \sum_{ij} \Sigma_{i=1}^{n-1} \Sigma_{j=1}^{2} \Sigma_{2}^{1} \Sigma_{2}^{0} (c; \lambda)$$

$$\theta_2 = 2 \sum_{ij} \Sigma_{i=1}^{n-1} \Sigma_{j=1}^{2} \Sigma_{2}^{1} \Sigma_{2}^{1} \Sigma_{2}^{0} (c; \lambda)$$

$$- \sum_{ij} \Sigma_{i=1}^{n-1} \Sigma_{j=1}^{2} \Sigma_{2}^{1} \Sigma_{2}^{1} \Sigma_{2}^{0} (c; \lambda)$$

We may now compare the two mean square errors by substituting for $n_1$, $n_2$ and $n$ in the expression,

$$\text{m.s.e.}(\hat{\mu}) - \text{m.s.e.}(\hat{\mu}_{\mu_0}) = \left( \frac{V^*}{n} + \theta_2 \right) - \left( \frac{K_1}{n_1} + \frac{K_2}{n_2} + \theta_1 \right).$$

Double sampling gives a smaller m.s.e. if the expression is positive.

The detailed expression is complicated and would not be given here.
F. Bias of $\hat{\mu}_{lr}$ when $\Sigma$ is Unknown

When $\Sigma$ is unknown, the linear regression preliminary test estimator becomes

$$
\hat{\mu}_{lr} = \begin{cases} 
\bar{y} - S_{12} S^{-1}_{22} \bar{x}_2 & \text{if } m_1 n_1 (\bar{x}_1 S^{-1}_{22} \bar{x}_1) \leq T_0^2 \\
\bar{y} + S_{12} S^{-1}_{22} (\bar{x}_1 - \bar{x}_2) & \text{if } m_1 n_1 (\bar{x}_1 S^{-1}_{22} \bar{x}_1) > T_0^2 
\end{cases} 
$$

(3.33)

where $m_1 = n_1 - 1$, $T_0^2$ is the $100(1-\alpha)$th percentile of the Hotelling's $T^2$ distribution with $m_1$ degrees of freedom and we define

$$
S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}
$$

where

$$
\bar{y} = \frac{1}{n_2} \sum_{i=1}^{n_2} y_i
$$

$$
\bar{x}_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} x_i
$$

$$
\bar{x}_2 = \frac{1}{n_2} \sum_{i=1}^{n_2} x_i
$$

$$
S_{11} = \frac{n_2}{n_2} \sum_{i=1}^{n_2} (y_i - \bar{y})^2
$$
\[ S_{12} = \sum_{i=1}^{n_2} (y_i - \bar{y})(x_i - \bar{x}_2)' \]

\[ S_{22} = \sum_{i=1}^{n_1} (x_i - \bar{x}_1)(x_i - \bar{x}_1)' \]

In this section, we shall obtain the bias of \( \hat{\mu}_r \) when \( \Sigma \) is unknown. If we denote the rejection region for the preliminary test \( \{ m_1 n_1 (x_i s^{-1} \bar{x}_1) : m_1 n_1 (x_i s^{-1} \bar{x}) > \tau_0^2 \} \) by \( \tilde{G} \),

then

\[
E(\hat{\mu}_r) = E(\bar{y} - S_{12} s^{-1} \bar{x}_2 | \tilde{G}) P(\tilde{G}) + E(\bar{y} + S_{12} s^{-1} (\bar{x}_1 - \bar{x}_2) | \tilde{G}) P(\tilde{G})
\]

\[
= E(\bar{y} - S_{12} s^{-1} \bar{x}_2) + E(S_{12} s^{-1} \bar{x}_1 | \tilde{G}) P(\tilde{G})
\]

(3.34)

Now \( (\bar{x}_1, \bar{x}_2, \bar{y}) \) has a normal distribution as in section B and is independent of \( (S_{22}, S_{11}, S_{12}) \) which has a Wishart distribution.

\[ E(\bar{y}) = \mu \]

and so if we write \( E(\hat{\mu}_r) = \mu + B_2 \), we see that the bias is

\[ B_2 = E(S_{12} s^{-1} \bar{x}_1 | \tilde{G}) P(\tilde{G}) - E(S_{12} s^{-1} \bar{x}_2) \]

Since \( S_{12} s^{-1} \) and \( \bar{x}_2 \) are independent, we know

\[ E(S_{12} s^{-1} \bar{x}_2) = E(S_{12} s^{-1}) \cdot E(\bar{x}_2) \]
where

\[ E(S_{12}S_{22}^{-1}) = E\{E(S_{12}S_{22}^{-1}|X)\} \]

\[ = E[E[S_{12}|X]S_{22}^{-1}] \]

\[ = E[E[ \sum_{i=1}^{n_2} (y_i - \bar{y})(X_i - \bar{X}_2)'|X]S_{22}^{-1}] \]

\[ E(y_1|X) = u + \Sigma_{12}E^{-1}(X_1 - \mu_x) \]

\[ \Rightarrow E(S_{12}S_{22}^{-1}) = E[\Sigma_{12}E^{-1}S_{22}S_{22}^{-1}] = \Sigma_{12}E^{-1} \]

and

\[ E(S_{12}S_{22}^{-1}X_2) = \Sigma_{12}E^{-1}u_x \]

Hence

\[ B_2 = E[S_{12}S_{22}^{-1}X_1|\tilde{G}]P(\tilde{G}) - \Sigma_{12}E^{-1}u_x \quad (3.35) \]

It remains to evaluate the first term.

Let \( f(\bar{X}_1) \) be the multivariate normal density of \( \bar{X}_1 \) and \( g(S_{22}, S_{12}, S_{11}) \) the joint density of \( S_{22}, S_{12} \) and \( S_{11} \), then

\[ E[S_{12}S_{22}^{-1}X_1|\tilde{G}]P(\tilde{G}) \]

\[ = \int \cdots \int S_{12}S_{22}^{-1}f(\bar{X}_1)g(S_{22}, S_{12}, S_{11})d\bar{X}_1 dS_{22}dS_{12}dS_{11} \quad (3.36) \]

We make the following transformations as we did in (2.31).
\( W_1 = TB'BT' \)
\[
W_2 = [S_{12} - \Sigma_{12}T'TB'B]\Sigma_{11.2}^{-1/2}B^{-1}
\]
\[
W_3 = (S_{11}^2 - S_{12}B^{-1}B^{-1}S_{21})\Sigma_{11.2}
\]

Substituting in (3.36) we have

\[
E\{S_{12}S_{22}^{-1}\mid \tilde{G}\}P(\tilde{G}) \tag{3.38}
\]

\[
= \int \cdots \int \frac{1}{\tilde{G}} T + \Sigma_{12}T'T\tilde{X}_1 f(\tilde{X}_1) E(W_1, W_2, W_3) dW_3 dW_2 dW_1 d\tilde{X}_1
\]

where as before \( W_1 \sim W(I, n_1-1), W_2 \sim N(0, I), W_3 \sim W(1, n_1-K) \)

and they are independent. The joint density is

\[
g(W_1, W_2, W_3) = c_0 e^{-\frac{1}{2} \text{tr}(W_1'W_2 + W_2'W_3 + W_1')} |W_3|^{\frac{1}{2}(n_1-p-3)} |W_1|^{\frac{1}{2}(n_1-p-2)}
\]  
\[ \tag{3.39} \]

The region of integration is given by

\[
\tilde{G} = \{n_1m_1(\tilde{X}_1'T'W_1^{-1}\tilde{X}_1);(\tilde{X}_1'T'W_1^{-1}\tilde{X}_1)n_1m_1 > T^2\}.
\]

Hence (3.38) becomes
But \( E(W_2) = 0 \) so that from the independence of the \( W_i \)'s, the first term is zero. Hence (3.38) is equal to

\[
\frac{\Sigma_{12}^{T, T}}{\frac{1}{2} p(n_1 - 1) \frac{1}{2} p(p - 1) \prod_{i=1}^{P} \Gamma[\frac{1}{2}(n_1 - 1)]} \int_{\mathcal{G}} e^{-\frac{1}{2} \text{tr} W_1} |W_1|^\frac{1}{2}(n_1 - p - 2) \bar{x}_1 f(\bar{x}_1) d\bar{x}_1 d\bar{x}_1
\]

\( \bar{x}_1 \sim N(\mu_x, \frac{1}{n_1} T^{-1} T^{-1}) \)
We let \( Z = T\bar{x}_1 \) and \( \nu_x = T\bar{y}_x \). Therefore \( Z \sim N(\nu_x, \frac{1}{n_1}) \).

Therefore \( \Gamma = \{ n_1 m_1 (Z' W^{-1} Z) : n_1 m_1 (Z' W^{-1} Z) > t_0^2 \} \). Hence we wish to evaluate

\[
\frac{\Sigma_{12} T' T T^{-1}}{2^p(n-1) \pi^p(n-1)} \prod_{i=1}^{\frac{1}{2} \text{tr} W_1} |W_1|^{\frac{1}{2}(n_1-p-2)} \int \cdots \int_{\Gamma [\frac{1}{2}(n_1-1)]} Zg(Z) dZ dW_1
\]

and

\[
F' = n_1 (Z' W^{-1} Z) \frac{(n_1-p)}{p}
\]

has the noncentral \( F \) distribution with \( p \) and \( n_1-p \) degrees of freedom and noncentrality parameter \( \lambda = n_1 \nu'_x \nu_x \). Therefore

\[
P(\Gamma) = P(F' > F_p, n_1-p(\alpha))
\]

\[
= P(G > \frac{p}{n_1-p} F_p, n_1-p(\alpha)).
\]

where

\[
G = \frac{p}{n_1-p} F'
\]

Let

\[
c = \frac{p}{n_1-p} F_p, n_1-p(\alpha),
\]

then
123

\[ R = P(\tilde{\alpha}) = \int_{c}^{\infty} e^{-\frac{1}{2} \lambda} \sum_{j=0}^{\infty} \frac{1}{j!} (\frac{\lambda}{2})^{j} G_{(p+\frac{n_1-p}{2}, \frac{1}{2})} (p+\frac{1}{2}j, \frac{n_1-p}{2}) \]  

(3.41)

where

\[ G_{(p+\frac{n_1-p}{2}, \frac{1}{2})} (p+\frac{1}{2}j, \frac{n_1-p}{2}) \]

is as defined in (2.35). Differentiating (3.41) w.r.t. \( v_{x}^{(1)} \), we have

\[ \frac{3R}{3v_{x}^{(1)}} = \int_{c}^{\infty} e^{-\frac{1}{2} \lambda} \sum_{j'=0}^{\infty} \frac{2n_1 v_{x}^{(1)}}{2} \frac{1}{j'!} (\frac{\lambda}{2})^{j'} G_{(p+\frac{2+j'}{2}, \frac{n_1-p}{2})} (p+\frac{1}{2}j', \frac{n_1-p}{2}) \]

(3.42)

or

\[ \frac{3R}{3v_{x}^{(1)}} = n_1 v_{x}^{(1)} [1-G_{p+2, n_1-p}(c;\lambda)] - n_1 v_{x}^{(1)} P(\tilde{\alpha}) \]

(3.42)

where

\[ G_{p+2, n_1-p}(c;\lambda) \]

is the cumulative distribution of the noncentral \( G \) distribution with \( p+2, n_1-p \) degrees of freedom and noncentrality parameter \( \lambda \).
Next we make use of the distributions of $Z$ and $W_1$. From the independence, we may write

$$\bar{R} = P(\bar{G}) = \prod_{j=1}^{\frac{n_1}{2}} e^{-\frac{n_1}{2}(Z(j) - \nu_x(j))^2}$$

(3.43)

Differentiating (3.43) w.r.t. $\nu^{(1)}_x$, we obtain

$$\frac{\partial \bar{R}}{\partial \nu^{(1)}_x} = \prod_{j=1}^{\frac{n_1}{2}} e^{-\frac{n_1}{2}(Z(j) - \nu_x(j))^2}$$

(3.44)
Therefore equating (3.42) and (3.44), we have

\[
- \frac{1}{2} \text{tr} W_1 \frac{1}{2(n_1-2)} \int \ldots \int \exp \left\{ -\frac{1}{2} \left[ \frac{1}{2}(n_1-1) \right] \bar{\theta} \right\} \frac{1}{2} g(\bar{\theta}) g(Z) d\bar{\theta} dW_1
\]

\[= \nu_x^{(1)} [1-G^*, p+2, n_1-p(c; \lambda)] = I(Z^{(1)}) \quad \text{say.} \quad (3.45)\]

This gives (3.40) to be \( \Sigma_{12} \Sigma_{22}^{-1} I(Z) \) where \( I(Z) \) is a pxl vector with 1-th component = \( I(Z^{(1)}) \). From (3.45)

\[
I(Z) = \nu_x [1-G^*, p+2, n_1-p(c; \lambda)]
\]

Hence (3.40) is

\[
\Sigma_{12} \Sigma_{22}^{-1} \nu_x [1-G^*, p+2, n_1-p(c; \lambda)]
\]

We now obtain the bias of \( \hat{\mu}_{lr} \) to be

\[
B_2 = \Sigma_{12} \Sigma_{22}^{-1} \mu_x [1-G^*, p+2, n_1-p(c; \lambda)] - \Sigma_{12} \Sigma_{22}^{-1} \mu_x
\]

\[= -\Sigma_{12} \Sigma_{22}^{-1} \mu_x G^*, p+2, n_1-p(c; \lambda)\]

\[= -\Sigma_{12} \Sigma_{22}^{-1} \mu_x F^*, p+2, n_1-p(c_2; \lambda)\]

where

\[
c_2 = \frac{p}{p+2} F, p, n_1-p(\alpha)
\]
As a partial check, when \( c = 0 \), we always reject \( H_0 \) and the estimator reduces to \( \bar{y} + S_{12}S_{22}^{-1}(\bar{x}_1 - \bar{x}_2) \) which has zero bias. In this case, \( F_{p+2,n_1-p}(c_2;\lambda) = 0 \) and \( B_2 = 0 \).

When \( c = \infty \), we always accept \( H_0 \) and the estimator is \( \bar{y} - S_{12}S_{22}^{-1}\mu_x \) with \( \text{Bias} = -S_{12}S_{22}^{-1}\mu_x \). Here \( F_{p+2,n_1-p}(c_2;\lambda) = 1 \) and \( B_2 = -S_{12}S_{22}^{-1}\mu_x \).

In order to evaluate \( B_2 \), we let \( \Sigma_{22} = I \) and \( \sigma^2 = 1 \) wlog. The values of \(-B_2\) for \( p = 2 \) and \( n_1 = 15 \) are given in Table 3.5 for a few values of \( \alpha, \mu_x \) and \( \Sigma_{12} \). From Table 3.5, the following properties of the bias can easily be observed.

1. The bias is zero when the null hypothesis \( H_0: \mu_x = 0 \) is true.

2. For fixed \( n_1, \mu_x \) and \( \Sigma_{12} \), the bias generally decreases as \( \alpha \) increases.

3. The bias is zero when either \( \mu_x \) or \( \Sigma_{12} \) has identical components and the other has components which differ only in sign.

4. For fixed \( n, \Sigma_{12} \) and \( \alpha \) and some component of \( \mu_x \), the value of \( B_2 \) first increases and then decreases to zero as the other component of \( \mu_x \) increases from 0.0 to 1.0.

5. For fixed \( n, \alpha \) and \( \mu_x \), the value of the bias is an increasing function of \( \Sigma_{12} \).
Table 3.5. Values of $-B_2$ for $p = .2$ and $n_1 = .15$.

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G. The M.S.E. of \( \hat{\mu}_l \) when \( \Sigma \) is Unknown

In this section, we find the mean squared error of \( \hat{\mu}_l \).

\[
\text{M.S.E.}(\hat{\mu}_l) = V(\hat{\mu}_l) + (\text{Bias } \hat{\mu}_l)^2
\]

For \( \Sigma \) unknown, \( \hat{\mu}_l \) is given in (3.33) and hence

\[
E(\hat{\mu}_l^2) = E((\bar{y} - S_{12}S_{22}^{-1}\bar{x}_2)^2|\bar{g})P(\bar{g}) + E((\bar{y} - S_{12}S_{22}^{-1}\bar{x}_2)^2|\bar{g})P(\bar{g}) + 2E(S_{12}S_{22}^{-1}\bar{x}_2|\bar{g})P(\bar{g})
\]

\[
- 2E[S_{12}S_{22}^{-1}\bar{x}_2 \bar{x}_1' S_{22}^{-1}S_{21}]P(\bar{g}) = E[S_{12}S_{22}^{-1}\bar{x}_2 \bar{x}_1' S_{22}^{-1}S_{21}]P(\bar{g})
\]

\[
+ 2E(S_{12}S_{22}^{-1}\bar{x}_2 | \bar{g})P(\bar{g}) - 2E[S_{12}S_{22}^{-1}\bar{x}_2 \bar{x}_1' S_{22}^{-1}S_{21}]P(\bar{g})
\]

\[
+ E(\bar{y} - S_{12}S_{22}^{-1}\bar{x}_2)^2 .
\]  

(3.46)

Recall that

\[
\bar{g} = \{n_1m_1(\bar{x}_1' S_{22}^{-1}\bar{x}_1): n_1m_1(\bar{x}_1' S_{22}^{-1}\bar{x}_1) > T_0^2\}
\]

\[
= \{n_1m_1(Z' W_1^{-1}Z): n_1m_1(Z' W_1^{-1}Z) > T_0^2\}
\]

Then following arguments similar to those used to obtain (2.54), we obtain the first term of (3.46).

\[
E[S_{12}S_{22}^{-1}\bar{x}_2 \bar{x}_1' S_{22}^{-1}S_{21}]P(\bar{g}) = Q* + E_{12}S_{22}^{-1}T^{-1}M^*T_{12}^{-1}S_{22}^{-1}S_{21}
\]
where

\[ Q^* = \sum_{i=1}^{11.2} \frac{p}{n_1(n_1-p)} \int \frac{t}{d} \, dt, \]

\( t \) has a noncentral F distribution with \( p \) and \( n_1-p \) degrees of freedom and noncentrality parameter \( n_1 \Sigma^{-1}_{xx} \Sigma^{-1}_{x2} \) and \( M^* \) is a pxp matrix with \( i \)-th diagonal element

\[ = (v_x^{(i)})^2 [1-G^*_p, n_1-p(c;\lambda)] + \frac{1}{n_1} [1-G^*_p+2, n_1-p(c;\lambda)] \]

and the \((i,K)\)th off-diagonal element

\[ = v_x^{(i)} v_x^{(K)} [1-G^*_p, n_1-p(c;\lambda)] \]

Similarly, by arguments analogous to those used to obtain (2.55), the second term equals

\[ 2E(S_{12}S_{22}S_{21}^{-1}\bar{y}_{21} | \bar{G})P(\bar{G}) = 2\mu E_{12}S_{22}^{-1} \Sigma_{xx}^{-1} (1-G^*_p+2, n_1-p(c;\lambda)) \]

\[ + 2E_{12}S_{22}^{-1} M^*T_{12}^{-1} \Sigma_{xx}^{-1} 22^*21 \]

\[ - 2E_{12}S_{22}^{-1} \Sigma_{xx}^{-1} 22^*21 \Sigma_{xx}^{-1} 22^*21 \]

For the third term,

\[ E[S_{12}S_{22}^{-1}\bar{y}_{21}S_{12}^{-1}S_{21}^{-1}| \bar{G})P(\bar{G}) \]

\[ = \text{tr}E(S_{22}^{-1}S_{21}^{-1}S_{12}^{-1}\bar{y}_{21} \bar{y}_{21} | \bar{G})P(\bar{G}) \]
\[= \text{tr}\{E_{S_{22}S_{21}S_{12}}^{-1}[\bar{X}_1\bar{X}_2]\mid S, \bar{X}_1, \bar{X}_2\}P(\bar{G})\]

\[= \text{tr}\{E_{S_{22}S_{21}S_{12}}^{-1}[\bar{X}_1\bar{X}_2]\mid E(\bar{X}_1\bar{X}_2)\mid \bar{G}\}P(\bar{G})\]

\[= \text{tr}\{E_{S_{22}S_{21}S_{12}}^{-1}[\bar{X}_1\bar{X}_2 [\mu'_x + (\bar{X}_1 - \mu'_x)]]\mid \bar{G}\}P(\bar{G})\]

\[= \text{tr}\{E_{S_{22}S_{21}S_{12}}^{-1}[\bar{X}_1\bar{X}_2]\mid \bar{G}\}P(\bar{G})\]

\[= Q^* + \Sigma_{12} S_{22}^{-1}M^T \Sigma_{22}^{-1}\Sigma_{22}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1}\]

and hence the third term of (3.46)

\[= -2Q^* - 2\Sigma_{12} S_{22}^{-1}M^T \Sigma_{22}^{-1}\Sigma_{22}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1}\]

Finally the fourth term of (3.46) is

\[E(\bar{y} - S_{12} S_{22}^{-1} \bar{X}_2)^2 = V(\bar{y} - S_{12} S_{22}^{-1} \bar{X}_2) + \left[E(\bar{y} - S_{12} S_{22}^{-1} \bar{X}_2}\right]^2\]

\[= \frac{1}{n_2} \Sigma_{11.2} + \frac{\Sigma_{11.2}}{n_2(n_2-p-2)} + \frac{2n_2 \mu'_x \Sigma_{22}^{-1} \mu_x}{p}\]

\[+ (\mu - \Sigma_{12} S_{22}^{-1} \mu_x)^2\]

by using (2.58). Substituting into (3.46) we have

\[E(\hat{\mu}_{\lambda, r})^2 = \Sigma_{12} S_{22}^{-1}M^T \Sigma_{22}^{-1}\Sigma_{22}^{-1}\Sigma_{22}^{-1}\Sigma_{22}^{-1} - 2\Sigma_{12} S_{22}^{-1} \mu_x G^* + 2n_1-p(c; \lambda)\]

\[- 2\Sigma_{12} S_{22}^{-1} \mu_x \Sigma_{22}^{-1}\]

\[\frac{1}{n_2} \Sigma_{11.2} + \frac{\Sigma_{11.2}}{n_2(n_2-p-2)} + \frac{2n_2 \mu'_x \Sigma_{22}^{-1} \mu_x}{p}\]
\[ + \mu^2 + \Sigma_{12} \Sigma_{22}^{-1} u_x^1 \Sigma_{22}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1} \]

\[ V(\hat{\mu}_{\lambda r}) = E(\hat{\mu}_{\lambda r}^2) - [E(\hat{\mu}_{\lambda r})]^2 = \Sigma_{12} \Sigma_{22}^{-1} M^* T_{\lambda}^{-1} M^* \Sigma_{22}^{-1} \Sigma_{22}^{-1} \]

\[ - 2 \Sigma_{12} \Sigma_{22}^{-1} u_x^1 \Sigma_{22}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1} \left[ 1 - Q_p^{+2, n_1 - p (c; \lambda)} \right] - Q_p \]

\[ + \frac{1}{n_2} \Sigma_{11} \Sigma_{11}^{-1} + \frac{2n_2 \mu_x^1 \Sigma_{22}^{-1} \mu_x^1}{p} \]

\[ + \Sigma_{12} \Sigma_{22}^{-1} u_x^1 \Sigma_{22}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1} - \Sigma_{12} \Sigma_{22}^{-1} u_x^1 \Sigma_{22}^{-1} \Sigma_{22}^{-1} \Sigma_{22}^{-1} \left[ Q_p^{+2, n_1 - p (c; \lambda)} \right]^2 \]

(3.47)

In order to check the above results, we consider (3.33). When \( c = 0 \), we always reject \( H_0 \) and

\[ \hat{\mu}_{\lambda r} = \bar{y} - S_{12} \Sigma_{22}^{-1} \Sigma_{22}^{-1} \]

Therefore

\[ V(\hat{\mu}_{\lambda r} | c = 0) = V(\bar{y} - S_{12} \Sigma_{22}^{-1} \Sigma_{22}^{-1}) + V(S_{12} \Sigma_{22}^{-1} \Sigma_{22}^{-1}) \]

\[ - 2 \text{Cov}(S_{12} \Sigma_{22}^{-1} \Sigma_{22}^{-1}, S_{12} \Sigma_{22}^{-1} \Sigma_{22}^{-1}) \]

\[ + 2 \text{Cov}(\bar{y}, S_{12} \Sigma_{22}^{-1} \Sigma_{22}^{-1}) \]

\[ = V(\bar{y} - S_{12} \Sigma_{22}^{-1} \Sigma_{22}^{-1}) - V(S_{12} \Sigma_{22}^{-1} \Sigma_{22}^{-1}) \]

\[ + 2 \text{Cov}(\bar{y}, S_{12} \Sigma_{22}^{-1} \Sigma_{22}^{-1}) \]

where
\[
\frac{d}{\pi \sum \sum \sum_{i=1}^{n}} \left( 1 - \frac{d - \frac{1}{2}u}{d} \right) \frac{2u}{2 \cdot \pi \sum_{i=1}^{n}} + \frac{2 \cdot \pi \sum_{i=1}^{n}}{2} \frac{2u}{2} = (6 \eta) \text{ (6)}
\]

\[
(2 - \frac{d - \frac{1}{2}u}{d}) \frac{2u}{2 \cdot \pi \sum_{i=1}^{n}} + \frac{2 \cdot \pi \sum_{i=1}^{n}}{2} \frac{2u}{2} = \left( \frac{d}{\pi \sum \sum \sum_{i=1}^{n}} \right)^2 (8 \eta) \text{ (8)}
\]

\[
\frac{d}{\pi \sum \sum \sum_{i=1}^{n}} + \frac{2 \cdot \pi \sum_{i=1}^{n}}{2} \frac{2u}{2} + \frac{2 \cdot \pi \sum_{i=1}^{n}}{2} \frac{2u}{2} = (6 \eta) \text{ (6)}
\]

\[
(1 - \frac{d - \frac{1}{2}u}{d}) \frac{2u}{2 \cdot \pi \sum_{i=1}^{n}} + \frac{2 \cdot \pi \sum_{i=1}^{n}}{2} \frac{2u}{2} = \left( \frac{d}{\pi \sum \sum \sum_{i=1}^{n}} \right)^2 (8 \eta) \text{ (8)}
\]

\[
\begin{align*}
\text{Hence} & \quad 12 \sum \sum \sum_{i=1}^{n} \frac{u}{2} + \frac{d}{\pi \sum \sum \sum_{i=1}^{n}} + 1 \left( 1 - \frac{d - \frac{1}{2}u}{d} \right) \frac{2u}{2} = \\
& \quad (1 - \frac{d - \frac{1}{2}u}{d}) \frac{2u}{2} + \frac{2 \cdot \pi \sum_{i=1}^{n}}{2} \frac{2u}{2}
\end{align*}
\]

\[
\begin{align*}
(1 - \frac{d - \frac{1}{2}u}{d}) \frac{2u}{2} + \frac{2 \cdot \pi \sum_{i=1}^{n}}{2} \frac{2u}{2} & = (s \cdot T \frac{u}{2}) (T \frac{u}{2} - s \frac{u}{2}) \frac{1}{2} + (s \cdot T \frac{u}{2} - s \frac{u}{2}) \frac{1}{2} = (s \cdot T \frac{u}{2} - s \frac{u}{2}) \frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
\text{Now} & \quad 12 \sum \sum \sum_{i=1}^{n} \frac{u}{2} + \frac{d}{\pi \sum \sum \sum_{i=1}^{n}} + \left( 1 - \frac{d - \frac{1}{2}u}{d} \right) \frac{2u}{2} = \\
& \quad (s \cdot T \frac{u}{2} - s \frac{u}{2}) \frac{1}{2} + (s \cdot T \frac{u}{2} - s \frac{u}{2}) \frac{1}{2} = (s \cdot T \frac{u}{2} - s \frac{u}{2}) \frac{1}{2}
\end{align*}
\]
As a partial check, when \( c = 0 \), (3.47) becomes

\[
V(\hat{\mu}_{lr}) = \frac{1}{n_2} \sum_{11.2} + \sum_{11.2} \frac{p}{n_2(n_2-p-2)} \left[ 1 + \frac{2n_2\mu_x's^{-1}X}{p} \right]
\]

\[- \sum_{11.2} \frac{p}{n_1(n_1-p-2)} \left[ 1 + \frac{2n_1\mu_x's^{-1}X}{p} \right] + \frac{1}{n_1} \sum_{12} \Sigma_{22}^{-1} \Sigma_{21}
\]

which is identical with (3.48), the variance of \( \hat{\mu}_{lr} \) when we always reject \( H_0 \). When \( c = \infty \), (3.47) reduces to

\[
\frac{1}{n_2} \sum_{11.2} + \sum_{11.2} \frac{p}{n_2(n_2-p-2)} \left[ 1 + \frac{2n_2\mu_x's^{-1}X}{p} \right]
\]

which is identical with (3.49), the variance of the preliminary test estimator when we always accept \( H_0 \).

H. Relative Efficiency (\( e_2 \))

As in section D, we compare the performance of the preliminary test estimator \( \hat{\mu}_{lr} \) with the usual linear regression estimator, \( \bar{y} + S_{12}S_{22}^{-1}(\bar{x}_1 - \bar{x}_2) \), when the information of \( \mu_x \) is ignored. We denote the relative efficiency of \( \hat{\mu}_{lr} \) to \( \bar{y} + S_{12}S_{22}^{-1}(\bar{x}_1 - \bar{x}_2) \) by \( e_2 \) and define

\[
e_2 = \frac{\text{M.S.E.}(\bar{y} + S_{12}S_{22}^{-1}(\bar{x}_1 - \bar{x}_2))}{\text{M.S.E.}(\hat{\mu}_{lr})} \quad (3.50)
\]
Since $\bar{y} + S_{12}S_{22}^{-1}(\bar{X}_1 - \bar{X}_2)$ is unbiased, its M.S.E. equals its variance which is given in (3.48), we denote it by $g_2$. Using (3.47), we obtain

$$M.S.E.(\hat{\alpha_r}) = \Sigma_{12}^{11.21.21}M.T.$$

- $2\Sigma_{12}^{11.21}M.T. \Sigma_{11.21}^{11.21} [1 - F_{p+2, n_1-p}(c_2; \lambda)]$

- $Q^* + \frac{1}{n_2} \Sigma_{11.21} + \Sigma_{11.21} \frac{p}{n_2(n_2-p-2)}[1 + \frac{2n_2u_x^x \Sigma_{11.21}^{11.21}}{p}]$

+ $\Sigma_{12}^{11.21}M.T. \Sigma_{11.21}^{11.21}$

- $\frac{1}{n_2} \Sigma_{11.21} + \frac{1}{n_1} \Sigma_{12}^{11.21} [1 - F_{p+2, n_1-p}(c_2; \lambda)]$

- $\Sigma_{12}^{11.21}M.T. \Sigma_{11.21}^{11.21} F^*_{p+4, n_1-p}(c_4; \lambda)$

+ $2\Sigma_{12}^{11.21}M.T. \Sigma_{11.21}^{11.21} F^*_{p+2, n_1-p}(c_2; \lambda)$

- $\Sigma_{11.21} \frac{p}{n_1(n_1-p)} \int_0^1 t^f(t)dt$

+ $\Sigma_{11.21} \frac{p}{n_2(n_2-p-2)}[1 + \frac{2n_2u_x^x \Sigma_{11.21}^{11.21}}{p}] = h_2$ say.

Therefore

$$e_2 = \frac{g_2}{h_2}$$

(3.51)
Wlog we let $\Sigma_{22} = I$, $\sigma^2 = 1$ and write

$$e_2 = \frac{g_2}{h_2}$$

where

$$g_2 = \frac{1}{n_2} (1-g) + \frac{1}{n_1} + (1-g) \frac{p}{n_2(n_2-p-2)} [1 + \frac{2n_2u_x'u_x}{p}]$$

$$h_2 = \frac{1}{n_2} (1-g) + \frac{1}{n_1} g[1-F_{p+2,n_1-p}(c_4;\lambda)]$$

$$- k_1 \frac{p}{n_1(n_1-p)} \int_0^\infty tf(t)dt$$

$$+ (1-g) \frac{p}{n_2(n_2-p-2)} [1 + \frac{2n_2u_x'u_x}{p}]$$

and

$$g = \Sigma_{12} \Sigma_{21} ; k_1 = \Sigma_{12} u_x ; c_2 = \frac{p}{p+2} F_{p,n_1-p}(\alpha)$$

$$c_4 = \frac{p}{p+4} F_{p,n_1-p}(\alpha) ; d = F_{p,n_1-p}(\alpha) ; \lambda = n_1 u_x'u_x .$$

In the computation of the values of $e_2$, we again use the incomplete Beta distribution to approximate the noncentral $F$ distribution. For $\int_0^\infty tf(t)dt$, we use the fact that
\[ \int_0^\infty t f(t) dt - \int_0^\infty f(t) dt = \int_0^\infty t f(t) dt \]  \hspace{1cm} (3.52)

since \( \int_0^\infty t f(t) dt \) exists and

\[ \int_0^\infty t f(t) dt = E(t) = \frac{n_1-p}{n_1-p-2} \left[ 1 + \frac{2n_1\mu'_x\mu_x}{p} \right] \]

Using (3.52), we write

\[ h_2 = \frac{1}{n_2} (1-g) + \frac{1}{n_1} g[1-F^*_p+2,n_1-p(c_2;\lambda)] \]

\[ - K_1^2 F^*_p+4,n_1-p(c_4;\lambda) + 2K_1^2 F^*_p+2,n_1-p(c_2;\lambda) \]

\[ -(1-g) \frac{p}{n_1(n_1-p-2)} \left[ 1 + \frac{2n_1\mu'_x\mu_x}{p} \right] + (1-g) \frac{p}{n_1(n_1-p)} \int_0^\infty t f(t) dt \]

\[ + (1-g) \frac{p}{n_2(n_2-p-2)} \left[ 1 + \frac{2n_2\mu'_x\mu_x}{p} \right] \]

For the purpose of comparison with the results of Han (1973b), we compute the values of \( e_2 \) for \( p = 1 \) and certain values of \( n_1, n_2, \rho, \mu_x \) and \( \alpha \). These values are shown in Table 3.6 and again reveal no significant difference from the values obtained by Han. Any differences are due to the approximations and rounding off errors in the computations.

From Table 3.6 we observe that \( e_2 \) assumes its maximum value when \( \mu_x = 0 \). It then decreases to a minimum and then increases to 1.0 as \( \mu_x \) increases. This is because for large
values of \( \mu_x \), we would always reject \( H_0 \) and use the usual linear regression estimator. For fixed \( n_1, n_2, \alpha \) and small values of \( \mu_x \), \( e_2 \) increases with \( \rho \) while for moderately large values of \( \mu_x \), it decreases as \( \rho \) increases. The values of \( e_2 \) is a decreasing function of \( \alpha \) for fixed \( n_1, n_2, \rho \) and small values of \( \mu_x \), while for moderately large values of \( \mu_x \), it is an increasing function of \( \alpha \).

The values of \( e_2 \) for \( p = 2 \) are given in Table 3.7 for some values of \( \Sigma_{12}, \Sigma_x, n_1, n_2 \) and \( \alpha \). From this table we observe the following.

1. For fixed values of \( n_1, n_2, \Sigma_{12} \) and \( \alpha \), the relative efficiency \( e_2 \) is maximum when the null hypothesis is true, i.e. when \( \mu_x = 0 \).
2. For fixed \( n_1 \) and \( n_2 \), the maximum value of \( e_2 \) is an increasing function of \( \Sigma_{12} \), but a decreasing function of \( \alpha \).
3. For fixed \( \alpha \), \( n_1, n_2, \Sigma_{12} \) and some component of \( \mu_x \), the relative efficiency decreases to a minimum and then increases to 1.0 as the other component increases.
4. For moderately large values of \( \mu_x \), \( e_2 \) is a decreasing function of \( \Sigma_{12} \) and increasing function of \( \alpha \).
Table 3.6. Values of $e_2$ for $p = 1$

$n_1 = 30, n_2 = 15$  $\alpha = 0.05$  $\alpha = 0.10$

<table>
<thead>
<tr>
<th>$\mu_X$</th>
<th>.5</th>
<th>.7</th>
<th>.5</th>
<th>.7</th>
</tr>
</thead>
<tbody>
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<td>1.2833</td>
<td>1.0755</td>
<td>1.2088</td>
</tr>
<tr>
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<td>1.1326</td>
<td>1.0320</td>
<td>1.0907</td>
</tr>
<tr>
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<td>0.8853</td>
<td>0.9489</td>
<td>0.8988</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8599</td>
<td>0.7338</td>
<td>0.8953</td>
<td>0.7916</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8380</td>
<td>0.6928</td>
<td>0.8924</td>
<td>0.7821</td>
</tr>
<tr>
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<td>0.9619</td>
<td>0.9132</td>
</tr>
<tr>
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<td>0.9174</td>
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<tr>
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<td>0.9982</td>
</tr>
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$\alpha = 0.25$  $\alpha = 0.50$

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<th>.5</th>
<th>.7</th>
</tr>
</thead>
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<td>1.0093</td>
<td>1.0231</td>
</tr>
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<td>1.0366</td>
<td>1.0029</td>
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</tr>
<tr>
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<td>0.9830</td>
</tr>
<tr>
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<td>0.9885</td>
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</table>
Table 3.6. (continued)

\[ n_1 = 50, n_2 = 10 \quad \alpha = 0.05 \quad \alpha = 0.10 \]

\[ \mu_X \quad .5 \quad .7 \quad .5 \quad .7 \]

\[
\begin{array}{ccccccc}
0.0 & 1.0395 & 1.1151 & 1.0305 & 1.0876 \\
0.1 & 1.0081 & 1.0264 & 1.0044 & 1.0149 \\
0.2 & 0.9528 & 0.8889 & 0.9636 & 0.9122 \\
0.3 & 0.9308 & 0.8382 & 0.9547 & 0.8899 \\
0.4 & 0.9498 & 0.8779 & 0.9730 & 0.9317 \\
0.5 & 0.9789 & 0.9458 & 0.9909 & 0.9761 \\
0.6 & 0.9947 & 0.9860 & 0.9982 & 0.9952 \\
0.7 & 0.9992 & 0.9978 & 0.9998 & 0.9994 \\
0.8 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
0.9 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
1.0 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
\end{array}
\]

\[ \alpha = 0.25 \quad \alpha = 0.50 \]

\[
\begin{array}{ccccccc}
0.0 & 1.0149 & 1.0417 & 1.0038 & 1.0105 \\
0.1 & 1.0008 & 1.0035 & 1.0000 & 1.0002 \\
0.2 & 0.9825 & 0.9563 & 0.9956 & 0.9888 \\
0.3 & 0.9832 & 0.9571 & 0.9966 & 0.9911 \\
0.4 & 0.9926 & 0.9807 & 0.9988 & 0.9969 \\
0.5 & 0.9982 & 0.9953 & 0.9998 & 0.9994 \\
0.6 & 0.9998 & 0.9993 & 1.0000 & 0.9999 \\
0.7 & 1.0000 & 0.9999 & 1.0000 & 1.0000 \\
0.8 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
0.9 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
1.0 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
\end{array}
\]
Table 3.7. Values of $e_2$ for $p = 2$.

<table>
<thead>
<tr>
<th>$\mu^1_x$</th>
<th>$\Sigma^1_{12}$</th>
<th>(0.5)</th>
<th>(0.5)</th>
<th>(0.7)</th>
<th>(0.7)</th>
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</thead>
<tbody>
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<td></td>
<td></td>
<td>(0.0)</td>
<td>(0.5)</td>
<td>(0.7)</td>
<td>(0.7)</td>
</tr>
<tr>
<td>(0.0,0,0)</td>
<td>1.0534</td>
<td>1.1822</td>
<td>4.0500</td>
<td>1.5020</td>
<td></td>
</tr>
<tr>
<td>(0.0,0,2)</td>
<td>1.0377</td>
<td>1.0190</td>
<td>1.1762</td>
<td>0.9871</td>
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</tr>
<tr>
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<td>0.5544</td>
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<td>(0.0,0,5)</td>
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<td>0.5261</td>
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<tr>
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<td>0.9746</td>
<td>0.8475</td>
<td>0.4443</td>
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<td>0.8987</td>
<td>0.5429</td>
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</table>

$\alpha = 0.10$

<table>
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<th>$\mu^1_x$</th>
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<th>(0.7)</th>
<th>(0.7)</th>
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<tbody>
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<td></td>
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<td>(0.5)</td>
<td>(0.7)</td>
<td>(0.7)</td>
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<tr>
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Table 3.7. (continued)

\( n_1 = 30, n_2 = 10 \quad \alpha = 0.25 \)

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<th>( 0.5 )</th>
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\( \alpha = 0.50 \)

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<td>0.8384</td>
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<tr>
<td>(0.2, 0.5)</td>
<td>0.9996</td>
<td>0.9906</td>
<td>0.9435</td>
<td>0.9938</td>
</tr>
<tr>
<td>(0.3, 0.3)</td>
<td>0.9951</td>
<td>0.9727</td>
<td>0.8637</td>
<td>1.0050</td>
</tr>
<tr>
<td>(0.3, 0.4)</td>
<td>0.9980</td>
<td>0.9843</td>
<td>0.9919</td>
<td>0.9992</td>
</tr>
<tr>
<td>(0.3, 0.5)</td>
<td>0.9994</td>
<td>0.9934</td>
<td>0.9581</td>
<td>0.9982</td>
</tr>
<tr>
<td>(0.4, 0.4)</td>
<td>0.9985</td>
<td>0.9915</td>
<td>0.9473</td>
<td>1.0006</td>
</tr>
<tr>
<td>(0.4, 0.5)</td>
<td>0.9995</td>
<td>0.9966</td>
<td>0.9765</td>
<td>0.9997</td>
</tr>
<tr>
<td>(0.5, 0.5)</td>
<td>0.9998</td>
<td>0.9987</td>
<td>0.9901</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
Table 3.7. (continued)

\[ n_1 = 30, \quad n_2 = 15 \quad \alpha = 0.05 \]

<table>
<thead>
<tr>
<th>( \mu_x )</th>
<th>( \Sigma_{12} )</th>
<th>( 0.5 )</th>
<th>( 0.5 )</th>
<th>( 0.7 )</th>
<th>( -0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.0, 0.0)</td>
<td>1.0897</td>
<td>1.0294</td>
<td>0.7086</td>
<td>0.9827</td>
<td></td>
</tr>
<tr>
<td>(0.0, 0.2)</td>
<td>1.0643</td>
<td>0.8768</td>
<td>0.7100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.0, 0.3)</td>
<td>1.0418</td>
<td>0.7943</td>
<td>0.5132</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.0, 0.4)</td>
<td>1.0223</td>
<td>0.5434</td>
<td>0.5991</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.2, 0.2)</td>
<td>1.0094</td>
<td>0.4568</td>
<td>0.5930</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.2, 0.3)</td>
<td>0.9490</td>
<td>0.3631</td>
<td>1.3712</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.2, 0.4)</td>
<td>0.9629</td>
<td>0.3421</td>
<td>0.8804</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.2, 0.5)</td>
<td>0.9770</td>
<td>0.3781</td>
<td>0.8109</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.3, 0.3)</td>
<td>0.8942</td>
<td>0.6246</td>
<td>0.3192</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.3, 0.4)</td>
<td>0.9273</td>
<td>0.6442</td>
<td>0.3254</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.3, 0.5)</td>
<td>0.9580</td>
<td>0.7115</td>
<td>0.9195</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.4, 0.4)</td>
<td>0.9152</td>
<td>0.6827</td>
<td>0.3524</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.4, 0.5)</td>
<td>0.9530</td>
<td>0.7590</td>
<td>0.4284</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.5, 0.5)</td>
<td>0.9615</td>
<td>0.8310</td>
<td>0.5252</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| \( \alpha = 0.10 \) |
|---|---|---|---|---|---|
| (0.0, 0.0) | 1.0749 | 1.2419 | 2.8992 | 1.5916 |
| (0.0, 0.2) | 1.0513 | 1.0131 | 1.0970 | 0.9625 |
| (0.0, 0.3) | 1.0317 | 0.8952 | 0.7387 | 0.7482 |
| (0.0, 0.4) | 1.0159 | 0.8479 | 0.6217 | 0.6763 |
| (0.0, 0.5) | 1.0064 | 0.8619 | 0.6281 | 0.7013 |
| (0.2, 0.2) | 0.9511 | 0.7659 | 0.5041 | 1.2494 |
| (0.2, 0.3) | 0.9625 | 0.7213 | 0.4320 | 1.0422 |
| (0.2, 0.4) | 0.9756 | 0.7377 | 0.4375 | 0.9128 |
| (0.2, 0.5) | 0.9870 | 0.8009 | 0.5096 | 0.8788 |
| (0.3, 0.3) | 0.9257 | 0.7081 | 0.4050 | 1.0903 |
| (0.3, 0.4) | 0.9545 | 0.7458 | 0.4378 | 0.9947 |
| (0.3, 0.5) | 0.9772 | 0.8193 | 0.5302 | 0.9533 |
| (0.4, 0.4) | 0.9513 | 0.7938 | 0.4926 | 1.0208 |
| (0.4, 0.5) | 0.9764 | 0.8641 | 0.6015 | 0.9916 |
| (0.5, 0.5) | 0.9828 | 0.9176 | 0.7140 | 1.0023 |
Table 3.7. (continued)

\[ \begin{array}{c|cccc}
\mu_1 & 0.5 & 0.5 & 0.7 & -0.5 \\
\hline
\Sigma_{12} & 0.0 & 0.5 & 0.7 & -0.5 \\
\hline
(0.0, 0.0) & 1.0448 & 1.1355 & 1.6682 & 1.2944 \\
(0.0, 0.2) & 1.0283 & 0.9979 & 1.0157 & 0.9568 \\
(0.0, 0.3) & 1.0160 & 0.9368 & 0.8255 & 0.8397 \\
(0.0, 0.4) & 1.0071 & 0.9252 & 0.7819 & 0.8225 \\
(0.0, 0.5) & 1.0025 & 0.9461 & 0.8246 & 0.8682 \\
(0.2, 0.2) & 0.9717 & 0.8522 & 0.6390 & 1.1066 \\
(0.2, 0.3) & 0.9817 & 0.8424 & 0.6090 & 1.0122 \\
(0.2, 0.4) & 0.9904 & 0.8753 & 0.6586 & 0.9605 \\
(0.2, 0.5) & 0.9960 & 0.9258 & 0.7624 & 0.9566 \\
(0.3, 0.3) & 0.9677 & 0.8519 & 0.6160 & 1.0316 \\
(0.3, 0.4) & 0.9837 & 0.8924 & 0.6866 & 0.9964 \\
(0.3, 0.5) & 0.9935 & 0.9404 & 0.7963 & 0.9855 \\
(0.4, 0.4) & 0.9850 & 0.9276 & 0.7628 & 1.0054 \\
(0.4, 0.5) & 0.9942 & 0.9626 & 0.8588 & 0.9976 \\
(0.5, 0.5) & 0.9964 & 0.9818 & 0.9236 & 1.0004 \\
\hline
\end{array} \]

\[ \begin{array}{c|cccc}
\alpha & 0.50 & 0.50 & 0.50 & 0.50 \\
\hline
\mu_1 & 0.5 & 0.5 & 0.5 & 0.5 \\
\hline
(0.0, 0.0) & 1.0170 & 1.0484 & 1.1829 & 1.0963 \\
(0.0, 0.2) & 1.0099 & 0.9961 & 0.9942 & 0.9769 \\
(0.0, 0.3) & 1.0050 & 0.9767 & 0.9290 & 0.9375 \\
(0.0, 0.4) & 1.0020 & 0.9778 & 0.9265 & 0.9428 \\
(0.0, 0.5) & 1.0006 & 0.9875 & 0.9544 & 0.9672 \\
(0.2, 0.2) & 0.9899 & 0.9426 & 0.8332 & 1.0303 \\
(0.2, 0.3) & 0.9945 & 0.9465 & 0.8368 & 1.0018 \\
(0.2, 0.4) & 0.9976 & 0.9655 & 0.8844 & 0.9892 \\
(0.2, 0.5) & 0.9992 & 0.9841 & 0.9405 & 0.9907 \\
(0.3, 0.3) & 0.9914 & 0.9561 & 0.8582 & 1.0072 \\
(0.3, 0.4) & 0.9964 & 0.9739 & 0.9077 & 0.9989 \\
(0.3, 0.5) & 0.9988 & 0.9887 & 0.9556 & 0.9973 \\
(0.4, 0.4) & 0.9972 & 0.9855 & 0.9443 & 1.0009 \\
(0.4, 0.5) & 0.9991 & 0.9940 & 0.9749 & 0.9996 \\
(0.5, 0.5) & 0.9995 & 0.9976 & 0.9893 & 1.0000 \\
\end{array} \]
Table 3.7. (continued)

\[\begin{array}{cccccc}
\mu' & \Sigma_{12} & 0.0 & 0.5 & 0.7 & -0.5 \\
\hline
(0.0,0.0) & 1.0352 & 1.1133 & 3.4684 & 1.3150 \\
(0.0,0.2) & 1.0219 & 0.9725 & 0.8563 & 0.8837 \\
(0.0,0.4) & 1.0116 & 0.9122 & 0.5854 & 0.7548 \\
(0.2,0.2) & 1.0044 & 0.9134 & 0.5627 & 0.7628 \\
(0.2,0.4) & 1.0012 & 0.9505 & 0.6778 & 0.8539 \\
(0.3,0.3) & 0.9659 & 0.8176 & 0.3762 & 1.1049 \\
(0.4,0.4) & 0.9789 & 0.8185 & 0.3609 & 0.9875 \\
(0.3,0.5) & 0.9689 & 0.8436 & 0.3878 & 1.0227 \\
(0.4,0.5) & 0.9903 & 0.8731 & 0.4412 & 0.9396 \\
(0.5,0.5) & 0.9968 & 0.9407 & 0.6229 & 0.9544 \\
(0.0,0.4) & 0.9689 & 0.8436 & 0.3878 & 1.0227 \\
(0.4,0.5) & 0.9903 & 0.8731 & 0.4412 & 0.9396 \\
(0.5,0.5) & 0.9968 & 0.9407 & 0.6229 & 0.9544 \\
\hline
\hline
\alpha = 0.10 & & & & & \\
\hline
(0.0,0.0) & 1.0295 & 1.0935 & 2.4876 & 1.2520 \\
(0.0,0.2) & 1.0167 & 0.9738 & 0.8560 & 0.8953 \\
(0.0,0.4) & 1.0079 & 0.9346 & 0.5582 & 0.8095 \\
(0.2,0.2) & 1.0027 & 0.9457 & 0.6792 & 0.8417 \\
(0.3,0.3) & 0.9862 & 0.8745 & 0.4653 & 0.9896 \\
(0.3,0.4) & 0.9947 & 0.9259 & 0.5887 & 0.9652 \\
(0.3,0.5) & 0.9986 & 0.9720 & 0.7829 & 0.9785 \\
(0.4,0.4) & 0.9958 & 0.9766 & 0.8085 & 1.0006 \\
(0.4,0.5) & 0.9990 & 0.9928 & 0.9277 & 0.9991 \\
(0.5,0.5) & 0.9996 & 0.9980 & 0.9770 & 1.0000 \\
\end{array}\]
Table 3.7. (continued)

\[ n_1 = 50, n_2 = 10 \quad \ldots \ldots \quad \alpha = 0.25 \]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\Sigma_{12} & \begin{array}{c}
\begin{array}{c}
0.5 \\
0.0
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
0.5 \\
0.5
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
0.7 \\
0.7
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
-0.5 \\
-0.7
\end{array}
\end{array} \\
\hline
\mu_x & 0 & 0 & 0 & 0 \\
\hline
(0.0, 0.0) & 1.0178 & 1.0550 & 1.5722 & 1.1397 \\
(0.0, 0.2) & 1.0086 & 0.9820 & 0.8923 & 0.9309 \\
(0.0, 0.3) & 1.0034 & 0.9684 & 0.8035 & 0.9016 \\
(0.0, 0.4) & 1.0009 & 0.9806 & 0.8589 & 0.9389 \\
(0.0, 0.5) & 1.0002 & 0.9936 & 0.9437 & 0.9791 \\
(0.2, 0.2) & 0.9876 & 0.9269 & 0.6285 & 1.0274 \\
(0.2, 0.3) & 0.9947 & 0.9474 & 0.6918 & 0.9947 \\
(0.2, 0.4) & 0.9984 & 0.9768 & 0.8277 & 0.9891 \\
(0.2, 0.5) & 0.9997 & 0.9937 & 0.9423 & 0.9951 \\
(0.3, 0.3) & 0.9942 & 0.9671 & 0.7744 & 1.0033 \\
(0.3, 0.4) & 0.9984 & 0.9869 & 0.8905 & 0.9984 \\
(0.3, 0.5) & 0.9997 & 0.9967 & 0.9676 & 0.9989 \\
(0.4, 0.4) & 0.9992 & 0.9952 & 0.9545 & 1.0001 \\
(0.4, 0.5) & 0.9998 & 0.9989 & 0.9881 & 0.9999 \\
(0.5, 0.5) & 1.0000 & 0.9997 & 0.9967 & 1.0000 \\
\hline
\end{array}
\]

\[ \alpha = 0.50 \]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\Sigma_{12} & \begin{array}{c}
\begin{array}{c}
0.5 \\
0.0
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
0.5 \\
0.5
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
0.7 \\
0.7
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
-0.5 \\
-0.7
\end{array}
\end{array} \\
\hline
\mu_x & 0 & 0 & 0 & 0 \\
\hline
(0.0, 0.0) & 1.0067 & 1.0203 & 1.1617 & 1.0492 \\
(0.0, 0.2) & 1.0028 & 0.9927 & 0.9528 & 0.9724 \\
(0.0, 0.3) & 1.0009 & 0.9906 & 0.9333 & 0.9695 \\
(0.0, 0.4) & 1.0002 & 0.9957 & 0.9654 & 0.9861 \\
(0.0, 0.5) & 1.0000 & 0.9990 & 0.9906 & 0.9966 \\
(0.2, 0.2) & 0.9963 & 0.9770 & 0.8495 & 1.0071 \\
(0.2, 0.3) & 0.9987 & 0.9867 & 0.9024 & 0.9985 \\
(0.2, 0.4) & 0.9997 & 0.9956 & 0.9624 & 0.9979 \\
(0.2, 0.5) & 1.0000 & 0.9991 & 0.9915 & 0.9993 \\
(0.3, 0.3) & 0.9988 & 0.9932 & 0.9447 & 1.0006 \\
(0.3, 0.4) & 0.9997 & 0.9979 & 0.9809 & 0.9997 \\
(0.3, 0.5) & 1.0000 & 0.9996 & 0.9960 & 0.9999 \\
(0.4, 0.4) & 0.9999 & 0.9994 & 0.9940 & 1.0000 \\
(0.4, 0.5) & 1.0000 & 0.9999 & 0.9988 & 1.0000 \\
(0.5, 0.5) & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
\hline
\end{array}
\]
IV. THE REGRESSION ESTIMATOR WITH
A CERTAIN SHRUNKEN ESTIMATOR FOR THE MEAN
OF THE AUXILIARY VARIABLE

A. Introduction

Let \( \mu \) be the mean of \( Y \) and \( \mu_X \) be the mean of the pxl vector of auxiliary variables \( X \). We consider in this chapter a regression estimator of \( \mu \) by using a shrunken estimator of the form \( c \bar{X} : 0 < c \leq 1 \) for \( \mu_X \), when prior information about \( \mu_X \) is available, i.e. \( \mu_X \) is close to \( \mu_0 \), instead of the usual minimum variance unbiased linear estimator \( \bar{X} \). We first consider the case \( p = 1 \) and following Thompson (1968a), we find the optimal value of \( c \) which minimizes the m.s.e. of \( \hat{\mu}^* \), the regression estimator of \( \mu \) which is defined below. The m.s.e. of \( \hat{\mu}^* \) will be derived and the efficiency of the preliminary test estimator of Chapter II relative to \( \hat{\mu}^* \) will be discussed. Since \( \mu_0 \) is known, without loss of generality, we let \( \mu_0 = 0 \).

Let \( \hat{\mu}_X = c\bar{X} \) and assume \( \sigma^2 \), \( \sigma^2_y \), \( \rho \) known, then \( \hat{\mu}^* \) is defined as \( \hat{\mu}^* = \bar{Y} - \beta c\bar{X} \) where

\[
\beta = \frac{\sigma_{xy}}{\sigma_x^2}
\]

m.s.e.\( (\hat{\mu}^*) = E(\bar{Y} - \beta c\bar{X} - \mu)^2 \) \hspace{1cm} (4.1)

In order to find \( c \) to minimize (4.1), we differentiate w.r.t. \( c \) and equate to zero. Thus
\[ \frac{3}{3\sigma_0} E(\overline{y} - \beta \overline{cX} - \mu)^2 = 0 \]

and since integrand is absolutely integrable,

\[ E \frac{3}{3\sigma} (\overline{y} - \beta \overline{cX} - \mu)^2 = 0 \]

\[ E 2(\overline{y} - \beta \overline{cX} - \mu)(-\beta \overline{X}) = 0 \]

\[ E \overline{yX} - \beta c \overline{EX}^2 - \mu \overline{EX} = 0 \]

\[ \beta c (\mu^2_x + \frac{\sigma^2_x}{n}) = \frac{1}{n} \beta \sigma^2 \]

Therefore

\[ c = \frac{1}{n} \frac{\sigma^2_x}{\mu^2_x + \frac{\sigma^2_x}{n}} \quad \text{(4.2)} \]

Since \( \mu_x \) is unknown, we may estimate it by \( \overline{X} \) as in Thompson (1968a). Therefore

\[ \hat{c} = \frac{1}{n} \frac{\sigma^2_x}{\overline{X}^2 + \frac{\sigma^2_x}{n}} \quad \text{(4.3)} \]

Hence the regression estimator of \( \mu \) using a shrunken estimator for \( \mu_x \) is

\[ \hat{\mu}^* = \overline{y} - \frac{3}{n} \frac{\sigma^2_x \overline{X}}{\overline{X}^2 + \frac{\sigma^2_x}{n}} \]
or

\[
\hat{\mu}^* = \bar{y} - \frac{\beta \sigma^2_{XY}}{n\bar{x}^2 + \sigma^2_x} \tag{4.4}
\]

The case \( p = 2 \) can be treated similarly even though the derivations are more difficult. This case will not be treated here. The case \( p \geq 3 \) will be treated in section C of the present chapter.

B. The M.S.E. of \( \hat{\mu}^* \) and Relative Efficiency (e_3)

\[
\text{m.s.e.}(\hat{\mu}^*) = E[(\bar{y} - \frac{\beta \sigma^2_{XY}}{n\bar{x}^2 + \sigma^2_x} - \mu)^2] \tag{4.5}
\]

\[
= E(\bar{y} - \mu)^2 - 2\beta E \frac{\sigma^2_{XY}}{n\bar{x}^2 + \sigma^2_x} + 2\beta \mu E \frac{\sigma^2_x}{n\bar{x}^2 + \sigma^2_x} + \beta^2 E \left( \frac{\sigma^2_{XY}}{n\bar{x}^2 + \sigma^2_x} \right)^2
\]

The second term can be evaluated as

\[
-2\beta \sigma^2_x \frac{\bar{X} \bar{Y}}{n\bar{x}^2 + \sigma^2_x} = -2\beta \sigma^2_x E \left[ \frac{\bar{X} \bar{Y}}{n\bar{x}^2 + \sigma^2_x} | \bar{X} \right]
\]

\[
= -2\beta \sigma^2_x E \frac{\bar{X}}{n\bar{x}^2 + \sigma^2_x} E[\bar{Y} | \bar{X}]
\]

\[
= -2\beta \sigma^2_x E \frac{\bar{X}}{n\bar{x}^2 + \sigma^2_x} \left[ \mu + \beta (\bar{X} - \mu_x) \right]
\]
Therefore

\[
\text{m.s.e.}(\hat{\mu}) = E(\bar{y} - \mu)^2 + 2\beta^2 \mu \sigma^2 E \frac{\bar{x}}{n\bar{x}^2 + \sigma^2_x} - 2\beta^2 \sigma^2 E \frac{\bar{x}^2}{n\bar{x}^2 + \sigma^2_x}
\]

\[+ \beta^2 \sigma^4 \frac{\bar{x}^2}{(n\bar{x}^2 + \sigma^2_x)^2}\]

\[= \frac{1}{n} \sigma^2 + 2\beta^2 \mu \sigma^2 E \frac{\bar{x}}{n\bar{x}^2 + \sigma^2_x} - 2\beta^2 \sigma^2 E \frac{\bar{x}^2}{n\bar{x}^2 + \sigma^2_x}
\]

\[+ \beta^2 \sigma^4 \frac{\bar{x}^2}{(n\bar{x}^2 + \sigma^2_x)^2}\]

We may now use the Gauss-Hermite quadrature to evaluate the above expected values. The relevant approximation given in equation 25.4.46 and Table 25.10 of Davis and Polonsky (1964) is

\[
\int_{-\infty}^{\infty} e^{-x^2} f(x) dx = \sum_{i=1}^{k} W_i f(x_i) + R_k
\]

where \(x_i\) are the \(i\)-th zeros of Hermite polynomials \(H_k(x)\), which are the related orthogonal polynomials. The weights

\[
W_i = \frac{2^{k-1}k! \sqrt{\pi}}{k^2[H_{k-1}(x_i)]^2}
\]
The remainder

\[ R_k = \frac{k! \sqrt{\pi}}{2^k (2k)^k} f^{2k}(\xi) \quad (-\infty < \xi < \infty) \]

To use (4.8) we make the following transformation.

\[ \bar{x} = y \sim N(\mu_x, \frac{\sigma^2_x}{n}) \]

Therefore

\[ \frac{\bar{x}}{n\bar{x}^2 + \sigma^2_x} = \int_{-\infty}^{\infty} \frac{\sqrt{n}}{\sqrt{2\pi \sigma^2_x}} e^{-\frac{1}{2} \left( \frac{y-\mu_x}{\sigma_x} \right)^2} dy \]

Let

\[ \frac{y-\mu_x}{\sigma_x} \sqrt{\frac{n}{2}} = x \]

\[ \Rightarrow y = \sqrt{\frac{2}{n}} (x \sigma_x) + \mu_x \]

\[ \Rightarrow dy = \sigma_x \sqrt{\frac{2}{n}} dx \]

Therefore

\[ \frac{\bar{x}}{n\bar{x}^2 + \sigma^2_x} = \int_{-\infty}^{\infty} \frac{\sqrt{\frac{2}{n}} (x \sigma_x) + \mu_x}{\sqrt{n} \left[ \sqrt{\frac{2}{n}} (x \sigma_x) + \mu_x \right]^2 + \sigma^2_x} e^{-x^2 \sqrt{\frac{2}{n}} \sigma_x dx} \]
Therefore

\[ E \left( \frac{\bar{x}}{\bar{x}^2 + \sigma^2} \right) = \int_{-\infty}^{\infty} \frac{\sqrt{\frac{2}{\pi}} (x \sigma_x + \mu_x)}{n \sqrt{\frac{2}{\pi}} (x \sigma_x + \mu_x)^2 + \sigma_x^2} \frac{1}{\sqrt{\pi}} e^{-x^2} \, dx \]  \hspace{1cm} (4.9)

Similarly

\[ E \left( \frac{\bar{x}^2}{n \bar{x}^2 + \sigma^2} \right) = \int_{-\infty}^{\infty} \frac{(\sqrt{\frac{2}{\pi}} (x \sigma_x + \mu_x)^2)}{n \sqrt{\frac{2}{\pi}} (x \sigma_x + \mu_x)^2 + \sigma_x^2} \frac{1}{\sqrt{\pi}} e^{-x^2} \, dx \]  \hspace{1cm} (4.10)

and

\[ E \left( \frac{\bar{x}^2}{(n \bar{x}^2 + \sigma_x^2)^2} \right) = \int_{-\infty}^{\infty} \frac{(\sqrt{\frac{2}{\pi}} (x \sigma_x + \mu_x)^2)}{[n \sqrt{\frac{2}{\pi}} (x \sigma_x + \mu_x)^2 + \sigma_x^2]^2} \frac{1}{\sqrt{\pi}} e^{-x^2} \, dx \]  \hspace{1cm} (4.11)

Efficiency of the preliminary test estimator (\( \hat{\mu} \)) relative to \( \hat{\mu}^* \) is

\[ e_3 = \left( \frac{1}{M.S.E.(\hat{\mu})} \right)^\frac{1}{M.S.E.(\hat{\mu}^*)} = \frac{M.S.E.(\hat{\mu}^*)}{M.S.E.(\hat{\mu})} \]  \hspace{1cm} (4.12)

Therefore, using (2.24) and noting that \( \beta = \Sigma_{12} x_{22}^{-1} \), we have
The values of $e_3$ for $n = 9$, and $k = 20$ are given in Table 4.1 for certain choices of $\mu_x$, $\rho$ and $\alpha$. From the table we observe that

1. $e_3$ has a maximum greater than unity at $\mu_x = 0$. Again this corresponds to the case when the null hypothesis of the preliminary test is true.

2. For fixed $n$, $\mu_x$ and $\alpha$, $e_3$ is in general a decreasing function of $\rho$.

3. For fixed $n$, $\mu_x$ and $\rho$, $e_3$ is also generally a decreasing function of $\alpha$.

4. For fixed $n$, $\rho$ and $\alpha$, $e_3$ first decreases to a minimum, then increases to above unity and then finally drops back to unity as $\mu_x$ increases.
Table 4.1. Values of $e_3$ for $n = 9$ and $k = 20$.

<table>
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<th>$\mu_x$</th>
<th>.7</th>
<th>.9</th>
<th>.7</th>
<th>.9</th>
<th>.7</th>
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<td>1.0282</td>
<td>1.0170</td>
<td>1.0282</td>
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C. The Shrunken Regression Estimator for $p \geq 3$

In this section we consider the shrunken regression estimator for $p \geq 3$. Suppose $\Sigma$ is known. Wlog we let $\Sigma_{22} = I$ and $\sigma^2 = 1$ and consider the regression estimator

$$\hat{\mu}_1 = \bar{y} - \Sigma_{12}\bar{x}(1 - \frac{p-2}{\bar{x}'\bar{x}}) \quad (4.14)$$

where following James and Stein (1960), we use

$$\bar{x}(1 - \frac{p-2}{\bar{x}'\bar{x}})$$

as an estimator of $\mu_x$.

We shall now derive the m.s.e. of $\hat{\mu}_1$ and the efficiency of the preliminary test estimator, $\hat{\mu}$, relative to $\hat{\mu}_1$. We shall denote this relative efficiency by $e_4$.

$$\text{m.s.e.}(\hat{\mu}_1) = E[\bar{y} - \Sigma_{12}\bar{x}(1 - \frac{p-2}{\bar{x}'\bar{x}}) - \mu]^2 \quad (4.15)$$

$$= E(\bar{y} - \mu)^2 - 2\Sigma_{12}E\bar{y}\bar{x}(1 - \frac{p-2}{\bar{x}'\bar{x}}) + 2\mu\Sigma_{12}E\bar{x}(1 - \frac{p-2}{\bar{x}'\bar{x}})$$

$$+ \Sigma_{12}E[(1 - \frac{p-2}{\bar{x}'\bar{x}})\bar{x}(1 - \frac{p-2}{\bar{x}'\bar{x}})]\Sigma_{21}$$

Now

$$E\bar{y}\bar{x} = E[E\bar{y}\bar{x}|\bar{x}]$$

$$= E[\bar{x}[\mu + \Sigma_{12}(\bar{x} - \mu_x)]]$$

$$= \mu\mu_x - \mu_x\mu_x'\Sigma_{21} + E(\bar{x}\bar{x}')\Sigma_{21}$$
Similarly

\[
\frac{\bar{X}}{\bar{X}'X} = E\frac{\bar{X}}{\bar{X}'X} = \frac{\bar{X}}{\bar{X}'X} [\mu + \Sigma_{12}(\bar{X}-\bar{X}')] \\
= \mu E\frac{\bar{X}}{\bar{X}'X} - E\left(\frac{\bar{X}}{\bar{X}'X}\right)\mu'\Sigma_{21} + E\left(\frac{\bar{X}X'}{\bar{X}'X}\right)\Sigma_{21}
\]

Therefore

\[
m.s.e.(\hat{\mu}_1) = \frac{1}{n} + 2\Sigma_{12}\mu_x'\Sigma_{21} - 2(p-2)\Sigma_{12}E\left(\frac{\bar{X}}{\bar{X}'X}\right)\mu_x'\Sigma_{21} \\
- \Sigma_{12}E(\bar{X}X')\Sigma_{21} + (p-2)^2\Sigma_{12}E\left(\frac{\bar{X}X'}{\bar{X}'X}\right)^2\Sigma_{21}
\]

Using (2.24) and wlog letting \( \Sigma_{22} = I, \sigma^2 = 1 \), the efficiency of the preliminary test estimator relative to \( \hat{\mu}_1 \)

\[
\frac{1}{n} + 2\Sigma_{12}\mu_x'\Sigma_{21} - 2(p-2)\Sigma_{12}E\left(\frac{\bar{X}}{\bar{X}'X}\right)\mu_x'\Sigma_{21} \\
\Sigma_{12}E(\bar{X}X')\Sigma_{21} + (p-2)^2\Sigma_{12}E\left(\frac{\bar{X}X'}{\bar{X}'X}\right)^2\Sigma_{21}
\]

To evaluate \( e_4 \), we need to evaluate

\[
(1) \quad E\frac{\bar{X}}{\bar{X}'X}
\]
(2) \[ EXX' \text{, and} \]

(3) \[ E \frac{XX'}{X'X}^2 \]

where \( X \sim N(u_x, \frac{1}{n}I) \). Let \( X = U \) so that \( X_i = U_i \sim N(\mu_i, \frac{1}{n}) \) where

\[
u_x = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix}
\]

then

\[
E \frac{X}{X'X} = \begin{pmatrix} E \frac{U_1}{U_1^2} \\ \frac{1}{p} \sum_{i=1}^{p} U_i^2 \\ \vdots \\ E \frac{U_p}{U_p^2} \\ \frac{1}{p} \sum_{i=1}^{p} U_i^2 \end{pmatrix}
\]
We now consider the $i$-th component of (1) and note that

$$U'U = \Sigma U_i^2 = \|U\|^2 \sim \chi_p^2(n \|u_x\|^2)$$

$$\sim \chi_{p+2K}^2$$

where

$$K \sim \text{Poisson}(\frac{n}{2} \|u_x\|^2)$$

i.e.

$$p(K=k) = \frac{e^{-\frac{n}{2} \|u_x\|^2} \left(\frac{n}{2} \|u_x\|^2\right)^k}{k!}$$

Therefore

$$E\left(\frac{1}{\|U\|^2}\right) = E\left(\frac{1}{\chi_{p+2K}^2}\right)$$

$$= EE\left(\frac{1}{\chi_{p+2K}^2} | K\right)$$

$$= \frac{1}{p-2+2k}$$

Therefore

$$E\left(\frac{1}{\|U\|^2}\right) = E\left(\frac{1}{p-2+2k}\right) = \sum_{k=0}^{\infty} \frac{1}{p-2+2k} \sum_{i=1}^{p} \mu_i^2 \left(\frac{n}{2} \sum_{i=1}^{p} \mu_i^2\right)^k.$$  

Alternatively using the independence and marginal density of each component, we may also write
Differentiating (4.18) w.r.t. \( \mu_1 \) we have

\[
\frac{\partial}{\partial \mu_1} E\left(\frac{1}{\|\mathbf{u}\|^2}\right) = \sum_{k=0}^{\infty} \frac{1}{p-2+2k} \left(-n\mu_1\right) \frac{\frac{n}{2} \sum_j \mu_j^2 \left(\frac{n}{2} \sum_j \mu_j^2\right)^k}{k!}
\]

\[
+ \sum_{k=0}^{\infty} \frac{1}{p-2+2k} \left(-n\mu_1\right) \frac{-\frac{n}{2} \sum_j \mu_j^2 \left(\frac{n}{2} \sum_j \mu_j^2\right)^{k-1}}{k!}
\]

\[
= n\mu_1 \sum_{k=0}^{\infty} \frac{1}{p-2+2(k+1)} \frac{\frac{n}{2} \sum_j \mu_j^2 \left(\frac{n}{2} \sum_j \mu_j^2\right)^k}{k!}
\]

\[
- n\mu_1 \sum_{k=0}^{\infty} \frac{1}{p-2+2k} \frac{\frac{n}{2} \sum_j \mu_j^2 \left(\frac{n}{2} \sum_j \mu_j^2\right)^k}{k!}
\]

Therefore

\[
\frac{\mu}{\partial \mu_1} E\left(\frac{1}{\|\mathbf{u}\|^2}\right) = n\mu_1 E\frac{1}{p+2k} - n\mu_1 E\left(\frac{1}{\|\mathbf{u}\|^2}\right)
\]
Similarly differentiating (4.19) w.r.t. \( u \), we have

\[
\frac{\partial}{\partial u} E\left( \frac{1}{||u||^2} \right) = \int \cdots \int \frac{p}{n^2} \prod_{l=1}^{p} \frac{1}{U_l^2} \left( \mu_{i1} - \mu_{i1} \right) \cdot \frac{-n}{2} \sum_{j=1}^{p} \left( \frac{1}{2} \frac{1}{(2\pi)^{p/2}} \right) \cdot dU_1 \cdots dU_p
\]

\[
= n E\left( \frac{U_{i1} - \mu_{i1}}{||u||^2} \right)
\]

Equating (4.20) and (4.21), we obtain

\[
E\left( \frac{U_{i1}}{||u||^2} \right) = \mu_{i1} E\left( \frac{1}{p+2K} \right)
\]

Hence we may denote

\[
E\left( \frac{\bar{X}}{\bar{X}' \bar{X}} \right) = \mu_{i1} E\left( \frac{1}{p+2K} \right)
\]

by \( L \) where \( L \) is a \( px1 \) vector whose \( i \)-th component

\[
= \mu_{i1} E\left( \frac{1}{p+2K} \right)
\]

\[
= \mu_{i1} \sum_{k=0}^{\infty} \frac{1}{p+2K} \cdot \frac{-n}{2} \sum_{j=1}^{p} \left( \frac{1}{2} \frac{1}{(2\pi)^{p/2}} \right) \cdot \frac{n}{1} \cdot \frac{\mu_{j2}^2}{k!}
\]

\[
= \mu_{i1} \sum_{k=0}^{\infty} \frac{1}{p+2K} \cdot \frac{-n}{2} \sum_{j=1}^{p} \left( \frac{1}{2} \frac{1}{(2\pi)^{p/2}} \right) \cdot \frac{n}{1} \cdot \frac{\mu_{j2}^2}{k!}
\]
Now the second term is

\[ E(\tilde{X}X') = V(\tilde{X}) + u_x u'_x \]

\[ = \frac{1}{n} I + u_x u'_x \]  \hspace{1cm} (4.23)

The third term,

\[ \frac{\tilde{X}X'}{(\tilde{X}X')^2} , \]

is a p×p matrix with i-th diagonal element

\[ E \frac{u_i^2}{(\sum_{j=1}^{p} u_j^2)^2} \]

and i-\(\ell\)th off-diagonal element

\[ E \frac{u_i u_\ell}{(\sum_{j=1}^{p} u_j^2)^2} \]

to evaluate

\[ E \frac{u_i^2}{(\sum_{j=1}^{p} u_j^2)^2} , \]

we may twice differentiate alternative expressions for

\[ E \left( \frac{1}{\|u\|^2} \right)^2 \]
w.r.t. $\mu_1$ and then equate the results. Thus using an expression similar to (4.18), we have

$$\frac{\partial^2}{\partial \mu_1^2} E\left(\frac{1}{\|u\|^2}\right)^2 = nE\left(\frac{1}{p+2k}\right)^2$$

$$+ n\mu_1 \sum_{k=1}^{\infty} \left(\frac{1}{p+2k}\right)^2 e^{\frac{n}{2} \sum_{j=1}^{p} \mu_j^2(n_{j}^2 + 2k + 2k^2)k^{-1}}$$

$$+ n\mu_1 \sum_{k=0}^{\infty} \left(\frac{1}{p+2k}\right)^2 (-n\mu_1) e^{-\frac{n}{2} \sum_{j=1}^{n} \mu_j^2(n_{j}^2 + 2)k}$$

$$= nE\left(\frac{1}{p+2k}\right)^2 - n\mu_1\left\{ n\mu_1 E\left(\frac{1}{p+2k}\right)^2 - n\mu_1 E\left(\frac{1}{\|u\|^2}\right)^2 \right\}$$

$$= nE\left(\frac{1}{p+2k}\right)^2 + (n\mu_1)^2 E\left(\frac{1}{p+2k}\right)^2 - (n\mu_1)^2 E\left(\frac{1}{\|u\|^2}\right)^2$$

$$- nE\left(\frac{1}{\|u\|^2}\right)^2 - (n\mu_1)^2 E\left(\frac{1}{p+2k}\right)^2 + (n\mu_1)^2 E\left(\frac{1}{\|u\|^2}\right)^2$$

(4.24)

Next, using an expression similar to (4.19), we have

$$\frac{\partial^2}{\partial \mu_1^2} E\left(\frac{1}{\|u\|^2}\right)^2 = n\frac{\partial}{\partial \mu_1} E\left(\frac{U_1}{\|u\|^2}\right)^2 - n\frac{\partial}{\partial \mu_1} \left\{ \mu_1 E\left(\frac{1}{\|u\|^2}\right)^2 \right\}$$

$$= nE\mu_1 \left(\frac{U_1}{\|u\|^2}\right)^2 - nE\left(\frac{1}{\|u\|^2}\right)^2 - n\mu_1 E\left(\frac{U_1}{\|u\|^2}\right)^2$$

$$+ n\mu_1 nE\left(\frac{U_1}{\|u\|^2}\right)^2 - n\mu_1 E\left(\frac{1}{\|u\|^2}\right)^2$$
= \frac{n^2 E U_1^2}{(\|U\|^2)^2} - n^2 \mu_1 E \frac{U_1}{(\|U\|^2)^2} - nE\left(\frac{1}{\|U\|^2}\right)^2

- n^2 \mu_1 E \frac{U_1}{(\|U\|^2)^2} + n^2 \mu_1^2 E \left(\frac{1}{\|U\|^2}\right)^2

Therefore equating (4.24) and (4.25), we have

\begin{align*}
nE\left(\frac{1}{p+2k}\right)^2 + n^2 \mu_1 E \left(\frac{1}{p+2+2k}\right)^2 - 2n^2 \mu_1^2 E \left(\frac{1}{p+2k}\right)^2 - nE\left(\frac{1}{\|U\|^2}\right)^2 \\
+ n^2 \mu_1^2 E \left(\frac{1}{\|U\|^2}\right)^2
\end{align*}

= \frac{n^2 E U_1^2}{(\|U\|^2)^2} - 2n^2 \mu_1 E \left(\frac{1}{p+2k}\right)^2 - nE\left(\frac{1}{\|U\|^2}\right)^2 + n^2 \mu_1^2 E \left(\frac{1}{\|U\|^2}\right)^2

\begin{align*}
n^2 E \frac{U_1^2}{(\|U\|^2)^2} &= nE\left(\frac{1}{p+2k}\right)^2 + n^2 \mu_1 E \left(\frac{1}{p+2+2k}\right)^2 \\
E\frac{U_1^2}{(\|U\|^2)^2} &= \frac{1}{n} E\left(\frac{1}{p+2k}\right)^2 + \mu_1 E \left(\frac{1}{p+2+2k}\right)^2
\end{align*}

(4.26)

Similarly to evaluate

\begin{align*}
E \frac{U_1 U_\lambda}{(\sum_{j=1}^{p} U_j^2)^2}
\end{align*}

we differentiate each of the alternative expressions for

\begin{align*}
E\left(\frac{1}{\|U\|^2}\right)^2
\end{align*}
first w.r.t. \( \mu_1 \) and then w.r.t. \( \mu_k \) and equate the results.

Thus using

\[
E \left( \frac{1}{||u||} \right)^2 = E \left( \frac{1}{p-2+2k} \right)^2
\]

\[
= \sum_{k=0}^{\infty} \left( \frac{1}{p-2+2k} \right)^2 \frac{\sum_{j=1}^{p} \mu_j^2 \left( \sum_{j=1}^{p} \mu_j^2 \right)^k}{k!}
\]

we have

\[
\frac{\partial^2}{\partial \mu_k \partial \mu_1} E \left( \frac{1}{||u||} \right)^2 = n \mu_1 \sum_{k=0}^{\infty} \left( \frac{1}{p+2k} \right)^2 \left( \sum_{j=1}^{p} \mu_j \left( \sum_{j=1}^{p} \mu_j^2 \right)^k \right)
\]

\[
+ n \mu_1 \sum_{k=1}^{\infty} \left( \frac{1}{p+2k} \right)^2 \left( \frac{n}{2} \sum_{j=1}^{p} \mu_j \left( \sum_{j=1}^{p} \mu_j^2 \right)^{k-1} \right)
\]

\[
- n \mu_1 \sum_{k=0}^{\infty} \left( \frac{1}{p-2+2k} \right)^2 \left( \frac{-n}{2} \sum_{j=1}^{p} \mu_j \left( \sum_{j=1}^{p} \mu_j^2 \right)^k \right)
\]

\[
+ n \mu_1 \sum_{k=1}^{\infty} \left( \frac{1}{p-2+2k} \right)^2 \left( \frac{-n}{2} \sum_{j=1}^{p} \mu_j \left( \sum_{j=1}^{p} \mu_j^2 \right)^{k-1} \right)
\]

\[
= n^2 \mu_1 \mu_k E \left( \frac{1}{p+2+2k} \right)^2 - 2n^2 \mu_1 \mu_k E \left( \frac{1}{p+2+2k} \right)^2
\]

\[
+ n^2 \mu_1 \mu_k E \left( \frac{1}{||u||} \right)^2
\]

\[(4.27)\]
Next using
\[
E\left(\frac{1}{\|\mathbf{u}\|}\right)^2 = \int_0^\infty \frac{p}{n} \sum_{j=1}^{p} \exp \left( -\frac{n}{2} \sum_{j=1}^{p} (U_j - \mu_j)^2 \right) \frac{1}{(2\pi)^{p/2}} \, du_1 \ldots du_p,
\]
we have
\[
\frac{\partial^2}{\partial \mu_\ell \partial \mu_1} E\left(\frac{1}{\|\mathbf{u}\|}\right)^2 = n E \frac{\mu_1 (U_\ell - \mu_\ell)}{\|\mathbf{u}\|^2} - n \mu_1 n E \frac{(U_\ell - \mu_\ell)}{\|\mathbf{u}\|^2}.
\]
\[
= n^2 E \frac{U_\ell U_1}{\|\mathbf{u}\|^2} - n^2 \mu_\ell E \frac{U_1}{\|\mathbf{u}\|^2} - n^2 \mu_1 E \frac{U_\ell}{\|\mathbf{u}\|^2} + n^2 \mu_1 \mu_\ell E \frac{1}{\|\mathbf{u}\|^2}.
\]
(4.28)

Further we note that from (4.22) we can deduce that
\[
E \frac{U_1}{\|\mathbf{u}\|^2} = \mu_1 E \left(\frac{1}{p+2K}\right)^2
\]
and
\[
E \frac{U_\ell}{\|\mathbf{u}\|^2} = \mu_\ell E \left(\frac{1}{p+2K}\right)^2
\]
Thus (4.28) is equivalent to
\[
\frac{\partial^2}{\partial \mu_\ell \partial \mu_1} E\left(\frac{1}{\|\mathbf{u}\|}\right)^2 = n^2 E \frac{U_\ell U_1}{\|\mathbf{u}\|^2} - 2n^2 \mu_1 \mu_\ell E \left(\frac{1}{p+2K}\right)^2
\]
\[
+ n^2 \mu_1 \mu_\ell E \left(\frac{1}{\|\mathbf{u}\|^2}\right)^2
\]
(4.29)
Therefore equating (4.27) and (4.29), we have

\[ n^2 \mu_1 \mu_\ell E\left( \frac{1}{p+2+2K} \right)^2 - 2n^2 \mu_1 \mu_\ell E\left( \frac{1}{p+2K} \right)^2 + n^2 \mu_1 \mu_\ell E\left( \frac{1}{\|U\|} \right)^2 \]

\[ = n^2 E\frac{U_1 U_\ell}{\|U\|^2} - 2n^2 \mu_1 \mu_\ell E\left( \frac{1}{p+2K} \right)^2 + n^2 \mu_1 \mu_\ell E\left( \frac{1}{\|U\|} \right)^2 \]

\[ \implies E\frac{U_1 U_\ell}{\|U\|^2} = \mu_1 \mu_\ell E\left( \frac{1}{p+2+2K} \right)^2 \quad (4.30) \]

Denote \( \frac{XX'}{E(X'X)^2} \) by \( M \), then \( M \) is a \( p \times p \) matrix whose \( i \)-th diagonal element

\[ E\frac{U_1^2}{\|U\|^2} = \frac{1}{n} E\left( \frac{1}{p+2K} \right)^2 + \mu_1^2 E\left( \frac{1}{p+2+2K} \right)^2 \]

\[ = \frac{1}{n} \sum_{k=0}^{\infty} \left( \frac{1}{p+2K} \right)^2 e^{-\frac{n}{2} \sum_{j=1}^{p} \frac{\mu_j^2}{2} n \sum_{j=1}^{p} \mu_j^2 k} \]

\[ + \mu_1^2 \sum_{k=0}^{\infty} \left( \frac{1}{p+2+2K} \right)^2 e^{-\frac{n}{2} \sum_{j=1}^{p} \frac{\mu_j^2}{2} n \sum_{j=1}^{p} \mu_j^2 k} \]
and the $i$-th off-diagonal element

\[
E \frac{U_i U_k}{\|U\|^2} = \mu_i \mu_k E \left( \frac{1}{p+2+2k} \right)^2 \mu_i \mu_k \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \mu_j^2 \prod_{j=1}^{n} \mu_j^2}{k!}
\]

Thus substituting

\[
E \left( \frac{X}{X'X} \right), \quad \frac{XX'}{E(XX')^2}
\]

and

\[
E(XX')
\]

into (4.16), we may write

\[
e_4 = \frac{h(a)}{k(a)} \quad (4.31)
\]

where

\[
h(a) = \frac{1}{n} + \Sigma_{12} L_{xx} L_{xx} E_{21} - 2(p-2) \Sigma_{12} L_{xx} E_{21}
\]

\[
+ (p-2)^2 \Sigma_{12} M_{21} - \frac{1}{n} \Sigma_{12} E_{21}
\]
and

\[ k(a) = \frac{1}{\pi} - \frac{1}{\pi} \sum_{i=1}^{\Sigma \frac{E_{2i} H_p}{2} + \frac{q}{2}} (q; \lambda) - \frac{1}{\pi} \sum_{i=1}^{\Sigma \frac{E_{2i} H_p}{2} + \frac{q}{2}} (q; \lambda) \]
V. BIBLIOGRAPHY


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VII. APPENDIX

To justify the result in (3.7) we consider, wlog, the differentiation of $T$ w.r.t. $\delta$.

$$\frac{\partial T}{\partial \delta} = \frac{\partial}{\partial \delta} \int_{\infty}^{\infty} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{2^j} h_{p+2j}(t) dt$$

(A.1)

Let

$$g(t,\delta) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{2^j} h_{p+2j}(t)$$

(A.2)

We can differentiate under the integral sign in (A.1) by the Lebesgue Dominated Convergence Theorem if $\forall t$,

$$\left| \frac{g(t,\delta+s) - g(t,\delta)}{s} \right| \leq G(t)$$

for every $|s| \leq s_0$ where $G(t)$ is integrable over $(c,\infty)$. Using triangular inequality,

$$|g(t,\delta) - g(t,\delta+s)| \leq |g(t,\delta) - g_0(t,\delta+s)|$$

$$+ |g_0(t,\delta+s) - g(t,\delta+s)|$$

where

$$g_0(t,\delta+s) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{1}{2^j} h_{p+2j}(t)$$
then

\[ \left| \frac{g(t, \delta) - g(t, \delta + s)}{s} \right| \leq \sum_{j=0}^{\infty} e^{-\frac{1}{2}s} \left| \frac{1}{s} \right| \frac{1}{j!} \frac{(\delta + s)^j}{2^j} + \sum_{j=0}^{\infty} e^{-\frac{1}{2}(\delta + s)} \left| \frac{1}{s} \right| \frac{1}{j!} \frac{(\delta + s)^j}{2^j} h_{p+2j}(t) \]

(A.3)

\[ \left| e^{\frac{-1}{2}s} - 1 \right| \leq \frac{e^{\frac{1}{2}s_0}}{s_0} \quad \text{for } |s| \leq s_0 \]

so that the integral of the first term of right side of (A.3)

\[ \leq \frac{e^{\frac{1}{2}s_0}}{s_0} \int_{-1}^{1} \sum_{j=0}^{\infty} e^{-\frac{1}{2}\delta} \frac{1}{j!} \frac{(\delta + s)^j}{2^j} h_{p+2j}(t) \, dt = \frac{e^{\frac{1}{2}s_0}}{s_0} [1 - H_p(c; \delta)] < \frac{e^{\frac{1}{2}s_0}}{s_0} < \infty \]

Next \( e^{-\frac{1}{2}s} \leq \frac{1}{2^j} |s| \leq \frac{1}{2^j} \)

Also, if we let \( f(x) = x^j, f'(x) = jx^{j-1} \).

\((\delta + s)^j = f(\delta + s) = f(\delta) + sf'(\delta + \theta s)\)

by the Mean Value Theorem = \( \delta^j + sj(\delta + \theta s)^{j-1} \).

\[ \left| \frac{(\delta + s)^j - \delta^j}{s} \right| \leq \left| j(\delta + \theta |s|)^{j-1} \right| \leq j(\delta + \theta s_0)^{j-1} \]

\[ \leq j(\delta + s_0)^{j-1} \text{ for } j \geq 1; \ (0 < \theta = \theta(s) < 1) \]
This implies that the integral of the second term of (A.3)

\[
\begin{align*}
&\leq e^{s_0} \sum_{j=1}^{\infty} e^{-\frac{1}{2}\delta} \frac{(\delta+s_0)^{j-1}}{2^j j!} \int h_{p+2j}(t)dt \\
&= \frac{1}{2} e^{s_0} \sum_{j=1}^{\infty} e^{-\frac{1}{2}\delta} \frac{(\delta+s_0)^{j-1}}{2^j (j-1)!} \int h_{p+2j}(t)dt \\
&= e^{s_0} \sum_{j=0}^{\infty} e^{-\frac{1}{2}\delta} \frac{1}{j!} \left( \frac{\delta+s_0}{2} \right)^j \int h_{p+2j+2}(t)dt \\
&= e^{s_0} \left[ 1 - H_{p+2} (c; \delta+s_0) \right] < e^{s_0} < \infty
\end{align*}
\]

Therefore

\[
\frac{\partial T}{\partial \delta} = \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{\delta}{2} \right)^j \int h_{p+2j}(t)dt
\]

Let

\[
g_j(t, \delta) = e^{-\frac{1}{2}\delta} \frac{1}{j!} \left( \frac{\delta}{2} \right)^j h_{p+2j}(t) \quad (A.4)
\]

In order to differentiate under the summation, we must show that for every fixed \( t \),

\[
\left| \frac{g_j(t, \delta) - g_j(t, \delta+s)}{s} \right| \leq U_j(t), \quad |s| \leq s_1
\]
where

\[ \sum_{j=0}^{\infty} U_j(t) < \infty. \]

\[ h_{p+2j}(t) = \frac{e^{-\frac{t}{2}(\frac{t}{2})^2}}{2 \Gamma\left(\frac{p+2j}{2}\right)} \]

\[ \Gamma\left(\frac{p+2j}{2}\right) = \int_{0}^{\infty} e^{-x} x^{\frac{p+2j}{2} - 1} \, dx \]

\[ \int_{0}^{\frac{t}{2}} e^{-x} x^{\frac{p+2j}{2} - 1} \, dx \]

\[ = e^{-\frac{t}{2}(\frac{t}{2})^2} \frac{p+2j}{2} \frac{e^{\frac{t}{2}(\frac{t}{2})^2}}{2} \]

Therefore

\[ h_{p+2j}(t) \leq \frac{e^{-\frac{t}{2}(\frac{t}{2})^2}}{2 \Gamma\left(\frac{p+2j}{2}\right)} \left(\frac{p+2j}{2}\right) = \frac{1}{t} \frac{p+2j}{2} \]  \hfill (A.5)
By triangular inequality,

$$|g_j(t, \delta) - g_j(t, \delta+s)| \leq |g_j(t, \delta) - g_{j0}(t, \delta+s)|$$

$$+ |g_{j0}(t, \delta+s) - g_j(t, \delta+s)|$$

where

$$g_{j0}(t, \delta+s) = e^{-\frac{1}{2}(\delta+s)} \frac{1}{j!} (\frac{\delta}{2})^j h_{p+2j}(t)$$

By similar arguments as above and using (A.5),

$$\left| \frac{g_j(t, \delta) - g_j(t, \delta+s)}{s} \right|$$

$$\leq k_1 \sum_{j=0}^{\infty} (p+2j) e^{-\frac{1}{2}(\delta+s)} \frac{1}{j!} (\frac{\delta}{2})^j$$

$$+ k_2 \sum_{j=0}^{\infty} (p+2j) e^{-\frac{1}{2}(\delta+s)} \frac{1}{j!} (\frac{\delta+s}{2})^j = \sum_{j=0}^{\infty} U_j(t) < \infty$$

since each term is some moment of a Poisson distribution and

$k_1, k_2$ are fixed constants.

To justify the result in (3.9), we let

$$\left(\frac{n_1}{\sqrt{2\pi}}\right)^p \prod_{j=1}^{P} e^{-\frac{n_1}{2}(z(j)-\gamma_x(j))^2} \gamma_{x}(j) = \mathcal{T}dT . \quad (A.6)$$
Therefore (3.8)

\[
= \int \cdots \int \frac{n_1}{2} (z(1) - \gamma_x(1))^2 e^{-\frac{1}{2} (z(1) - \gamma_x(1))^2} T dZ(1) dT
\]

Let

\[
\frac{n_1}{2} (z(1) - \gamma_x(1))^2 e^{-\frac{1}{2} (z(1) - \gamma_x(1))^2} T = g(T, \gamma_x(1)).
\]

We must show that for every fixed $T$

\[
\left| \frac{g(T, \gamma_x(1)+h) - g(T, \gamma_x(1))}{h} \right| \leq \Phi(T), |h| \leq h_0
\]

(A.7)

where

\[
\int \cdots \int \Phi(T) dZ(1) dT < \infty
\]

Now

\[
= \frac{n_1}{2} [z(1) - (\gamma_x(1)+h)]^2 e^{-\frac{1}{2} [z(1) - (\gamma_x(1)+h)]^2} - e^{-\frac{1}{2} (z(1) - \gamma_x(1))^2} \frac{n_1}{2} (z(1) - \gamma_x(1))
\]

= \frac{n_1}{2} e^{-\frac{1}{2} (z(1) - \gamma_x(1))^2} \left[ \frac{n_1}{2} h(z(1) - \gamma_x(1)) - \frac{n_1}{2} h^2 \right]

(A.8)
If $f_1(h) = e^{kh}$, $f'_1(h) = ke^{kh}$. Let

$$\frac{n_1}{e^{\frac{h}{2}}}(Z(1) - \gamma_x(1)) = f_1(h) = f_1(0) + hf'_1(\theta_1h)$$  \hspace{1cm} (A.9)

$$= 1 + \frac{n_1\theta_1h}{2}(Z(1) - \gamma_x(1))e^{-\frac{n_1\theta_1h}{2}(Z(1) - \gamma_x(1))}$$

Let

$$\frac{n_1}{e^{\frac{h}{2}}} = f_2(h) = f_2(0) + hf_2'(\theta_2h)$$

$$= 1 - n_1\theta_2h^2e^{-\frac{n_1\theta_2h^2}{2}}$$  \hspace{1cm} (A.10)

Multiplying (A.9) and (A.10) we have

$$\frac{n_1}{e^{\frac{h}{2}}}(Z(1) - \gamma_x(1)) e^{-\frac{n_1\theta_2h^2}{2}}$$

$$= 1 + \frac{n_1\theta_1h}{2}(Z(1) - \gamma_x(1))e^{-\frac{n_1\theta_1h}{2}(Z(1) - \gamma_x(1))} - n_1\theta_2h^2e^{-\frac{n_1\theta_2h^2}{2}}$$

$$- \frac{n_1^2\theta_2h^3}{2}(Z(1) - \gamma_x(1))e^{-\frac{n_1^2\theta_2h^2}{2}} + \frac{n_1\theta_1h}{2}(Z(1) - \gamma_x(1))$$

Therefore for $|h| < h_1$, the expression in (A.8)
\[ \leq e^{-\frac{1}{2} [Z^{(1)} - \gamma_x^{(1)}]^2} \{ 1 + \frac{h_1 n_1}{2} \left| Z^{(1)} - \gamma_x^{(1)} \right| e^{-\frac{1}{2} [Z^{(1)} - \gamma_x^{(1)}]^2} \}
\]

+ \frac{n_1 h_1^2}{2} \left| Z^{(1)} - \gamma_x^{(1)} \right| e^{-\frac{1}{2} [Z^{(1)} - \gamma_x^{(1)}]^2} \}

= e^{-\frac{1}{2} [Z^{(1)} - \gamma_x^{(1)}]^2} + c_1 \left| Z^{(1)} - \gamma_x^{(1)} \right| e^{-\frac{1}{2} [Z^{(1)} - \gamma_x^{(1)}]^2}

+ \frac{n_1 h_1^2}{2} \left| Z^{(1)} - \gamma_x^{(1)} \right| e^{-\frac{1}{2} [Z^{(1)} - \gamma_x^{(1)}]^2}

where

\[ c_1 = \frac{n_1 h_1}{2} (1 + n_1 h_1^2) \]

and \( \psi(\gamma_x^{(1)}) \) is some function of \( \gamma_x^{(1)} \). Finally we note

\[
(1 + c_2) \prod_{j=1}^{P} \frac{\sqrt{n_1}}{\sqrt{2\pi}} e^{-\frac{n_1}{2} (Z^{(j)} - \gamma_x^{(j)})^2} dZ^{(j)}
\]

+ \prod_{j=1}^{P} \frac{\sqrt{n_1}}{\sqrt{2\pi}} e^{-\frac{n_1}{2} [Z^{(1)} - \gamma_x^{(1)}]^2} \prod_{j \neq 1}^{P} \frac{\sqrt{n_1}}{\sqrt{2\pi}} e^{-\frac{n_1}{2} (Z^{(j)} - \gamma_x^{(j)})^2}

\[ dZ^{(1)} dZ^{(j)} \]
\[(1+e_2) \int \cdots \int \frac{1}{\sqrt{2\pi}} e^{-\frac{n_1}{2} (Z(i) - \gamma_x(i))^2} T \, dZ(1) \, dT \]

\[+ c_1 \int \cdots \int |Z(i) - \gamma_x(i)| \frac{1}{\sqrt{2\pi}} e^{-\frac{n_1}{2} (Z(i) - \psi(\gamma(i)))^2} T \, dZ(1) \, dT \]

\[= \int \cdots \int \Phi(T) \, dZ(1) \, dT < \infty \]