The dynamical system of iterated Cevian Tribbles

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The dynamical system of iterated Cevian Tribbles

by

Emily Ann Carroll

A thesis submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
MASTER OF SCIENCE

Major: Mathematics

Program of Study Committee:
Arka Ghosh, Co-major Professor
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Iowa State University
Ames, Iowa
2016
DEDICATION

I would like to dedicate this thesis to my husband Ryan Carroll, my father John Petty, and to my late grandmother Mary Petty, without whose support I could not have even begun this work.
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ABSTRACT

Ceva’s Theorem gives a necessary and sufficient condition for three lines through the vertices of a triangle to intersect at a single point. We investigate what happens when that condition is not met, which means the three lines form a triangle inside the original (called a Tribble), and the process is iterated. By Cantor’s Intersection Theorem, we know that the Tribbles will converge to a point within the initial triangle as long as the side lengths of the Tribbles go to zero. We consider different ways to iterate this process. We establish an $n^{th}$ term test for convergence when the Cevian ratios are deterministic sequences. We prove that when the Cevian ratios used to iterate are chosen at random, the Tribbles always converge. We initiate study on the distribution of limit points. With the aid of a simulation in MATLAB that produces graphical plots for the numerical estimations of Tribble limit points, we begin to visualize and describe the distribution of limit points.
CHAPTER 1. OVERVIEW AND BACKGROUND

This chapter contains an overview of the entire work followed by an introduction to the basic definitions and background theorems. Section 1.1 contains the overview and may be skipped, in which case the reader should begin with Section 1.2, which introduces important formal definitions.

1.1 Overview

First we establish some definitions, classical theorems, and notation. Then we define what it means for Tribbles to converge and give some sufficient conditions for convergence and divergence in certain cases. We prove convergence in the most basic deterministic case, then find a condition on a sequence to cause the Tribbles to diverge to a triangle. Next we bring randomness into the picture, first proving that when Cevian ratios are chosen at random we always have convergence. We begin study on this topic with a “base case,” constructing Tribbles using Cevian ratios chosen randomly from two values with equal probability, and use a simulation to compute the limit points numerically. The points are plotted so that we can begin to visualize the distributions. A preliminary look at the results of the simulation lead us to believe that the behavior is predictable for certain choices of Cevian ratios.

• Definitions and Classical Theorems

Line segments that connect the corner of a triangle to any point on the opposite side are called Cevians. Ceva’s Theorem gives a necessary and sufficient condition for three Cevians to all intersect in at single point. Let \(x, y, z\) represent the three Cevian ratios, which is the ratio of the distances from the corners of the triangle to the Cevian, each corresponding to a different side (more thorough definition to come later). When three Cevians are drawn, they
either intersect at a single point within the triangle, or they form a smaller triangle. This point
or new triangle is called a Tribble. For pictures and formal definitions, see Section 1.2. For a
definition of the Cevian ratios $x, y, z$, see Section 1.3.

**Theorem 1.4** (Ceva’s Theorem). *Suppose we construct a Tribble using Cevian ratios $x, y, z$. The Tribble formed by these three Cevians is a single point if and only if*

$$xyz = 1 \quad (1.1)$$

**Theorem 1.5** (Routh’s Theorem). *Suppose a Tribble is constructed with Cevian ratios are $x, y, z$. Let $A_0$ be the area of the initial triangle and $A_1$ be the area of the Tribble. Define:*

$$R(x, y, z) = \frac{(xyz - 1)^2}{(xy + y + 1)(yz + z + 1)(zx + x + 1)} \quad (1.2)$$

*Then the following equation holds:*

$$A_1 = A_0 R(x, y, z) \quad (1.3)$$

The function $R(x, y, z)$ is called the Routh function. Note that when $x, y, z > 0$, we have
$0 \leq R(x, y, z) < 1$.

For more in-depth discussions of Routh’s Theorem, Ceva’s Theorem, and their importance
in Geometry, see [1].

*Introduction to Tribble Convergence*

We call the initial Triangle $T_0$, and when a Tribble is constructed inside of $T_n$, we call it $T_{n+1}$. We specify $T_n$ by the Cevian ratios used to construct it, $x_n, y_n, z_n$. We say that the Tribbles converge if the infinite intersection of them is a single point. When the infinite intersection is a triangle or a line instead, we say the Tribbles diverge to a line or a triangle. The following theorem follows easily from Cantor’s Intersection Theorem.
Theorem 2.3 (Tribble Intersection Theorem). If the corresponding side lengths of the Tribbles go to zero, then the Tribbles converge.

For a more thorough introduction to Tribble Convergence, see Chapter 2.

• Deterministic Cases

Lemma 3.1 and Corollary 3.2 When \( x_n, y_n, z_n > 0 \) are constant (with respect to \( n \)), the side lengths go to zero, so the Tribbles converge.

Repeated Cevian ratios are discussed in Section 3.1.

Proposition 3.1 (\( n^{th} \) Term Test for Tribbles). Consider the area \( A_n \) of the \( n^{th} \) Tribble (which can be computed from the Routh function).

1. If \( A_n \to c > 0 \), then the Tribbles diverge to a triangle.

2. If \( A_n \to 0 \), then the test is inconclusive (as the Tribbles may either diverge to a line segment or converge).

In trying to find sequences that give rise to diverging Tribbles, we proved the following:

Proposition 3.2 (Certain Tribbles to Diverge to a Triangle). Let \( x_n = y_n = z_n \) such that \( x_n > 0 \) and \( x_n \neq 1 \) for all \( n \). Tribbles iterated using such a sequence diverge to a triangle if and only if \( \sum_{k=1}^{\infty} \frac{1}{x_n} \) converges.

Tribble convergence in the case of sequences for Cevian ratios is discussed in Section 3.2.

• Independent Random Cevian Ratios

Now suppose that the following is a collection of independently and identically distributed random variables used as previously described to iterate Tribbles.

\[ \{x_n, y_n, z_n : n \in \mathbb{N}\} \]

We are interested in how the distribution that generates \( x_n, y_n, z_n \) determines the distribution of the limit points of the Tribbles, if they converge. In this sense, choosing a probability
distribution with which to choose \(x_n, y_n, z_n\) gives rise to a probability distribution defined on the interior of the initial triangle. It turns out, as long as the Cevian ratios are chosen independently, the Tribbles converge.

**Theorem 4.2** (Tribble’s Theorem). Suppose Cevian ratios are chosen independently and identically from a distribution defined on \((0, \infty)\). Then the Tribbles converge.

Proof of Tribble’s Theorem and more discussion about random Cevian ratios, is in Chapter 4.

- **The Coin Toss Case**

  Let \(r\) and \(s\) be any positive real numbers. Let \(DU\{r, s\}\) denote the probability distribution such that \(P(X = r) = P(X = s) = 0.5\). Now, for each \(n \in \mathbb{N}\), choose \(x_n, y_n, z_n\) independently using \(DU\{r, s\}\). When we iterate Tribbles using such \(x_n, y_n, z_n\), we call it the Coin Toss case. Discussion of this process begins in Section 4.2.

- **Simulating the Coin Toss Case**

  Computing a formula for the distribution of Tribble limit points is beyond the scope of this work and may be impossible, but we used a simulation in MATLAB to attempt to visualize what they look like when \(x_n, y_n, z_n\) \(DU\{r, s\}\). First, we describe how the simulation works in Section 4.2.1. To summarize, the program chooses Cevian ratios from some \(r\) and \(s\) randomly, constructs a Tribble, and repeats until the area is small, where it plots a point. It computes 10,000 points in this way so that we can start to visualize the distribution of limit points. The results are discussed in Section 4.2.2, and the outputs can be found in Appendix B.

- **Main Results**

  We have initiated study on the new topic of iterated Tribbles, a dynamical system which arises when Cevian ratios violate the Ceva condition and the process is repeated. The main results proven thus far are

  1. When the same three Cevian ratios are repeated and the Tribbles iterated, the Tribbles converge.
2. When the Cevian ratios are generated by a sequence depending on \( n \), the Tribbles may or may not converge. We have begun discussion on tests for convergence.

3. When the Cevian ratios are chosen independently from a probability distribution, the Tribbles converge.

1.2 Definitions

**Definition 1.1.** Any line segment that joins a vertex of a triangle with some point on the opposite side is called a Cevian. An example is shown in Figure 1.1.

![Figure 1.1 An example of a triangle (dotted gray) with a Cevian (solid black).](image)
Definition 1.2. When three Cevians are constructed inside of a triangle (one from each corner), the resulting triangle or single point in the interior of the triangle is called a Tribble. See Figure 1.2.

![Figure 1.2 Examples of Tribbles, one single-point Tribble and one triangular Tribble.](image1)

1.3 Notation

Suppose we construct a Triangle with Tribble as shown in Figure 1.3.

![Figure 1.3 A triangle and single-point Tribble with all points labeled. This labeling would also apply if the Tribble was Triangular.](image2)

We often refer to the triangle \(abc\) as the initial triangle. We usually label the closed set bound by its edges as \(T_0\), and denote its area by \(A_0\). Similarly, a Tribble constructed within \(T_0\) is referred to as \(T_1\) (again, a closed set) and has area \(A_1\), and so on.
Let $\text{dist}$ represent Euclidean distance. Define

\[
x = \frac{\text{dist}(a, c')}{\text{dist}(c', b)}
\]

\[
y = \frac{\text{dist}(b, a')}{\text{dist}(a', c)}
\]

\[
z = \frac{\text{dist}(c, b')}{\text{dist}(b', a)}
\]

We often refer to $x, y, z$ as the **Cevian ratios**. Observe that they can be any positive real number, which allows a location of a Cevian along the side of a triangle to be expressed and determined uniquely by a positive real number, up to orientation. That is, the notation given here is counterclockwise, but we could also use clockwise. In this way, a Tribble can be specified by its three Cevian ratios.

Values of Cevian ratios that are near zero will produce a Cevian which is close to one corner, and large values will produce a Cevian that is close to the other corner. If the Cevian ratio is 1, then the Cevian is a median.\(^1\) This leads us to the following observation:

**Observation 1.3** (Symmetry Property of Cevians). Let $x$ be any positive real number. Suppose the Cevian ratio for the Cevian running from $c'$ to $c$ in Figure 1.3 (without loss of generality) is $x$. Let $d_1 = \text{dist}(a, c')$ under this assumption. Suppose instead that the Cevian ratio is \(\frac{1}{x}\). Let $d_2 = \text{dist}(c', b)$ under this assumption. Observe that $d_1 = d_2$.

Observe that because of this symmetry property, it matters little whether we use clockwise or counterclockwise notation. It also matters little what the actual coordinates of the initial triangle are, because we can obtain one triangle by an invertible linear transformation of another, and so anything that we can prove for a particular triangle will also be true for any other triangle.

---

\(^1\)A line segment that joins the corner of a triangle to the midpoint of the opposite side.
1.4 Classical Theorems

**Theorem 1.4** (Ceva’s Theorem). Consider any triangle with vertices a, b, and c, and Cevians from a to a’, from b to b’ and from c to c’ (See Figure 1.3). Suppose the respective Cevian ratios are x, y, z. The Tribble formed by these three Cevians is a single point if and only if

\[ xyz = 1 \]  

(1.4)

We refer to equation (1.4) as the **Ceva Condition**. We did not find a name in the literature for the triangle formed when the the Ceva condition is violated (a Cevian triangle is something different), and so the nomenclature “Tribble” is new as of this work.²

**Theorem 1.5** (Routh’s Theorem). Consider any triangle with vertices a, b, and c, and Cevians from a to a’, from b to b’ and from c to c’ (See Figure 1.3). Suppose the respective Cevian ratios are x, y, z. Let \( A_0 \) be the area of the initial triangle and \( A_1 \) be the area of the Tribble. Define:

\[ R(x, y, z) = \frac{(xyz - 1)^2}{(xy + y + 1)(yz + z + 1)(zx + x + 1)} \]  

(1.5)

Then the following equation holds:

\[ A_1 = A_0 R(x, y, z) \]  

(1.6)

Observe that Routh’s Theorem implies Ceva’s Theorem, since \( A_1 = 0 \) if and only if \( R(x, y, z) = 0 \) if and only if \( xyz - 1 = 0 \) which is equivalent to the Ceva Condition. We refer to the rational function (1.5) as the **Routh Function**. Since it is necessarily true that \( A_1 < A_0 \) and areas are always nonnegative, we may also deduce that for any \( x, y, z \), we have \( R(x, y, z) \in [0, 1) \).

More information on Ceva’s Theorem and Routh’s Theorem can be found in [1].

²Tribble is the name of a cat, who is named for a creature in a TV show.
CHAPTER 2. INTRODUCTION TO TRIBBLE CONVERGENCE

Let $T_0$ be a triangle, and let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ be positive real numbers. Suppose a Tribble $T_1$ is constructed with $x_1, y_1, z_1$ for Cevian ratios. Now, construct Tribble $T_{n+1}$ inside triangle $T_n$ using Cevian ratios $x_n, y_n, z_n$, infinitely iterating this process. Let us establish what it means for Tribbles constructed in this way to converge.

Definition 2.1. Suppose we have an initial triangle $T_0$, with nested Tribbles $\{T_n\}_{n=0}^{\infty}$ (that is $T_{n+1} \subseteq T_n$). We say the Tribbles converge if

$$\bigcap_{n=0}^{\infty} T_n := \lim_{m \to \infty} \bigcap_{n=0}^{m} T_n$$

exists and is a single point.

The infinite intersection of Tribbles will always be either a triangle, a line, or a single point. Our notion of convergence is defined to be only when the intersection is a point. So, when the intersection of Tribbles is a triangle or a line, we say the Tribbles diverge.

Theorem 2.2 (Cantor’s Intersection Theorem). Suppose $(X, d)$ is a nonempty complete metric space, and let $\{C_n\}_{n=1}^{\infty}$ be a collection of closed, nested subsets of $X$ such that $\text{diam}(C_n) \to 0$, where $\text{diam}(S) := \sup\{d(x, y) : x, y \in S\}$. Then $\bigcap_{n=0}^{m} C_n$ is a single point.

Theorem 2.3 (Tribble Convergence Theorem). Suppose we have an initial triangle $T_0$, with nested Tribbles $\{T_n\}_{n=0}^{\infty}$ constructed with Cevian ratios $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$. The Tribbles converge if and only if the Cevian ratios are such that the lengths of all sides of the Tribbles are converging to zero.

Proof. If at least one side length does not converge to zero, then the Tribbles cannot converge.

Suppose Tribbles are constructed within $\mathbb{R}^2$ equipped with the standard metric. Suppose that the lengths of the sides of Tribbles $\{T_n\}_{n=0}^{\infty}$ are all converging to zero. For each $n$, let
$b_n$ denote the length of the longest side of $T_n$, thus $b_n \to 0$. Observe that $diam(T_n) = b_n$. Therefore, $\{T_n\}_{n=0}^{\infty}$ is a set of nested closed sets with their diameter converging to zero, so by 2.2, the infinite intersection is a single point.

Theorem 2.3 gives a necessary and sufficient condition for which the Tribbles converge. However, it is not obvious what conditions on $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ will cause the side lengths to converge to zero, as we have not found an easy way to compute the side lengths of a Tribble given its Cevian ratios.
CHAPTER 3. DETERMINISTIC CASES

3.1 Constant $x_n, y_n, z_n$

Let $x, y, z$ be fixed positive real numbers. If $x, y, z$ satisfy the Ceva condition, then any Tribble constructed with these Cevian ratios will be a single point, so there is nothing to iterate. Thus, we want to assume that $x, y, z$ violate the Ceva condition. For all $n$, let $x_n = x$, $y_n = y$, $z_n = z$. That is, we continue to iterate Tribbles using the same three Cevian ratios each time. An example of such a construction is shown in Figure 3.1.

![Figure 3.1](image)

Figure 3.1 An example of iterated Tribbles with repeated ratios. Here, $x = 3$, $y = \frac{5}{3}$, $z = 2$, with clockwise notation. The boundaries of initial triangle $T_0$ and Tribbles $T_1, T_2, T_3$ are shown in blue.

3.1.1 Proving Convergence

By Routh’s theorem, we know immediately that the area of the Tribbles goes to zero. To see this, observe that since the Cevian ratios are fixed, $c := R(x, y, z)$ is a constant. Since $A_n = cA_{n-1}$, we have $A_n = c^nA_0 \to 0$, since $c \in [0,1)$. However, this is not sufficient to establish that the Tribbles converge, since it does not prove that the side lengths go to zero.
Lemma 3.1. When Tribbles are iterated with $x_n = x$, $y_n = y$, $z_n = z$, the lengths of the sides of the Tribbles go to zero.

Proof. We can make a geometric argument to show that the side lengths go to zero. Since every triangle can be obtained from an invertible linear transformation of any other, we might as well assume the initial triangle is equilateral.

Without loss of generality, consider just the bottom edges of the triangle with Tribble (recall that we denote these $T_n$ and $T_{n+1}$) in Figure 3.2, and assume the Cevian ratio along that bottom edge is $x$. Let $a_n, b_n, c_n$ be the distances as shown. That is, $x = \frac{a_n}{b_n}$.

Observe that since $c_n = a_n + b_n$, we can write $b_n = \alpha c_n$, where $\alpha = \frac{1}{x+1}$. Observe also that $c_{n+1} < b_n$.\footnote{It is clear in this picture, but may not be true in extreme cases if one of the angles forming the initial triangle is obtuse. However, it is sufficient to prove the result for this triangle, where it is clear that $c_{n+1} < b_n$.} Thus, $c_{n+1} < \alpha c_n$, and we can continue in this way to get

$$c_{n+1} < \alpha c_n < \alpha^2 c_{n-1} < \ldots < \alpha^{n+1} c_0$$

Since $x > 0$, we know that $\alpha = \frac{1}{x+1} \in (0, 1)$. Thus $c_{n+1} < \alpha^{n+1} c_0 \to 0$, and so we can conclude that the length of the bottom edges of the Tribbles are converging to zero. Similarly, both of the other two side lengths converge to zero.\hfill \Box
**Corollary 3.2** (Convergence for Constant $x_n, y_n, z_n$). Let $x, y, z$ be fixed positive real numbers that violate the Ceva condition. Let $T_0$ be any triangle. Construct Tribble $T_{n+1}$ inside triangle $T_n$ using ratios $x_n = x$, $y_n = y$, $z_n = z$, infinitely iterating this process. Then the Tribbles converge.

The argument presented in 3.1 and 3.2 merely proves existence of the limit; it is not constructive. However, in this case, we can use a center-of-mass argument to compute the limit directly.

### 3.1.2 Computing the Limit

- If we put weights $(yz, 1, y)$ at the vertexes $(a, b, c)$ of the triangle, then the intersection $c_1$ of the lines $aa'$ and $bb'$ will be the barycenter of the triangles. Identifying $a, b, c$ with vectors $\vec{a}'$, $\vec{b}'$, $\vec{c}'$ starting at the origin and end at the corresponding points, we therefore obtain

\[
c_1 = \frac{1}{1 + y + yz} \left( yz \cdot \vec{a}' + \vec{b}' + y \cdot \vec{c}' \right)
\]

Similarly,

\[
b_1 = \frac{1}{1 + x + xy} \left( \vec{a}' + x \cdot \vec{b}' + xy \cdot \vec{c}' \right)
\]

and

\[
a_1 = \frac{1}{1 + z + zx} \left( z \cdot \vec{a}' + zx \times \vec{b}' + \vec{c}' \right)
\]

In particular, since $\vec{a}' - \vec{c}' = \overrightarrow{c_0a_0}$ and $\vec{b}' - \vec{c}' = \overrightarrow{c_0b_0}$,

\[
\frac{\overrightarrow{a_1b_1}}{\frac{1 - xyz}{(1 + x + xy)(1 + z + zx)} (\vec{a}' + x \cdot \vec{b}' - (1 + x) \cdot \vec{c}')} = \frac{1 - xyz}{(1 + x + xy)(1 + z + zx)} (\overrightarrow{c_0a_0} + x \cdot \overrightarrow{c_0b_0})
\]

Similarly,

\[
\frac{\overrightarrow{b_1c_1}}{\frac{1 - xyz}{(1 + x + xy)(1 + y + yz)} (\overrightarrow{a_0b_0} + y \cdot \overrightarrow{a_0c_0})}
\]
and

\[\overrightarrow{c_1a} = \frac{1 - xyz}{(1 + x + xy)(1 + z + zx)} (z \cdot \overrightarrow{b_0a_0} + \overrightarrow{b_0c_0})\]

In particular,

\[|a_1b_1| < \frac{|1 - xyz| \cdot (1 + x)}{(1 + x + xy)(1 + z + zx)} \cdot diam(T_0).\]

Clearly,

\[\frac{(1 - xyz)(1 + x)}{(1 + x + xy)(1 + z + zx)} = \frac{1 + x}{1 + x + xy} \cdot \frac{1 - xyz}{1 + z + zx} < 1.\]

Finally,

\[\frac{(xyz - 1)(1 + x)}{(1 + x + xy)(1 + z + zx)} < 1\]

if and only if

\[(xyz - 1)(1 + x) < (1 + x + xy)(1 + z + zx)\]

if and only if

\[xyz - 1 + x^2yz - x < 1 + x + z + 2zx + x^2z + xy + xyz + x^2yz\]

if and only if

\[0 < 2 + 2x + z + 2zx + x^2z + xy.\]

Since the latter inequality holds true, so it the first one in the chain. Thus \(diam(T_1) < \gamma(x, y, z)diam(T_0)\), where \(\gamma(x, y, z) \in (0, 1)\) is a function of the parameters \(x, y, z\).

- The above analysis shows that the barycenters of \(T_0\) and \(T_1\) will coincide if the following conditions hold true for some constants \(\alpha, \beta, \gamma > 0\):

1. The weight of \(a_0\) is \(\alpha z + \beta + \gamma y z\)
2. The weight of \(b_0\) is \(\alpha zx + \beta x + \gamma\)
3. The weight of \(c_0\) is \(\alpha + \beta xy + \gamma y\)
To propagate this result further, to the next iteration of Tribbles, we need to satisfy also the following conditions:

1. The weight of $a_1$ is $k(\alpha z + \beta + \gamma yz)$
2. The weight of $b_1$ is $k(\alpha zx + \beta x + \gamma)$
3. The weight of $c_1$ is $k(\alpha + \beta xy + \gamma y)$

for some $k > 0$. In other words, we want to have

1. $\alpha(1 + z + zx) = k(\alpha z + \beta + \gamma yz)$
2. $\beta(1 + x + xy) = k(\alpha zx + \beta x + \gamma)$
3. $\gamma(1 + y + yz) = k(\alpha + \beta xy + \gamma y)$

Matching sums of the terms on the left and on the right-hand side we obtain that necessarily $k = 1$. Moreover, it is easy to see that the rows in the left-hand side are dependent. In fact, their sum is zero. Thus we can keep just two first rows to find a positive solution. Considering $\gamma > 0$ as a given parameter, we obtain the following for $\alpha$ and $\beta$:

$$\alpha(1 + zx) - \beta = \gamma yz$$
$$-\alpha zx + \beta(1 + xy) = \gamma$$

Solving the system we obtain

$$\alpha = \gamma \frac{yz(1 + xy) + 1}{(1 + zx)(1 + xy) - zx} > 0,$$

and

$$\beta = \gamma \frac{(1 + zx) + z^2 xy}{(1 + zx)(1 + xy) - zx} > 0.$$
1. The weight of $a_0$ is
\[
yz(1 + xy) + 1 \left/ \frac{1 + xy + x^2yz}{1 + xy + x^2yz} \right. + yz
\]
\[
= \frac{z + yz^2 + y^2xz^2 + 1 + zx + z^2yx + yz + y^2xz + x^2y^2z^2}{1 + xy + x^2yz}
\]
\[
= \frac{1 + z(+)yz^2 + zx + yz + xyz(x + y + z) + x^2y^2z^2}{1 + xy + x^2yz} + \ldots
\]

2. The weight of $b_0$ is $\alpha zx + \beta x + \gamma$

3. The weight of $c_0$ is $\alpha + \beta xy + \gamma y$

### 3.2 Sequences

When $x_n, y_n, z_n$ are nonconstant sequences (depending on $n$), the Tribbles may or may not converge. Recall that $A_n$ denotes the area of the $n$th Tribble, and $A_0$ is the area of the initial triangle. Then by Theorem 1.5, we have

\[
A_n = R(x_n, y_n, z_n)A_{n-1} = R(x_n, y_n, z_n)R(x_{n-1}, y_{n-1}, z_{n-1})A_{n-2}
\]

and so on, thus

\[
A_n = A_0 \prod_{k=1}^{n} R(x_k, y_k, z_k) \quad (3.1)
\]

If the Tribbles converge, then the area must go to zero. However, if the area goes to zero, then the Tribbles may diverge to a line segment rather than a singleton point, and so in this case, the area will still go to zero. However, if the Tribbles diverge to a Triangle, then $A_n$ will go to a nonzero positive number (namely the area of that triangle). This gives us an “$n$th term test for Tribbles.”

**Proposition 3.1** ($n$th Term Test for Tribbles). Let $A_n$ denote the area of the $n$th Tribble. Consider the formula for $A_n$, equation (3.1) above.

1. If $A_n \to c > 0$, then the Tribbles diverge to a triangle.

2. If $A_n \to 0$, then the test is inconclusive (as the Tribbles may either diverge to a line segment or converge).
Proposition 3.2. Certain Tribbles to Diverge to a Triangle] Let \( x_n = y_n = z_n \) such that \( x_n > 0 \) and \( x_n \neq 1 \) for all \( n \). Tribbles iterated using such a sequence diverge to a triangle if and only if \( \sum_{k=1}^{\infty} \frac{1}{x_n} \) converges.

Proof.

Claim 1: If \( x_n = y_n = z_n \) such that \( x_n > 0 \) and \( x_n \neq 1 \) for all \( n \), and \( \sum_{k=1}^{\infty} \frac{1}{x_n} \) converges, say to \( \beta \), then the Tribbles diverge to a triangle.

We prove that \( A_n \to c > 0 \) as \( n \to \infty \). First, we note that \( A_n = R(x_n, y_n, z_n)A_{n-1} \) and \( R(x_n, x_n, x_n) \in (0, 1) \). Now, we prove that the following infinite product converges.

\[
\prod_{k=1}^{\infty} R(x_k, y_k, z_k)
\]

which, plugging in \( x_n \) becomes

\[
\prod_{k=1}^{\infty} \frac{(x_n^3 - 1)^2}{(x_n^2 + x_n + 1)^3} \tag{3.2}
\]

Next, we find a lower bound for the infinite product. Expand to get

\[
\prod_{k=1}^{\infty} \frac{x_n^6 - 2x_n^3 + 1}{x_n^6 + 3x_n^5 + 6x_n^4 + 7x_n^3 + 3x_n + 1}
\]

\[
\geq C_1 \prod_{k=1}^{\infty} \frac{x_n^6 - 2x_n^3}{x_n^6 + (3 + 6 + 7 + 6 + 3 + 1)x_n^5}
\]

where \( C_1 \) is a number in \([0, 1]\) that compensates for potentially finitely many terms for which this inequality doesn’t hold. Note that since \( \sum_{k=1}^{\infty} \frac{1}{x_n} \) converges, we know that \( x_n \to +\infty \). Therefore, except for perhaps finitely many \( n \), \( x_n > 1 \), so \( x_n^5 > x_n^4, x_n^3, ... \)

\[
= C_1 \prod_{k=1}^{\infty} \frac{x_n^6 - 2x_n^3}{x_n^6 + 26x_n^5}
\]

\[
= C_1 \prod_{k=1}^{\infty} \frac{x_n^3 - 2}{x_n^3 + 26x_n^2}
\]

\[
= C_1 \prod_{k=1}^{\infty} \left(1 - \frac{26x_n^2 + 2}{x_n^3 + 26x_n^2}\right) \tag{3.3}
\]

Now, since \( x_n \to +\infty \), we know that

\[
0 < \frac{26x_n^2 + 2}{x_n^3 + 26x_n^2} \to 0
\]
Observe that when $x$ is small and positive, we have $e^{-2x} \leq 1 - x$. Let $exp(x)$ denote $e^x$. And so, using $C_2$ to compensate for finitely many terms for which the inequality does not hold, we can continue the inequality from (3.3):

$$\geq C_1 C_2 \prod_{k=1}^{\infty} exp\left(-2 \cdot \frac{26x_n^2 + 2}{x_n^3 + 26x_n^2}\right) \tag{3.4}$$

Next, observe that (except at finitely many terms, hence $C_3$ below)

$$\frac{26x_n^2 + 2}{x_n^3 + 26x_n^2} \leq \frac{26x_n^2 + 2x_n^2}{x_n^3} = \frac{28}{x_n}$$

Therefore, we have

$$exp\left(-2 \cdot \frac{26x_n^2 + 2}{a_n^2 + 26x_n^2}\right) \geq exp\left(- \frac{56}{x_n}\right)$$

And so continuing from (3.4), if we let $C = C_1C_2C_3$ (which is between 0 and 1) we have

$$\geq C \cdot \prod_{k=1}^{\infty} exp\left(- \frac{56}{x_n}\right)$$

$$= C \cdot exp\left(- 56 \sum_{k=1}^{\infty} \frac{1}{x_n}\right)$$

$$= C \cdot e^{-56\beta} > 0$$

Therefore, we have shown that the partial product $A_n$ is bound below by $C \cdot e^{-56\beta}$. Since the Routh function outputs only values in $[0, 1)$, we know that $A_n$ is decreasing. Therefore, there exists $s$ such that $A_n \to c > 0$ and by Proposition 3.1, this implies the Tribbles diverge to a triangle, so Claim 1 is proven.

**Claim 2:** If $x_n = y_n = z_n$ such that $x_n > 0$ and $x_n \neq 1$ for all $n$, and the Tribbles diverge to a triangle, then $\sum_{k=1}^{\infty} \frac{1}{x_n}$ converges.

Assume that $x_n = y_n = z_n$ and the Tribbles diverge to a triangle.

Now, the initial triangle might as well be equilateral, so every iterated Tribble is also equilateral by symmetry. Invertible linear transformations cannot transform triangles into lines - not even in the limit. So here, it suffices to prove that the product of $R(x_n, y_n, z_n)$ goes to 0 to prove convergence of Tribbles.
Observe that since \( x^3 - 1 = (x - 1)(x^2 + x + 1) \), the product (3.2) is equal to
\[
\prod_{k=1}^{\infty} \left(1 - \frac{3x_n}{x_n^2 + x_n + 1}\right)
\]
Therefore, we know that this sum converges:
\[
\sum_{k=1}^{\infty} \ln \left(1 - \frac{3x_n}{x_n^2 + x_n + 1}\right)
\]
Observe that we have the following inequality.
\[
1 - \frac{3x_n}{x_n^2 + x_n + 1} > 1 - \frac{3x_n}{x_n^2} = 1 - \frac{3}{x_n}
\]
Which means that
\[
\ln \left(1 - \frac{3x_n}{x_n^2 + x_n + 1}\right) > \ln \left(1 - \frac{3}{x_n}\right)
\]
Since the sum of the left side converges, the sum of the right must also. That is, this sum converges:
\[
\sum_{k=1}^{\infty} \ln \left(1 - \frac{3}{x_n}\right)
\]
Now, for any \( a > 0 \), we know that \( 0 < a \leq -\ln(1 - a) \). So, if we let \( a = 1/x_n \) and take the sum, we have
\[
\sum_{k=1}^{\infty} \frac{3}{x_n} \leq \sum_{k=1}^{\infty} -\ln \left(1 - \frac{3}{x_n}\right) < \infty
\]
Therefore \( \sum_{k=1}^{\infty} \frac{1}{x_n} \) converges, so Claim 2 is proven. \( \square \)
Example 3.3 (Tribbles Diverging to a Triangle).

Let $x_n = y_n = z_n = (n + 1)!$. Figure 3.3 shows a Triangle with Tribbles iterated using this sequence. Although only a few Tribbles are easily visible, there are in fact 99 Tribbles iterated in this picture.

![Figure 3.3](image)

Figure 3.3 An illustration of iterated Tribbles constructed with $x_n = y_n = z_n = (n + 1)!$ for $n$ from 1 to 99. The Tribbles diverge to a triangle.
CHAPTER 4. INDEPENDENT RANDOM CEVIAN RATIOS

4.1 Tribble’s Theorem

Now suppose that the following is a collection of independently and identically distributed random variables used as previously described to iterate Tribbles.

\[ \{x_n, y_n, z_n : n \in \mathbb{N}\} \]

We are interested in how the distribution that generates \(x_n, y_n, z_n\) determines the distribution of the limit points of the Tribbles, if they converge. In this sense, choosing a probability distribution with which to choose \(x_n, y_n, z_n\) gives rise to a probability distribution defined on the interior of the initial triangle. Computing a formula for that distribution is beyond the scope of this work and may be impossible, but we attempt to visualize what the distribution of Tribble limit points looks like when Cevian ratios are chosen by the discrete uniform distribution on \{r, s\} in Section 4.2.

**Lemma 4.1** (Finite Choice). *Whenever the Cevian ratios are chosen from finite set \{r_1, r_2, ..., r_k\} (regardless of whether \(x_n, y_n, z_n\) are sequences with values in that set, or whether they are chosen from a probability distribution on that set), the Tribbles converge.*

We can prove this by a slight modification of the proof of Lemma 3.1: since \(a_n/b_n \in \{r_1, r_2, ..., r_k\}\), we can let \(\alpha = max\{(r_j + 1)^{-1} : 1 \leq j \leq k\}\). The rest of the proof follows exactly the same. In fact, this shows something stronger, that the Tribbles must converge as long as the Cevian ratios are chosen (by sequence or by distribution) from a set \(A\) of positive numbers such that \(max\{1/(r+1) : r \in A\}\) exists. This exists if and only if the set \(A\) contains its maximum.
Proposition 4.1 (Maximum Element). Whenever the Cevian ratios are chosen from a set that contains its maximum (regardless of whether $x_n, y_n, z_n$ are sequences with values in that set, or whether they are chosen from a probability distribution on that set), the Tribbles converge.

Now, even when we choose Cevian ratios using a probability distribution on a finite set, there are uncountably many possibilities for where the limit point of the Tribbles can be located within the interior of the initial triangle. This is because each possible limit point is one-to-one correspondence with some sequence on that finite set.¹

One might also wonder if the condition proposed in Proposition 4.1 is a necessary condition for Tribble convergence, or is it merely sufficient? It followed easily from arguments presented before, but upon further study, we found that independence is sufficient, regardless of what the support of the distribution may be.

Theorem 4.2 (Tribble’s Theorem). Suppose Cevian ratios are chosen independently and identically from a distribution defined on $(0, \infty)$. Then the Tribbles converge.

Proof. Recall that it suffices to prove that the side lengths of the Tribbles go to 0. Let $d_n = \text{diam}(T_n)$, and recall that the diameter of a triangle is the length of its longest side. Thus, we want to show that $d_n \to 0$.

Let $\gamma_n = \frac{d_n}{d_{n-1}}$. Observe that

$$
\frac{1}{n} \ln \left( \prod_{k=1}^{n} \gamma_n \right) = \frac{1}{n} \sum_{k=1}^{n} \ln \left( \gamma_n \right)
$$

By Ergodic’s Theorem (see [2]), as $n \to \infty$, this expression goes to (where $E$ denotes expectation)

$$
E(\ln \gamma_1)
$$

¹The set of irrational numbers is in one-to-one correspondence with a subset of the set of sequences on \{0, 1, 2, ..., k\} when they are written in base $k$. 
Since $T_n \subset T_{n-1}$, we know that $d_n < d_{n-1}$, and so $\ln \gamma_1 < 0$. Therefore, $E(\ln \gamma_1) = -a$ for some $a > 0$. In summary, we have

$$\frac{1}{n} \ln \left( \prod_{k=1}^{n} \gamma_n \right) \to -a$$

Therefore, for $n$ sufficiently large, we have

$$\frac{1}{n} \ln \left( \prod_{k=1}^{n} \gamma_n \right) < -\frac{a}{2}$$

$$\Rightarrow \ln \left( \prod_{k=1}^{n} \gamma_n \right) < -\frac{na}{2}$$

$$\Rightarrow \prod_{k=1}^{n} \gamma_n < \exp \left( -\frac{na}{2} \right)$$

And since $a$ is positive, the right side goes to 0 as $n \to \infty$. Therefore we know that

$$\prod_{k=1}^{n} \gamma_n \to 0$$

Finally, by our choice of $\gamma$, observe that

$$\prod_{k=1}^{n} \gamma_n = \frac{d_n}{d_0}$$

Thus we have shown that $d_n \to 0$, which by Theorem 2.3, is sufficient to show that the Tribbles converge. \hfill \Box

**Remark** (Independence). *The assumption of independence is important here; without it, the Tribbles may converge or diverge.*

For example, suppose $a_n$ follows a uniform distribution on the set $[1, 2]$ and $x_n = y_n = z_n = a_n(n + 1)!$. Because the Cevian Ratios are depending on $n$, they are not independent. Because $(n + 1)!$ grows fast enough for the Tribbles to diverge to a triangle and $a_n \geq 1$, Tribbles iterated with $x_n = y_n = z_n = a_n(n + 1)!$ will diverge as well.

The Tribbles may also converge with non-independently chosen Cevian Ratios. Suppose $a_n, b_n, c_n$ independently and identically follow a distribution on a set $A$ that contains its maximum. Let $x_n = a_n/n$, $y_n = b_n/n$, $z_n = c_n/n$. Since the set $\{a/n : a \in A, n \in \mathbb{N}\}$ contains that same maximum, we know that Tribbles iterated in such a way will converge.
4.2 The Coin Toss Case

Let $r$ and $s$ be any positive real numbers. Let $DU\{r, s\}$ denote the probability distribution such that $P(X = r) = P(X = s) = 0.5$. Now, for each $n \in \mathbb{N}$, choose $x_n, y_n, z_n$ independently using $DU\{r, s\}$. When we iterate Tribbles using such $x_n, y_n, z_n$, we call it the Coin Toss case. Choosing $x_n, y_n, z_n$ from $r$ and $s$ in this way gives rise to a probability distribution defined on the interior of the initial triangle, the distribution of Tribble limit points. We used a simulation in MATLAB to compute these limits numerically and produce graphics that allow us to begin to understand what these distributions look like.

4.2.1 How the Simulation Works

Figure 4.1 shows some psuedocode that describes how the simulation works, and the actual code can be found in Appendix A. Recall that $T_0$ is the initial triangle, $T_n$ is the $n^{th}$ Tribble, and $A_n$ is the area of $T_n$.

<table>
<thead>
<tr>
<th>Set points of initial triangle $T_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set area = $A_0$</td>
</tr>
<tr>
<td><strong>for $N = 1$ to $10,000$</strong></td>
</tr>
<tr>
<td>Set $n = 1$</td>
</tr>
<tr>
<td><strong>while area &gt; $\epsilon = 1$</strong></td>
</tr>
<tr>
<td>Choose $x_n, y_n, z_n$ from $DU{r, s}$</td>
</tr>
<tr>
<td>Compute corners of $T_n$</td>
</tr>
<tr>
<td>Set area = $A_n$</td>
</tr>
<tr>
<td>Increment $n$ by 1</td>
</tr>
<tr>
<td>set $x_n, y_n, z_n = 1$</td>
</tr>
<tr>
<td>Compute and plot the point $T_{n+1}$</td>
</tr>
</tbody>
</table>

Figure 4.1 Psuedocode for the Coin Toss Tribble Simulation

The output of the code is a plot showing the borders of the initial triangle and the numerical estimations for the limit point of each of the 10,000 trials. The initial triangle that we used was large, with area 2500, and the points that we plotted were very thick, so triangles of area
1 would have looked very small in the picture. If one wanted to construct Tribbles inside of a smaller triangle, we could easily edit one line of code to set the tolerance to something more appropriate. However, in doing so, we would get a very similar picture—the same up to a linear transformation.

In summary, for each trial, Tribbles are iterated by selecting Cevian ratios at each stage independently from $DU\{r, s\}$. When the size of a Tribble is less than the tolerance $\epsilon = 1$, we stop iterating, plot a point, and begin another trial beginning with the same initial triangle, choosing Cevian ratios anew, randomly from the same two choices. We do this 10,000 times, so the plot produced shows 10,000 limit points.

The points on the plot were set to low transparency to help visualize the distribution. That is, portions of the triangle in which the Tribbles frequently converged show up as dark in the picture. So, visually, we can think of the relative darkness of a portion of the triangle as corresponding with how likely the limit point is to land there. The transparency level chosen depended on what made the picture look the most clear, so the relative darkness from one image to the other may not be comparable. We ran this simulation for various choices of $\{r, s\}$ in a for-loop and saved the results as images. The images can be found in Appendix B.

### 4.2.2 Results of the Simulation

**Summary**

We organized the results into a plot of $\ln(r)$ vs $\ln(s)$ for each $\{r, s\}$ by plotting the ordered pair $(\ln r, \ln s)$ (and those equivalent by symmetry properties, see below) in colors which correspond to the various Tribble limit distribution types. This plot is shown in Figure 4.2. First we show the datapoints alone, but it is difficult to look at just a plot full of colored points, so it may be easier to look at the second image. This one has the regions surrounding the points shaded in the same color. The shading surrounding the points was more or less automatically generated by Adobe Photoshop, so this plot does not give any information about what types of distributions we might observe in the “gray areas” (for lack of a better word) between the various colors. Its interpolative power is untested, although we believe that we can interpolate
within the blue and cyan regions, as we will discuss shortly. Refer to Table 4.1 for a description of the distribution types used in these plots.

**Explanation**

To visually organize the various types of distributions we found when running the simulation, we want to plot them in $\mathbb{R}^2$, using $r$ and $s$ as the $x$ and $y$ values\(^2\), and using a color to group them by type. So, a point $(x, y)$ on the plot in blue would signify that we ran the simulation using $r = x, s = y$ and that the resulting distribution looked similar to the results produced by the other blue points.

Recall Observation 1.3. From this property, the distribution of Tribble limit points given by $DU\{r, s\}$ will appear similar to that by $DU\{\frac{1}{r}, \frac{1}{s}\}$. Several images in Appendix B illustrate this claim. Also, since $DU\{r, s\} = DU\{s, r\}$ this choice will trivially give an identical distribution of limits as well.

Now, when attempting to describe how the choice of $\{r, s\}$ determines the distribution of Tribble limit points, we would want to plot the point $(r, s)$ in $\mathbb{R}^2$ using a color which matches that of other points that seem to produce similar distributions. By the above argument, if we plot the point $(r, s)$ in a certain color, then the points $(s, r)$, $(\frac{1}{r}, \frac{1}{s})$ and $(\frac{1}{s}, \frac{1}{r})$ will be the same color since each of those four choices produce the same Tribbe limit distribution. Now, since all Cevian ratios are positive, all points on such a plot would reside in the first quadrant.

Also, since the choice $r$ behaves similarly to $\frac{1}{r}$, the plot should be scaled in such a way that the points with one or more coordinate less than 1 should not appear more clustered together than those with values greater than 1. For these reasons, we used a log-log plot. That is, for each $\{r, s\}$ whose Tribble limits we simulated, we plot

\[
\begin{align*}
(\ln r, \ln s), & \quad (\ln s, \ln r), \quad (−\ln r, −\ln s), \quad (−\ln s, −\ln r)
\end{align*}
\]

in $\mathbb{R}^2$ for each $\{r, s\}$ that we used in the simulation.\(^3\) These four points take the same color, as will any point that appeared—by a qualitative determination—to produce a similar distribution. The method used to categorize the various Tribble limit distribution types was literally

---

\(^2\)rather, a function of them, as discussed shortly

\(^3\)where $\ln$ denotes the natural logarithm.
Table 4.1 Descriptions of various Tribble limit distribution types. For examples of each type, see Appendix B.

<table>
<thead>
<tr>
<th>Type</th>
<th>Nickname</th>
<th>Color</th>
<th>Typical Example</th>
<th>Occurs When...</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Blob</td>
<td>Blue</td>
<td>{0.05, 1}</td>
<td>(r) and (s) are both much greater than one (or much less)</td>
<td>A curved triangular shape near the center.</td>
</tr>
<tr>
<td>2</td>
<td>Tiger Stripe</td>
<td>Cyan</td>
<td>{10, 0.01}</td>
<td>(r) is much less than one and (s) is much greater (or vice versa)</td>
<td>Most points are found within stripes emanating from the corners.</td>
</tr>
<tr>
<td>3</td>
<td>Flower</td>
<td>Green</td>
<td>{2.1, 0.01}</td>
<td>(r) and (s) are both close to 1</td>
<td>All points are found within small disconnected areas near the center.</td>
</tr>
<tr>
<td>4</td>
<td>Other</td>
<td>Magenta</td>
<td>None</td>
<td>(r) is close to 1 and (s) is not (or vice versa)</td>
<td>Varies.</td>
</tr>
</tbody>
</table>
to print them out and sort them into piles that looked similar. Table 4.1 shows the names and descriptions of the various Tribble limit distribution types, along with observations about what choices of \(\{r, s\}\) produced that type.

### 4.2.3 Limitations of the Simulation

Behavior when exactly one of \(r\) and \(s\) is close to 1 varied greatly. During our first attempt at categorizing the Tribble limit distribution types, Type 4 (Other) had been separated into three different groups. However, once we plotted each \((r, s)\) in colors by type, we found that there were points that appeared almost on top of one another yet the distributions looked very different, so we concluded that this methodology was insufficient for such points. Perhaps scaling the plot differently, running more trials, or further analysis is needed, but these are endeavors for future work.

This is also a limitation of the simulation itself. If \(r\) is close to 1 and \(s\) is not, then the probability of constructing a Tribble with Cevian ratios \(r, r, r\) is \(\frac{1}{8}\). Since \(r\) is close to 1, the Cevians will all be close to medians, and so the resulting Tribble is very small. This would cause the iteration process to terminate with fewer iterations than normal, even though there may still be interesting behavior inside of this small Tribble. Indeed, this may be a flaw in examining the distribution of Tribble limits when \(r\) and \(s\) are both close to 1, but maybe less so, since our approximation of the distribution would not look far too different from the actual distribution when viewed at the same scale.

### 4.2.4 More on Coin Toss Tribble Limit Distribution Types

To begin to see why Types 1, 2, and 3 take on the patterns they do, we can look at the eight primary Tribbles; that is, the eight possibilities for \(T_1\). The limit point must be inside one (or more) of the eight primary Tribbles. The probability of the limit point being in a given primary Tribble is at least \(\frac{1}{8}\). Thus, we can get a very crude sense of what the distribution of limit points looks like by looking at the eight primary Tribbles and how they overlap. Also, sketching the eight primary Tribbles allows us to see how each possibility will “pull” the limit point away from the barycenter of the initial Triangle.
Figure 4.2  A plot of $\ln(r)$ vs. $\ln(s)$ in colors by distribution types described in Table 4.1, with and without shading.
Type 1

Figure 4.3 shows some primary Tribbles for \{10, 2\}. If \(r, r, r\) or \(s, s, s\) are chosen for \(T_{n+1}\), which happens 25% of the time, the limit is pulled toward the center of \(T_n\). Otherwise, the other 75% of the time, the limit is pulled slightly off center by the permutations of \(s, s, r\) and \(r, r, s\) (one is smaller, so it pulls harder than other). These are both relatively large Tribbles (compared to some other examples which follow), so the pull is not very strong. Thus the limit points are clustered around the middle in a blob shape.

Figure 4.3 Examples of primary Tribbles for \{10, 2\}. The Cevian ratios that produced the Tribble are shown at the top of the figures.

Type 2

Example primary Tribbles for Type 2 are shown in Figure 4.4. When \(r\) is much greater than 1 and \(s\) is much smaller than 1, the eight primary Tribbles are as follows:

- Permutations of \(r, r, s\) and \(s, s, r\) will form long, narrow Tribbles close to the sides of the initial Triangle. There are 6 of these.

- \(s, s, s\) and \(r, r, r\) are very large Tribbles that take up most of the area of the initial triangle.

The very large Tribbles produced by \(s, s, s\) and \(r, r, r\) have little effect on the limit, and only occur 25% of the time. The long narrow Tribbles produced by \(r, r, s\) and \(s, s, r\) occur 75% of the time, and pull the limit hard toward the corners because of their size.

So, when we imagine iterating Tribbles in this way, 75% of the time, the \(n^{th}\) Tribble is squished close to a side of the \((n - 1)^{th}\) Tribble, and the other 25%, the \(n^{th}\) Tribble takes up most of the space inside the \((n - 1)^{th}\) Tribble, and so the limit points tend to be near the sides
of the initial Triangle, and the probability of them landing more toward the center goes to zero. After all, the only way to have a limit point near the center is if only \( r, r, r \) and \( s, s, s \) have been chosen many times. The probability of any Tribble being formed by \( r, r, r \) or \( s, s, s \) is \( \frac{1}{4} \), since these are 2 out of 8 mutually exclusive possibilities, and so by independence, the probability that \( T_1, T_2, ..., T_n \) were all formed by \( r, r, r \) or \( s, s, s \) is \( 4^{-n} \) which quickly goes to zero.

**Type 3**

When \( r \) and \( s \) are both close to 1, the numerator of the Routh function \((xyz - 1)^2\) (here \( x, y, z \in \{r, s\} \)) is close to zero, and so the \( n^{th} \) Tribble is very small and close to the center of the \((n - 1)^{th} \) Tribble. This is similar to Type 1, but the smaller Tribbles pull harder. The small, off center Tribbles pull the limit points apart more, which causes the flower-like pattern to form. Some example primary Tribbles are shown in Figure 4.5.

![Figure 4.4](image1.png)  
**Figure 4.4** Examples of primary Tribbles for \( \{20, .03\} \). The Cevian ratios that produced the Tribble are shown at the top of the figures.

![Figure 4.5](image2.png)  
**Figure 4.5** Examples of primary Tribbles for \( \{2, 1.01\} \). The Cevian ratios that produced the Tribble are shown at the top of the figures.
Type 4

Assume that $r \gg 1$ and $s \approx 1$.

- When $r, r, r$ is chosen, we get a large Tribble which has little effect on the limit.
- When $s, s, s$ is chosen, we get a very small Tribble in the middle, which is likely small enough to cause the simulation to terminate at this step.
- When $r, r, s$ is chosen, we get a Tribble which takes up most of one “half” of the Triangle, since Cevians with ratio $r$ are close to the side of the preceding triangle, and Cevians with ratio $s$ are close to medians.
- When $s, s, r$ is chosen, it is similar to the above, but chopped in “half” by another median.

These cases are illustrated in Figure 4.6. Thinking about the “pull” these four types of primary Tribbles seemed insufficient to understand the varied behavior of Type 4 distributions, so further study is needed.

Figure 4.6 Examples of primary Tribbles for $\{0.9, 50\}$. The Cevian ratios that produced the Tribble are shown at the top of the figures.
4.2.5 Conclusion

In conclusion, when $r$ and $s$ are sufficiently far away from 1, the behavior of the Tribble limit points is relatively predictable. That is, we do conjecture that Figure 4.2 can be used for interpolation within the blue and cyan regions. Future works may want to quantify how far away from 1 we would need $r$ and $s$ to be in order to get this predictability, but for now our crude, anecdotal estimate is that between .5 and 2 should be considered close to 1. When $r$ and $s$ are both close to 1, we see a clustering of limit points near the center of the initial triangle, and when one of $r$ and $s$ is close to 1 and the other is far from 1, the results are rather unpredictable and call for more study.
CHAPTER 5. SUMMARY AND CONCLUSION

5.1 Summary

We began simply by assuming that the Ceva condition is violated and iterating the construction of Tribbles in different ways, using repeated Cevian ratios, sequences, and probability distributions. The Tribble Convergence Theorem (Theorem 2.3) gives us a necessary and sufficient condition for convergence that allowed us to prove our main results. These main results give us the sense that Tribbles will converge as long as the Cevian ratios are not specifically designed to grow toward $+\infty$ or shrink toward 0 fast enough. This is only attainable when the Cevian ratios used to construct $T_n$ are non-independent from $n$.

5.2 Conclusion

We have initiated study on the new topic of iterated Tribbles, a dynamical system which arises when Cevian ratios violate the Ceva condition and the process is repeated. The main results proven thus far are

1. When the same three Cevian ratios are repeated and the Tribbles iterated, the Tribbles converge.

2. When the Cevian ratios are generated by a sequence depending on $n$, the Tribbles may or may not converge. We have begun discussion on tests for convergence.

3. When the Cevian ratios are chosen independently from a probability distribution, the Tribbles converge.
5.3 Open-Ended Questions

1. When Cevian ratios are chosen from $DU\{r, s\}$, our simulation failed to say anything meaningful when exactly one of $r$ and $s$ was close to 1. Can further analysis give us more insight? It would also be interesting to quantify exactly how far from 1 we need $r$ and $s$ to be in order to predict the Tribble limit point distribution type.

2. It would be interesting to run the simulation by choosing Cevian ratios from $DU\{r_1, r_2, ..., r_k\}$ and see if similar Tribble limit distribution types occur, and how they can be predicted from $\{r_1, r_2, ..., r_k\}$.

3. Next, one might want to choose Cevian ratios from continuous distributions.

4. What other conditions can we put on sequences of Cevian ratios to force convergence or divergence?

5. What is the rate of convergence?

6. What is the fractal dimension of the set of limits?

7. We could use game theory to interpret Tribbles. The game has three players. Imagine that three people are going to share a triangular cake. Each player chooses a Cevian ratio, which corresponds to the ratio of how much cake they feel the other two players should get. Whenever the three chosen ratios violate the Ceva condition, the players choose new ratios for the Tribble inside. Does this game have a Nash equilibrium?

8. What if Tribbles are constructed on spheres or hyperbolic geometries?
APPENDIX A. MATLAB CODE FOR THE COIN TOSS SIMULATION

The code is inefficiently written and has a very long runtime. If it is to be used in future work, we recommend improving on the efficiency. The code here will produce the numerical estimations of the limit points for ratios chosen from 100 and 101. The program used in practice was slightly different in that the function “chooseRatios” took $r$ and $s$ as arguments, so that we could run this within for loops to make many choices for $r$ and $s$. However, the code shown here should give the reader a sufficient understanding of how the program runs.

```matlab
function InfiniteTribbles

% Settings
M = [0, 100, 50; 0, 0, 50]; % It's M so that the initial triangle can always be recovered.

% Tolerance
```

% The variable M will get written over during each trial.
epsilon = 1;

% Number of Trials
N = 100000;

% Running N trials

% First draw the initial triangle M.
hold on
fill(M(1,:),M(2,:), 'white')

for n=1:N
    % Reseting things.
    clear M area A
    M = MM;
    area = polyarea(M(1,:),M(2,:));
    A = area;

    % A while/do loop to make the Tribbles until they are small enough.
    k=2;
    while area > epsilon
        % Pick the ratios.
        [r12,r23,r13] = chooseRatios();
        % Find the corners of the next Tribble.
        M(:,:,k) = makeTribble(M(:,:,k-1),r12,r23,r13);
%scatter(M(1,:),n),M(2,:),n))

%Find the area of the Tribble.
area = polyarea(M(1,:),k),M(2,:),k));
A(k)=area;
k=k+1;
end % while loop

%Put a marker on the inside of the last Tribble we found.
r12=1; r23=1; r13=1;
M(:,k) = makeTribble(M(:,(k-1)),r12,r23,r13);
scatter(M(1,1,k),M(2,1,k),'black','filled');

end % for loop

savefig('demoThingy')

end %%%%%%%%%%%%%%%%%%%%%%%%%%%%%% %

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Other Functions

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Edit this function to choose how the ratios are picked.
function [r12,r23,r13] = chooseRatios
function r = pickOne
    c = randi(2,1,1);
    if c==1
        r=6; %%%
    elseif c==2
        r=2; %%%
    end %ifs
end % function pickOne

r12 = pickOne();
r23 = pickOne();
r13 = pickOne();

end

% Inputs the ordered pairs of the three corners of a triangle ( stored in A)
% and outputs the ordered pairs of a Tribble inside, following the % specified ratios.
function B = makeTribble(A,r12,r23,r13)

% Read in the corners of the triangles from the matrix A.
    x1=A(1,1); y1 = A(2,1);
    x2=A(1,2); y2 = A(2,2);
    x3=A(1,3); y3 = A(2,3);

% Find the edgepoints (where Cevian meets a side of the triangle )
\[ [z_1, w_1] = \text{findEdgepoint}(x_3, y_3, x_2, y_2, r_{23}); \]
\[ [z_2, w_2] = \text{findEdgepoint}(x_1, y_1, x_3, y_3, r_{13}); \]
\[ [z_3, w_3] = \text{findEdgepoint}(x_2, y_2, x_1, y_1, r_{12}); \]
\% scatter([z_1, z_2, z_3], [w_1, w_2, w_3], 'black', 'filled')
\% plot([x_1, z_1], [y_1, w_1]); plot([x_2, z_2], [y_2, w_2]); plot([x_3, z_3], [y_3, w_3]);

\% Find the 3 corners of the Tribble
\% the m's and b's are slope and intercept of the cevians
\[ [m_1, b_1] = \text{findCevian}(x_1, y_1, z_1, w_1); \]
\[ [m_2, b_2] = \text{findCevian}(x_2, y_2, z_2, w_2); \]
\[ [m_3, b_3] = \text{findCevian}(x_3, y_3, z_3, w_3); \]

\% the x-coords (s) and y-coords (t) of the intersection points
\[ s_1 = (b_3-b_2)/(m_2-m_3); \quad t_1 = m_2*s_1+b_2; \% i=2, j=3 \]
\[ s_2 = (b_3-b_1)/(m_1-m_3); \quad t_2 = m_1*s_2+b_1; \% i=1, j=3 \]
\[ s_3 = (b_2-b_1)/(m_1-m_2); \quad t_3 = m_1*s_3+b_1; \% i=1, j=2 \]
\% scatter([s_1, s_2, s_3], [t_1, t_2, t_3])

%Output
\[ B = [s_1, s_2, s_3; t_1, t_2, t_3]; \]

\% Given two points \((x_i, y_i)\) and \((x_j, y_j)\), find a point between them such that
% the ratio of the distances is r_{ij}.

function [zk, wk] = findEdgepoint(xi, yi, xj, yj, r_{ij})
    clear zk wk;
    zk = (r_{ij} \times xj + xi) / (1 + r_{ij});
    wk = (r_{ij} \times yj + yi) / (1 + r_{ij});
end

% Given two points, find the slope mk and intercept bk of the line through%
% them.

function [mk, bk] = findCevian(xk, yk, zk, wk)
    mk = (yk - wk) / (xk - zk);
    bk = yk - xk * (yk - wk) / (xk - zk);
end

% Getting the nice picture:
  % Save plot as EPS
  % Open it in Illustrator
  % Click on one of the points
  % Go to Select -> Same -> Fill Color
  % Set Opacity to very small
APPENDIX B. OUTPUTS OF COIN TOSS SIMULATION

Here is an image for every choice of \( \{r, s\} \) used in the simulation, arranged by type. The two numbers that appear at the top of the plot are the values of \( r \) and \( s \) that the Cevian ratios are chosen from. Each image shown here corresponds to a datapoint in Figure 4.2.

**Type 1 - Blob**

**Occurs when:** \( r \) and \( s \) are both much greater than one (or much less)

**Color in Figure 4.2:** Blue
Type 2 - Tiger Stripe

Occurs when: $r$ is much less than one and $s$ is much greater (or vice versa)

Color in Figure 4.2: Cyan
Type 3 - Flower

**Occurs when:** \( r \) and \( s \) are both close to 1

**Color in Figure 4.2:** Green
Type 4 - Other

**Occurs when:** $r$ is close to 1 and $s$ is not (or vice versa)

**Color in Figure 4.2:** Magenta
Discarded

A few that we tried produced graphs that provided little to no information, so they were not included in Figure 4.2.
BIBLIOGRAPHY
