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Stability of multiple-loop nonlinear time-varying systems

David William Porter
Iowa State University

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David William Porter

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# TABLE OF CONTENTS

## CHAPTER 1: INTRODUCTION  
1
- Stability in Functional Analysis Setting  
  1
- Gain and Sector Conditions  
  2
- Multiple-Loop System  
  3
- Single-Loop System  
  5
- Outline  
  5

## CHAPTER 2: NOTATION  
6

## CHAPTER 3: PREVIOUS RESULTS  
8
- A Frequency-Domain Result  
  8
- Stability Results Involving Gain and Sector Conditions  
  13
- Interpretations of Sector Conditions  
  26

## CHAPTER 4: MAIN RESULTS  
33
- System Configuration  
  33
- A Gain Result  
  36
- Transforming the System  
  40
- A Sector Result  
  49
- A Margin of Boundedness $\delta$  
  54
- Continuity  
  63

## CHAPTER 5: APPLICATIONS  
68

## CHAPTER 6: CONCLUSION  
105

## LITERATURE CITED  
108

## ACKNOWLEDGMENTS  
110

## APPENDIX A  
111
- Several Linear Spaces  
  111
APPENDIX B 114

Relations 114

APPENDIX C 115

Completion of Proof for Theorem 7 115

APPENDIX D 117

Completion of Proof for Theorem 8 117
CHAPTER 1: INTRODUCTION

Stability in Functional Analysis Setting

The central problem of stability theory is to ascertain qualitative features of system behavior in the absence of knowledge of specific system solutions. Typically an intuitive notion of the type of behavior desired is expressed in the form of a precise mathematical definition of stability. Then conditions on system parameters are sought which are sufficient to guarantee the system displays this type of stability. Here it is desired that a system be characterized by one of the following two types of behavior:

(1) The system is not explosive.

(2) The system is not critically sensitive to noise.

Concepts from functional analysis provide an appropriate vehicle for translating these notions of desired behavior into stability definitions.

First a suitable model of a system is required. Here a system is viewed in a "black box" sense as a mathematical relation which connects input functions from an input space with output functions from an output space. A relation is not explosive if inputs of finite "size" correspond only to outputs of finite "size". The notion of "size" is given a mathematical interpretation as a norm on a function space. Then a relation is not explosive if each set of bounded inputs corresponds to a set of bounded outputs. Such a relation is termed bounded. A problem occurs here due to the fact that the usual spaces from analysis contain only bounded functions. These spaces are unacceptable for use as output spaces since it would be required at the outset that bounded
inputs lead to bounded outputs. This difficulty is obviated by employing an enlargement of the typical normed space, the extended space, as the space of input and output functions.

The other type of stable behavior considered here is lack of critical sensitivity to noise. Intuitively this type of behavior is displayed by a relation where inputs arbitrarily "close" to each other lead to outputs arbitrarily "close" to each other. This notion of stability is made precise by utilizing the norm of the difference of two functions in a function space as a measure of their "closeness". Then lack of critical sensitivity to noise is seen to be equivalent mathematically to continuity. Another useful physical interpretation of continuity is that it precludes the jump phenomenon.

Gain and Sector Conditions

Boundedness results are presented here in terms of gain or sector conditions on certain relations. The gain of a relation is roughly defined as the maximum ratio of the norm of the output to the norm of the input. This is an appealing definition in view of the notion of gain employed in the linear theory. The sector condition is a generalization of the gain condition which allows the boundedness results to find much wider application.

Incremental counterparts to gain and sector conditions are employed to arrive at continuity results. Loosely speaking, the incremental gain of a relation is defined as the maximum ratio of the norm of the deviation in the output to the norm of the deviation in the input. A generalization of the incremental gain condition leads to the incremental sector condition.
Interesting practical interpretations of sector conditions can be given for certain function spaces. For instance, consider relations having input and output spaces which are extensions of the space of square integrable functions. A memoryless nonlinearity satisfies a certain sector condition if its graph lies within a region of the plane enclosed by two lines passing through the origin. If the further restriction is made that the slope of the nonlinearity lies between two particular constants, then a certain incremental sector condition is satisfied. For a linear time-invariant system a sector condition and its incremental counterpart are equivalent. Further, it is found that if the Nyquist diagram is situated appropriately relative to a particular circle in the complex plane then a certain sector condition is satisfied.

Multiple-Loop System

The idea of a multiple-loop system is translated into precise mathematical terms as a set of simultaneous functional equations. A block diagram corresponding to these equations takes the form of an interconnection of a number of relations. The input supplied to each relation is composed of a general system input plus a weighted sum of outputs provided by other relations. The set of inputs and outputs of the relations whose interconnection produces the multiple-loop system is viewed as the set of general system outputs of the multiple-loop system. Stability of the multiple-loop system is interpreted in terms of stability of the collection of relations which connect the general system input with each of the general system outputs. If each of
these relations is stable, then the multiple-loop system is referred to as stable.

Here stability results are presented in terms of the interconnection structure of a multiple-loop system and in terms of the gains of the relations which are interconnected. For a single-loop system these results lead to the intuitively appealing conclusion that an open-loop gain product less than unity implies closed-loop stability.

It is found that a certain transformation of a multiple-loop system allows gain stability conditions to be generalized to sector stability conditions. Due to this transformation, the theory finds much wider application than at first seems possible. One illustration of sector results is provided by considering a multiple-loop system which is an interconnection of an arbitrary number of memoryless nonlinearities with a number of linear time-invariant relations. For such a system, stability conditions can be found involving Nyquist diagrams of the linear parts and requiring the nonlinearities to satisfy certain sector conditions. In the case of a single loop these results reduce to previously obtained results [8], [16] reminiscent of the Nyquist criterion. Further manipulation leads to the familiar Popov conditions [16].

From the manner in which results are proven, it is clear that if the stability conditions are satisfied then bounds on system outputs or deviations in system outputs can actually be calculated. If tighter restrictions are placed on system parameters, then tighter bounds are obtained. In this sense the margin by which a system satisfies stability conditions is a measure of "how stable" that system is. Hence, some
feeling can be obtained for the "degree of stability" of a multiple-loop system.

Single-Loop System

All stability results obtained prior to this investigation pertain only to a single-loop system. The results presented here are most closely related to results presented by Zames [15] and Sandberg [10]. In fact, it is found for the special case of a single-loop that the multiple-loop stability conditions specialize to conditions Zames [15] presents.

From a certain perspective the multiple-loop formulation is no more general than the single-loop. After all, any multiple-loop system can be represented as a single loop possessing open-loop elements which are multiple-input multiple-output. Then the single-loop theory applies. A disadvantage of this approach is that it tends to hide the influence the actual structure of the interconnection has on the problem. An advantage of the single-loop view is that fewer stability conditions are imposed on the system. However, these conditions are in general more difficult to verify than ones found from the multiple-loop view.

Outline

In Chapter 2 some necessary nomenclature is established. Prior results pertinent to this investigation are discussed in Chapter 3. In Chapter 4 new results are presented. A detailed description of what is meant here by a multiple-loop system is given followed by several stability theorems. Several applications of the theory are presented in Chapter 5. The conclusion is provided by Chapter 6.
CHAPTER 2: NOTATION

The symbol $\in$ denotes set inclusion. Union is denoted by $\cup$ and intersection by $\cap$. The supremum and maximum are denoted, respectively, by $\sup$ and $\max$. The symbol $j$ is used for $\sqrt{-1}$ and $s$ always denotes a complex number with real part $\sigma$ and imaginary part $\omega$. The conjugate of a complex number $a$ is denoted by $\overline{a}$. $\mathbb{R}^n$ denotes an $n$-dimensional Euclidean space. For the vectors $x, y \in \mathbb{R}^n$ the notation $x \preceq y$ is used to indicate each component of $x$ is less than or equal to the corresponding component of $y$.

The notation $f: X \rightarrow Y$ refers to the mapping $f$ from the set $X$ into the set $Y$. The notation $\{x: A\}$ is interpreted as the set of all $x$ such that condition $A$ is satisfied. The cartesian product of two spaces is defined by $X \times Y = \{(x, y): x \in X \text{ and } y \in Y\}$.

The identity matrix is denoted by $I$, a matrix with $i,j^{th}$ element $a_{ij}$ is denoted by $[a_{ij}]$, and a diagonal matrix with $i^{th}$ diagonal element $a_i$ is denoted by $[\text{diag } a_i]$. The transpose of the matrix $A$ is denoted by $A^T$ and the conjugate-transpose by $A^*$. The positive square root of the maximum eigenvalue of $A^*A$ is denoted by $\sqrt{\lambda_1(A)}$. For a square matrix $A$, the determinant is denoted by $|A|$ and the inverse by $A^{-1}$.

Following the notation of [1], a minor of the matrix $A$ is given by

$$A\begin{pmatrix} i_1, i_2, \ldots, i_p \\ k_1, k_2, \ldots, k_p \end{pmatrix} = \begin{bmatrix} a_{i_1k_1} & a_{i_1k_2} & \cdots & a_{i_1k_p} \\ a_{i_2k_1} & a_{i_2k_2} & \cdots & a_{i_2k_p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{ik_1} & a_{ik_2} & \cdots & a_{ik_p} \end{bmatrix}$$
where \( i_1 < i_2 < \ldots < i_p \) and \( k_1 < k_2 < \ldots < k_p \). The principal minors are those for which \( i_j = k_j \) for each \( j = 1, 2, \ldots, p \). For an \( m \times m \) matrix \( A \) the successive principal minors are

\[
A \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ A \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \ldots, \ A \begin{pmatrix} 1 & 2 & \ldots & m \\ 1 & 2 & \ldots & m \end{pmatrix}.
\]
CHAPTER 3: PREVIOUS RESULTS

Relevant past investigations of stability in a functional analysis setting are discussed here. All prior efforts have been directed towards analysis of single-loop systems. The first result discussed deals with a system having an open loop composed of a multiple-input multiple-output linear time-invariant part in cascade with a bank of nonlinearities. For this system Sandberg [10] provides a stability result having a frequency-domain interpretation. Next some stability results concerned with boundedness and continuity of a certain pair of functional equations are discussed. These results which are due to Zames [15] are phrased in terms of gain and sector conditions. A discussion of several interpretations of sector conditions concludes this chapter.

There is one detail which should be made clear at the outset. All results discussed here are posed in such a manner as to separate questions of existence and uniqueness of solutions from questions of stability of solutions. This is certainly a logical separation and means for a stable system that whatever possibly nonunique solutions exist display the appropriate properties. Existence and uniqueness can often be established by use of the appropriate fixed point theorem [3], [4], [7].

A Frequency-Domain Result

Here a stability result is given for the system represented in Fig. 1. The block L represents a multiple-input multiple-output
linear time-invariant system while the block \( N \) represents a bank of nonlinearities.

![Diagram](image)

**Fig. 1.** Single-loop system having open loop composed of linear part in cascade with bank of nonlinearities.

First a suitable space for input and output functions is defined. Assume the functions to be dealt with are \( n \)-vector-valued functions of time. Consider the space

\[
L_{2n}[0,\infty) = \{f : f \text{ is measurable and } \int_0^\infty f^T(t)f(t)dt < \infty \}.
\]

The space desired is the extension of this space given by

\[
E_n = \{f : f \text{ is measurable and } \int_0^t f^T(v)f(v)dv < \infty \text{ for all } t \in [0,\infty) \}.
\]

Now \( N \) and \( L \) are described further and the system equations given. \( N \) is characterized by a function \( q : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) where

\[
q(f(t),t) = [q_1(f_1(t),t), q_2(f_2(t),t), \ldots, q_n(f_n(t),t)]^T
\]

and the \( q_i \) are real-valued functions with the following properties for each \( i \):

1. \( q_i(0,t) = 0 \) for all \( t \in [0,\infty) \).
(2) There exist real numbers $a$ and $b$ such that $a \leq \frac{q_i(w,t)}{w} \leq b$ for all $w \neq 0$ and all $t \in [0, \infty)$.

(3) $q_i(w(t),t)$ is a measurable function of $t$ whenever $w(t)$ is measurable.

$L$ is characterized by an $n \times n$ matrix weighting function $k(t)$. It is assumed each element of this matrix is in $L_1[0, \infty)$. Then $L$ takes any input $u \in L_2[0, \infty)$ into an output $h \in L_2[0, \infty)$ by the integral equation

$$h(t) = \int_0^t k(t-v)u(v)dv.$$ 

It is not assumed $L$ is described by an ordinary differential equation. However, if this is the case it appears at first that initial conditions can not be accounted for. This is not true because examination of the block diagram shows the negative of the initial condition response can be added to $g$ and in this manner included in the analysis. Now the system equations represented by Fig. 1 are seen to be

$$g(t) = f(t) + \int_0^t k(t-v)q(f(v),v)dv.$$ 

The following theorem pertains to this system and is a special case of an abstract result presented by Sandberg [10].

**Theorem 1:** Let $g \in L_2[0, \infty)$ and $f \in E_n$ satisfy the system equations. Define

$$K(s) = \int_0^\infty k(t)e^{-st}dt$$ 

for $s > 0$. 
Suppose that
\[
(1) \quad \left| 1 + \frac{1}{2}(a + b)K(s) \right| \neq 0 \quad \text{for} \quad \sigma > 0, \quad \text{and}
\]
\[
(2) \quad \frac{1}{2}(b - a) \sup_{\omega} E\{[1 + \frac{1}{2}(a + b)K(j\omega)]^{-1}K(j\omega)\} < 1.
\]
Then \( f \in L_{2n}[0, \infty) \).

It is interesting to observe that the \textit{a priori} restriction of \( f \in \mathbb{E}_n \) is tantamount to assuming no finite escape time. The next theorem is also given in [10].

**Theorem 2:** Assume the hypotheses of Theorem 1 are satisfied, \( g(t) \to 0 \) as \( t \to \infty \), and the elements of \( k(t) \) are each in \( L_2[0, \infty) \). Then \( f(t) \to 0 \) as \( t \to \infty \).

Sandberg shows that for \( n = 1 \) conditions (1) and (2) of Theorem 1 admit an interpretation in the complex plane. In [8] it is found that for \( b > 0 \) conditions (1) and (2) are satisfied if one of the following is true:

1. For \( a > 0 \) the locus of \( K(j\omega) \) for \( \omega \in (-\infty, \infty) \) lies outside the circle with center \((\frac{1}{2}(a^{-1} + b^{-1}), 0)\) and radius \(\frac{1}{2}(a^{-1} - b^{-1})\), and this locus does not encircle the point \((\frac{1}{2}(a^{-1} + b^{-1}), 0)\).
2. For \( a = 0 \) the real part of \( K(j\omega) \) is greater than \(-b^{-1}\) for all \( \omega \).
3. For \( a < 0 \) the locus of \( K(j\omega) \) for \( \omega \in (-\infty, \infty) \) is contained within the circle with center \((\frac{1}{2}(a^{-1} + b^{-1}), 0)\) and radius \(\frac{1}{2}(b^{-1} - a^{-1})\).

The above are illustrated in Fig. 2 where the locus must lie in the shaded region for the appropriate condition to be true.
Fig. 2. Frequency-domain stability conditions.

(a) \( a > 0 \).

(b) \( a = 0 \).

(c) \( a < 0 \).

Now assume \( n = 1 \) and the block \( L \) is described by the ordinary differential equation

\[
\sum_{j=0}^{m} a_j h^{(j)} = \sum_{j=0}^{m-1} b_j u^{(j)}
\]

where the \( a_j \) and \( b_j \) are real constants, the superscript denotes the order of the differentiation, and \( a_m \neq 0 \). Assume there is no general input to the system so that \( g \) is the negative of the initial condition response of \( L \). Further, assume the zeros of the polynomial \( \sum_{j=0}^{m} a_j s^j \) are strictly in the left half plane. Then by Theorem 2, if a solution exists, all that is needed to infer \( f(t) \to 0 \) as \( t \to \infty \) is that \( K(j\omega) \) satisfy one of the conditions illustrated in Fig. 2 and the nonlinearity
q satisfy the earlier listed conditions. Here then is an answer to the often asked question: Under what conditions will the system response go to zero from arbitrary initial conditions?

There is clearly much similarity between the above and the familiar Nyquist criterion from the linear theory. In fact, if \( a = b \) the above conditions are identical with the Nyquist criterion. This connection with the linear theory gives reason to believe that these results are at least in the right "ball park".

Further stability results involving frequency-domain conditions are obtained in [9] for an \( L_\infty \) type of stability. Also systems modeled by difference equations are considered. In [11] results are given which provide continuity, exponential bounds, and ultimate periodicity of system responses.

Stability Results Involving Gain and Sector Conditions

Boundedness and continuity results obtained for a certain pair of functional equations are discussed here. These results presented by Zames [15] are phrased in terms of gain and sector conditions. First a suitable space for input and output functions of time is defined. Then the precise mathematical model used for a system is discussed, and definitions are given for boundedness and continuity. Finally, definitions of gain and sector conditions are given and stability theorems presented.

All input and output functions are real-valued and defined on the time interval \( T \) which is of the form \( [t_0, \infty) \) or \( (-\infty, \infty) \). The notion of
truncation of such functions is employed to define extensions of the usual spaces of analysis.

Definition: For a real-valued function $x$ defined on $T$ the truncation at time $t \in T$ is given by

$$x_t(t) = \begin{cases} x(\tau) & \text{for } t < \tau \\ 0 & \text{for } t \geq \tau \end{cases}.$$

Appendix A contains the definition of a normed linear space. The following definition deals with a special kind of normed linear space.

Definition: $X$ is a space of real-valued functions on $T$ possessing the following properties:

1. $X$ is a normed linear space where if $x \in X$ the norm of $x$ is denoted by $\|x\|$.  
2. If $x \in X$ then $x_t \in X$ for all $t \in T$.  
3. If $x$ is such that $x_t \in X$ for all $t \in T$, then
   
   (a) $\|x_t\|$ is a nondecreasing function of $t \in T$, and
   
   (b) $\lim_{t \to \infty} \|x_t\|$ is finite if and only if $x \in X$ where
   
   $$\lim_{t \to \infty} \|x_t\| = \|x\|$$ if $x$ does belong to $X$.

Many of the common function spaces satisfy the conditions placed on $X$. For instance, these conditions are satisfied by the $L^p$ spaces for $p = 1, 2, \ldots, \infty$. Appendix A contains a discussion of $L^p$ spaces. From the viewpoint of applications, the $L^2$ space of square integrable functions and the $L^\infty$ space of bounded functions are of particular interest.
Now an extension of the space $X$ is defined which serves as a suitable space for input and output functions in the stability problem formulation.

**Definition:** The extended space $X_e$ is the linear space of real-valued functions of time each having all finite truncations in $X$. Thus,

$$X_e = \{x : x \text{ is a real-valued function on } T \text{ and } x_t \in X \text{ for all } t \in T\}.$$ 

An extended norm is defined for $x \in X_e$ by $|x|_e = |x|$ if $x \in X$ and $|x|_e = \infty$ if $x \not\in X$.

It should be noted that despite the definition of the extended norm the linear space $X_e$ is not a normed linear space.

Since the extended space contains "explosive" functions, it becomes a suitable space for inputs and outputs where $X$ is not. Use of $X$ for inputs and outputs would require knowing *a priori* that the system is not explosive. This would result in assuming stability to prove stability.

The definition of $X_e$ makes the significance of assumptions (2) and (3) in the definition of $X$ clear. Assumption (2) guarantees $X_e$ is an enlargement of $X$ by implying $X \subseteq X_e$. If $x \in X_e$ then assumption (3) allows determination of whether or not $x$ has finite norm by examining $\lim_{t \to \infty} ||x_t||$.

This fact is crucial to the proofs of stability theorems presented later.

The precise mathematical model of a system employed here is that of a relation defined below.
Definition: A relation \( H \) on \( X_e \) is a subset of the product space \( X_e \times X_e \). If \( (x,y) \in H \) then \( y \) is said to be an image of \( x \) under \( H \) and is often denoted by \( Hx \). The notation \( Hx(t) \) refers to the value of an image of \( x \) under \( H \) at time \( t \). The domain of \( H \) is defined by

\[
\text{Do}(H) = \{x: \text{ there exists a } y \text{ so that } (x,y) \in H\},
\]

and the range of \( H \) is defined by

\[
\text{Ra}(H) = \{y: \text{ there exists an } x \text{ so that } (x,y) \in H\}.
\]

If \( A \) is a subset of \( X_e \), the image of \( A \) under \( H \) is defined by

\[
HA = \{y: (x,y) \in H \text{ and } x \in A \cap \text{Do}(H)\}.
\]

Appendix B contains further discussion of relations. If \( H \) and \( K \) are relations and \( c \) is a real constant, then the sum \( H + K \), the product \( cH \), and the composition product \( KH \) are defined in the usual way. Further, the inverse relation \( H^{-1} \) always exists, and the identity relation is denoted by \( I \). A relation which is single-valued and has the entire \( X_e \) space as domain is termed an operator.

It is interesting to observe that use of \( X_e \) in defining a relation essentially requires the relation to have no finite "escape time". This is due to the fact that an output truncated at a finite time must have a finite norm.

Systems for which it makes sense to speak of initial conditions can be modeled as a relation in basically two ways. A single relation can be used which is multiple-valued having each output correspond to a different initial condition. An alternative is to use a different relation corresponding to each initial condition. If the system is
linear and described by a set of ordinary differential equations, then a relation can be utilized to account only for the forced response and the initial condition response can be simply added to the output.

Now the stability properties of boundedness and continuity are defined.

Definition: A subset \( S \) of \( X_e \) is said to be bounded if there exists a real number \( D \) such that if \( x \in S \) then \( ||x||_e < D \). A relation \( H \) on \( X_e \) is bounded if the image of every bounded subset of \( X_e \) is itself a bounded subset of \( X_e \).

Note this definition is stronger than simply saying the image of each input of finite norm is itself of finite norm. In the latter case it might be possible to have a sequence \( x_n, n = 1, 2, \ldots, \) with \( ||x_n|| = 1 \) and \( ||Hx_n|| = n \) for each \( n \).

Definition: A relation \( H \) on \( X_e \) is continuous if for each \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( x \in Do(H), y \in Do(H), \) and \( ||x-y||_e < \delta \) then \( ||Hx-Hy||_e < \varepsilon \).

It is interesting to observe that \( x-y \) can have a finite norm even if \( x \) and \( y \) both have infinite norms. This leads to the fact that for a continuous relation explosive inputs which are arbitrarily "close" to each other lead to outputs which are also arbitrarily "close" to each other. It is also interesting to observe that continuity is incompatible with the jump phenomenon. Further it should be noted that a continuous relation must be single-valued.
In [15] Zames investigates boundedness and continuity of the single-loop system illustrated in Fig. 3. The functional equations describing this system are

\[ e_1 = a_1x + w_1 + y_2 \]
\[ e_2 = a_2x + w_2 + y_1 \]
\[ y_1 = H_1e_1 \]
\[ y_2 = H_2e_2 \]

where it is assumed that:

- \( H_1 \) and \( H_2 \) are relations on \( X_e \).
- \( x \) in \( X_e \) is an input.
- \( a_1 \) and \( a_2 \) are real constants.
- \( w_1 \) and \( w_2 \) are fixed biases in \( X \).
- \( e_1, e_2, y_1, \) and \( y_2 \) all in \( X_e \) are outputs.

Fig. 3. Block diagram of single-loop system.
For stability purposes interest is focused on the relations which connect the input $x$ to each of the outputs $e_1, e_2, y_1$ and $y_2$. These relations are designated by $E_1, E_2, F_1$ and $F_2$, respectively. $E_1$ is defined by

$$E_1 = \{(x,e_1):(x,e_1) \in X_e \times X_e \text{ and there exists } e_2, y_1, y_2, H_1 e_1, \text{ and } H_2 e_2 \text{ such that (1) is true}\}.$$  

The relations $E_2, F_1,$ and $F_2$ are similarly defined.

From earlier discussion it is seen that the biases $w_1$ and $w_2$ can be used to account for initial condition responses.

At this point it is interesting to observe how the use of a relation as the basic system model makes it possible to avoid questions of existence and uniqueness of solutions. Examining $E_1$, for instance, it is clear the domain of $E_1$ is not required to be the entire $X_e$ space. Hence, it is not required that there exist a solution corresponding to each input. Further, for an input $x$ which is in the domain of $E_1$, it is not required the corresponding $e_1$ be unique.

Certainly in most problems it is desired that there exist unique solutions. However, it may be extremely difficult to mathematically determine this. By formulating the problem in terms of relations this poses no difficulty for the stability analysis. This sort of situation occurs, for instance, if the relation $H_2$ is a hysterises nonlinearity and the relation $H_1$ is linear, time-invariant, and modeled by a set of ordinary differential equations.
Another reasonable approach to the stability problem is to use an operator as the basic system model. Since an operator is single-valued and has the entire $X_e$ space as domain, this approach requires the a priori assumption of existence of unique solutions.

Now the notion of gain is made precise for a relation.

**Definition:** For a relation $H$ with all $(Hx)_t = 0$ whenever $x_t = 0$, $x \in \text{Do}(H)$, and $t \in T$, the gain is defined by

$$g(H) = \sup \frac{|| (Hx)_t ||}{|| x_t ||}$$

where the supremum is taken over all $x \in \text{Do}(H)$, all $Hx \in \text{Ra}(H)$, and all $t \in T$ for which $x_t \neq 0$.

The following inequalities are obtained directly from the definition of gain and are crucial to proofs of stability theorems:

$$||(Hx)_t|| \leq g(H) ||x_t|| \text{ for } x \in \text{Do}(H) \text{ and } t \in T,$$

$$||Hx||_e \leq g(H) ||x||_e \text{ for } x \in \text{Do}(H).$$

The second follows from the first on letting $t \to \infty$.

Now a theorem proven by Zames [15] which provides boundedness conditions for the single loop system is presented.

**Theorem 3:** The relations $E_1$, $E_2$, $F_1$, and $F_2$ associated with the single-loop system are bounded if $g(H_1)g(H_2) < 1$. 
The incremental counterpart of the definition of gain is supplied by the following definition.

**Definition:** For a relation $H$ with all $(Hx-Hy)_t = 0$ whenever $(x-y)_t = 0$, $x$ and $y \in Do(H)$, and $t \in T$, the incremental gain is defined by

$$g(H) = \sup \frac{|| (Hx-Hy)_t ||}{|| (x-y)_t ||}$$

where the supremum is taken over all $x, y \in Do(H)$, all $Hx, Hy \in Ra(H)$, and all $t \in T$ for which $(x-y)_t \neq 0$.

The following inequalities similar to those for the nonincremental case are satisfied:

$$|| (Hx-Hy)_t || \leq g(H) || (x-y)_t || \text{ for } x \text{ and } y \in Do(H) \text{ and } t \in T,$$

$$|| Hx-Hy ||_e \leq g(H) || x-y ||_e \text{ for } x \text{ and } y \in Do(H).$$

Now a continuity theorem given in [15] is presented.

**Theorem 4:** The relations $E_1$, $E_2$, $F_1$, and $F_2$ associated with the single-loop system are continuous if $g(H_1)g(H_2) < 1$.

It is often true that the problem can be presented in such a manner that zero is in the domains of both $H_1$ and $H_2$ and has a unique image of zero under both relations. In this situation the condition of Theorem 4 is also sufficient for boundedness. This is seen by setting $y = 0$ in the definition of incremental gain. Then it is clear that $g(H_1)g(H_2) < 1$ implies $g(H_1)g(H_2) < 1$. 
A certain transformation of the single-loop system results in a significant generalization of Theorems 3 and 4. It is found many more systems can be examined than at first appears possible. The effect of the transformation on stability conditions is to change them from gain restrictions to conicity restrictions. Following is the definition of a conic relation.

Definition: A relation H on $X_e$ is interior conic with center parameter c and radius parameter $r \geq 0$ if

$$||H(x)t - cx_t|| \leq r||x_t||$$

for all $x \in Do(H)$, all $H \in Ra(H)$, and all $t \in T$. H is exterior conic with center parameter c and radius parameter $r \geq 0$ if the above inequality is reversed.

If X is an inner product space another notation defined below can be employed to specify the nature of a conic relation. The definition of an inner product space is given in Appendix A.

Definition: Assume H is a relation on the extension of an inner product space. H is inside the sector $\{a,b\}$ if $a \leq b$ and

$$<(Hx)_t - ax_t, (Hx)_t - bx_t> \leq 0$$

for all $x \in Do(H)$, $H \in Ra(H)$, and $t \in T$. H is outside the sector $\{a,b\}$ if the inequality is reversed.

For the special case of an inner product space the specific correspondence between conicity conditions and sector conditions is indicated
by the following two statements. A relation \( H \) is interior (exterior) conic with center parameter \( c \) and radius parameter \( r \) if \( H \) is inside (outside) the sector \( [c-r, c+r) \). Conversely, a relation \( H \) is inside (outside) the sector \( [a,b) \) if \( H \) is interior (exterior) conic with center parameter \( \frac{1}{2}(a+b) \) and radius parameter \( \frac{1}{2}(b-a) \).

It is of interest to consider the situation where \( b \) goes to infinity in the definition of a sector. For \( a = 0 \) this limiting case is covered by the following definition of positivity.

Definition: A relation \( H \) on the extension of an inner product space is positive if

\[
\langle x^*_t, (Hx)_t \rangle \geq 0
\]

for all \( x \in \text{Dom}(H) \), all \( Hx \in \text{Ran}(H) \), and all \( t \in T \).

An incremental counterpart for each of the three preceding definitions is provided by the following definitions.

Definition: A relation \( H \) on \( X \) is incrementally interior conic with center parameter \( c \) and radius parameter \( r \geq 0 \) if

\[
|| (Hx-Hy)_t - c(x-y)_t || \leq r || (x-y)_t ||
\]

for all \( x, y \in \text{Dom}(H) \), all \( Hx, Hy \in \text{Ran}(H) \), and all \( t \in T \). \( H \) is incrementally exterior conic with center parameter \( c \) and radius parameter \( r \geq 0 \) if the above inequality is reversed.

Definition: Assume \( H \) is a relation on the extension of an inner product space. \( H \) is incrementally inside the sector \( [a,b) \) if \( a \leq b \) and
\[(Hx-Hy)_t - a(x-y)_t, (Hx-Hy)_t - b(x-y)_t \leq 0\]

for all \(x, y \in Do(H)\), all \(Hx, HyeRa(H)\), and all \(t \in T\). \(H\) is incrementally outside the sector \([a, b]\) if the above inequality is reversed.

**Definition:** Assume \(H\) is a relation on the extension of an inner product space. \(H\) is incrementally positive if

\[<(x-y)_t, (Hx-Hy)_t > \geq 0\]

for all \(x, y \in Do(H)\), all \(Hx, HyeRa(H)\), and all \(t \in T\).

It is easily found that for the special case of an inner product space the same type of correspondence exists between the incremental versions of conicity and sector conditions as for the nonincremental versions.

Now two theorems are presented which provide sufficient boundedness and continuity conditions phrased in terms of sector conditions.

**Theorem 5:** Let the open-loop relations \(H_\perp\) and \(H_2\) of the single-loop system be conic. Suppose for constants \(\gamma\) and \(\epsilon\) where one is positive and one is zero that

1. \(-H_2\) is inside the sector \([a+\gamma, b-\gamma]\) where \(b > 0\), and
2. \(H_\perp\) satisfies one of the following conditions:
   - Case 1a: If \(a > 0\) then \(H_\perp\) is outside the sector
     \[\left\{-\frac{1}{a} - \epsilon, -\frac{1}{b} + \epsilon\right\}.\]
Case 1b: If \( a < 0 \) then \( H_1 \) is inside the sector
\[
\left\{ -\frac{1}{b} + \varepsilon, -\frac{1}{a} - \varepsilon \right\}.
\]

Case 2: If \( a = 0 \) then \( H_1 + \left( \frac{1}{b} - \varepsilon \right) I \) is positive
and if \( \gamma = 0 \) then \( g(H_1) < \infty \).

Then the relations \( E_1, E_2, F_1, \) and \( F_2 \) associated with the single-loop system are each bounded.

Theorem 6: Suppose all hypotheses of Theorem 5 are replaced by their incremental counterparts. Then the relations \( E_1, E_2, F_1, \) and \( F_2 \) associated with the single-loop system are each continuous.

For the special case of an inner product space it is easily found that the gain theorems can be obtained from the sector theorems. To show this assume \( g(H_1)g(H_2) < 1 \). Theorem 5 can be utilized to find boundedness is implied. In this manner the results of Theorem 3 are obtained. First note that from
\[
\| (H_1x)_t \| \leq g(H_1) \| x_t \| \text{ for all } x \in \text{Do}(H_1), \text{ all } H_1x \in \text{Ra}(H_1), \text{ and all } t \in T
\]
it is inferred that \( H_1 \) is interior conic with center parameter zero and radius parameter \( g(H_1) \). Now this implies \( H_1 \) is inside \( \{-g(H_1), g(H_1)\} \). Similarly it is found that \( -H_2 \) is inside \( \{-g(H_2), g(H_2)\} \).

Now define \( \varepsilon \) by
\[
\varepsilon = \frac{1}{g(H_2)} - g(H_1).
\]
\( \varepsilon \) is positive since \( g(H_1)g(H_2) < 1 \).
This results in $H_1$ being inside $(-\frac{1}{g(H_2)} + \epsilon, \frac{1}{g(H_2)} - \epsilon)$. But setting $\gamma = 0$ in Theorem 5, it is seen from Case 1b that the relations $E_1, E_2, F_1, F_2$ associated with the single-loop system are bounded. Using similar reasoning it is found that Theorem 4 can be obtained from Theorem 6.

Interpretations of Sector Conditions

Several interpretations of sector conditions for particular types of relations are available. Here some results presented by Zames [15], [16] are discussed. For a certain class of linear time-invariant operators on $L_{2e}[0,\infty)$ it is found sector conditions can be phrased in terms of conditions imposed on the Nyquist diagram. In general, for any relation on $L_{2e}[0,\infty)$, it is found certain conditions having an interpretation in the output versus input plane are sufficient for satisfaction of sector conditions. These conditions referred to as instantaneous sector conditions find particular application to memoryless nonlinearities, nonlinearities which are time varying, and hysteresis nonlinearities.

First consider the class of linear time-invariant operators defined below.

Definition: $Q$ is the class of operators on $L_{2e}[0,\infty)$ satisfying an equation of the type

$$Hx(t) = h_\infty x(t) + \int_0^t h(t-v)x(v)dv$$
where \( h_\infty \) is a constant, the impulse response \( h \in L^1_\infty [0, \infty) \), and for some \( \sigma_0 < 0 \) the function \( h(t) e^{-\sigma_0 t} \) lies in \( L^1_\infty [0, \infty) \).

The Laplace transform of members of \( Q \) provides the means by which sector conditions can be interpreted in the complex plane.

Definition: The Laplace transform \( \tilde{H}(s) \) of \( H \in Q \) is given by

\[
\tilde{H}(s) = h_\infty + \int_0^\infty h(t) e^{-st} dt \text{ for } \sigma \geq 0.
\]

Of course any linear time-invariant system modeled by a set of ordinary differential equations possesses a Laplace transform as defined above. Further, this is regardless of whether or not the impulse response lies in \( L^1_\infty [0, \infty) \), but the transform may not be defined for all \( \sigma \geq 0 \). It is easily found that such a system has a corresponding integral equation in the class \( Q \) if and only if the poles of the Laplace transform lie strictly in the left half complex plane.

The following lemma proven in [16] phrases sector conditions in terms of the behavior of the Nyquist diagram in the complex plane.

Definition: The Nyquist diagram of \( H \in Q \) is the locus of \( \tilde{H}(j\omega) \) for \( \omega \in (-\infty, \infty) \).

Lemma 1: Let \( H \) be an operator in \( Q \) and let \( c \) and \( r > 0 \) be constants.

1. If \( \tilde{H}(s) \) satisfies the inequality

\[
|\tilde{H}(j\omega) - c| \leq r \text{ for all } \omega \in (-\infty, \infty),
\]

2. It is easily found that such a system has a corresponding integral equation in the class \( Q \) if and only if the poles of the Laplace transform lie strictly in the left half complex plane.
then $H$ is incrementally interior conic with center parameter $c$ and radius parameter $r$.

(2) If $\bar{H}(s)$ satisfies the inequality

$$|\bar{H}(j\omega) - c| \geq r \text{ for all } \omega \in (-\infty, \infty)$$

and if the Nyquist diagram of $H$ does not encircle the point $(c,0)$, then $H$ is incrementally exterior conic with center parameter $c$ and radius parameter $r$.

(3) If $\text{Re}\{ar{H}(j\omega)\} \geq 0$ for all $\omega \in (-\infty, \infty)$, then $H$ is incrementally positive.

Due to the linearity of relations in $Q$, the incremental and non-incremental sector conditions become equivalent. Hence, Lemma 1 is true in the nonincremental case also.

Now sector conditions are given an interpretation in the output versus input plane through the following definition of instantaneous sector conditions.

**Definition:** Assume $H$ is a relation on $L_2[0,\infty)$. Each of the following must be true for all $x \in \text{Do}(H)$, all $Hx \in \text{Ra}(H)$, and all $t \in T$.

1. $H$ is instantaneously inside the sector $\{a,b\}$ if

$$Hx(t) = 0 \text{ whenever } x(t) = 0 \text{ and if } a \leq \frac{Hx(t)}{x(t)} \leq b \text{ for } x(t) \neq 0.$$
(2) \( H \) is instantaneously outside the sector \( \{a,b\} \) if
\[
a \leq b \text{ and either } \frac{Hx(t)}{x(t)} \leq a \text{ or } \frac{Hx(t)}{x(t)} \geq b \text{ for } x(t) \neq 0.
\]

(3) \( H \) is instantaneously positive if \( x(t)Hx(t) \geq 0 \).

A graphical representation of the above conditions is provided by Fig. 4. It is easily seen that if the point \( (x(t), Hx(t)) \) always lies in the appropriate shaded region of the plane then the appropriate instantaneous sector condition is satisfied.

Of particular interest here is the memoryless nonlinearity defined below.

![Fig. 4. Interpretation of sector conditions in output versus input plane.](image)

(a) Inside \( \{a,b\} \).

(b) Outside \( \{a,b\} \).

(c) Positive.
Definition: A relation $H$ on $X^e$ is memoryless if there exists a real-valued function $N$ such that the equation $Hx(t) = N[x(t)]$ is always satisfied.

If $N$ is a function of $t$ also, then a time-varying nonlinearity results. Further, a hysteresis nonlinearity can be thought of as corresponding to a multiple-valued $N$. The above nonlinearities lend themselves particularly well to analysis in terms of instantaneous sector conditions.

The following lemma establishes the usefulness of the instantaneous conditions.

Lemma 2: If the relation $H$ on $L^2_{[0,\infty)}$ is instantaneously inside (outside) the sector $\{a,b\}$, then $H$ is inside (outside) the sector $\{a,b\}$. Also, an instantaneously positive relation is positive. Further, the converse of the above is true for a memoryless relation.

Incremental counterparts to the instantaneous sector conditions are provided by the following definition.

Definition: Assume $H$ is a relation on $L^2_{[0,\infty)}$. Each of the following statements must be true for all $x,y \in \text{Do}(H)$, all $Hx,Hy \in \text{Ra}(H)$, and all $t \in T$.

1. $H$ is instantaneously incrementally inside the sector $\{a,b\}$ if $Hx(t) = Hy(t)$ whenever $x(t) = y(t)$ and if $a \leq \frac{Hx(t)-Hy(t)}{x(t)-y(t)} \leq b$ for $x(t) \neq y(t)$.
(2) \( H \) is instantaneously incrementally outside the sector 
\( \{a, b\} \) if \( a \leq b \) and either 
\[
\frac{H_x(t) - H_y(t)}{x(t) - y(t)} \leq a \quad \text{or} \quad \frac{H_x(t) - H_y(t)}{x(t) - y(t)} \geq b \text{ for } x(t) \neq y(t).
\]

(3) \( H \) is instantaneously incrementally positive if
\[
[x(t) - y(t)][H_x(t) - H_y(t)] \geq 0.
\]

For a memoryless relation \( H \) on \( L_2^e[0,\infty) \), Fig. 5 provides an illustration of (1) and (2) above.

It is easily shown that \( H \) is instantaneously incrementally inside the sector \( \{a, b\} \) if for each point \( P \) of the graph of \( N \) the rest of the

Fig. 5. Incremental sector conditions in output versus input plane.
(a) Incrementally inside \( \{a, b\} \).
(b) Incrementally outside \( \{a, b\} \).
The following lemma is the incremental counterpart of Lemma 2.

**Lemma 3:** If all conditions of Lemma 2 are replaced by their incremental counterparts, then the lemma remains true.

The interpretations of sector conditions presented here result in a frequency-domain stability condition for a single-loop having an open loop composed of a linear relation in Q and a time-varying nonlinearity. By using Lemmas 1 and 2 in conjunction with Theorem 5, it is easily seen that essentially the same result is obtained as cited earlier due to Sandberg for the \( n = 1 \) case. In [16] Zames utilizes this result with a certain transformation to obtain the familiar Papov stability conditions. Further, these results can be extended to \( L_\infty \)-stability [14].
CHAPTER 4: MAIN RESULTS

New stability results obtained in a functional analysis setting are presented here. Specifically, conditions sufficient to guarantee boundedness and continuity of a multiple-loop nonlinear time-varying system are derived. First a precise mathematical model of a multiple-loop system is given in the form of a particular interconnection of relations. Then a boundedness theorem is presented which involves the interconnection structure and the gains of the relations interconnected. Next a particular transformation of a multiple-loop system is discussed. This leads to a generalization of boundedness results by allowing gain conditions to be replaced by sector conditions. Then a set of conditions are given which guarantee boundedness of a single-loop system. A system satisfying these conditions is referred to as having a margin of boundedness \( \delta \). It is found these conditions are useful for the analysis of multiple-loop systems. This chapter is concluded by presentation of continuity results obtained through application of boundedness results to a special system.

System Configuration

A portion of the block diagram of a multiple-loop system is shown in Fig. 6. The purpose of this figure is to indicate a multiple-loop system is an interconnection of relations each having an input composed of a general system input \( a_i x \) plus a fixed bias \( w_i \) plus a weighted sum of outputs of other relations.
In mathematical terms the model of a multiple-loop system is provided by the set of $m$ simultaneous functional equations

$$e_i = a_i x + w_i + \sum_{j=1}^{m} b_{ij} y_j$$  \hspace{1cm} \text{for } i = 1, 2, \ldots, m \hspace{1cm} (2a)$$

$$y_i = H_{i \cdot} e_i$$  \hspace{1cm} \text{for } i = 1, 2, \ldots, m \hspace{1cm} (2b)$$

where the following are true:

- Each $H_{i \cdot}$ is a relation on $X_e$.
- $x \in X_e$ is the system input.
Each $a_i$ is a constant.
Each $w_i$ is a fixed bias in $X$.
Each $b_{ij}$ is a constant.
Each $e_{i}\in X_e$ is a system output.
Each $y_{i}\in X_e$ is a system output.

It should be noted that just as in the case of a single-loop system the bias terms can be used to account for initial condition responses.

For purposes of stability investigations attention is focused on relations which connect the input $x$ with each of the outputs. $E_i$ connects $x$ with $e_i$, and $F_i$ connects $x$ with $y_i$. More precisely for $i = 1, 2, \ldots, m$

$$E_i = \{(x, e_i) : (x, e_i) \in X_e \times X_e \text{ and there exist } e_j \text{ for all } j \neq i, \quad y_j \text{ for all } j, \text{ and } H_{j}e_j \text{ for all } j \text{ all in } X_e \text{ such that equations (2) are satisfied}\}$$

and

$$F_i = \{(x, y_i) : (x, y_i) \in X_e \times X_e \text{ and there exist } e_j \text{ for all } j, \quad y_j \text{ for all } j \neq i, \text{ and } H_{j}e_j \text{ for all } j \text{ all in } X_e \text{ such that equations (2) are satisfied}\}.$$
A Gain Result

The following theorem gives sufficient conditions for boundedness of a multiple-loop system.

Theorem 7: All relations $E_i$ and $F_i$ associated with the multiple-loop system (2) are bounded if $g(H_i) < \infty$ for all $i$ and the successive principal minors of the matrix

$$I - [b_{ij}g(H_j)]$$

are all positive.

Remark 1: Gain enters into the proof of Theorem 7 only through the inequality $||(H_i x)_t|| \leq g(H_i)||x_t||$. Hence, it might as well be assumed that each $H_i$ is conic with center parameter zero and radius parameter $r_i$. If $X$ is an inner product space, this is equivalent to $H_i$ being inside the sector $\{-r_i, r_i\}$. In this situation the boundedness condition would be the successive principle minors of $I - [b_{ij}r_j]$ are all positive.

Proof of Theorem 7: It is sufficient to show that each relation $E_i$ is bounded since this implies each relation $F_i$ is also bounded. This follows from

$$||y_i||_e = ||H_i e_i||_e \leq g(H_i)||e_i||_e$$

and the condition that $g(H_i) < \infty$. Clearly $F_i$ is bounded if $E_i$ is bounded.
Now boundedness of $E_i$ is established for each $i$. Let $x, e_i, y_i,$ and $H_i e_i$ be functions in $X_e$ for each $i$ which satisfy equations (2).

Truncate equations (2a) at $t \in T$ giving for each $i$

$$e_{it} = a_i x_t + w_{it} + \sum_{j=1}^{m} b_{ij} y_{jt}.$$  

Noting all truncated functions lie in $X$, it is found that for each $i$

$$||e_{it}|| \leq |a_i| ||x_t|| + ||w_{it}|| + \sum_{j=1}^{m} |b_{ij}| ||y_{jt}||.$$  

From the definition of gain,

$$||e_{it}|| \leq |a_i| ||x_t|| + ||w_{it}|| + \sum_{j=1}^{m} |b_{ij}| |g(H_j)||e_{jt}||.$$  

Now translate this into matrix notation by using the following definitions:

$$e_t = [||e_{1t}||, ||e_{2t}||, \ldots, ||e_{mt}||]^T,$$

$$h = [|a_1|, |a_2|, \ldots, |a_m|]^T,$$

$$w_t = [|w_{1t}|, |w_{2t}|, \ldots, |w_{mt}|]^T.$$  

Then for all $t \in T$

$$e_t \leq h ||x_t|| + w_t + [b_{ij}|g(H_j)]e_t.$$  

This gives

$$(I - [b_{ij}|g(H_j)])e_t \leq h ||x_t|| + w_t.$$
Now clearly if \( (I - [b_{ij} g(H_j)])^{-1} \) exists and has all of its elements nonnegative, then for all \( t \in T \)

\[
e_t \leq (I - [b_{ij} g(H_j)])^{-1} \{ h||x_t|| + w_t \}.
\]

It is shown in Appendix B that the hypotheses of the theorem are sufficient to guarantee this. It follows that there exist constants \( f_i > 0 \) and \( k_{ij} > 0 \) such that for each \( i \) and for all \( t \in T \)

\[
||e_{it}|| \leq f_i ||x_t|| + \sum_{j=1}^{m} k_{ij} ||w_{jt}||.
\]

Now assume \( x \) belongs to a bounded subset of \( X_c \). Since each \( w_i \in X \) and since \( ||e_{it}||, ||x_t||, \) and \( ||w_{jt}|| \) are all nondecreasing functions of \( t \), letting \( t \to \infty \) for each \( i \) implies

\[
||e_i|| \leq f_i ||x|| + \sum_{j=1}^{m} k_{ij} ||w_j||.
\]

Since the \( w_j \) are fixed and since there exists a constant \( D \) such that \( ||x|| < D \), it is clear that each relation \( E_i \) is bounded.

A system composed of a single loop of \( m \) relations in cascade is a special case of a multiple-loop system. Theorem 7 provides interesting boundedness conditions for such a system. The \( B \) matrix is founded to be of the form
\[ B = \begin{bmatrix}
0 & 0 & \ldots & 0 & b_{1m} \\
b_{21} & 0 & \ldots & 0 & 0 \\
0 & b_{32} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & b_{m,m-1} & 0
\end{bmatrix}. \]

From this it follows that

\[ I - \begin{bmatrix} |b_{1j}|g(H_j) \end{bmatrix} = \begin{bmatrix}
1 & 0 & \ldots & 0 & -|b_{1m}|g(H_m) \\
-|b_{21}|g(H_1) & 1 & \ldots & 0 & 0 \\
0 & -|b_{32}|g(H_2) & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & -|b_{m,m-1}|g(H_{m-1}) & 1
\end{bmatrix}. \]

It is easily found the first \( m-1 \) successive principal minors of the above matrix are each unity. The boundedness condition which comes from the last successive principal minor is

\[ |b_{1m}|g(H_1)|b_{21}|g(H_2)|b_{32}|g(H_3)|\ldots|b_{m,m-1}|g(H_{m-1}) < 1. \]

Hence, it is found even for a single loop of several relations that an open loop gain product less than unity implies boundedness. Certainly Theorem 3 of Chapter 3 is a special case of the above.

It is worth-while to note from the proof of Theorem 7 that specific bounds on system outputs can be found in terms of a bound on the system input \( x \). Hence, if quantitative information is desired the theory is capable of providing it.
For a system found to be bounded from Theorem 7, it is seen the system remaining after removal of any relation $H_i$ is also bounded. To show this assume the hypotheses of Theorem 7 are satisfied. From Theorem 11, stated in Appendix C, it is implied that all principal minors of the matrix $I - [|b_{ij}|g(H_j)]$ are positive. Now removal of the $i^{th}$ relation is equivalent to deletion of the $i^{th}$ row and column of this matrix. But this leaves a matrix which has all principal minors positive. Hence, the system with the $i^{th}$ relation removed is also bounded by Theorem 7.

In certain situations it is desired that stability be retained even if part of the system is disconnected. For such situations Theorem 7 is particularly well adapted. However, it would obviously be useless to try and use this theorem directly in the design of feedback compensation.

Transforming the System

A transformation of a multiple-loop system is discussed here which allows Theorem 7 to find much wider application than at first appears possible. From Remark 1 it is seen if $X$ is an inner product space that boundedness conditions obtained from Theorem 7 require each $H_i$ to be inside a symmetric sector $\{-r_i, r_i\}$. Through a transformation a boundedness theorem can be derived having hypotheses which require each $H_i$ either to be inside or outside a particular sector. Clearly the latter boundedness results encompass a wider variety of situations.
The basic approach employed here is to develop a transformed system of the same form as (2) having a set of solutions which contains the set of solutions of (2). Then boundedness of the transformed system implies boundedness of system (2). It is then found if an appropriate sector condition is satisfied by each \( H_i \) that the conditions of Theorem 7 are satisfied for the transformed system. These sector conditions then guarantee boundedness of the set of equations (2).

Now equations (2a) are placed in a matrix format by making the following definitions:

\[
\begin{align*}
e &= [e_1, e_2, \ldots, e_m]^T, \\
a &= [a_1, a_2, \ldots, a_m]^T, \\
w &= [w_1, w_2, \ldots, w_m]^T, \\
y &= [y_1, y_2, \ldots, y_m]^T, \\
B &= [b_{ij}].
\end{align*}
\]

This results in the following equations:

\[
\begin{align*}
e &= ax + w + By, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (2a) \\
y_i &= H_i e_i \quad \text{for } i = 1, 2, \ldots, m. \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (2b)
\end{align*}
\]

Now the equations of a multiple-loop system referred to as the transformed system are given. First let \( A \) and \( C \) be disjoint subsets of the real line such that \( A \cup C = \{1, 2, \ldots, m\} \). Then for each
\( i \in A \cup C \) pick constants \( d_i \) and \( c_i \) such that \( d_i = 0 \) if \( i \notin A \) and \( c_i = 0 \) if \( i \notin C \). Next define the relation \( H_i' \) by

\[
H_i' = \begin{cases} 
H_i + d_i I & \text{if } i \in A \\
(H_i - 1 + c_i I)^{-1} & \text{if } i \in C 
\end{cases}
\]

Now assume the inverse of the matrix \( I + B [\text{diag } d_i] \) exists and make the following definitions:

\[
\begin{align*}
a' &= (I + B [\text{diag } d_i])^{-1} a, \\
w' &= (I + B [\text{diag } d_i])^{-1} w, \\
B' &= (I + B [\text{diag } d_i])^{-1} (B + [\text{diag } c_i]), \\
e' &= \begin{bmatrix} e_1' & e_2' & \ldots & e_m' \end{bmatrix}^T, \\
y' &= \begin{bmatrix} y_1' & y_2' & \ldots & y_m' \end{bmatrix}^T.
\end{align*}
\]

The equations of the transformed system corresponding to the system modeled by equations (2) are then given by the following:

\[
\begin{align*}
e' &= a' x + w' + B' y', & \text{(3a)} \\
y_i' &= H_i' e_i' \quad \text{for } i = 1, 2, \ldots, m. \quad \text{(3b)}
\end{align*}
\]

Clearly these equations are of the same form as equations (2).

Now assume \( x, e_i, y_i, \) and \( H_i e_i \) are functions in \( X_e \) for each \( i \) such that equations (2) are satisfied. It is shown now that this solution for equations (2) can be used to find a solution for equations (3). Imagine placing a feedback of \(-c_i I\) or a feed-forward of \(d_i I\)
around each relation $H_i$ in Fig. 6. This is illustrated in Fig. 7. By using standard block diagram manipulations to adjust the interconnection, essentially the same system is retained.

Fig. 7. Feedback and feed-forward around relations of Fig. 6.
(a) $i \in C$.
(b) $i \in A$.

Now the primed inputs and outputs of the single-loop systems of Fig. 7 are shown to satisfy equations (3). Specifically, these primed inputs and outputs are defined by the equations $e_i' = e_i + c_i y_i$ and $y_i' = y_i + d_i e_i$ for each $i$. In a matrix format this becomes

$$
\begin{bmatrix}
e' \\
y'
\end{bmatrix} = \begin{bmatrix} I & [\text{diag } c_i] \\
[\text{diag } d_i] & I
\end{bmatrix} \begin{bmatrix} e \\
y
\end{bmatrix}.
$$

Multiplication by the inverse matrix gives both $e$ and $y$ in terms of $e'$ and $y'$ through the equation

$$
\begin{bmatrix}
e \\
y
\end{bmatrix} = \begin{bmatrix} I & -[\text{diag } c_i] \\
-[\text{diag } d_i] & I
\end{bmatrix} \begin{bmatrix} e' \\
y'
\end{bmatrix}.
$$
Substituting for $e$ and $y$ in equation (2a) gives the equation

$$e' - [\text{diag } c_i]y' = ax + w + B(-[\text{diag } d_i]e' + y').$$

After rearranging this equation, it is found

$$(I + B[\text{diag } d_i])e' = ax + w + (B + [\text{diag } c_i])y'.$$

Multiplication by $(I + B[\text{diag } d_i])^{-1}$ gives

$$e' = a'x + w' + B'y'.$$

Hence, (3a) is satisfied.

Now assume $i \in A$. Then $y'_i = y_i + d_i e_i = H_i e_i + d_i e_i$. Thus, there exists $(H_i + d_i I)e_i$ such that $y'_i = (H_i + d_i I)e_i$. Since $H_i' = H_i + d_i I$ and $e_i' = e_i$, there exists $H_i' e_i'$ such that $y'_i = H_i' e_i'$. For this situation then (3b) is satisfied.

Now assume $i \notin C$. Then $y_i = H_i e_i = H_i (e_i' - c_i y_i)$. This implies there exists $H_i^{-1} y_i$ such that $H_i^{-1} y_i = e_i' - c_i y_i$. This means there exists $(H_i^{-1} + c_i I) y_i$ such that $(H_i^{-1} + c_i I) y_i = e_i'$. But this in turn implies there exists $(H_i^{-1} + c_i I)^{-1} e_i'$ such that $(H_i^{-1} + c_i I)^{-1} e_i' = y_i$. Since $H_i' = (H_i^{-1} + c_i I)^{-1}$ and $y'_i = y_i$, it is then known there exists $H_i' e_i'$ such that $y'_i = H_i' e_i'$. Thus, (3b) is also satisfied in this situation.

Hence, it has been shown that for each solution of equations (2) there is a corresponding solution of equations (3). In this sense the set of solutions of (2) is a subset of the set of solutions of (3). From this it is deduced that boundedness of the multiple-loop transformed system (3) implies boundedness of the multiple-loop system (2).
Using the obvious definitions for the relations $E_i'$ and $F_i'$ associated with the multiple-loop system (3), assume each $E_i'$ and $F_i'$ is bounded. Let $x$ belong to a bounded subset of $X_e$, and let $x, e_i, y_i,$ and $H_{i1}$ be functions in $X_e$ for each $i$ which satisfy equations (2). There exists a solution of equations (3) with $e_i' = e_i + c_i y_i$ and $y_i' = y_i + d_i e_i$. This leads to the equations $e_i = e_i' - c_i y_i'$ and $y_i = y_i' - d_i e_i'$. But $e_i'$ and $y_i'$ belong to bounded subsets of $X_e$ for each $i$. Hence, $$|e_i| \leq |e_i'| + |c_i| |y_i'|$$ and $$|y_i| \leq |y_i'| + |d_i| |e_i'|.$$ From this it is clear each $E_i$ and $F_i$ associated with the multiple-loop system (2) is bounded.

The following example illustrates the transformation through the use of block diagrams.

Example 1: A multiple-loop system comprised of three relations is shown in Fig. 8. The transformed system is represented in Fig. 9 where the primed blocks can be envisioned in terms of the single-loop systems shown.

The constants $c_3, d_1, \text{and } d_2$ are set equal to zero. The $B$ matrix is given by

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix}.$$ 

This results in
Fig. 8. Multiple-loop system of Example 1.
Fig. 9. Transformed system for Example 1.
\[(I + B[\text{diag } d_1])^{-1} = \begin{bmatrix} 1 & 0 & -b_{13}d_3 \\ 0 & 1 & -b_{23}d_3 \\ 0 & 0 & 1 \end{bmatrix} \]

Hence,

\[
a' = \begin{bmatrix} 1 & 0 & -b_{13}d_3 \\ 0 & 1 & -b_{23}d_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 - b_{13}d_3a_3 \\ a_2 - b_{23}d_3a_3 \\ a_3 \end{bmatrix},
\]

\[
w' = \begin{bmatrix} w_1 - b_{13}d_3w_3 \\ w_2 - b_{23}d_3w_3 \\ w_3 \end{bmatrix},
\]

and

\[
B' = \begin{bmatrix} 1 & 0 & -b_{13}d_3 \\ 0 & 1 & -b_{23}d_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & b_{12} & b_{13} \\ b_{21} & c_2 & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix} = \begin{bmatrix} c_1 - b_{13}d_3b_{31} & b_{12} - b_{13}d_3b_{32} & b_{13} \\ b_{21} - b_{23}d_3b_{31} & c_2 - b_{23}d_3b_{32} & b_{23} \\ b_{31} & b_{32} & 0 \end{bmatrix}.
\]

Also, of course, \(H_1' = (H_1^{-1} + c_1I)^{-1}\), \(H_2' = (H_2^{-1} + c_2I)^{-1}\), and \(H_3' = H_3 + d_3I\). An examination of the two figures shows one can be obtained from the other by standard block diagram manipulations.
A Sector Result

For purposes of convenience slightly different notation is used here when specifying the nature of a conic relation. The incremental counterpart of this notation is obvious.

Definition: A relation \( H \) is conic with constants \((a,b)\) if for \( a \leq b \) the relation is inside \( \{a,b\} \) and for \( a > b \) the relation is outside \( \{b,a\} \).

The following theorem guarantees boundedness of a multiple-loop system (2) if certain sector conditions are satisfied. Note \( X \) is assumed to be an inner product space.

Theorem 8: Suppose for each \( i \) that \( H_i \) is conic with constants \((a_i,b_i)\) where \( a_i \neq b_i \). Let \( A \) and \( C \) be disjoint subsets of the real line such that \( A \cup C = \{1,2,\ldots,m\} \). Define the constants \( d_i \) and \( c_i \) by

\[
d_i = \begin{cases} \frac{-1}{2}(b_i + a_i) & \text{if } i \in A \\ 0 & \text{if } i \in C \end{cases}
\]

and

\[
c_i = \begin{cases} 0 & \text{if } i \in A \\ \frac{b_i + a_i}{2b_ia_i} & \text{if } i \in C \end{cases}
\]

Let the matrix \( B' \) be specified by

\[
B' = [b_{ij}'] = (I + B[\text{diag } d_i])^{-1}(B + [\text{diag } c_i])
\]
where it is assumed the indicated inverse exists. Further, define
\[ \eta_i > 0 \] by
\[ \eta_i = \begin{cases} \frac{1}{2} (b_i - a_i) & \text{if } i \in A \\ - \left( \frac{2b_i a_i}{b_i - a_i} \right) & \text{if } i \in C \end{cases} \]

Now if the successive principal minors of the matrix
\[ I - [b_{ij}]_{i,j} \]
are all positive then all relations \( E_i \) and \( F_i \) associated with the multiple-loop system (2) are bounded.

Remark 2: The limiting cases for \( i \in C \) of \( b_i \to \infty \) or \( a_i \to -\infty \) can be rigorously dealt with as explained in Appendix D. For \( b_i \to \infty \) the theorem remains true if the hypotheses are changed to read \( H_i - a_i I \) is positive and the definitions of \( c_i \) and \( \eta_i \) are changed to \( c_i = -\frac{1}{2a_i} \) and \( \eta_i = -2a_i \). Similarly for \( a_i \to -\infty \) the hypotheses are changed to \( -H_i + b_i I \) is positive and the definitions of \( c_i \) and \( \eta_i \) become \( c_i = -\frac{1}{2b_i} \) and \( \eta_i = 2b_i \).

Proof of Theorem 8: Consider the transformed system (3) associated with system (2) for the constants \( d_i \) and \( c_i \) defined in Theorem 8. It is shown in Appendix C that the relation \( H_i \) being conic with constants \( (a_i, b_i) \) for each \( i \) is sufficient to guarantee \( H_i \)' is inside \( \{-\eta_i, \eta_i\} \). The equations of the transformed system are of the same form as equations (2). Also, from the definition of \( w' \), it is seen that since \( w_i \in X \) for
all $i$ that $w_i \in \mathcal{X}$ for all $i$. Hence, applying Theorem 7, the transformed system is found to be bounded since the successive principal minors of $I - [|b_{ij}| |n_j|]$ are all positive. But this implies system (2) is bounded.

The restriction in the theorem that $n_i > 0$ imposes interesting constraints on $a_i$ and $b_i$. For $i \in A$ it is implied that $a_i < b_i$. If $i \in C$, then either $a_i < 0$ or $b_i > 0$. Within this constraint all situations are acceptable except for $b_i > a_i > 0$ and $a_i < b_i < 0$. This is illustrated in Fig. 10 for the special case of instantaneous sector conditions discussed in Chapter 3.

An illustration of Theorem 8 is provided by considering a multiple-loop system formed from the interconnection of linear time-invariant operators in $Q$ with time-varying nonlinearities. If boundedness conditions are available from Theorem 8, they take the form of conicity requirements on the operators in $Q$ and the time-varying nonlinearities. Under these conicity conditions inputs from a bounded subset of the $L_2$ space correspond to outputs in bounded subsets of the $L_2$ space. Assume the $i^{th}$ time-varying nonlinearity is a relation $H_i$ on $L_2 e[0,\infty)$ which satisfies the equation $H_i x(t) = N_i[x(t),t]$ where $N_i$ is a real-valued function. Then, assuming $i \in A$, the conicity requirement that $H_i$ be conic with constants $(a_i, b_i)$ is satisfied if the following instantaneous conditions are true:

\[
a_i \leq \frac{N_i(x,t)}{x} \leq b_i \text{ for all } x \neq 0 \text{ and all } t \in [0,\infty),
\]

\[
N_i(0,t) = 0 \text{ for all } t \in [0,\infty).
\]
Fig. 10. Instantaneous form of conditions imposed by Theorem 8.

(a) Typical if $i \in A$.  
(b) Never possible if $i \in C$.  
(c) Never possible if $i \in C$.  
(d) Never possible for any $i$.

This follows from Lemma 2 of Chapter 3. The conicity requirements on
the operators in $Q$ are given an interpretation in the complex plane by
Lemma 1 of Chapter 3. The $i^{th}$ linear time-invariant operator is conic
with constants $(a_i, b_i)$ if one of the following is true:

1. For $a_i < b_i$ the Nyquist diagram of the operator
   lies inside the circle in the complex plane which
intersects the real axis at the points \((a_i, 0)\) and \((b_i, 0)\).

(2) For \(a_i > b_i\), the Nyquist diagram lies outside the circle in the complex plane which intersects the real axis at the points \((b_i, 0)\) and \((a_i, 0)\). Further, the Nyquist diagram does not encircle the point \(\left(\frac{1}{2}(b_i + a_i), 0\right)\).

The above stability results and those of Theorem 1 due to Sandberg apply to the same general type of system. However, the problem formulation is different since the system dealt with in Theorem 1 is cast in the form of a single loop. Despite this, the only significant difference between the results is in the conditions placed on the linear time-invariant parts. In Theorem 1 a single condition is given involving the supremum over \(\omega\) of the positive square root of the maximum eigenvalue of a matrix which is a function of \(\omega\). Using Theorem 8 results in several conditions involving Nyquist diagrams. This illustrates the fact that a single-loop approach as compared with a multiple-loop approach results in fewer stability conditions which are in general more difficult to verify.

As for Theorem 7, satisfaction of the conditions of Theorem 8 allows specific bounds on system outputs to be found in terms of a bound on the input. This is seen by assuming the conditions of Theorem 8 are satisfied and referring to the discussion of the transformation employed in the proof of the theorem. For each solution of (2) there exists a corresponding primed solution of the appropriate
system (3). Assume $\mathbf{x} \in \mathbf{X}$. Then it is known $||e_i|| \leq ||e_i'|| + |c_i| ||y_i'||$. Since $\mathbf{H}_i$ is inside $\{-\eta_1, \eta_1\}$, it is found $||y_i'|| = ||\mathbf{H}_i' e_i'|| \leq \eta_1 ||e_i'||$. This leads to $||e_i|| \leq (1 + |c_i| \eta_1) ||e_i'||$.

Now the conditions of Theorem 7 are satisfied for system (3). Thus, from the latter portion of the proof of Theorem 7, it is seen constants $f_i' > 0$ and $k_{ij}' > 0$ can be calculated such that

$$||e_i|| \leq (1 + |c_i| \eta_1)(f_i'||x|| + \sum_{j=1}^{m} k_{ij}' ||w_j'||).$$

Similarly, it is found a bound on $||y_i||$ can be found in terms of $||x||$.

The theory is found to be capable of providing a feeling for the "degree of stability" possessed by a system. Assuming first that the conditions of Theorem 8 are satisfied, it is clear that making restrictions on system parameters more stringent results in tighter bounds on system responses. In this sense the margin by which boundedness conditions are satisfied is a measure of "how stable" a system is.

A Margin of Boundedness $\delta$

The following defines a condition on a single-loop system which is later found to be helpful in the application of Theorem 8 to multiple-loop systems.

Definition: The single-loop system (1) possessing open-loop relations $\mathbf{H}_1$ and $\mathbf{H}_2$ has a margin of boundedness $\delta$ if one of the following is true for some $0 < \delta < 1$:
Case la: $H_1$ is conic with constants $(a,b)$ where $b < a < 0$, and $-H_2$ is inside the sector

$$\left\{ -\frac{1}{b} - \delta\left(\frac{b-a}{2ba}\right), -\frac{1}{a} + \delta\left(\frac{b-a}{2ba}\right) \right\}.$$  

Case lb: $H_1$ is conic with constants $(a,b)$ where $a < 0$ and $b > 0$. Further, $-H_2$ is inside the sector

$$\left\{ -\frac{1}{b} - \delta\left(\frac{b-a}{2ba}\right), -\frac{1}{a} + \delta\left(\frac{b-a}{2ba}\right) \right\}.$$  

Case 2: $H_1 - aI$ is positive where $a < 0$, and $-H_2$ is inside the sector $\{-\delta\left(\frac{1}{2a}\right), -\frac{1}{a} + \delta\left(\frac{1}{2a}\right)\}$.

The above definition is given an instantaneous interpretation by Fig. 11 for relations on $L_{2e}[0,\infty)$. If each of the points $(x(t), H_1x(t))$ and $(x(t), -H_2x(t))$ always lie in the appropriate shaded regions in the figure, then the single-loop system has a margin of boundedness $\delta$. Actually, from the figure, it is seen that Case 2 can be obtained from Case la in the limit as $b \to -\infty$ or from Case lb in the limit as $b \to \infty$.

The motivation for the above definition is found to be that a single-loop system having a margin of boundedness $\delta$ satisfies the conditions of Theorem 8 within that margin. This is shown by noting for Cases la and lb that $H_1$ is conic with constants $(a_1,b_1)$ and $H_2$ is conic with constants $(a_2,b_2)$ where $a_1 = a$, $b_1 = b$, $a_2 = \frac{1}{a} - \delta\left(\frac{b-a}{2ba}\right)$, and $b_2 = \frac{1}{b} + \delta\left(\frac{b-a}{2ba}\right)$. The conicity of $H_2$ is inferred from the conicity of $-H_2$ by property (2) in Appendix D. In Case 2, $H_1 - a_1I$ is positive and $H_2$ is conic with constants $(a_2,b_2)$ where $a_1 = a$, $a_2 = \frac{1}{a} - \delta\left(\frac{1}{2a}\right)$, and
Fig. 11. Instantaneous conditions for a margin of boundedness $\delta$.
(a) Case la: $b < a < 0$. (b) Case lb: $b > 0$ and $a < 0$. (c) Case 2: $a < 0$. 
\( b_2 = \delta \left( \frac{1}{2a} \right) \). Now it is shown for each case that the conditions of Theorem 8 and Remark 2 are satisfied. Let \( A = \{2\} \) and \( C = \{1\} \). The matrix \( B \) is found to be

\[
B = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
\]

Then \( B' \) is easily found to be

\[
B' = \begin{bmatrix}
c_1 - d_2 & 1 \\
1 & 0
\end{bmatrix}.
\]

Now for Cases 1a and 1b

\[
c_1 - d_2 = -\left( \frac{b_1 + a_1}{2b_1 a_1} \right) + \frac{1}{2}(b_2 + a_2) = -\left( \frac{b+a}{2ba} \right) + \frac{1}{2}(\frac{1}{a} + \frac{1}{b}) = 0.
\]

Further, for Case 2

\[
c_1 - d_2 = -\frac{1}{2a_1} + \frac{1}{2}(b_2 + a_2) = -\frac{1}{2a} + \frac{1}{2}(\frac{1}{a}) = 0.
\]

Hence, \( B' = B \) indicating the feedback in the transformation is canceled by the feed-forward. Now it is found that

\[
I - [b_{ij}' \mid \eta_j] = \begin{bmatrix}
1 & -\eta_2 \\
-\eta_1 & 1
\end{bmatrix}
\]

so the single-loop system is bounded if

\[
1 - \eta_1 \eta_2 > 0.
\]

For Cases 1a and 1b
Further, for Case 2

\[ n_1 n_2 = -2a \left[ \frac{1}{2} (b_2 - a_2) \right] = -a \left[ -\frac{1}{a} + 2\delta \left( \frac{1}{2a} \right) \right] = 1 - \delta. \]

Therefore, in each case

\[ 1 - n_1 n_2 = \delta > 0. \]

Hence, the boundedness condition is satisfied within a margin \( \delta \) implying the single-loop system is bounded.

The problem which arises concerning use of Theorem 7 for the design of feedback compensation does not occur for Theorem 8. This is easily seen by considering the single-loop system with a margin of boundedness \( \delta \). Assume the conditions of Case 1a are satisfied. The system is then bounded by Theorem 8, but removal of the relation \( H_2 \) does not leave a system which is required to have a finite gain.

It can be shown by using the idea of a margin of boundedness \( \delta \) that new results presented here specialize to results presented by Zames [15]. Specifically, it is found that Theorem 5 of Chapter 3 is obtained by applying Theorem 8 of this chapter to a single-loop system. Suppose the conditions of Theorem 5 are satisfied for some \( \gamma > 0 \). The case \( \gamma = 0 \) need not be considered since this is shown by Zames to be a special case of \( \gamma > 0 \). Now compare each case of Theorem 5 with the corresponding case in the definition of a margin of boundedness \( \delta \). Clearly a \( \delta \) can be found so the system being examined has a margin of boundedness \( \delta \). But
this implies the system is bounded from Theorem 8. Hence, Theorem 8 can be utilized to prove Theorem 5.

Now two examples are given which indicate how the interconnection structure influences the final form of the stability conditions. In order that this influence might be most easily discerned, Theorem 7 is utilized rather than the more complicated Theorem 8. Then it is discussed how the form of the interconnection can be used as a guide for the application of Theorem 8. In this connection it is found the idea of a margin of boundedness $\delta$ is quite useful.

Example 2: Consider the multiple-loop system represented by the block diagram of Fig. 12 where $H_1$, $H_2$, and $H_3$ are all relations on $X_e$ of finite gain.

First the $B$ matrix is found to be

$$B = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$
This gives

\[
1 - \left| b_{1j} g(H_j) \right| = \begin{bmatrix}
1 & -g(H_2) & -g(H_3) \\
-g(H_1) & 1 & 0 \\
-g(H_1) & 0 & 1
\end{bmatrix}
\]

Then calculating the successive principal minors leads to the boundedness conditions:

\[
1 - g(H_1)g(H_2) > 0,
\]

\[
1 - g(H_1)g(H_2) - g(H_1)g(H_3) > 0.
\]

Since gains are positive, the first condition can be eliminated because it is implied by the second. The second condition also implies

\[
1 - g(H_1)g(H_3) > 0.
\]

From this it is clear that if the multiple-loop system of Fig. 12 is bounded from Theorem 7, then each subloop of this system has an open-loop gain less than unity. Hence, from Theorem 7 each subloop is bounded. However, the converse of this is not true unless each subloop satisfies the conditions of Theorem 7 within a certain margin. Define \( \delta_1 \) and \( \delta_2 \) by

\[
1 - g(H_1)g(H_2) = \delta_1
\]

and

\[
1 - g(H_1)g(H_3) = \delta_2.
\]

Then if \( \delta_1 + \delta_2 > 1 \) the multiple-loop system is bounded.
Example 3: The system examined here is shown in Fig. 13 where \( H_1, H_2, \) and \( H_3 \) are each relations on \( X_e \) of finite gain. The constants \( k_1 \) and \( k_2 \) are positive. The \( B \) matrix is easily found to be

\[
B = \begin{bmatrix}
0 & -1 & -1 \\
1 & 0 & k_1 \\
1 & k_2 & 0
\end{bmatrix}.
\]

Then it is seen that

\[
I - \begin{bmatrix}
|b_{ij}|g(H_j)
\end{bmatrix} = \begin{bmatrix}
1 & -g(H_2) & -g(H_3) \\
-g(H_1) & 1 & -k_1g(H_3) \\
-g(H_1) & -k_2g(H_2) & 1
\end{bmatrix}.
\]

Calculation of the successive principal minors of the above matrix gives the boundedness conditions:

\[
1 - g(H_1)g(H_2) > 0,
\]

\[
1 - g(H_1)g(H_2) - g(H_1)g(H_3) - k_1k_2g(H_3)g(H_2) - k_1g(H_1)g(H_2)g(H_3) + k_2g(H_1)g(H_2)g(H_3) > 0.
\]

The first condition can be eliminated since it is implied by the second. Actually the second condition implies each of the following:

\[
1 - g(H_1)g(H_2) > 0, \quad (4a)
\]

\[
1 - g(H_1)g(H_3) > 0, \quad (4b)
\]

\[
1 - k_1k_2g(H_3)g(H_2) > 0, \quad (4c)
\]
Fig. 13. Block diagram for Example 3.

\[ 1 - k_1 g(H_1) g(H_2) g(H_3) > 0, \quad (4d) \]
\[ 1 - k_2 g(H_1) g(H_2) g(H_3) > 0. \quad (4e) \]

Hence, it is clear that if the multiple-loop system of Fig. 13 is bounded from Theorem 7, then each subloop of the system has open-loop gain less than unity. Thus, from Theorem 7 each subloop is bounded. Again the converse of this is not true unless the subloops each satisfy the conditions of Theorem 7 within a certain margin. If the right hand side of each inequality in (4) is replaced with a positive number such that the sum of these numbers is greater than four, then the multiple-loop system of Fig. 13 is bounded.

The purpose of the above two examples is to indicate, that as far as Theorem 7 is concerned, boundedness conditions for a multiple-loop
system can often be stated in terms of margins by which subloops satisfy boundedness conditions. This information can be used to guide application of Theorem 8 since this theorem actually involves application of Theorem 7 to a transformed system. It can often be seen how boundedness conditions on subloops of the transformed system reflect back into the original system. Here the condition that a subloop of the original system has a margin of boundedness $\delta$ becomes useful. This is due to the fact that satisfaction of this condition implies a corresponding subloop of the transformed system satisfies a boundedness condition within a margin $\delta$. Boundedness conditions for the original system then involve $\delta$. This allows an organized approach and often reveals tradeoffs in conditions.

**Continuity**

By devising a system relating changes in inputs to changes in outputs, continuity results can be obtained almost directly from boundedness results. This approach produces the following theorem.

*Theorem 9:* If the conditions of Theorem 8 are replaced by their incremental counterparts, then all relations $E_i$ and $F_i$ associated with the multiple-loop system (2) are continuous.

*Remark 3:* The limiting cases of $b_i \to \infty$ and $a_i \to -\infty$ for $i \in \mathbb{C}$ are handled in the same manner as for Theorem 8 but by requiring incremental conditions to be satisfied.
Proof: First a system relating changes in inputs to changes in outputs is presented. Define the relation $G_i$ on $X_e$ by

$$G_i = \{(e,y) : \text{there exist } w \text{ and } v \text{ in } \text{Do}(H_i) \text{ such that } e = (w-v) \text{ and there exists } H^w_i \text{ and } H^v_i \text{ such that } y = H^w_i - H^v_i\}.$$ 

$G_i$ then relates changes in the input of $H_i$ to changes in its output. Consider the following system:

\begin{align*}
\hat{e}_i &= a_i \hat{x} + \sum_{j=1}^{m} b_{ij} \hat{y}_j \quad \text{for } i = 1, 2, \ldots, m, \quad (5a) \\
\hat{y}_i &= G_i \hat{e}_i \quad \text{for } i = 1, 2, \ldots, m. \quad (5b)
\end{align*}

Let $x_1$, $e_{i1}$, $y_{i1}$, and $H_i e_{i1}$ be functions in $X_e$ satisfying (2) for each $i$. Further, let $x_2$, $e_{i2}$, $y_{i2}$, and $H_i e_{i2}$ be functions in $X_e$ also satisfying (2) for each $i$. Now for $\hat{x} = x_1 - x_2$, $\hat{e}_i = e_{i1} - e_{i2}$, and $\hat{y}_i = y_{i1} - y_{i2}$ there exists $G_i \hat{e}_i$ in $X_e$ such that equations (5) are satisfied. This is easily seen by subtracting the equations corresponding to the input $x_2$ from those corresponding to the input $x_1$. The $w_i$ are eliminated since they are fixed regardless of the input. Hence, equations (5) relate changes in the input $\hat{x}$ to changes in the outputs $\hat{e}_i$ and $\hat{y}_i$ through the relations $\hat{E}_i$ and $\hat{F}_i$.

Assume the conditions of Theorem 9 are satisfied. $H_i$ is incrementally conic with constants $(a_i, b_i)$. This implies $G_i$ is conic with constants $(a_i, b_i)$. To see this let $e \in \text{Do}(G_i)$, $G_i e \in \text{Ra}(G_i)$, and $t \in T$. Then by the definition of $G_i$ there exists a $w$ and $v$ in $\text{Do}(H_i)$ such
that \( e = w - v \) and there exists \( H^w \) and \( H^v \) such that \( G^e = H^w - H^v \).

Therefore,

\[
<(G^e)_t - a^e_t, (G^e)_t - b^e_t> = <(H^w - H^v)_t - a'(w-v)_t, (H^w - H^v)_t - b'(w-v)_t>.
\]

Hence, incremental conicity requirements on \( H^i \) imply corresponding conicity requirements on \( G^i \).

Applying Theorem 8 to system (5), it is seen that \( \hat{e}_i^i \) and \( \hat{y}_i^i \) are bounded for all \( i \). This means bounded changes in the input produce only bounded changes in the output. This is close to what is desired. To actually obtain continuity, the proofs of Theorems 8 and 7 must be examined. The proof of Theorem 8 involves showing an appropriate transformed system (3) is bounded from Theorem 7. Denote inputs and outputs for the transformed system corresponding to (5) by \( \hat{e}_i^i \) and \( \hat{y}_i^i \). Since the \( w_i \) in equations (5) are zero, the equation for \( w' \) in equations (3) indicates \( w'_i \) in the transformed system is zero. Hence, from the last portion of the proof of Theorem 7, it is seen there exists \( f_i > 0 \) such that for each \( i \)

\[
||\hat{e}_i^i|| \leq f_i ||\hat{x}|| \quad \text{and} \quad ||\hat{y}_i^i|| \leq \eta_i f_i ||\hat{x}||
\]

for any solution of the transformed system with \( \hat{x} \in X \).

It was shown when discussing the transformation that for each solution of (5) there is a solution of the transformed system such that for each \( i \)
Thus, for each solution of (5) with \( \hat{x} \in \mathcal{X} \) it is seen

\[
|\tilde{e}_1| \leq (f_1 + |c_1|n_1f_1)|\hat{x}| \quad \text{and} \quad |\tilde{y}_1| \leq (n_1f_1 + |d_1|f_1)|\hat{x}|.
\]

Now returning to the beginning of the proof to the solutions corresponding to inputs \( x_1 \) and \( x_2 \), it is seen that for \( x_1 - x_2 \in \mathcal{X} \),

\[
|e_{11} - e_{12}| \leq (f_1 + |c_1|n_1f_1)|x_1 - x_2|
\]

and

\[
|y_{11} - y_{12}| \leq (n_1f_1 + |d_1|f_1)|x_1 - x_2|.
\]

Pick \( \varepsilon > 0 \). Let \( \delta_i = \frac{\varepsilon}{f_i + |c_i|n_i f_i} \). Then \( |x_1 - x_2| < \delta_i \) implies

\[
|E_i x_1 - E_i x_2| = |e_{11} - e_{12}| \leq (f_i + |c_i|n_i f_i)|x_1 - x_2| < \varepsilon.
\]

Hence, \( E_i \) is continuous for each \( i \). Similarly all \( F_i \) are continuous.

Now consider a multiple-loop system formed from the interconnection of linear time-invariant operators in \( \mathcal{Q} \) with memoryless nonlinearities. If Theorem 9 yields continuity conditions, they take the form of incremental conicity requirements. Then satisfaction of these requirements implies inputs arbitrarily close to each other in the \( L_2 \) space correspond to outputs arbitrarily close to each other in the \( L_2 \) space. Boundedness conditions were discussed earlier for a similar system in connection with Theorem 8. It is found the incremental conicity requirements on the operators in \( \mathcal{Q} \) have the same interpretation here as earlier. However, the requirements on the nonlinearities are...
interpreted differently. Assume the $i^{th}$ memoryless nonlinearity is a
relation $H_i$ on $L_2[0,\infty)$ which satisfies the equation $H_i x(t) = N_i[x(t)]$
where $N_i$ is a real-valued differentiable function. Then, assuming
$i \in \mathcal{A}$, the requirement that $H_i$ be incrementally conic with constants
$(a_i, b_i)$ is satisfied if $a_i \leq \frac{dN_i(x)}{dx} \leq b_i$ for all $x$. This is true by
Lemma 3 of Chapter 3.

From the latter portion of the proof of Theorem 9, it is clear
satisfaction of the hypotheses of the theorem actually implies more
than continuity. The theory is capable of providing quantitative
information in the form of specific bounds on deviations in system
outputs in terms of deviation in the system input. Under initial
satisfaction of the conditions of Theorem 9, a further tightening of
restrictions on system parameters clearly results in tighter bounds
on deviations in outputs. Thus, a similar situation exists as for
Theorem 8 in that the margin within which conditions of Theorem 9 are
satisfied can be viewed as a measure of "how stable" a system is.
CHAPTER 5: APPLICATIONS

In this chapter the stability of several systems is investigated through the use of Theorems 8 and 9. For each system examined, it is found the form the interconnection structure takes is a helpful guide for application of the theory. This is reflected by the fact that in each instance general stability conditions always require certain subloops to have a margin of boundedness \( \delta \).

For each system investigated, boundedness and continuity are interpreted in terms of the \( L_2 \) norm. This is done because analysis in terms of this norm allows results to be obtained in the most direct manner. This permits the emphasis to be placed on changes in the theory required to go from a single-loop to a multiple-loop system. Several extensions of the single-loop theory have been made which are interesting to examine relative to the multiple-loop theory. For instance, Zames [14] presents a theorem for \( L_\infty \)-boundedness which is comparable with Theorem 8 and has a frequency-domain interpretation for a single-loop with a linear part and a nonlinearity.

Due to the fact that the \( L_2 \) norm is used exclusively, it must be assumed in all cases that all system inputs and outputs are square integrable over any finite time-interval. From an engineering viewpoint this is a trivial restriction since it is almost always true for any physical system of interest.

Results presented in this chapter are relevant to the Lyapunov type of stability as well as the type of stability defined in a functional analysis setting. This seems reasonable from consideration of
a dynamical system for which a bounded set of inputs leads to a bounded set of outputs in the sense of the $L_2$ norm. Roughly speaking, a zero input then corresponds to an output which becomes small in the remote future regardless of initial conditions. But this is close to the idea of asymptotic stability. Willems [13] makes this more precise by proving that global asymptotic stability in the sense of Lyapunov results if the state space is accessible and observable in some sense.

The first system considered here is a particular interconnection of three specific linear time-invariant systems with three specific time-varying nonlinearities. This is followed by examination of a network possessing passive components. Finally, a particular nonlinear time-varying differential equation involving time delay is analyzed.

Example 1: Consider the multiple-loop system shown in Fig. 14. Application of Theorem 8 shows this system is bounded in the sense of the $L_2$ norm.

First a more detailed description of the system illustrated in the block diagram is given. Clearly this system is an interconnection of three linear time-invariant systems, a time-varying gain, a piecewise linear nonlinearity, and a hysteresis nonlinearity. Let $h(t)$ be the inverse Laplace transform of $\frac{s+20}{(s+1)(s+2)}$. Then the block labeled $\frac{s+20}{(s+1)(s+2)}$ represents a system having input $u$ and output $v$ which satisfy the integral equation.
Fig. 14. Multiple-loop system of Example 1.

\[ v(t) = z(t) + \int_0^t h(t-\tau)u(\tau)d\tau \]

for \( t \geq 0 \) where \( z(t) \) accounts for initial conditions. It is assumed \( z(t) \) lies in \( L_2[0,\infty) \). The blocks labeled \( \frac{s+2}{(s+1)(s+2)} \) and \( \frac{s+4}{(s+1)(s+2)(s+5)} \) represent systems modeled by similar integral equations having initial condition responses in \( L_2[0,\infty) \). The
assumption is made that each block in Fig. 14 has inputs and outputs which are square integrable over any finite time-interval.

Now consider any input or output of any block in Fig. 14. Boundedness implies a bound on the "size" of this input or output can be given in terms of a bound on the "size" of the input \( r \). For example, consider the output \( c \). Now for each \( D \) and each set of initial conditions a \( C \) can be calculated so that when a solution exists

\[
\int_0^\infty r(t)^2 \, dt < D
\]

implies

\[
\int_0^\infty c(t)^2 \, dt < C.
\]

This means, roughly speaking, that inputs which become small rapidly enough in the remote future lead to outputs which become small in the remote future regardless of initial conditions.

The following modification of the system in Fig. 14 is shown to be continuous by Theorem 9. Imagine replacing each of the nonlinearities of the system in Fig. 14 with a slope restricted nonlinearity. Specifically, replace the graphs of the piecewise linear nonlinearity and the hysteresis nonlinearity with, respectively, the real-valued differentiable functions \( N_1 \) and \( N_2 \). Assume the inequalities \( .53 \leq \frac{dN_1(x)}{dx} \leq 1.6 \)

and \( 0 \leq \frac{dN_2(x)}{dx} \leq .5 \) are satisfied for all \( x \).

Now consider any input or output of any block in the modified system of Fig. 14. Continuity implies a bound on the "size" of the
deviation in this input or output can be found in terms of a bound on
the "size" of the deviation in the input. Being more specific, consider
the output c. Let r and r' be two general system inputs for which under
identical initial conditions the, respective, outputs c and c' exist. Then
if
\[ \int_0^\infty [r(t) - r'(t)]^2 dt < \infty \]
a constant K can be calculated which is independent of initial conditions
and for which
\[ \int_0^\infty [c(t) - c'(t)]^2 dt \leq K \int_0^\infty [r(t) - r'(t)]^2 dt. \]
Loosely speaking, this means inputs which become close rapidly enough
in the remote future lead under identical initial conditions to outputs
which become close in the remote future. The fact that the system is
continuous also means the jump phenomenon cannot be displayed.

General stability results which specialize to those given above
can be obtained for the multiple-loop system of Fig. 15. These results
are found from application of Theorems 8 and 9 and are cast in a form
which reveal tradeoffs.

First consider in more detail the system depicted by Fig. 15.
The blocks labeled H_1, H_2, and H_3 are members of the class of linear
time-invariant operators Q. Zero-input responses of the systems
modeled by these operators are accounted for by the functions z_1, z_2,
and z_3 which are all assumed to be in L_2([0, \infty)). The blocks labeled
H_4, H_5, and H_6 represent time-varying nonlinearities which are modeled
by relations on L_2_e([0, \infty)). The constant k is assumed to be nonnegative.
This multiple-loop system clearly has equations of the form of (2)
where the input $r$ corresponds to $x$ and the $w_1$ functions are linear combinations of $z_1$, $z_2$, and $z_3$.

Use of Theorem 8 yields the following boundedness conditions. Assume the single-loop system possessing open-loop relations $H_1$ and $-H_4$ has a margin of boundedness $\delta$ where $H_1$ is conic with constants $(a_1, b_1)$. In other words, assume subloop 1 has a margin of boundedness $\delta$. This means $H_1$ is conic with constants $(a_1, b_1)$ and $H_4$ is inside $\left\{ -\frac{1}{b_1}, -\frac{b_1-a_1}{2b_1a_1} \right\}$ for some $0 < \delta < 1$ where either $b_1 < a_1 < 0$ or $a_1 < 0$ and $b_1 > 0$. Further, assume $H_2$ is inside $\{-b_2, b_2\}$, $H_3$ is inside $\{-b_3, b_3\}$, $H_5$ is inside $\{0, b_5\}$, and $H_6$ is inside $\{0, b_6\}$ where $b_2$, $b_3$, $b_5$, and $b_6$ are all positive. Then the system of Fig. 15 is bounded if

$$\delta(1 - b_2b_3b_5) + kb_2(b_3 + b_5) \frac{2b_1a_1}{b_1-a_1} > 0.$$
It is interesting to observe that a necessary condition for satisfaction of the above inequality is \(b_2b_3b_5 < 1\). Further, if this condition is satisfied a \(k \geq 0\) can always be found for which boundedness is guaranteed. This is particularly interesting since from Theorem 7 the condition \(b_2b_3b_5 < 1\) guarantees boundedness of subloop 2 comprised of relations \(H_2\), \(H_3\), and \(H_5\). Apparently if subloops 1 and 2 satisfy boundedness conditions and the coupling \(k\) between them is weak enough, then the entire multiple-loop system is bounded.

Now the boundedness conditions cited above for the system of Fig. 15 are employed to show the system of Fig. 14 is bounded. Making a comparison of the two figures it is seen the block labeled \(H_1\) corresponds with the block labeled \(\frac{s+20}{(s+1)(s+2)}\). Now the block in Fig. 14 is stated above to be modeled by an integral equation having Laplace transform \(\frac{s+20}{(s+1)(s+2)}\). Since the poles of this transform lie strictly in the left half plane, this system belongs to the class of linear operators \(Q\). Further, the initial condition response belongs to \(L_2[0,\infty)\). Similarly, the blocks labeled \(H_2\) and \(H_3\) correspond, respectively, with the blocks labeled \(\frac{4}{(s+2)(s+4)}\) and \(\frac{s+4}{(s+1)(s+2)(s+5)}\). Also the block labeled \(H_5\) corresponds with the time-varying gain \(.25(1-cost)\). Finally, the blocks labeled \(H_4\) and \(H_6\) correspond with, respectively, the piecewise linear nonlinearity and the hysteresis nonlinearity. Hence, with \(k = .5\) it is seen the system of Fig. 14 fits the form of the system shown in Fig. 15. Now let \(\overline{H}_1(s) = \frac{s+20}{(s+1)(s+2)}, \overline{H}_2(s) = \frac{4}{(s+2)(s+4)}, \overline{H}_3(s) = \frac{s+4}{(s+1)(s+2)(s+5)},\) the relation \(H_5\) satisfy the equation \(H_5x(t) = .25(1-cost)x(t),\) and the relations \(H_4\) and \(H_6\) be described by the graphs
shown in Fig. 14. From the Nyquist diagrams shown in Fig. 16, it is clear from Lemma 1 of Chapter 3 that \( H_1 \) is outside \([-5.33, -0.5]\) and both \( H_2 \) and \( H_3 \) are inside \([-0.5, 0.5]\). Further, it is clear from Lemma 2 of Chapter 3 that \( H_5 \) is inside \([0, 0.5]\), \( H_4 \) is inside \([0.53, 1.6]\), and \( H_6 \) is inside \([0, 0.5]\). Now pick \( \delta = 0.375 \). Then for \( a_1 = -0.5 \), \( b_1 = -5.33 \), \( b_2 = b_3 = 0.5 \), and \( b_5 = b_6 = 0.5 \)

\[
\delta(1 - b_2 b_3 b_5) + k b_2 (b_3 + b_6) \frac{2 b_4 a_4}{b_1 - a_1} > 0.
\]

Further,

\[
- \frac{1}{b_1} - \frac{b_1 - a_1}{2b_1 a_1} = 0.526
\]

and

\[
- \frac{1}{a_1} + \frac{b_1 - a_1}{2b_1 a_1} = 1.66.
\]

Hence, the boundedness conditions cited above for the system of Fig. 15 are satisfied by the system of Fig. 14.

Now consider the modification discussed above of the system of Fig. 14. The modified system is still of the form of Fig. 15 but with the relations \( H_4 \) and \( H_6 \) satisfying, respectively, the equations \( H_4 x(t) = N_1[x(t)] \) and \( H_6 x(t) = N_2[x(t)] \). This system is found to be continuous from Theorem 9 by showing the incremental counterparts of the conditions satisfied by the system of Fig. 14 are satisfied here. From Lemma 1, \( H_1 \) is incrementally outside \([-5.33, -0.5]\) and both \( H_2 \) and \( H_3 \) are incrementally inside \([-0.5, 0.5]\). Further, from Lemma 3 of Chapter 3, \( H_5 \) is incrementally inside \([0, 0.5]\), \( H_4 \) is incrementally inside \([0.53, 1.6]\) and \( H_6 \) is incrementally inside \([0, 0.5]\). Hence, the modified system is continuous.
Now the boundedness conditions stated above for the system of Fig. 15 are verified through the use of Theorem 8. Assume these conditions are satisfied. Then each $H_i$ is conic with constants $(a_i, b_i)$ where

$$a_4 = - \frac{1}{b_1} - \delta(\frac{b_1-a_1}{2b_1a_1}), \quad b_4 = - \frac{1}{a_1} + \delta(\frac{b_1-a_1}{2b_1a_1}), \quad a_2 = -b_2, \quad a_3 = -b_3,$$

and

$$a_5 = a_6 = 0.$$  

In Theorem 8 select $A = \{2, 3, 4, 5, 6\}$ and $C = \{1\}$. Then
\[ d_2 = d_3 = 0, \quad d_5 = -\frac{b_5}{2}, \quad \text{and} \quad d_6 = -\frac{b_6}{2}. \quad \text{Further,} \quad \eta_1 = -\frac{2b_1a_1}{b_1-a_1}, \]
\[ \eta_2 = b_2, \quad \eta_3 = b_3, \quad \eta_5 = \frac{b_5}{2}, \quad \text{and} \quad \eta_6 = \frac{b_6}{2}. \]

Now from the block diagram of Fig. 15 the B matrix is found to be

\[
B = \begin{bmatrix}
0 & 0 & -1 & -1 & 0 & -1 \\
k & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

It is easily found that

\[
(I + B[\text{diag}d])^{-1} = \begin{bmatrix}
1 & 0 & 0 & d_4 & 0 & d_6 \\
0 & 1 & 0 & 0 & d_5 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

From this, \(B'\) is calculated to be

\[
B' = (I + B[\text{diag}d])^{-1} \cdot (B + [\text{diag}c])^{-1} = \begin{bmatrix}
c_1 + d_4 & d_6 & -1 & -1 & 0 & -1 \\
k & 0 & d_5 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]
Due to the fact that the single-loop system possessing open-loop relations $H_1$ and $-H_4$ has a margin of boundedness $\delta$, it is easily found the element in the upper left hand corner of $B'$ is zero. Also $1 - n_1n_4 = \delta$.

Now further manipulations give

$$I - \left| \begin{array}{cccc} 1 & n_2d_6 & -n_3 & -n_4 & 0 & -n_b \\ -kn_1 & 1 & n_2d_5 & 0 & -n_5 & 0 \\ 0 & -n_2 & 1 & 0 & 0 & 0 \\ -n_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -n_3 & 0 & 1 & 0 \\ 0 & -n_2 & 0 & 0 & 0 & 1 \end{array} \right|.$$ 

Calculation of the successive principal minors leads to the following five inequalities whose satisfaction guarantees boundedness:

$$1 + kn_1n_2d_6 > 0,$$

$$(1 + n_2n_3d_5) + kn_1n_2(d_6 - n_3) > 0,$$

$$(1 - n_4n_1)(1 + n_2n_3d_5) + kn_1n_2(d_6 - n_3) > 0,$$

$$(1 - n_4n_1)(1 + 2n_2n_3d_5) + kn_1n_2(d_6 - n_3) > 0,$$

$$(1 - n_4n_1)(1 + 2n_2n_3d_5) + kn_1n_2(2d_6 - n_3) > 0.$$ 

Observing that $d_5$ and $d_6$ are negative, it is easily seen the last inequality implies all the ones preceding it. Using the expressions for the $n_i$ and $d_i$ and noting again $1 - n_1n_4 = \delta$, the last inequality above becomes
But this is exactly the condition assumed satisfied at the beginning. Hence, boundedness is obtained.

Example II: Boundedness conditions are given in this example for two types of systems which have the same equations in functional form. First a network formed from the interconnection of a time-varying nonlinear conductance, a time-varying nonlinear resistance, and two passive elements is considered. Then a system is examined which is an interconnection of two linear time-invariant systems and two nonlinearities.

The network to be examined is shown in Fig. 17. The voltages and currents labeled $e_i$ are considered to be inputs to the appropriate elements while the voltages and currents labeled $y_i$ are considered to be outputs. The element labeled $H_1$ has a current input and a voltage output while the element $H_2$ has a voltage input and a current output. Because of this, $H_1$ is referred to as an impedance element and $H_2$ is referred to as an admittance element. These two elements are assumed to be passive. This means for each element that if $i$ denotes the current through the element and $v$ denotes the voltage drop across the element then $\int_0^t i(\tau)v(\tau) d\tau \geq 0$ for all $t \geq 0$. The elements labeled $H_3$ and $H_4$ are, respectively, a time-varying nonlinear conductance and a time-varying nonlinear resistance. These elements are characterized by the functions $N_1$ and $N_2$ through the equations

$$y_3(t) = N_1[e_3(t), t]$$
Fig. 17. Network for Example II.

and

\[ y_4(t) = N_2[e_4(t), t]. \]

It is assumed for each element that all inputs and outputs are square integrable over any finite time-interval. Certainly from an engineering viewpoint this is a trivial assumption.

Now Theorem 8 can be utilized to obtain the following results.

Assume for each time \( t \) the graph of each of the functions \( N_1 \) and \( N_2 \) lies within the appropriate shaded region of Fig. 18 for some \( \epsilon > 0 \) and \( a_3 > 0 \) where \( b_3 \) and \( b_4 \) are arbitrarily large. Being more precise, assume there exist constants \( \epsilon > 0, a_3 > 0, b_3, \) and \( b_4 \) such that the following conditions are satisfied by the functions \( N_1 \) and \( N_2 \):

\[
\frac{N_1(x,t)}{x} \leq b_3 \text{ for all } x \neq 0 \text{ and all } t \epsilon [0,\infty),
\]

\[
N_1(0,t) = 0 \text{ for all } t \epsilon [0,\infty),
\]

\[
\frac{1}{a_3} + \epsilon \leq \frac{N_2(x,t)}{x} \leq b_4 \text{ for all } x \neq 0 \text{ and all } t \epsilon [0,\infty),
\]

\[
N_2(0,t) = 0 \text{ for all } t \epsilon [0,\infty).
\]
Then for each $e_i$ of the network corresponding to a current input $r$ with $\int_0^\infty r^2(t)dt < \infty$ a constant $K_i$ can be calculated so that

$$\int_0^\infty e_i^2(t)dt \leq K_i \int_0^\infty r^2(t)dt.$$  

Similarly for each $y_i$ corresponding to an input $r$ with $\int_0^\infty r^2(t)dt < \infty$ a constant $L_i$ can be calculated such that

$$\int_0^\infty y_i^2(t)dt \leq L_i \int_0^\infty r^2(t)dt.$$  

It is interesting to observe that by varying the constant $a_3$ the condition on element $H_3$ can be relaxed if a more stringent condition is placed on element $H_4$ and conversely. Hence, a tradeoff in conditions is revealed here.

Now consider the system shown in Fig. 19 formed from the interconnection of two linear systems with two nonlinearities. The nonlinearities are memoryless and characterized by the functions $N_1$ and $N_2$. 

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Fig. 18. Conditions on elements $H_3$ and $H_4$. (a) Nonlinear conductance. (b) Nonlinear resistance.
The constant gain $k$ in the outer feedback loop is nonnegative. Systems modeled by differential equations are represented by the blocks labeled $\overline{H}_1(s)$ and $\overline{H}_2(s)$. It is assumed these functions of $s$ are in the form

$$\overline{H}_1(s) = \frac{\sum_{j=0}^{n_1} c_{j1} s^j}{\sum_{j=0}^{n_1} d_{j1} s^j}$$

and

$$\overline{H}_2(s) = \frac{\sum_{j=0}^{n_2} c_{j2} s^j}{\sum_{j=0}^{n_2} d_{j2} s^j}$$

where $d_{n_11} \neq 0$ and $d_{n_22} \neq 0$. Then it is assumed the systems represented by the blocks labeled $\overline{H}_1(s)$ and $\overline{H}_2(s)$ each have input $u$ and output $v$ which satisfy, respectively, the differential equations...
\[ \sum_{j=0}^{n_1} d_{j1} v^{(j)} = \sum_{j=0}^{n_1} c_{j1} u^{(j)} \]

and

\[ \sum_{j=0}^{n_2} d_{j2} v^{(j)} = \sum_{j=0}^{n_2} c_{j2} u^{(j)} \]

The superscript denotes differentiation of that order. The two polynomials \( \sum_{j=0}^{n_1} d_{j1} s^j \) and \( \sum_{j=0}^{n_2} d_{j2} s^j \) each have all zeros strictly in the left half plane. The assumption is made, of course, that each block in Fig. 19 has all inputs and outputs square integrable over any finite time-interval.

Application of Theorem 8 produces the following. Assume there exist constants \( b_1 < a_1 < 0 \) and \( b_2 < a_2 < 0 \) such that the loci of \( H_1(j\omega) \) and \( H_2(j\omega) \) lie within the appropriate shaded regions of Fig. 20 for \( \omega \in (-\infty, \infty) \) and do not encircle the unshaded regions. Further, assume there exist constants \( 0 < \delta_1 < 1 \) and \( 0 < \delta_2 < 1 \) such that the graphs of the nonlinearities lie within the appropriate shaded region of Fig. 20. Now if

\[ \delta_1 \delta_2 > \frac{4b_1 a_1 b_2 a_2 k}{(b_1-a_1)(b_2-a_2)} , \]

then in the \( L_2 \) sense a bound on the "size" of any input or output of any block of Fig. 19 can be calculated in terms of a bound on the "size" of the input \( r \). Being more specific, consider the output \( c \) corresponding
Fig. 20. Graphical conditions imposed on components of system in Fig. 19.

to an input $r$. For each $D$ and each set of initial conditions there exists a $C$ such that

$$\int_0^\infty r^2(t) dt < D$$

implies

$$\int_0^\infty c(t)^2 dt < C.$$
Loosely speaking, this means an input \( r \) which becomes small rapidly enough in the remote future leads to an output \( c \) which becomes small in the remote future.

It is interesting to observe the conditions imposed on the system of Fig. 19 are in a form which reveal tradeoffs. For instance, if \( \delta_2 \) is decreased while increasing \( \delta_1 \) proportionately, then the conditions remain satisfied. This corresponds to relaxing the condition on the nonlinearity \( N_2 \) at the expense of the condition on the nonlinearity \( N_1 \).

Now boundedness results in the sense of the \( L_2 \) norm are presented for a set of simultaneous functional equations. These results can be shown to specialize to the results given above for the network of Fig. 17 and the system represented in Fig. 19. Consider the equations

\[
\begin{align*}
    e_1 &= r + w_1 - ky_2 - y_3 \\
    e_2 &= w_2 + y_1 - y_4 \\
    e_3 &= w_3 + y_1 \\
    e_4 &= w_4 + y_2 \\
    y_i &= H_i e_i \quad \text{for } i = 1, 2, 3, 4
\end{align*}
\]

where each \( H_i \) is a relation on \( L_2 e[0,\infty) \), each \( w_i \in L_2[0,\infty) \), and the constant \( k \) is nonnegative. These equations are clearly of the form of equations (2) where \( r \) corresponds to \( x \).

The block diagram corresponding to equations (6) is shown in Fig. 21. This system is clearly in a suitable form for application of Theorem 8. Let subloop 1 denote the loop having relations \( H_1 \) and \( H_3 \).
and subloop 2 denote the loop having relations $H_2$ and $H_4$. From the structure of the interconnection, it seems a reasonable approach is to assume subloop 1 has a margin of boundedness $\delta_1$ and subloop 2 has a margin of boundedness $\delta_2$. Then if the transformed system has the same structure, boundedness conditions will involve the additional loop corresponding to the loop containing $H_1$, $H_2$, and $k$.

Utilizing the above approach leads to the following results. Assume the single-loop system possessing open-loop relations $H_1$ and $-H_3$ has a margin of boundedness $\delta_1$. Further, assume the single-loop system possessing open-loop relations $H_2$ and $-H_4$ has a margin of boundedness $\delta_2$. Then boundedness is assured if

$$\delta_1 \delta_2 > n_1 n_2 k$$

where for $i = 1, 2$.
\[
\eta_i = \begin{cases} 
- \frac{2b_i a_i}{b_i - a_i} & \text{if } H_i \text{ is conic with} \\
\text{constants } (a_i, b_i) \\
- 2a_i & \text{if } H_i - a_i \text{ is positive}
\end{cases}
\]

In the transformed system the loop containing \(H_1', H_2', \) and \(k\) corresponds to the loop containing \(H_1, H_2,\) and \(k.\) This is found true later due to the fact \(B' = B.\) The \(\eta_1\) and \(\eta_2\) defined above are found from Theorem 8 to be such that \(H_1'\) is inside \(-\eta_1, \eta_1\) and \(H_2'\) is inside \(-\eta_2, \eta_2\). Hence, the influence of the loop involving \(H_1', H_2',\) and \(k\) is indicated by the term \(\eta_1 \eta_2 k\) in the above inequality.

For the special case \(k = 0\) the outer feedback loop is broken so boundedness of the entire system is implied by boundedness of subloops 1 and 2. The above boundedness conditions reflect this by becoming \(\delta_1 \delta_2 > 0.\) But this is already true from the definition of a margin of boundedness \(\delta.\) Effectively the above conditions reduce to requiring the conditions of Theorem 5 of Chapter 3 be satisfied for both subloop 1 and subloop 2.

Now the results presented for the network of Fig. 17 are found to be a special case of the boundedness results presented for equations (6). First observe that with each \(w_i\) set equal to zero and \(k = 1\) the equations of the network are of the same form as equations (6). Assumptions made on the circuit elements guarantee each \(H_i\) is a relation on \(L_{2e[0, \infty)}.\)

Now it is shown conditions satisfied by elements of the network guarantee boundedness. Since \(H_1\) and \(H_2\) are passive, both \(\int_0^t e_i(t) y_i(t) dt\)
and \( \int_0^t e_2(\tau)y_2(\tau)d\tau \) are nonnegative for all \( t \in [0, \infty) \). But in the \( L_2 \) space
\[
\int_0^t e_1(\tau)y_1(\tau)d\tau = \langle e_1, (H_1e_1)_t \rangle \quad \text{and} \quad \int_0^t e_2(\tau)y_2(\tau)d\tau = \langle e_2, (H_2e_2)_t \rangle.
\]

Hence, \( H_1 \) and \( H_2 \) are both positive relations on \( L_2([0, \infty)) \). Now, referring to Fig. 18, select \( 0 < \delta_1 < 1, 0 < \delta_2 < 1, a_1 < 0, \) and \( a_2 < 0 \) such that

\[
-\delta_1\left(\frac{1}{2a_1}\right) = a_3, \quad -\delta_2\left(\frac{1}{2a_2}\right) = \frac{1}{a_3} + \varepsilon, \quad \frac{1}{a_1} + \delta_1\left(\frac{1}{2a_1}\right) = b_3, \quad \text{and}
\]
\[
-\frac{1}{a_2} + \delta_2\left(\frac{1}{2a_2}\right) = b_4. \quad \text{This can be done since the first two equalities effectively determine the ratios } \frac{\delta_1}{a_1} \text{ and } \frac{\delta_2}{a_2} \text{ while the last two equalities determine particular values of } a_1 \text{ and } a_2. \quad \text{Now since } H_1 \text{ and } H_2 \text{ are both positive relations, the relations } H_1-a_1I \text{ and } H_2-a_2I \text{ are also both positive. Further, from Lemma 2 of Chapter 3, } H_3 \text{ is inside } (-\delta_1\left(\frac{1}{2a_1}\right),
\]
\[
-\frac{1}{a_1} + \delta_1\left(\frac{1}{2a_1}\right) ) \text{ and } H_4 \text{ is inside } (-\delta_2\left(\frac{1}{2a_2}\right), -\frac{1}{a_2} + \delta_2\left(\frac{1}{2a_2}\right)). \quad \text{Hence,}
\]

from Case 2 of the definition of a margin of boundedness \( \delta \) it is seen subloops 1 and 2 have, respectively, margins of boundedness \( \delta_1 \) and \( \delta_2 \). All that is needed now to infer boundedness is that \( \delta_1\delta_2 = n_1n_2k = 4a_1a_2 \).

But \( -\delta_1\left(\frac{1}{2a_1}\right) = a_3 \) and \( -\delta_2\left(\frac{1}{2a_2}\right) = \frac{1}{a_3} + \varepsilon \) implies
\[
\delta_1\delta_2 = [1 - \varepsilon\delta_1\left(\frac{1}{2a_1}\right)]4a_1a_2 > 4a_1a_2.
\]

Thus the network is found to be bounded in the sense of the \( L_2 \) norm. The specific bounds given on the "size" of each \( e_i \) and \( y_i \) follow from the manner in which Theorem 8 is proven and the fact that each \( w_i \) is zero.
Now it is shown the results given for the system of Fig. 19 are obtained as a special case of the results given for the system of Fig. 21. Let the relations $H_3$ and $H_4$ be used to model, respectively, the nonlinearities $N_1$ and $N_2$. Further, let the relations $H_1$ and $H_2$ model, respectively, the blocks labeled $H_1(s)$ and $H_2(s)$. From the discussion of the class of operators $Q$ in Chapter 3, it is clear $H_1$ and $H_2$ are members of $Q$ having, respectively, Laplace transforms $H_1(s)$ and $H_2(s)$. Further, from the differential equation models of $H_1$ and $H_2$, it is clear initial condition responses are in $L_2[0,\infty)$. The functions $w_i$ can be used to account for initial condition responses and are also in $L_2[0,\infty)$. It is now easily found the system of Fig. 19 has functional equations of the form of equations (6).

Now it is shown conditions guaranteeing boundedness of equations (6) are satisfied under conditions imposed on the system of Fig. 19. From Lemmas 1 and 2 of Chapter 3, it is seen the conditions placed on the Nyquist diagram of $H_1$ and the graph of $N_1$ imply $H_1$ is conic with constants $(a_1, b_1)$ where $b_1 < a_1 < 0$ and $H_3$ is inside the sector

$$\{-\frac{1}{b_1} - \delta_1\frac{b_1-a_1}{2b_1a_1}, -\frac{1}{a_1} + \delta_1\frac{b_1-a_1}{2b_1a_1}\}$$

where $0 < \delta_1 < 1$. This implies, from Case 1b of the definition of a margin of boundedness $\delta$, that the single-loop system possessing open-loop relations $H_1$ and $-H_3$ has a margin of boundedness $\delta_1$. Similarly, it is found the single-loop system possessing open-loop relations $H_2$ and $-H_4$ has a margin of boundedness $\delta_2$. Now boundedness in the sense of the $L_2$ norm is assured if
\[ \delta_1 \delta_2 > \eta_1 \eta_2 k = \frac{4b_1a_1b_2a_2k}{(b_1-a_1)(b_2-a_2)} \].

But this is exactly the condition under which results for the system of Fig. 19 are given.

Now the boundedness results presented for equations (6) are obtained from Theorem 8. Assume the conditions given are satisfied.

Select \( A = \{3,4\} \) and \( C = \{1,2\} \). The matrix \( B \) is found from equations (6) to be

\[
B = \begin{bmatrix}
0 & -k & -1 & 0 \\
1 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}.
\]

Then the matrix \( B' \) is easily found to be

\[
B' = \begin{bmatrix}
c_1 + d_3 & -k & -1 & 0 \\
1 & c_2 + d_4 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}.
\]

Now it is not difficult to find that \( c_1 + d_3 = c_2 + d_4 = 0 \) due to the fact that subloops 1 and 2 have margins of boundedness \( \delta_1 \) and \( \delta_2 \), respectively. Note this means the transformed system has the same form as the original system since \( B' = B \). This leads to
Calculation of the successive principal minors produces the following sufficient conditions for boundedness:

\[
1 - kn_1 n_2 > 0,
\]

\[
(1 - n_3 n_1) - kn_1 n_2 > 0,
\]

\[
(1 - n_3 n_1)(1 - n_4 n_2) - kn_1 n_2 > 0.
\]

Clearly the first two conditions can be eliminated since they are implied by the third. Now, due to the conditions on subloops 1 and 2, it is found that \(1 - n_3 n_1 = \delta_1\) and \(1 - n_4 n_2 = \delta_2\). Hence, the system is bounded if

\[
\delta_1 \delta_2 < n_1 n_2 k.
\]

But this condition is assumed to be satisfied.

Example III: Some properties of the solutions of the following system of differential equations involving time delay are investigated here:

\[
\dot{x}_1(t) = -x_1(t) - .25x_2(t) - 4.5x_3(t) - .25x_1(t-.758)
\]

\[
- .25N_1[x_1(t-.758),t] + .25r(t)
\]
\[
\dot{x}_2(t) = -1.25x_2(t) - 4.5x_3(t) - 0.25x_1(t-0.758) \\
- 0.25N_2[18x_3(t) + x_2(t), t] + 0.25r(t)
\]

(7)

\[
\dot{x}_3(t) = x_2(t) - 2x_3(t).
\]

It is assumed the time-varying nonlinearities \( N_1 \) and \( N_2 \) are continuous functions of both arguments and that the input \( r \) is a continuous function of time.

Strictly speaking, the question of existence of solutions to (7) need not be considered in a stability investigation. However, information concerning this question is usually desired and is readily available in this particular situation. In Halanay [2] it is shown that given a continuous function \( v_o(t) \) equal to \( x_1(t) \) on \([-0.758, 0]\) and given values for \( x_2(0) \) and \( x_3(0) \), a continuous differentiable solution exists for equation (7) on \([0, \infty)\). This solution can be constructed by replacing \( x_1(t-0.758) \) in (7) by \( v_o(t-0.758) \) in the time interval \([0, 0.758]\). By the above continuity assumptions there exists a solution on \([0, 0.758]\). This solution in turn can be used to produce a solution on \([0.758, 1.516]\). Repeating this process \textit{ad infinitum} produces a solution on \([0, \infty)\).

Application of Theorem 8 yields the following results concerning solutions of equations (7). Assume the nonlinearities \( N_1 \) and \( N_2 \) satisfy the following conditions:

\[
2.33x^2 \leq xN_1(x, t) \leq 5.67x^2 \text{ for all } x \text{ and all } t \in [0, \infty),
\]

\[
2.75x^2 \leq xN_2(x, t) \leq 6.00x^2 \text{ for all } x \text{ and all } t \in [0, \infty),
\]

\[
N_1(0, t) = N_2(0, t) = 0 \text{ for all } t \in [0, \infty).
\]
Then for each number $D$ and each set of initial conditions, numbers $A_1$, $A_2$, and $A_3$ can be calculated such that for each input $r$ with

$$\int_0^\infty r^2(t)dt < D$$

a corresponding solution of equations (7) satisfies the inequalities

$$\int_0^\infty x_i^2(t)dt < A_i \text{ for } i = 1, 2, 3.$$

Theorem 9 can be utilized to obtain further information concerning properties of solutions to equations (7). Let $N_1$ and $N_2$ satisfy the following conditions for all $x$ and $y$ and for all $t \in [0, \infty)$.

$$2.33(x-y)^2 \leq (x-y)[N_1(x,t) - N_1(y,t)] \leq 5.67(x-y)^2,$$

$$2.75(x-y)^2 \leq (x-y)[N_2(x,t) - N_2(y,t)] \leq 6.00(x-y)^2.$$  

These conditions are satisfied if, for instance, each $N_1$ and $N_2$ is differentiable in the first argument with $2.33 \leq \frac{\partial N_1(x,t)}{\partial x} \leq 5.67$ and $2.75 \leq \frac{\partial N_2(x,t)}{\partial x} \leq 6.00$ for all $x$ and all $t \in [0, \infty)$. Now let $r$ and $r'$ be inputs satisfying the condition $\int_0^\infty [r(t) - r'(t)]^2dt < \infty$ and corresponding under identical initial conditions with the, respective, solutions of equations (7) $[x_1(t), x_2(t), x_3(t)]^T$ and $[x_1'(t), x_2'(t), x_3'(t)]^T$. Then constants $K_1$ can be calculated independent of initial conditions such that for $i = 1, 2, 3$

$$\int_0^\infty [x_i(t) - x_i'(t)]^2dt \leq K_1 \int_0^\infty [r(t) - r'(t)]^2dt.$$
Now, in order to obtain the above results, equations (7) must be put in functional form. It is desired to view the nonlinearities \( N_1 \) and \( N_2 \) as relations on \( L_2[0,\infty) \). Hence, expressions are needed for the inputs \( x_1(t-.758) \) and \( 18x_3(t) + x_2(t) \) of these relations. These expressions are obtained from the well-known result of the theory of linear differential equations which gives the solution of the nonhomogeneous problem in terms of solutions of the homogeneous problem.

Let \([x_1(t), x_2(t), x_3(t)]^T\) be a solution of equations (7) corresponding to the input \( r \) for the initial conditions \( v_0(t) \) on \([-0.758,0]\), \( x_2(0) \), and \( x_3(0) \). Then from the first equation in (7) it is clear that for \( t \in [0,\infty) \)

\[
x_1(t) = e^{-t}v_0(0) + \int_0^t e^{-(t-\tau)}[-0.25x_2(\tau) - 4.5x_3(\tau) - 0.25x_1(\tau-.758) + 0.25r(\tau)]d\tau.
\]

Now define the unit step function to be

\[
u(t) = \begin{cases} 
0 & \text{for } t < 0 \\
1 & \text{for } t \geq 0 
\end{cases}
\]

Referring to the discussion of how a solution of equations (7) is constructed, it is seen that for \( t \in [0,\infty) \)

\[
x_1(t-.758) = \begin{cases} 
v_0(t-.758) & \text{for } 0 \leq t \leq .758 \\
e^{-(t-.758)}v_0(0) & \text{for } t > .758 
\end{cases}
\]

\[
+ \int_0^t u(t-.758-\tau)e^{-(t-.758-\tau)}[-0.25x_2(\tau) - 4.5x_3(\tau) - 0.25x_1(\tau-.758)]d\tau 
- 0.25N_1[x_1(\tau-.758),\tau] + 0.25r(\tau)]d\tau \tag{8}
\]
Now defining the matrix

\[
A = \begin{bmatrix}
-1 & 0 \\
1 & -2
\end{bmatrix}
\]

it is seen from the last two equations of (7) that for \( t \in [0, \infty) \)

\[
\begin{bmatrix}
x_2(t) \\
x_3(t)
\end{bmatrix} = e^{At} \begin{bmatrix}
x_2(0) \\
x_3(0)
\end{bmatrix} +
\int_0^t e^{A(t-\tau)} [-0.25x_2(\tau) - 4.5x_3(\tau) - 0.25x_1(\tau-0.758) \\
- 0.25N_2[18x_3(\tau) + x_2(\tau), \tau] + 0.25r(\tau), 0]^T d\tau.
\]

Calculating \( e^{At} \) it is found that

\[
18x_3(t) + x_2(t) = (19e^{-t} - 18e^{-2t})x_2(0) + 18e^{-2t}x_3(0)
\]

\[
+ \int_0^t \left[ 19e^{-(t-\tau)} - 18e^{-2(t-\tau)} \right] [-0.25x_2(\tau) - 4.5x_3(\tau) \\
- 0.25x_1(\tau-0.758) - 0.25N_2[18x_3(\tau) + x_2(\tau), \tau] + 0.25r(\tau)] d\tau.
\]

From the above it is clear each solution of equations (7) must satisfy the integral equations (8) and (9). This suggests the following. Let \( H_1 \) and \( H_2 \) be relations on \( L_2e[0, \infty) \) which satisfy, respectively, the equations

\[
H_1x(t) = N_1[x(t), t]
\]

and

\[
H_2x(t) = N_2[x(t), t].
\]
Since only solutions of equations (7) are of interest, it is assumed that the domains of $H_1$ and $H_2$ are each the class of continuous functions. Next, let $H_3$ and $H_4$ be operators on $L_2[0,\infty)$ which satisfy, respectively, the equations

$$H_3 x(t) = \int_0^t u(t-\tau) e^{-(t-\tau)/2} x(\tau) d\tau$$

and

$$H_4 x(t) = \int_0^t [19e^{-(t-\tau)/2} - 18e^{-(t-\tau)/2}] x(\tau) d\tau.$$ 

Obviously $H_3$ and $H_4$ are both members of the class $Q$ of linear time-invariant operators. Now consider the set of functional equations

$$e_1 = w_1 + y_3$$

$$e_2 = w_2 + y_4$$

$$e_3 = .25r + w_3 - .25y_1 - .25y_3 - .25y_4$$

$$e_4 = .25r + w_4 - .25y_2 - .25y_3 - .25y_4$$

$$y_i = H_i e_1 \text{ for } i = 1,2,3,4$$

where each $w_i \in L_2[0,\infty)$. Clearly these equations are in the form of equations (2).

It is now found for appropriate definitions of each $e_i$ and $w_i$ that solutions of (7) satisfy the above functional equations. Let $[x_1(t), x_2(t), x_3(t)]^T$ be a solution corresponding to the input $r$ and the initial conditions $v_0(t)$ on $[-.758, 0]$, $x_2(0)$, and $x_3(0)$. Define $z_1(t)$ and $z_2(t)$ by
\[
\begin{align*}
z_1(t) &= \begin{cases} 
v_0(t-.758) & \text{for } 0 \leq t \leq .758 \\
e^{-(t-.758)}v_0(0) & \text{for } t \geq .758 \
\end{cases} 
\end{align*}
\]

and

\[
z_2(t) = (19e^{-t} - 18e^{-2t})x_2(0) + 18e^{-2t}x_3(0).
\]

Then define each \( e_1 \) and \( w_1 \) by the following:

\[
e_1(t) = x_1(t-.758),
\]

\[
e_2(t) = 18x_3(t) + x_2(t),
\]

\[
e_3(t) = -0.25x_2(t) - 4.5x_3(t) - 0.25x_1(t-.758) - 0.25N_1[x_1(t-.758),t] + 0.25r(t),
\]

\[
e_4(t) = -0.25x_2(t) - 4.5x_3(t) - 0.25x_1(t-.758) - 0.25N_2[18x_3(t) + x_2(t),t] + 0.25r(t),
\]

\[
w_1(t) = z_1(t),
\]

\[
w_2(t) = z_2(t),
\]

\[
w_3(t) = w_4(t) = -0.25z_1(t) - 0.25z_2(t).
\]

Clearly, since \( z_1 \) and \( z_2 \) are in \( L_2[0,\infty) \), each \( w_1 \) is in \( L_2[0,\infty) \). Now it is seen from equations (8) and (9) that the functional equations are satisfied. For stability purposes only solutions for which \( r \), each \( e_1 \), and each \( y_1 \) belong to \( L_2e[0,\infty) \) are considered. In this
situation it is seen from the earlier discussion of solutions of equations (7) that \( r, e^1, \) and each \( y_1 \) are continuous. Hence, these functions all belong to \( L^2_e[0, \infty) \). Thus, in a certain sense, the set of solutions of (7) are a subset of the set of solutions of the functional equations. This means stability properties of the functional equations can be used to investigate properties of solutions to equations (7).

Now Theorem 8 is employed to obtain boundedness conditions for the functional equations. The operators \( H_3 \) and \( H_4 \) have, respectively, the Laplace transforms

\[
\bar{H}_3(s) = \frac{e^{-0.758s}}{s+1}
\]

and

\[
\bar{H}_4(s) = \frac{s+20}{(s+1)(s+2)} ,
\]

Referring to Lemma 1 of Chapter 3, it is found from the Nyquist diagrams of Fig. 22 that \( H_3 + 0.5I \) is a positive relation and \( H_4 \) is outside the sector \((-5.33, -0.5)\). Hence \( H_3 - a_3 I \) is positive and \( H_4 \) is conic with constants \((a_4, b_4)\) where \( a_3 = -0.5, a_4 = -0.5, \) and \( b_4 = -5.33 \). Assume for the present that relations \( H_1 \) and \( H_2 \) are conic with, respectively, constants \((a_1, b_1)\) and \((a_2, b_2)\). Now let \( A = \{1, 2\} \) and \( C = \{3, 4\} \). The \( B \) matrix is easily found from the functional equations to be

\[
B = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-0.25 & 0 & -0.25 & -0.25 \\
0 & -0.25 & -0.25 & -0.25 \\
\end{bmatrix}.
\]
The $B'$ matrix is given by

$$B' = (I + B[\text{diag}d_1])^{-1}(B + [\text{diag}c_1]) =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-.25 & 0 & .25d_1 - .25 + c_3 & -.25 \\
0 & -.25 & -.25 & .25d_2 - .25 + c_4
\end{bmatrix}$$

Fig. 22. Nyquist diagrams for Example III.
Now the diagonal terms of this matrix suggest the following. Let the single-loop system possessing open-loop relations $H_3$ and $-.25H_1$ have a margin of boundedness $\delta_1$. Similarly, let the single-loop system possessing open-loop relations $H_4$ and $-.25H_2$ have a margin of boundedness $\delta_2$. Note, since the conic nature of $H_3$ and $H_4$ is known, specification of $\delta_1$ and $\delta_2$ determines the conic nature of $H_1$ and $H_2$. It is easily found that the above conditions imply $\frac{1}{2}(c_1 - c_3 + c_4) = 0$. This means the transformed system has the same interconnection structure as the original system since it is now true that $B' = B$. It should be observed at this point that the above assumptions imply $1 - .25n_1n_3 = \delta_1$ and $1 - .25n_2n_4 = \delta_2$.

Now further manipulations produce

\[
I - [b_j']\eta_j = \begin{bmatrix}
1 & 0 & -n_3 & 0 \\
0 & 1 & 0 & -n_4 \\
-.25n_1 & 0 & 1 - .25n_3 & -.25n_4 \\
0 & -.25n_2 & -.25n_3 & 1 - .25n_4
\end{bmatrix}
\]

Calculation of the successive principal minors gives the following boundedness conditions:

\[
1 - .25n_1n_3 - .25n_3 > 0,
\]

\[
(1 - .25n_1n_3)(1 - .25n_2n_4) - .25n_3(1 - .25n_2n_4) - .25n_4(1 - .25n_1n_3) > 0.
\]

The first condition can be eliminated since it is implied by the second. Further, from expressions given for $n_3$ and $n_4$ in Theorem 8 and Remark 2
and from the above expressions for \( \delta_1 \) and \( \delta_2 \), it is found boundedness is guaranteed if

\[
\delta_1 \delta_2 - .25\delta_2 - .275\delta_1 > 0.
\]

From the above, the results cited earlier for the differential equations (7) can be shown true. To see this consider the particular situation of \( \delta_1 = .582 \) and \( \delta_2 = .551 \). It is easily found that

\[
\delta_1 \delta_2 - .25\delta_2 - .275\delta_1 > 0.
\]

Hence, the functional equations are bounded. It is seen from the definition of a margin of boundedness \( \delta \) that it is required \( .25H_1 \) be inside the sector \( \{\delta_1, 2-\delta_1\} \) and \( .25H_2 \) be inside the sector \( \{.187 + .907\delta_2, 2-.907\delta_2\} \). For the above values of \( \delta_1 \) and \( \delta_2 \) this means it is required that \( H_1 \) be inside \( \{2.33, 5.67\} \) and \( H_2 \) be inside \( \{2.75, 6.00\} \).

Now assume that in equations (7) the nonlinearities \( N_1 \) and \( N_2 \) satisfy the following conditions:

\[
2.33x^2 \leq xN_1(x,t) \leq 5.67x^2 \text{ for all } x \text{ and all } t \in [0,\infty),
\]

\[
2.75x^2 \leq xN_2(x,t) \leq 6.00x^2 \text{ for all } x \text{ and all } t \in [0,\infty),
\]

\[
N_1(0,t) = N_2(0,t) = 0 \text{ for all } t \in [0,\infty).
\]

Hence, from Lemma 2 of Chapter 3, the relations \( H_1 \) and \( H_2 \) are inside the, respective, sectors \( \{2.33, 5.67\} \) and \( \{2.75, 6.00\} \). Now select a set of initial conditions. This corresponds to fixing the \( w_1 \) functions. Then pick a number \( D \) and a continuous input \( r \) such that
\[ \int_0^\infty r^2(t)dt < D. \]

It follows there exist constants \( K \) and \( L \) such that for a corresponding solution \( [x_1(t), x_2(t), x_3(t)]^T \) of (7)

\[ \int_0^\infty e_1^2(t)dt = \int_0^\infty x_1(t-.758^2)dt < K \]

and

\[ \int_0^\infty e_4^2(t)dt = \int_0^\infty [-.25x_2(t) - 4.5x_3(t) - .25x_1(t-.758) \]

\[ - .25N_2[18x_3(t) + x_2(t), t] + .25r(t)]^2dt < L. \]

The first inequality implies there exists an \( A_1 \) such that

\[ \int_0^\infty x_1(t)^2dt < A_1. \]

From the equation given earlier for \( [x_2(t), x_3(t)]^T \), it is easily found that

\[ x_2(t) = e^{-t}x_2(0) + \int_0^t e^{-(t-\tau)}e_4(\tau)d\tau \]

and

\[ x_3(t) = (e^{-t}e^{-2t})x_2(0) + e^{-2t}x_3(0) \]

\[ + \int_0^t [e^{-(t-\tau)}e^{-(t-\tau)}]e_4(\tau)d\tau. \]

Hence, both \( x_2 \) and \( x_3 \) are in the form of a sum of a fixed function in \( L_2[0, \infty) \) with the output of an operator which is in the class \( Q \). It is easily seen that each operator has a Nyquist diagram which lies within a circle in the complex plane with center at the origin. Thus, by Lemma 1 of Chapter 3, it is seen each operator has a finite gain. Since
the $||e_4||^2 < L$, it is then clear there exists constants $A_2$ and $A_3$ such that

$$\int_0^\infty x_2^2(t)dt < A_2$$

and

$$\int_0^\infty x_3^2(t)dt < A_3.$$

Now assume the nonlinearities $N_1$ and $N_2$ satisfy the following conditions for all $x$ and $y$ and for all $t \in [0, \infty)$:

$$2.33(x-y)^2 \leq (x-y)[N_1(x,t) - N_1(y,t)] \leq 5.67(x-y)^2,$$

$$2.75(x-y)^2 \leq (x-y)[N_2(x,t) - N_2(y,t)] \leq 6.00(x-y)^2.$$ Then from Lemmas 1 and 3 it is clear the incremental counterparts of the above boundedness conditions are satisfied. This implies continuity by Theorem 9. Now fix the $w_1$ functions by selecting a set of initial conditions. Next let $r$ and $r'$ be two continuous inputs with corresponding solutions $[x_1(t), x_2(t), x_3(t)]^T$ and $[x_1'(t), x_2'(t), x_3'(t)]^T$, respectively. Assuming

$$\int_0^\infty [r(t) - r'(t)]^2dt < \infty,$$

it is found using the obvious definitions for $e_1'$ and $e_4'$ that there exists $B$ and $C$ such that

$$\int_0^\infty [e_1(t) - e_1'(t)]^2dt \leq B \int_0^\infty [r(t) - r'(t)]^2dt$$

and

$$\int_0^\infty [e_4(t) - e_4'(t)]^2dt \leq C \int_0^\infty [r(t) - r'(t)]^2dt.$$
The first inequality obviously implies there exists a $K_1$ such that

$$\int_0^\infty \left[ x_1(t) - x_1'(t) \right]^2 dt \leq K_1 \int_0^\infty \left[ r(t) - r'(t) \right]^2 dt.$$ 

Now examine the equations given above for $x_2(t)$ and $x_3(t)$. Since initial conditions are fixed, it is seen that

$$x_2(t) - x_2'(t) = \int_0^t e^{-(t-\tau)} \left[ e_4(\tau) - e_4'(\tau) \right] d\tau$$

and

$$x_3(t) - x_3'(t) = \int_0^t \left[ e^{-(t-\tau)} - e^{-2(t-\tau)} \right] \left[ e_4(\tau) - e_4'(\tau) \right] d\tau.$$ 

Since the above integral operators have finite gain and since $||e_4 - e_4'||^2 \leq C ||r - r'||^2$, it is clear there exist constants $K_2$ and $K_3$ such that

$$\int_0^\infty \left[ x_2(t) - x_2'(t) \right]^2 dt \leq K_2 \int_0^\infty \left[ r(t) - r'(t) \right]^2 dt$$

and

$$\int_0^\infty \left[ x_3(t) - x_3'(t) \right]^2 dt \leq K_3 \int_0^\infty \left[ r(t) - r'(t) \right]^2 dt.$$
CHAPTER 6: CONCLUSION

Conditions sufficient to guarantee boundedness or continuity of a multiple-loop nonlinear time-varying system are presented here. Boundedness results are derived in terms of the interconnection structure of the multiple-loop system and in terms of gains of the relations interconnected to form the system. The range of application of these results is greatly expanded through a certain transformation which leads to a result involving sector conditions. Continuity results are found to be available from application of boundedness conditions to a system which relates changes in inputs to changes in outputs. This leads to results identical with boundedness results but with sector conditions replaced by their incremental counterparts.

An interesting illustration of the theory is provided by examining a system formed by the interconnection of an arbitrary number of memoryless nonlinearities with a number of linear time-invariant relations. It is found inputs belonging to a bounded subset of the $L_2$ space always correspond to outputs which belong to a bounded subset of the $L_2$ space under the following conditions:

1. The Nyquist diagram of each linear relation either lies outside an appropriate circle in the complex plane and does not encircle this circle or the Nyquist diagram lies wholly within an appropriate circle.

2. The graph of each nonlinearity lies in a region of the plane enclosed by two appropriate straight lines passing through the origin.
The exact meaning of appropriate in (1) and (2) is determined by the interconnection structure. By changing only (2), the above boundedness conditions become continuity conditions. Inputs which are arbitrarily "close" in the sense of the $L_2$ norm lead to outputs arbitrarily "close" in the sense of the $L_2$ norm if (2) reads: The slope of each nonlinearity has appropriate upper and lower bounds.

In a certain sense the general theory is found capable of providing some feeling of "how stable" a multiple-loop system is. If boundedness conditions are satisfied, then specific bounds on system outputs in terms of bounds on system inputs can be obtained. Further, if continuity conditions are satisfied, then specific bounds on deviations in system outputs in terms of deviations in system inputs are available. If conditions on the multiple-loop system are tightened, then bounds on system responses and deviations in system responses become tighter. Hence, the margin by which stability conditions are satisfied is somewhat of an indication of the "degree of stability" for the system.

Much can be inferred about stability conditions which the theory can provide solely by examining the form of the interconnection structure. Experience indicates the relative positions of the subloops can be used to guide application of the theory. In applications presented here it is found that if the subloops of the transformed system each satisfy stability conditions by a certain margin then conditions for the entire system can be phrased in terms of these margins. This often reveals tradeoffs in conditions on relations which allow stability to be retained.
Several applications of the theory to specific multiple-loop non-linear time-varying systems are presented. Three particular systems considered are the following:

(1) a certain interconnection of three specific linear time-invariant relations with a linear time-varying relation, a piecewise linear relation, and a hysteresis nonlinearity,

(2) a network formed from a passive impedance, a passive admittance, a nonlinear time-varying resistance, and a nonlinear time-varying conductance,

(3) a third order nonlinear time-varying differential equation involving time delay.
LITERATURE CITED


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Several Linear Spaces

Several types of linear spaces are discussed here. A thorough discussion of these spaces is found in many texts [4], [6], [12]. First a definition is given for a linear space. In this definition \( R \) denotes either the field of real numbers or the field of complex numbers. A number belonging to \( R \) is referred to as a scalar.

Definition: Let \( X \) be a set of elements for which the algebraic operations of addition and scalar multiplication are defined. If \( x, y \in X \) then the addition operation produces a unique element of \( X \) denoted by \( x + y \). Further, if \( a \in R \) and \( x \in X \) then the scalar multiplication operation produces a unique element of \( X \) denoted by \( ax \). The set \( X \) along with the two algebraic operations is a linear space if the following are true:

1. \( x + y = y + x \) for all \( x, y \in X \).
2. \( x + (y + z) = (x + y) + z \) for all \( x, y, z \in X \).
3. There is a unique element of \( X \) denoted by 0 such that \( x + 0 = x \) for all \( x \in X \).
4. For each \( x \in X \) there exists a unique element of \( X \) denoted by \( -x \) such that \( x + (-x) = 0 \).
5. \( a(x + y) = ax + ay \) for all \( a \in R \) and all \( x, y \in X \).
6. \( (a + b)x = ax + bx \) for all \( a, b \in R \) and all \( x \in X \).
7. \( a(bx) = (ab)x \) for all \( a, b \in R \) and all \( x \in X \).
8. \( 1x = x \) for all \( x \in X \).
9. \( 0x = 0 \) for all \( x \in X \).
In the above definition if \( R \) denotes the field of real numbers, then the linear space is referred to as a real linear space. Similarly if \( R \) denotes the field of complex numbers, then the linear space is referred to as a complex linear space. It should be noted no distinction is made between the number zero and the zero element of \( X \). Which is being referred to is always clear from context.

Now a special kind of linear space, a normed linear space, is defined.

**Definition:** A normed linear space is a linear space \( X \) on which a real-valued function referred to as the norm is defined. The value of the norm at \( x \in X \) is denoted by \( ||x|| \), and the following properties must be satisfied:

1. \( ||x+y|| \leq ||x|| + ||y|| \) for all \( x, y \in X \).
2. \( ||ax|| = |a| ||x|| \) for all \( a \in \mathbb{R} \) and all \( x \in X \).
3. \( ||x|| \neq 0 \) if \( x \neq 0 \).

Now a definition is given for an inner product space.

**Definition:** A complex linear space \( X \) is an inner product space if there exists on \( X \times X \) a complex-valued function called the inner product. The value of the inner product at \( (x, y) \in X \times X \) is denoted by \( \langle x, y \rangle \), and the following properties must be satisfied:

1. \( \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \) for all \( x, y, z \in X \).
2. \( \langle x, y \rangle = \overline{\langle y, x \rangle} \) for all \( x, y \in X \).
3. \( \langle ax, y \rangle = a \langle x, y \rangle \) for all \( a \in \mathbb{R} \) and all \( x, y \in X \).
4. \( \langle x, x \rangle \geq 0 \) for all \( x \in X \) and \( \langle x, x \rangle \neq 0 \) if \( x \neq 0 \).
For the case of a real linear space, the above definition is different. The only changes required are that the inner product be real-valued and the bar over $\langle y, x \rangle$ be removed in property (2).

Now $L_p$ spaces which are of interest for stability purposes are defined. A condition which is violated at most on a set of measure zero is referred to as being true almost everywhere.

**Definition:** For $1 \leq p < \infty$ the space is defined as the space of all real measurable functions $x(t)$ such that $|x(t)|^p$ is Lebesgue integrable over $T$. The $L_\infty$ space is the space of real measurable functions on $T$ such that for each $x(t)$ there exists an $M$ so that $|x(t)| \leq M$ almost everywhere on $T$.

In the above definition no distinction is made between functions which agree almost everywhere. Also the notation $L_p([t_0, \infty))$ or $L_p(-\infty, \infty)$ is often used to denote an $L_p$ space for $T = [t_0, \infty)$ or $T = (-\infty, \infty)$.

Each of the $L_p$ spaces is a normed linear space. If $x \in L_p$ for finite $p$, then the norm of $x$ is given by $\|x\| = \left( \int_T |x(t)|^p dt \right)^{1/p}$.

For $p = \infty$ the norm of $x \in L_\infty$ is the infimum of the set of all $M$ such that $|x(t)| \leq M$ almost everywhere on $T$. This infimum is called the essential supremum and denoted by $\text{ess sup } |x(t)|$. Hence for $x \in L_\infty$ it is seen $\|x\| = \text{ess sup } |x(t)|$.

The $L_2$ space is distinguished from the other $L_p$ spaces by the fact that it is an inner product space. If $x, y \in L_2$ the inner product is defined by $\langle x, y \rangle = \int_T x(t)y(t)dt$. 
APPENDIX B

Relations

Following Kelley [5] definitions are given here for certain manipulations of relations.

Definition: If $H$ and $K$ are relations on $X_e$ and $c$ is a real constant then:

$H + K = \{(x,y) : x \in \text{Do}(H) \cap \text{Do}(K) \text{ and there exist images } Hx \text{ and } Kx \text{ such that } y = Hx + Kx\}.$

$cH = \{(x,y) : x \in \text{Do}(H) \text{ and there exists an image } Hx \text{ such that } y = c(Hx)\}.$

$KH = \{(x,y) : \text{there exists } z \text{ such that } (x,z) \in H \text{ and } (z,y) \in K\}.$

$H^{-1} = \{(x,y) : (y,x) \in H\}.$

$I = \{(x,y) : x \in X_e \text{ and } y = x\}.$

It is of interest to note that despite the fact addition and scalar multiplication are defined, the space of all relations on $X_e$ is not a linear space.
APPENDIX C

Completion of Proof for Theorem 7

It is shown here the hypotheses of Theorem 7 are sufficient to guarantee the matrix $I - [b_{ij}] g(H_j)$ has an inverse with all non-negative elements. The following two theorems presented in Gantmacher [1, pp. 66 and 71 of Vol. II] are found to be useful.

Theorem 10: A matrix $A$ having all elements nonnegative always has a nonnegative eigenvalue $r$ such that the moduli of all the eigenvalues of $A$ do not exceed $r$. To this "maximal" eigenvalue $r$ there corresponds an eigenvector $y$ such that $y \geq 0$ and $y \neq 0$. Further, the adjoint matrix $B(\lambda) = (\lambda I - A)^{-1} |\lambda I - A|$ has all elements nonnegative for $\lambda \geq r$.

Theorem 11: If a matrix $G$ has all off diagonal elements negative or zero and the successive principal minors are positive, then all principal minors are positive.

Now, as in Theorem 7, assume the successive principal minors of $I - [b_{ij}] g(H_j)$ are all positive. Since the last successive principal minor is the determinant of this matrix, the matrix is nonsingular and has an inverse. Now from Theorem 10 it is clear the matrix $[b_{ij}] g(H_j)$ has a "maximal" eigenvalue $r$. Further, the matrix $(I - [b_{ij}] g(H_j))^{-1}$ has all elements nonnegative if $r \leq 1$. But $|I - [b_{ij}] g(H_j)|$ is the last successive principal minor of
I - \[|b_{ij}|g(H_j)\] and is positive. Hence, if \(r \leq 1\) then \(I - \[|b_{ij}|g(H_j)\]\) has an inverse with all nonnegative elements.

Now it only need be shown that \(r \leq 1\). Since \(r\) is an eigenvalue, it is found

\[0 = |rI - \[|b_{ij}|g(H_j)\]| = |(I - \[|b_{ij}|g(H_j)\]) + (r - 1)I|\]

This means \(1-r\) is an eigenvalue of the matrix \(I - \[|b_{ij}|g(H_j)\]\). Now from the theory of matrices it is known the characteristic equation for the \(n \times n\) matrix \(B\) can be written

\[|B - \lambda I| = (-\lambda)^n + \sum_{k=1}^{n} S_k (-\lambda)^{n-k} = 0\]

where each \(S_k\) is the sum of all principal minors of order \(k\) of the matrix \(B\). Thus, letting \(B = I - \[|b_{ij}|g(H_j)\]\) and \(\lambda = 1 - r\) results in

\[(r-1)^n + \sum_{k=1}^{n} S_k (r-1)^{n-k} = 0\]

where each \(S_k\) is the sum of all principal minors of order \(k\) of \(I - \[|b_{ij}|g(H_j)\]\). But from Theorem 11 it is clear all principal minors of \(I - \[|b_{ij}|g(H_j)\]\) are positive. Hence, each \(S_k > 0\). Now it is clear the above characteristic equation cannot be satisfied for \(r > 1\). Hence, \(r \leq 1\).
APPENDIX D

Completion of Proof for Theorem 8

It is shown here that the conditions imposed on $H_i$ in Theorem 8 are sufficient under the transformation to imply $H_i'$ is inside $\{-\eta_i, \eta_i\}$.

First a few properties of conic relations are listed:

1. If $H$ is conic with constants $(a, b)$, then for any real number $k$ the relation $H + kI$ is conic with constants $(a + k, b + k)$.

2. If $H$ is conic with constants $(a, b)$ and $k > 0$, then $kH$ is conic with constants $(ka, kb)$. If $k < 0$, then $kH$ is conic with constants $(kb, ka)$.

3. If $H$ is conic with constants $(a, b)$ where $ab \neq 0$, then $H^{-1}$ is conic with constants $(\frac{1}{b}, \frac{1}{a})$. The limiting cases of $b \to \infty$ and $a \to -\infty$ are dealt with rigorously in the following manner. If $H-aI$ is positive where $a \neq 0$, then $H^{-1}$ is conic with constants $(0, \frac{1}{a})$. Further, if $-H + bI$ is positive where $b \neq 0$, then $H^{-1}$ is conic with constants $(\frac{1}{b}, 0)$.

Similar properties to the above are proven in [15]. However, the notation is somewhat different.

Assume $i \in A$. Then $H_i' = H_i + d_iI$. Since $H_i$ is conic with constants $(a_i, b_i)$, it is seen by property (1) above that $H_i'$ is conic with constants $(a_i + d_i, b_i + d_i)$. Now $d_i = -\frac{1}{2} (b_i + a_i)$ results in $a_i + d_i = -\frac{1}{2} (b_i - a_i)$ and $b_i + d_i = \frac{1}{2} (b_i - a_i)$. Since $\eta_i = \frac{1}{2} (b_i - a_i) > 0$, it is found that $H_i'$ is inside $\{-\eta_i, \eta_i\}$.
Now assume \( i \in C \). Then \( H_i' = (H_i^{-1} + c_i I)^{-1} \). The relation \( H_i \) is conic with constants \((a_i, b_i)\) where \( a_i b_i \neq 0 \) due to the fact that

\[
\eta_i = -\left(\frac{2b_i a_i}{b_i - a_i}\right) > 0.
\]

Hence, by property (3) above, \( H_i^{-1} \) is conic with constants \( \left(\frac{1}{b_i}, \frac{1}{a_i}\right) \). Now by property (1) the relation \( H_i^{-1} + c_i I \) is conic with constants \( \left(\frac{1}{b_i} + c_i, \frac{1}{a_i} + c_i\right) \). Since \( c_i = -\frac{b_i + a_i}{2b_ia_i} \), it is seen that \( \frac{1}{b_i} + c_i = -\frac{2a_i}{2b_i a_i} \) and \( \frac{1}{a_i} + c_i = \frac{b_i - a_i}{2b_i a_i} \). Since \( b_i \neq a_i \), property (3) can be used again and reveals that \( (H_i^{-1} + c_i I)^{-1} \) is conic with constants \( \left(\frac{2b_i a_i}{b_i - a_i}, \frac{2b_i a_i}{b_i - a_i}\right) \). Now because \( \eta_i = -\left(\frac{2b_i a_i}{b_i - a_i}\right) > 0 \), it is seen that \( H_i' \) is inside \( \{-\eta_i, \eta_i\}\).

Now consider the remark made following the proof of Theorem 8 concerning the limiting cases for \( i \in C \) of \( b_i \to \infty \) and \( a_i \to -\infty \). The case \( b_i \to \infty \) is examined first. Assume \( H_i - a_i I \) is positive. Now \( H_i' = (H_i^{-1} + c_i I)^{-1} \). The constant \( a_i \neq 0 \) since \( \eta_i = -2a_i > 0 \). Hence, by property (3) \( H_i^{-1} \) is conic with constants \((0, \frac{1}{a_i})\). By property (1), \( H_i^{-1} + c_i I \) is conic with constants \((c_i, \frac{1}{a_i} + c_i)\). Since \( c_i = -\frac{1}{2a_i} \), it is clear that \( H_i^{-1} + c_i I \) is conic with constants \((\frac{1}{2a_i}, \frac{1}{2a_i})\).

Hence, \( (H_i^{-1} + c_i I)^{-1} \) is conic with constants \((2a_i, -2a_i)\). Since \( \eta_i = -2a_i > 0 \), it is true that \( H_i' \) is inside \( \{-\eta_i, \eta_i\}\). Similarly it can be shown that if \( -H_i + b_i I \) is positive then for the modified definitions of \( c_i \) and \( \eta_i \) the relation \( H_i' \) is inside \( \{-\eta_i, \eta_i\}\).