Behavior and analysis of steel liners for prestressed concrete reactor vessels

Orhan Gürbüz
Iowa State University
INFORMATION TO USERS

This material was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or “target” for pages apparently lacking from the document photographed is “Missing Page(s)”. If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in “sectioning” the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again — beginning below the first row and continuing on until complete.

4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from “photographs” if essential to the understanding of the dissertation. Silver prints of “photographs” may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.

5. PLEASE NOTE: Some pages may have indistinct print. Filmed as received.

Xerox University Microfilms
300 North Zeeb Road
Ann Arbor, Michigan 48106
GÜRBÜZ, Orhan, 1939-
BEHAVIOR AND ANALYSIS OF STEEL LINERS FOR PRESTRESSED CONCRETE REACTOR VESSELS.

Iowa State University, Ph.D., 1974
Engineering, civil

University Microfilms, A XEROX Company, Ann Arbor, Michigan
Behavior and analysis of steel liners for prestressed concrete reactor vessels

by

Orhan Gürbüz

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of DOCTOR OF PHILOSOPHY Department: Civil Engineering Major: Structural Engineering

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa

1974
<table>
<thead>
<tr>
<th>CHAPTER I. INTRODUCTION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>CHAPTER II. DESIGN BASES</td>
<td>7</td>
</tr>
<tr>
<td>Design Analyses</td>
<td>7</td>
</tr>
<tr>
<td>Stability Considerations</td>
<td>9</td>
</tr>
<tr>
<td>Stress Analysis Methods</td>
<td>13</td>
</tr>
<tr>
<td>Design Loads and Local Effects</td>
<td>24</td>
</tr>
<tr>
<td>Behavior of Liner Components</td>
<td>30</td>
</tr>
<tr>
<td>CHAPTER III. ONE-DIMENSIONAL INITIAL STRESS METHOD OF ANALYSIS</td>
<td>36</td>
</tr>
<tr>
<td>Assumptions</td>
<td>36</td>
</tr>
<tr>
<td>Formulation of the Problem</td>
<td>37</td>
</tr>
<tr>
<td>Solution Technique</td>
<td>40</td>
</tr>
<tr>
<td>Criteria for Solution Accuracy</td>
<td>44</td>
</tr>
<tr>
<td>Comparison to Another Method</td>
<td>44</td>
</tr>
<tr>
<td>CHAPTER IV. ELASTO PLASTIC FINITE ELEMENT METHOD OF ANALYSIS</td>
<td>46</td>
</tr>
<tr>
<td>Assumptions</td>
<td>47</td>
</tr>
<tr>
<td>Formulation of the Problem</td>
<td>48</td>
</tr>
<tr>
<td>Solution Technique</td>
<td>54</td>
</tr>
<tr>
<td>Criteria for Solution Accuracy</td>
<td>63</td>
</tr>
<tr>
<td>CHAPTER V. COMPARISON OF ONE- AND TWO-DIMENSIONAL ANALYSES</td>
<td>67</td>
</tr>
<tr>
<td>Purpose and Scope</td>
<td>67</td>
</tr>
<tr>
<td>Description of Study</td>
<td>71</td>
</tr>
<tr>
<td>Discussion of Results</td>
<td>76</td>
</tr>
<tr>
<td>CHAPTER VI. STUDY OF THE EFFECT OF LOCAL VARIATIONS</td>
<td>82</td>
</tr>
<tr>
<td>Description of Study</td>
<td>82</td>
</tr>
</tbody>
</table>
CHAPTER I.

INTRODUCTION

Prestressed Concrete Reactor Vessels (PCRV's) are structures which contain reactor systems directly without any other intervening pressure barrier. As such, they are continually subject to the pressure of the primary coolant which is usually a gas. The use of gas-cooled PCRV's has found wide acceptance recently. Today, there are about thirty vessels in operation, under construction or, being planned, all over the world.

A PCRV is usually cylindrical in shape and consists of five major components: (1) the concrete structure; (2) the post-tensioning system; (3) the nonprestressed reinforcement; (4) the liner assembly; (5) the thermal control system. The subject matter of this thesis is confined to the fourth component; the others will be discussed only to the extent that they affect the behavior of the liner.

One of the important problems in PCRV design is the provision for leak-tightness of the vessel. Since the radioactive coolant must be contained within the vessel cavity, the cavity is always lined with a suitable material. In the United States and abroad, steel liners are used as the primary leakage barrier. There are numerous penetrations in a typical PCRV. These penetrations house mechanical equipment and provide access to reactor interior and, are also lined with steel (Fig. 1).

The liner assembly consists of the liner plates, closely spaced anchors and, cooling tubes. The liner plate is essentially a thin shell, rigidly connected to the surrounding concrete by means of
continuous (e.g., angles, tees) or discreet (e.g., studs) anchors. Cooling tubes are either square or circular in cross section and are usually welded to the concrete side of the liner.

The PCRV liner is in biaxial compression throughout most of its design life. The compressive stress field is due mainly to prestressing loads, shrinkage and creep and, thermal loads. Under operating conditions internal pressure reduces the magnitude of the concrete-imposed compressive strains in the liner. But the resultant stresses are still compressive even under the maximum cavity pressure.

In the design of liners, the usual approach is to determine the liner thickness and cooling tube spacing (hence anchor spacing, since anchors are provided between the tubes) considering the construction and heat transfer requirements. Design analyses are then conducted to verify that liner assembly stresses and displacements are within the allowable limits under normal operating and accident conditions. For this purpose, the liner structure is considered as a separate entity, independent of the backing concrete but, subject to concrete-imposed strains and displacements. In design analysis, the problem is considered a stress problem rather than a stability problem. The latter is considered indirectly, as it affects the behavior of individual liner panels between anchor supports.

Stress analysis methods for liners were given in papers presented at Conference on Prestressed Concrete Vessels (1, 2, 3, 4) and, First and Second International Conference on Structural Mechanics in Reactor Technology (5, 6). In these and other methods, it is customary to analyze a one-dimensional section of the liner assembly under
appropriate loading and boundary conditions. A similar approach was taken in the design of the Fort St. Vrain PCRV liner, the only vessel constructed so far in the United States. In all one-dimensional stress analysis methods, the problem is formulated based on equilibrium at the nodes (the liner-anchor joint) and compatibility between the nodes. Because of material nonlinearity the problem is usually reduced to solving a set of nonlinear simultaneous equations. Local effects which tend to increase unbalanced forces between liner panels and thus increase anchor forces, must also be considered in the analysis. For this purpose a panel which is weaker in compression than other panels (due to causes such as lower yield point, less thickness, existence of lateral pressure, etc.) is usually termed a "weak" panel, the remainder being "strong" panels. Since such local variations may occur anywhere, it is necessary to consider the most critical weak panel location in the stress analysis so that the magnitude of absolute maximum stresses and displacements can be determined and evaluated in the light of code allowables.

The Prestressed Concrete Containment Vessels (PCCV's) are also lined, usually with a thin steel liner and thus present similar design problems. One-dimensional stress analysis methods used in the design of PCCV liners were described in Refs. (7, 8, 9). In Ref. (8) a two-dimensional analysis method, for the analysis of a PCCV or PCRV

---

1A containment vessel is a structure which contains a primary vessel such as PCRV. The main purpose of the containment vessel is to provide a leak-tight secondary barrier for the radioactive coolant, in case of an accident.
liner cross section perpendicular to the plane of the liner and including a portion of the backing concrete, was also briefly discussed.

The stability of liners in a rigid cavity has been investigated by numerous researchers (10, 11, 12, 13, 14). The analytical models used are of two types: (1) ring or flat strips without anchors (10, 11); (2) rectangular panels or curved strips with anchors (12, 13, 14). Analysis of the former models result in minimum buckling strains at which an alternate equilibrium position exists and thus the liner may buckle into that position with an external disturbance. Buckling may be prevented by providing anchors at a spacing less than the buckled length for a given strain. The latter models give minimum buckling strains and/or anchor stresses for a known anchor spacing.

The above summary indicates that, in the design of PCRV and PCCV liners with closely spaced anchors, it is usual to treat the liner analysis as a stress problem. There are procedures proposed, based on a stability approach, for determining an adequate anchoring system for the liners.

The PCRV liner stress analysis problem is considered in this dissertation. Specific objectives include:

1) To evaluate the adequacy of one-dimensional stress analysis methods,

2) To present a more refined, and two-dimensional, stress analysis method, and

3) To present an approximate method with which all local effects may be taken into account in design analysis.
As a basis for evaluating the appropriateness of the stress analysis approach, an overview of current liner design practice is presented in Chapter II. Available stability and stress analysis methods are reviewed and, loading conditions and component behavior are briefly discussed.

Considering the fact that most PCRV and PCCV liners have been designed on the basis of a one-dimensional analysis, mainly because of the simplicity and conservativeness of these methods, there is a need to evaluate the adequacy of this approach. Since there is no test available, this evaluation is to be based on a comparison with the more refined analysis. For this purpose, a one-dimensional analysis method, similar to Parker's formulation (5) but using a different solution technique was developed first (Chapter III). A more accurate two-dimensional analysis method was developed next using the finite element approach (Chapter IV). Computer programs were written for both methods. Then a parametric study, based on Fort St. Vrain design data, was conducted using selected parameters (Chapter V).

There is also a need for a technique with which the effect of all local variations can be taken into account in design analysis. Some local variations can be considered through modification of the weak panel characteristics while others need to be incorporated in the analytical procedure in some manner. Chapter VI summarizes a technique whereby the effects of various local variations may be studied. The characteristics of a weak panel needed in one-dimensional analysis may be developed using the approximate method presented in Appendix A.
The findings and conclusions of this study are intended to apply to general PCRV liner stress analysis. The stress analysis methods discussed in Chapters III and IV are also applicable to PCCV liners. However, due to greater anchor spacing-liner thickness ratio of the PCCV liner panels, the approximate method of strut analysis presented in Appendix A may not be applicable to the case of PCCV liners. Furthermore, some of the local variations discussed in Chapter VI simply do not exist for PCCV liners, although the procedure for taking local effects into account should be applicable to both types of liners.
CHAPTER II.
DESIGN BASES
Design Analyses

In the design of liners with anchors, the usual practice is to first select the liner thickness and cooling tube spacing considering the construction and cooling requirements (15). The anchor spacing, in the direction perpendicular to the cooling tubes, is in turn determined by the cooling tube spacing since stud anchors are usually placed between cooling tubes. The anchor spacing in the other direction is usually the same as above, thus forming a square anchor pattern.

Design analyses of liners with anchors involves consideration of both stability and stress problems. Since the liner is rigidly attached to the surrounding concrete, the stability of a panel (defined as a rectangular segment of the liner plate bounded by four anchors) rather than the total shell is of importance. Stability consideration in a liner design problem is treated in the next section.

For stress analysis it is possible to include the liner in the analysis of the entire vessel, representing it as a shell element subject to membrane forces (5, 15). However, it is not practical to include shear anchors in such an analysis (5). Furthermore, it is difficult to take into account local buckling and variations which affect behavior of the liner in the analysis of a complete vessel. For these reasons, stress analysis of the liners has been considered as a separate problem, divorced from the analysis of the vessel (2, 3, 5, 15, 16).
In this "uncoupled" approach, the first step is to determine liner "design" stresses or strains from an analysis of the vessel under different loading conditions. Finite difference (1, 3), dynamic relaxation (18), and finite element (16) methods have been used for this purpose. In the analysis of the vessel, the liner may or may not be taken into account (5, 16, 17). When the liner is included in the analysis, it is assumed that full strain compatibility exists between the liner and the concrete surface.

The second step in this approach involves stress analysis (also called load-dissipation analysis) of a segment of the liner. For this analysis, stiffness characteristics of liner components are required. Stiffness characteristics of anchors and cooling tubes are usually determined experimentally (5, 16). In British practice, the liner characteristics are mostly based on experimental results also (5). In the Fort St. Vrain design however, theoretical stiffnesses of the liner panels are used in the stress analysis.

The stress analysis results in the final design stresses or strains for the anchors, cooling tubes, and liner plates. A description of various stress analysis methods is given later in this chapter.

In the design analysis, it is important to accurately calculate the strains which will be imposed on the liner since the magnitude of these strains affects stresses and displacements in the liner components substantially. It is also important to determine accurately the behavior of liner components. Effects of various local variations must also be considered for adequate design (see Chapter VI).
Stability Considerations

Nature of the problem

Liners in prestressed concrete reactor and containment vessels are in biaxial compression under most loading conditions. For this reason, buckling of liners has been a major design consideration (3).

Compressive stresses are introduced into the liner in two different ways:

a) Compressive straining as a result of reduction in length or diameter (e.g., due to prestressing).

b) Compressive stressing due to cavity restraint when the liner temperature increases.

It has been shown that for a ring strip in a rigid cavity these two different ways of loading result in the same buckling strain (12). In view of this, no differentiation will be made in regards to how the strains are introduced in the following discussion.

In anchored liners, three types of panel buckling should be considered (13, 14):

a) Circumferential buckling (ring mode)

b) Axial buckling (strip mode)

c) Combined buckling (lobar mode).

From a practical point of view, possible effects of buckling of the anchored liner rather than the buckling phenomenon itself are of importance because:

a) Buckling may result in excessive lateral deflections which may interfere with functional requirements of vessel components.
b) Tensile stresses may develop in the buckled region which, in combination with increased brittleness due to irradiation, may lead to brittle fracture of the liner.

c) Inplane-load carrying capacity of buckled sections will be decreased due to bending, resulting in differential forces between buckled and unbuckled regions. These forces will cause shear forces in the anchors which must be evaluated for proper anchor design.

**Theoretical and experimental studies**

The case of stud-anchored liners has been studied theoretically and experimentally by Richer and his associates at Case Western Reserve University (12, 13, 14). Solutions for two models of liner elements have been obtained: a model of a two-dimensional cylindrical shell element (panel) supported by rigid studs and a ring model (with finite or infinite radius of curvature) representing a segment of a liner supported by elastic studs. The main results obtained from the first model are the minimum buckling strains while the second model, in addition, gives anchor forces.

In the panel model, the liner is assumed buckled in the ring, strip or lobar modes. It was found that the minimum buckling strain for the ring mode is always less than or equal to the minimum buckling strain for any other mode. The conclusion was that, in a cylindrical shell, the minimum buckling strain is independent of the axial spacing of studs. This conclusion is valid as long as the axial strain is equal to or less than the circumferential strain.
The minimum buckling strain in the ring mode for a specific example is given graphically in Fig. 3 based on data from Ref. (14). In this example the radius \( R \) to thickness \( h \) ratio, \( R/h \), is 750 with \( h \) equal to 0.5 in. It is to be noted that for lower \( R/h \) ratios the predicted minimum buckling strains would be higher. Similar data using the second model (e.g. liner strip with flexible anchors) are not available.

Chan and McMinn also gave an approximate equation with which the required spacing of anchors needed to prevent buckling can be determined (10). This approximate expression is based on buckling of the flat strip as a strut in the third mode and agrees fairly well with more refined theories. The expression is,

\[
a = \frac{2.5h}{\sqrt{c}} \tag{2.1}
\]

where \( a \) is anchor spacing, \( h \) is liner thickness, and \( c \) is the uniform design strain. For comparison the minimum buckling strain obtained from the above formula is also shown graphically in Fig. 2.

**Evaluation of the stability approach to liner design problem**

The elastic buckling studies briefly described above are not directly applicable to the PCRV liners for the following reasons: a) As shown in Fig. 2 the strains at which buckling is predicted are much greater than the yield strains of steels commonly used as liners. Therefore, the liners will yield long before any elastic buckling occurs. This is supported by the experiments on liner shell and ring models (13) in which plastic deformation was observed even though anchor spacing-liner thickness ratio was from 39 to 141.
b) Even if one assumes that elastic buckling analysis is still acceptable, the theoretical minimum buckling strains are still so much greater than the design strains for liners designed to date that the results of such theoretical analyses are of little practical value. The anchor forces predicted in Ref. (13) are also too small, much smaller than the shear anchor forces obtained from one-dimensional load dissemination analysis considering adverse variations between panels (19).

Although it is conceivable to extend the above-mentioned buckling analysis into the inelastic range, such a study probably will have a limited practical application due to the following reasons:

a) Stability methods cannot take into account some of the local variations (e.g., variations of yield point, liner thickness etc.) which should be considered in liner design.

b) In cylindrical PCRV's, the design strains are not uniform in the axial direction. In the new, multi-penetration designs even the circumferential design strains are nonuniform due to large penetrations which extend through most of the vessel height. As pointed out earlier (19), nonuniform design strains introduce differential panel forces which must be resisted by anchor forces. Available stability methods however, do not consider nonuniform strains.

c) From a practical point of view, the fabrication of a PCRV liner to an assumed geometric perfection is a difficult task. It may be expected that some liner panels will have negative curvature. If the liner has an initial inward deflection between two anchors, the liner strip then becomes a column with initial lateral deflections.
d) Liner forces are partially induced by the shear anchors which constitute eccentric loading on the panel.

Whether a panel is subject to initial inward deflection or eccentricity, such a panel will develop bending stresses when the concrete-imposed strains are applied. Then the problem is reduced to inelastic bending (due to a small length-thickness ratio) rather than that of stability.

As discussed previously, the anchor spacing is usually determined on the basis of the cooling tube spacing selected. This usually results in a rather close anchor spacing. In earlier liner designs, tests were conducted to insure that "buckling" was not a problem for the close anchor spacing determined (more properly, for the anchor spacing—liner thickness ratio selected). Since in most liner designs the anchor spacings usually stay within the range established by previous tests, "buckling" of a liner with closely spaced anchors is not really a problem. Since a stress analysis must be conducted for a liner with closely spaced anchors, it may be stated that a buckling study of a liner panel should be conducted in order to establish the panel characteristics as affected by unavoidable initial liner deflections.

**Stress Analysis Methods**

The term "stress analysis methods" as used here refers to those techniques with which a section of the liner assembly is analyzed to determine forces and displacements under an assumed loading condition.
Thus, it excludes techniques which are concerned with the stability conditions of the idealized model.

The main methods found in the literature are (19):

1. One-dimensional analysis:
   a) Parker's method (1, 5)
   b) Doyle and Chu's method (7)
   c) Bechtel's method in (8)

2. Two-dimensional analysis:
   The finite element method developed at the Franklin Institute Research Laboratories (8).

All methods in the category of one-dimensional analysis consider an idealized segment of the liner either in circumferential or in meridional direction. The model consists of a strip of liner and a series of anchors (Fig. 3). In the two-dimensional analysis, a section of the liner together with the anchors and surrounding concrete is considered.

It should be noted that the main differences among the one-dimensional methods mentioned above are solution techniques employed, assumed component behavior and, assumed boundary conditions. A brief description of available stress analysis methods is given in the following paragraphs.

For convenience, in the discussion hereafter segments of liner between anchors are referred to as "panels." Those panels which are assumed to have a lower yield point and modulus of elasticity are referred to as "weak" panels, the remainder being named "strong" panels (see Fig. 3a).
Description of individual methods

a) Parker's Method: The analysis procedure involves the following steps:

1. Determine the biaxial design strains from the vessel analysis.
2. Determine the weak panel characteristics and anchor characteristics from test results. Both weak and strong panel characteristics are idealized as elastic-perfectly elastic.
3. Using the recurrence relationship of the type

\[ F_{i+1} - c_1 F_i + c_2 F_{i-1} = c_3 F_i \]  \hspace{1cm} (2.2)

set up a set of simultaneous equations. In the recurrence formula: \( F_i \) is the final force in the ith panel; \( c_1, c_2, c_3 \) are constants related to panel and anchor stiffness and curvature; \( F_{\text{d}} \) is the ith panel design force based on design strains. This recurrence formula is obtained by substituting equilibrium relations and panel and anchor stiffnesses into the the compatibility equation for node displacements. (See Ref. (5) for derivation.)

4. Since the material properties are nonlinear, solve the simultaneous equations for panel forces by a step-by-step approach.
5. The anchor forces are then determined using equilibrium conditions at the nodes \( S_n = F_{i+1} - F_i \), where \( S_n \) is the force in the anchor between panels \( i \) and \( i+1 \).

As this brief description indicates, Parker's method is a flexibility approach in which the recurrence equations are compatibility conditions in terms of unknown panel forces.
b) Doyle and Chu's Method: Although this method has been used in the design of containment liners, it should be applicable to PCRV liners as well. The procedure is summarized as follows:

1. Assume that all panels are initially at yield and that one panel buckles. Since the load-axial deformation characteristics of panels initially at yield are not known (7), the strut analogy is used for the weak panel, i.e., the buckled panel.

2. Assume that the model is symmetrical with respect to the weak panel. Thus only one-half of the section need be analyzed (Fig. 3b).

3. The equilibrium conditions applied at each node result in the following set of equations:

\[
U_1 = U_2 + \frac{a(\sigma_i - \sigma_{bp})}{E} - \frac{a}{Eb} S_1
\]

\[
U_n = \frac{U_{n+1} + U_{n-1}}{2} - \frac{a}{2Eb} S_n
\]

where: \( U_n \) = displacement of the nth anchor, \( S_n \) = force in the nth anchor, \( \sigma_i \) = initial panel stress, \( \sigma_{bp} \) = final stress in the buckled panel, \( h \) = liner thickness, \( a \) = anchor spacing along the model, \( b \) = anchor spacing perpendicular to the model, and \( E \) = modulus of elasticity.

4. Since the above set of equations involve nonlinearity an iterative procedure is used to determine anchor displacements. Experimentally obtained anchor force-deformation characteristics are used.
5. The final force in any unbuckled panel is given by

\[ F_i = F_i - Ehb \frac{U_n - U_n^1}{a} \]  
(2.5)

where: \( F_i \) = final force in panel \( i \), \( F_i \) = initial force in panel \( i \), and the other variables are as defined above.

c) Bechtel's Method: The one-dimensional model used in Bechtel's relaxation method is shown in Fig. 4a. The method is developed for the design of containment liners. A brief summary of the development of theory is given below:

Assuming that anchor movements are completely prevented, the stress in the strong panel will be:

\[ \sigma_\theta = \frac{E}{1 - \nu^2} (\varepsilon_{c\theta} + \varepsilon_{c2}) \]  
(2.6)

where: \( \sigma_\theta \) = circumferential stress, \( \varepsilon_{c\theta} \) = circumferential design strain, \( \varepsilon_{c2} \) = axial design strain, and \( \nu \) = Poisson's ratio. The computed stress is assumed to exist regardless of whether it exceeds the actual yield point. This assumption implies that all strong panels remain elastic. The initial membrane force per unit width is

\[ N = h\sigma_\theta \]  
(2.7)

where: \( N \) = strong panel force, and \( h \) = thickness of plate.

If the anchor at one end of the weak panel is released, the displacement at this anchor (anchor 1) will be

\[ \delta_{11} = \frac{N}{K_C + K_B + K_R} \]  
(2.8)
where: \( \delta_{ij} \) = displacement at anchor i due to release of anchor j, 
\( K_C \) = anchor stiffness, \( K_B \) = weak panel stiffness, and \( K_R \) = strong panel stiffness.

Now assume that the second anchor (anchor 2) is released, leaving the first anchor free to deform. It can be shown that the total displacement at anchor 1 due to movements at anchors 1 and 2 will be

\[
\delta_{11} + \delta_{12} = \frac{N}{K_C + K_B + K_R} (1 + B) \tag{2.9}
\]

where:

\[
B = \frac{K_R^2}{(K_C + K_R)K + (K_C + K_R)(K_C + K_B + K_R)}
\]

In a similar manner, the total displacement at anchor 1 due to movements at anchors 1, 2, and 3 is found to be

\[
\delta_{11} + \delta_{12} + \delta_{13} = \frac{N}{K_C + K_B + K_R} (1 + B + B^2). \tag{2.10}
\]

In obtaining the last relationship, the following term appears in the denominator and is disregarded, conservatively, since it is greater than unity for the range of variables involved:

\[
A = \frac{N}{K_C + K_B + K_R} \left[ \frac{K_R(K_C + K_B)K + K_R(K_C + K_B + K_R) + K_R(K_C + K_B + K_R)}{(K_C + K_B)K + (K_C + K_B)(K_C + K_B + K_R)} \right]
\]

(2.11)

Considering an infinite number of anchors, the total displacement at anchor 1, \( \delta \), will be

\[
\delta = \frac{N}{K_C + K_B + K_R} \left[ 1 + \sum_{n=1}^{\infty} B^n \right] \tag{2.12}
\]

Now let
\[ N' = N \left[ 1 + \sum_{n=1}^{\infty} B^n \right] \]  
(2.13)

thus the problem is reduced to solving the following relationship:

\[ \delta = \frac{N'}{K_C + K_B + K_R} \]  
(2.14)

The actual computation involves the following steps:

1. Determine biaxial strains from vessel analysis.

2. Determine \( K_C \) and \( K_B \) from test results. (\( K_C \) is taken to be the initial slope of the load-displacement curve and \( K_B \) is twice the initial slope since both ends of the weak panel move toward each other.) Calculate \( K_R = \frac{Eh}{a} \) (ignoring the effect of Poisson's ratio).

3. Determine imaginary force \( N \) to be applied at anchor 1.

4. Find \( \delta \), displacement at anchor 1 due to movements of all anchors:

\[ \delta = \frac{N'}{K_C + K_B + K_R} \]  
(2.15)

In determining \( \delta \), first solve for \( \delta \) using initial stiffnesses for \( K_C \) and \( K_B \). If the computed value of \( \delta \) exceeds the elastic limit for any component, then a plastic analysis is required. For example, if \( \delta \) thus computed exceeds the elastic limit, \( \delta_e \), for the anchor, the above equation is rewritten as

\[ \delta_e K_C + (\delta - \delta_e)K_C' + \delta(K_B + K_R) = N' \]  
(2.16)

where \( K_C' \) is the anchor stiffness in the inelastic range, and the equation is solved for \( \delta \).
5. Using the displacement thus obtained, the anchor force in anchor 1 is obtained from

\[ S = K \delta \]  \hspace{1cm} (2.17)

if \( \delta \leq \delta_e \); or a similar relation, if \( \delta > \delta_e \). For example, for the case illustrated in step 4, the anchor force would be

\[ S = \delta_e K + (\delta - \delta_e)K' \]

(2.18)

As evidenced from the discussion here, Bechtel's method gives the anchor force in anchor 1 only. Forces at other anchors and in liner panels can be determined once the force in anchor 1 is known.

d) Finite Element Method: Unfortunately, details of the finite element method, developed at the Franklin Institute Research Laboratories, are not available. The significant differences between this approach and one-dimensional analysis appear to be the following:

1. One-dimensional analysis shows that the number of panels included in the model has an appreciable effect on the results. The predicted anchor forces and displacements increase with increasing number of panels (19). In the finite element approach, which was used in the design of the Fort Calhoun PCCV liner, only five panels and four anchors were considered. The accuracy of the method, using relatively fewer panels, should be evaluated.

2. In the finite element approach, stiffnesses of all elements are determined theoretically (assuming a lower modulus of elasticity and a lower yield point for the weak panel). In
the one-dimensional analysis experimentally determined anchor and weak panel characteristics are usually used.

3. The finite element method takes into account concrete behavior behind the liner and anchors. In one-dimensional methods, on the other hand, the effect of the concrete behavior is assumed to be indirectly taken into account by using experimentally determined component characteristics.

4. In the two-dimensional analysis, there is a need to use a yield criterion for steel and concrete because of the existence of biaxial stress field. In one-dimensional analysis such a problem, obviously, does not exist.

h) Other Methods: The stress analysis methods briefly described above are those which have been actually used in the design of reactor or containment vessel liners. There are other methods proposed for the design of liners which are not discussed in this study (3, 20).

Discussion of individual methods

a) Parker's method: This method of analysis is possibly the most versatile stress analysis method. In addition to variable design strains for panels, variations 1, 2, 3, 4, 7, and 9 (see next section) can be directly taken into account. These variations have been considered in the design of British PCRV's (5). It appears that the effect of variations 5 and 6 are not considered as was discussed in Ref. (21). The effect of local loads does not appear to be discussed in the literature; however, it is believed that local loads are taken
into account using theory of elasticity principles as was done in the design of the Fort St. Vrain liner.

One difficulty with Parker's method is in solving simultaneous equations which involve nonlinear coefficients. However, the theory can be reformulated using displacement principles and the resulting simultaneous equations solved by the initial stress method as described in Chapter III.

b) Doyle and Chu's method: This method of analysis is essentially a stiffness reformulation of Parker's approach (21). The basic difference is in the assumption of uniform initial panel forces made by Doyle and Chu. As a result of this assumption, their model is restricted to a liner segment symmetrical about a "weak" panel. Thus, this approach may be used in conducting stress analysis in the circumferential direction. Modification of the technique is necessary if it is applied in the meridional direction where design strain gradients exist.

c) Bechtel's method: The analysis technique used by Bechtel in the stress analysis of containment vessel liners is a relaxation method based on uniform design strains. In this method, local variations in liner thicknesses and yield points can be taken into account but not variations in anchor spacing and stiffnesses. The effect of assumed inward curvature is included by using experimental characteristics for the weak panel. The effect of lateral pressure is considered by defining an equivalent inplane force based on elastic relationships. The effects of all other variations were not considered, probably because they do not affect the containment vessel liners.
In using Bechtel's relaxation method, the following points should be kept in mind:

1. This method assumes uniform design strain. If there is strain gradient, modification of the method is necessary.

2. The strong panel force, \( N \), is determined using elastic relationships. This implies that all panels except the weak panel remain elastic regardless of the magnitude of the design strains. It has been shown that such an assumption may be too conservative (19).

3. In the derivation of the imaginary force, \( N' \), which is applied at the node between the weak and the adjacent strong panel, the term "A" was found to be greater than unity and was disregarded, conservatively. The term "A" in PCRV liners, however, can be shown to be always less than unity and thus computing the \( N' \) force from the relationship given above may be on the unsafe side. Therefore, care should be taken in evaluating this force.

4. The equation for determining \( N' \) is based on linear elastic relationships. Therefore, the accuracy of the method is not established in the case of nonlinear (or inelastic) component behavior. This fact is of importance since the behavior of anchors under shear loading is nonlinear.
Design Loads and Local Effects

It was mentioned previously that the usual procedure in liner design is to first determine the liner design strains and displacements from overall vessel analysis and then to analyze the liner assembly using design strains and displacements as the loading.

The types of loads considered in determining the liner design stresses or strains are:

a. dead loads
b. prestressing loads
c. creep and shrinkage of concrete
d. thermal loads
e. internal pressure.

The effect of the following loads on the liner assembly during the construction period are also considered:

f. live loads
g. wind loads.

It is to be noted that earthquake effects do not appear to be considered in past liner designs.

General discussion on these loads are given in the following paragraphs.

Dead loads and live loads

The effects of dead and live loads during construction period are seldom discussed in the literature. It is stated that these loads would largely depend upon the method of construction and, thus, stress analysis at various stages of the construction is essential (16).
Residual stresses induced in the liners during the construction (before the concrete has gained its strength) are not considered (2, 16), although compressive residual stress is considered to have beneficial effect on fatigue strength (2).

Dead load stresses in the completed vessel (after the concrete has gained its strength) were considered in the design of Fort St. Vrain PCRV liner. Maximum average strain was shown to be in the order of 20 μin./in., which may be considered small in comparison with strains due to other loads. Live load stresses were negligible in the completed structure but were considered during the construction period (16).

Wind loads

Wind loads are normally considered during construction of the vessel only.

Prestressing loads

The magnitude of the required prestress in a vessel is based on design criteria adopted for the vessel. The design criteria, including the magnitude of the design pressure differ substantially among vessels which have been constructed. Furthermore, different prestressing systems have been used. As a result, the magnitude and gradient of imposed liner strains under prestress vary considerably from design to design.
Creep loads

Creep is defined as the increase in concrete strain under sustained loads. Although creep and shrinkage are interrelated, their effects are usually considered separately.

The magnitude of creep-induced strains (and stresses) in the liner are considerable and time- and temperature-dependent. For example, in the Fort St. Vrain liner design analysis strains in the liner due to creep were twice (after 5 years) and three (after 30 years) times as much as the effective prestressing strain.

Shrinkage loads

The amount of concrete shrinkage is usually determined based on test results. As indicated above, shrinkage strain in the concrete is considered additive to creep strain. In the Fort. St. Vrain analysis, shrinkage-induced strains in the liner were assumed to be 150 \( \mu \text{in./in.} \) and the resulting stress (4000 psi) was simply added to analytical results obtained from other loads (16).

In British practice, strains in the liner due to concrete shrinkage, were assumed to be as high as 400 \( \mu \text{in./in.} \) (22, 23).

Thermal loads

Although the temperature in the liner plate is nonuniform, for design purposes a uniform "effective" (design) liner temperature is usually assumed. The effective liner temperature, together with the boundary conditions on the exterior surface of vessel, determine the thermal gradient across the vessel wall. The effect of this thermal gradient is to induce further compressive strains in the liner.
The magnitude of these strains have usually been determined from axisymmetric analysis of the vessel (5, 16).

Thermal effects due to restrained thermal expansion of the liner are also considered in liner design.

**Internal pressure**

The effect of internal pressure, ranging from vacuum to design pressure is considered in the design of liners. Internal pressure normally causes tensile stresses in the liner and thus, is not included in the loading combination from which maximum design compressive strains in the liner are determined. However, variation in liner design strains due to pressure fluctuation is important for fatigue considerations. The possibility of tensile stresses in the liner under internal pressure, in combination with other loads, have been investigated. In the Fort St. Vrain design it is concluded that the liner would be in biaxial compression throughout the vessel's design life (16). In the Hinkley Point B design, on the other hand, small tensile stresses in the liner were predicted under internal pressure, when the liner was assumed to be cold (5).

The list of loads discussed above essentially agrees with the list of loads\(^1\) which must be considered in liner design according to the proposed code (24). The only additional load which is included in certain loading combinations suggested in the proposed code is the earthquake load.

---

\(^1\) Loads which are not relevant to liner design are excluded.
Earthquake load is rarely discussed in the design of liners. In the Fort St. Vrain PCRV liner design, seismic stresses were computed for concrete alone, neglecting the effect of tendons, liner and reinforcement (16). The maximum stress in the concrete near the liner at the bottom haunch area was given as 270 psi for the working stress design condition (horizontal acceleration of 0.05 g and vertical acceleration of 0.033 g with 2 percent critical damping). If the modular ratio is assumed to be 8, the resulting principal stress in the adjacent liner will be about 2200 psi. This maximum stress corresponds to about 70 μin./in. biaxial strain in the liner.

Although the magnitude of strains due to seismic loads is relatively low, they should nevertheless be considered and included in design strains.

**Local effects**

Types of postulated local variations and their combinations with design strains under different loading conditions have varied in past liner designs. The variations considered by British designers and those considered in the Fort St. Vrain PCRV liner design are shown in Table 1. The types of local effects considered in stress analysis differs among past designs. The British designers have assumed the possibility of inelastic bending of individual panels and considered adverse effects of variation in liner properties and of initial inward curvature. In the Fort St. Vrain design these factors were not considered.
Table 1. Liner design criteria — local variations

<table>
<thead>
<tr>
<th>Item</th>
<th>British criteria</th>
<th>Fort St. Vrain criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Variation in liner thickness</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>2. Variation in liner yield point</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>3. Variation in spacing and stiffness of anchors</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>4. Initial inward curvature</td>
<td>Yes</td>
<td>No&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td>5. Concrete void behind liner</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>6. Water pressure behind liner</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>7. Local hot spots</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>8. Local loads</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>9. Loss of an anchor</td>
<td>Not known</td>
<td>Yes</td>
</tr>
</tbody>
</table>

<sup>a</sup>Effect of initial inward curvature is discussed qualitatively in Ref. 16.

In British practice water pressure and the possibility of a concrete void behind the liner are not considered. Cooling tube leakage is assumed not to be possible. If unacceptable voids are detected behind the liner, the voids are filled with grout via holes drilled in the liner.

The effect of local variations listed in Table 1, in general, is to increase forces and displacements in the liner components. The manner in which these effects may be taken into account and the resulting forces and displacements are discussed in Chapter VI.
Behavior of Liner Components

The stress analysis methods used in liner design were summarized in the preceding sections. It is to be noted that, in all methods, equilibrium or compatibility equations involve the stiffnesses of liner components. The manner in which these stiffnesses are determined and related assumptions used by previous investigators are summarized in this section.

Liner panels

a) **Strong panels**: Review of stress analysis methods indicate that three different assumptions have been used regarding the behavior of strong panels. These are:

1) Elastic-perfectly plastic [Parker (5) and this study]
2) Elastic under all loading conditions [Bechtel in (8)].
3) All panels yield before a weak panel is formed, implying that design strains beyond yield do not affect the behavior of the structure [Doyle and Chu (7)].

It has been shown that the second assumption results in the highest anchor forces which are conservative (19). It can be shown that if Bechtel's assumption is used in a case where design strains are greater than yield strain of the liner steel, the final strains in the panels away from a weak panel are higher than the yield strain. This indicates that these panels will actually be in yield, thus contradicting the assumption.
Doyle and Chu's assumption is correct when all panels yield under applied strains and then one panel (the weak one) buckles inelastically. However, in PCRV liners, such a behavior is unlikely. Any adverse variation will cause at least some panels to deflect laterally introducing bending moments. The latter in turn reduces the axial load capacity of the panel and since the yield load for the adjacent flat panel is larger, the adjacent panel can never yield. For this reason, Doyle and Chu's approach actually leads to anchor forces which are on the unsafe side for design strains greater than the biaxial yield strains, as was shown in (19).

b) Weak panels: The behavior of an assumed "weak" panel in the analytical models are determined either experimentally (5, 8) or, theoretically using strut analogy (2, 3, 7, 19).

The use of load-axial deformation relation of a column with rectangular cross section in place of that of an actual panel is called "strut analogy" in various publications. By introducing an eccentricity and/or an initial deflection, load-deformation relations for struts can be analytically determined (25-28).

In the course of this study a computer program was developed, based on Jezek's work (25, 26), for the analysis of fixed-ended and simply supported struts. In this program the effect of uniform lateral loads (simulating internal pressure and water pressure behind the liner) was included. Details of the theoretical development are included in Appendix A. It is to be noted that similar studies have been made in connection with the design of the Dungeness B Vessel in the United Kingdom (3), but no detailed formulation is available.
The validity of the assumption that stress-strain behavior of a liner panel can be based on the behavior of an equivalent fixed-end strut is indicated in Fig. 5. The test result is taken from Ref. (5). It is understood that panel width-thickness ratio was between 12 and 18. Strut analysis for these two ratios gives adequate results as shown in the figure.

It must be noted that the above statement is based on the results of one test only. No other test data seems to be available. Furthermore, the test results are given only up to a maximum strain of 2500 μin./in. (Fig. 5). In analysis however, the apparent strain may be much higher than this value (19). Therefore, there appears to be a need for more experimental research in this area in order to establish weak panel behavior and to verify the validity of strut analogy.

In the design analysis process the characteristics of such panels, whether obtained from tests or analytically, are usually idealized. In some British designs this idealization has resulted in elastic-perfectly plastic behavior, assuming no load dropoff, as shown in Fig. 6. It is to be noted that both initial stiffness and imaginary yield point of a weak panel are less than that of a strong panel. This is also shown in Fig. 6.

Other investigators have assumed load dropoff in the weak panel\(^1\), by making use of strut analogy (7, 19) or test results (2, 8).

\(^1\)The no-load-dropoff and load-dropoff curves (apparently based on test results) given in Refs. (5) and (15) seem to be contradictory. However, this difference could be completely attributable to length-thickness ratios of the panels tested. Load dropoff may tend to occur for larger anchor spacing-liner thickness ratios. Unfortunately, the test panel data are not given.
In the local effect studies discussed in Chapter VI, strut analogy (see Appendix A) was used to obtain weak panel characteristics. A trilinear approximation to one of these curves is also shown in Fig. 6.

**Stud anchors**

In most load dissipation analysis methods only inplane (shear-slip) characteristics of stud anchors have been considered (2, 5, 7, 15, 16, 19). It was stated that the actual stress field in the anchors is very complicated and thus its behavior is best determined by experimental studies (5). For this reason, in all designs experimentally obtained stud shear-deformation data has been used.

In this study the same approach has been taken. The load-displacement relationships of anchors were taken from Refs. (7) and (29), and approximated by four line segments as shown in Fig. 7. The same approximation was used in one- and two-dimensional methods given in Chapters III and IV.

**Cooling tubes**

In one-dimensional methods described in previous sections the weak panel is usually assumed to occur between adjacent stud anchors, not between adjacent cooling tubes or between a cooling tube and the nearest stud. The frictional force between the tubes and concrete is considered to be insufficient and unreliable to resist the movement of the cooling tube in the direction normal to the liner surface, if such tendency exists. Based on this reasoning, buckling or
inelastic bending between cooling tubes is never considered in British design (30).

The shearing resistance of cooling tubes in the plane of the liner has been taken into account in some British designs (30) and in the Fort St. Vrain design (16).

It appears to be debatable whether contribution of the cooling tubes in resisting differential panel forces (together with stud anchor) should be taken into account. It was suggested by some designers that shear resistance of the cooling tubes is not reliable because concrete adjacent to the cooling tubes cannot develop the predicted shear loads (30).

The problem may be better understood by considering the shear stresses in concrete between cooling tubes. If the friction between the liner and concrete is ignored it is clear that bearing forces on the concrete in contact with the tube must be equilibrated by concrete shearing stresses between two tubes. It is possible that, in some cases, the magnitude of predicted shearing stresses will be greater than that allowed by the proposed code (24).

Even if the calculation indicates that the concrete shear stress is within allowable, there is always the possibility of concrete void adjacent to tube wall, especially underneath horizontal cooling tubes.

It may be concluded therefore, that cooling tubes should not be considered as anchors in stress analysis. This will, of course, increase the calculated forces in stud anchors and is therefore conservative.
In this study shearing resistance of cooling tubes in the plane of the liner plate has been ignored.
CHAPTER III.

ONE-DIMENSIONAL INITIAL STRESS METHOD OF ANALYSIS

As indicated previously, in design analysis of PCRV liners, a unit width strip together with its anchors is usually used as the analysis model. It was also pointed out that among the methods developed for one-dimensional analysis Parker's flexibility approach is possibly the most versatile one.

The one-dimensional method of analysis discussed in this chapter uses a theoretical basis similar to Parker's (1, 5). The present theory has been formulated using a stiffness approach and the solution technique has been based on the so-called initial stress method of nonlinear analysis.

Assumptions

The analysis model consists of a one-dimensional segment of the liner with shear anchors assumed to act over the model width, as shown in Fig. 8. The following assumptions are made regarding the model and behavior of its components:

a) The liner panels are subject to design strains imposed by the backing concrete.

b) The friction between the liner plate and backing concrete is negligible.

c) The effect of liner curvature is neglected.
d) The stress-strain relationships of liner panels are either bilinear (strong panels) or trilinear (weak panels), as shown in Fig. 6.

e) The anchors provide only inplane resistance to displacements. Anchor characteristics are determined experimentally and approximated by four linear segments as shown in Fig. 7.

Assumptions a, b, c, and e are the same as those used in other one-dimensional methods (5, 7, 8, 19). In the case of strong panels, elastic (8) or elastic-perfectly plastic (5) behavior has been assumed. For weak panels, elastic-perfectly plastic behavior (5) or a relationship in which load dropoff occurs (8, 19) has been used. The trilinear idealization used in this study is based on strut analogy.

Formulation of the Problem

Equilibrium at the nth node (Fig. 8b) requires

\[ F_{i+1} - F_i - S_n = 0 \]  \hspace{1cm} (3.1)

where \( F_{i+1} \) and \( F_i \) are the panel forces in the \( i+1 \)th and \( i \)th panels, respectively. Compressive panel forces are considered to be positive. \( S_n \) represents the shear force in the anchor at the nth node, considered positive when the corresponding node displacement is positive.

Considering a completely elastic analysis, the stiffness of the \( i \)th panel per unit width, \( C_i \), is given by

\[ C_i = \frac{E_i h_i}{1 - \mu^2} \]  \hspace{1cm} (3.2)
where \( E \) is the modulus of elasticity, \( h \) is the liner thickness, and \( \mu \) is Poisson's ratio.

By letting \( K_n \) represent the anchor stiffness per unit width, and \( U_n \) the relative node displacement at the \( n \)th node (positive if in the negative \( x \)-direction), Eq. (1) can be rewritten as follows:

\[
C_{i+1}^r e_{i+1} - C_i^r e_i - K_n U_n = 0
\]  

(3.3)

where \( e \) represents the panel strain (compressive positive).

Solving for \( e_{i+1} \) gives

\[
e_{i+1} = \frac{C_i}{C_{i+1}} e_i + \frac{K_n}{C_{i+1}} U_n
e
\]  

(3.4)

Compatibility requires that, for panel \( i \)

\[
U_n - U_{n-1} = (e_i - e_{ci} a_i)
\]  

(3.5)

where \( a_i \) is the length of the \( i \)th panel, and \( e_{ci} \) is the concrete imposed design strain in the \( i \)th liner panel.

Similarly, for panel \( i+1 \),

\[
U_{n+1} - U_n = (e_{i+1} - e_{ci+1} a_{i+1})
\]  

(3.6)

Eliminating \( e \) from Eqs. (3.4)-(3.6),

\[
\frac{C_i}{a_i} U_{n-1} - \left( \frac{C_i}{a_i} + K_n \frac{C_{i+1}}{a_{i+1}} \right) U_n + \frac{C_{i+1}}{a_{i+1}} U_{n+1} = C_i e_{ci} - C_{i+1} e_{ci+1}
\]  

(3.7)

or in a more simplified form

\[
b_1 U_{n-1} + b_2 U_n + b_3 U_{n+1} = R_n^0
\]  

(3.8)

\(^1" Relative" implies that the displacement is additional displacement with respect to the deformed concrete and is caused by anchor deformation.
where $b_1$, $b_2$, and $b_3$ are elastic stiffness coefficients, and $R_n^0$ is defined as the residual node force at the $n$th node caused by the difference in panel forces acting at that node, and is positive if directed in positive $x$-direction.

If $m$ represents the number of interior nodes used in the model, using Eq. (3.8), $m$ simultaneous equations can be written which, in matrix form, will be

$$[B]\{U_n\} = \{R_n^0\} \tag{3.9}$$

where $[B]$ is a square matrix of elastic stiffness coefficients.

Equation (3.9) involve $m+2$ unknown displacements. The two additional equations are obtained from the boundary conditions\(^1\) which are

a) For fixed ends:

$$U_o = U_{m+1} = 0$$

b) For free ends [using Eq. (3.5) and the condition that panel force is zero]:

$$U_o = U_1 + \varepsilon_{1} a_1$$

$$U_{m+1} = U_m - \varepsilon_{m} a_j$$

The solution for nodal displacements is given by

\(^1\)In most linear problems the fixed end condition is the most practical one as discussed in later chapters. Free end condition may be assumed when the last panel is attached to a member with negligible inplane stiffness. In the case of flexible end attachments, an imaginary panel may be added to the last panel and the end of the imaginary panel is treated as a free end. The flexible attachment may then be treated as another shear anchor.
\[ U_n = [B]^{-1}r_n^0 \]  

(3.10)

providing all panels and anchors remain in the elastic range. After solving for nodal displacements, final panel and anchor forces may be obtained using elastic relationships.

**Solution Technique**

In the analysis of PCRV liners, however, the design strains may be such that a solution is necessary where one or more panels or anchors are stressed into the inelastic range. Because of the non-linearity involved, Eq. (3.9) is no longer valid. There are three different methods of nonlinear analysis (31, 32), which have been developed mainly for the finite element analysis of inelastic continua. These are:

a. Variable Stiffness Method,

b. Initial Strain Method, and

c. Initial Stress Method.

The first of these methods requires an incremental analysis in which, after each increment, the stiffness matrix is adjusted. This results in a costly computer program. The Initial Strain Method is not suited for the analysis where perfect plasticity is involved. For these reasons, the Initial Stress Method of analysis has been adopted in this study.

Application of the Initial Stress Method to the one-dimensional problem is best described in terms of a step-by-step procedure.
Step 1. Assuming everything remains elastic, establish the initial elastic stiffness matrix \([B]\).  

Step 2. Assuming all panels remain elastic, calculate the "apparent" (initial) residual node forces \([R^o_n]\).  

Step 3. a) Calculate the "apparent" (elastic) node displacements \([U^o_n]\) by Eq. (3.10).  

b) Calculate the "apparent" (elastic) panel strains by [using Eq. (3.5) and matrix formulation].  

\[
\{\varepsilon^o_i\} = \{\varepsilon^e_i\} + [A]\{U^o_n - U^o_{n-1}\}
\]  

(3.11)  

where \([A]\) is a diagonal matrix whose nonzero elements are \(1/a_i\).  

c) The "apparent" (elastic) panel force in the \(i\)th panel is given by \(F^i = \varepsilon^o_i C^i\). The actual panel force, \(F^i\), is determined using the actual stress-strain relationship:  

\[
F^i = f^i(\varepsilon^o_i)
\]  

(3.12)  

where \(f^i(\varepsilon^o_i)\) is the nonlinear stiffness function for panel \(i\).  

d) Similarly, the "apparent" (elastic) anchor force is given by \(S^o_n = U^o_n K^n\), where \(K^n\) is the initial anchor stiffness. The actual anchor force, \(S^o_n\), is given by  

\[
S^o_n = g^o_n(U^o_n)
\]  

(3.13)  

where \(g^o_n(U^o_n)\) is the nonlinear stiffness function for anchor \(n\).
Step 4. If one or more panels or anchors are beyond yield, the difference between actual and apparent forces will result in some fictitious forces (holding forces) which must be assumed to exist for equilibrium. Since these forces do not actually exist, they must be removed by the application of forces at the relevant nodes which are the opposites of the fictitious forces. The force to be applied at the nth node is given by

\[ R_n = (P_{i+1}' - F_{i+1}) - (P_i' - F_i) - (S_n' - S_n) \quad (3.14) \]

If all elements of \( R_n \) are zero the problem is elastic and no further computation is necessary.

Step 5. a) If \( R_n \) is not zero, the additional incremental displacements are given by

\[ \{\Delta U_n^1\} = [B]^{-1}\{R_n^1\} \quad (3.15) \]

where, again, the initial elastic stiffness matrix \([B]\) is used.

The total corrected node displacement is then given by

\[ \{U_n^1\} = \{U_n^0\} + \{\Delta U_n^1\} \quad (3.16) \]

b) Due to \( \{\Delta U_n^1\} \), there will be changes in panel strains given by

\[ \{\Delta \varepsilon_i^1\} = [A] \{\Delta U_n^1 - \Delta U_{n-1}^1\} \quad (3.17) \]

and the total panel strains are

\[ \{\varepsilon_i^1\} = \{\varepsilon_i^0\} + \{\Delta \varepsilon_i^1\} \quad (3.18) \]
c) From these new values of panel strains and node displacements, the new actual and apparent panel and anchor forces are computed as before.

d) A new set of residual nodal forces, \( \{ R_n^2 \} \), can now be obtained. If \( \{ R_n^2 \} \) is negligibly small, no further computation is necessary.

**Step 6.** If \( \{ R_n^2 \} \) is not negligible, repeat step 5, as the next iteration.

In general, Eq. (3.15) can be rewritten as

\[
\{ \Delta U_n^j \} = [B]^{-1} \{ R_n^j \} \tag{3.19}
\]

where \( j \) represents the \( j \)th iteration. The node displacements are continuously summed by

\[
\{ U_n^j \} = \{ U_n^{j-1} \} + \{ \Delta U_n^j \} \tag{3.20}
\]

and the panel strains by

\[
\{ e_i^j \} = \{ e_i^{j-1} \} + \{ \Delta e_i^j \} \tag{3.21}
\]

This process is repeated until

\[
\{ R_n^j \} = \{ (F'_{i+1} - F_{i+1}^{j-1}) - (F'_{i} - F_{i}^{j-1}) - (S'_{n} - S_{n}^{j-1}) \} \tag{3.22}
\]

becomes negligible. The results are sets of displacements, strains, panel forces, and anchor forces which satisfy all equilibrium and compatibility conditions, as well as the nonlinear panel and anchor characteristics.
Criteria for Solution Accuracy

The magnitudes of residual nodal forces, $R^j_n$, served as the natural criterion for convergence, since iteration is to be stopped when all values of $R^j_n$ are negligible. It was found in this study that, within the range of variables studied, the computed maximum anchor force is within 1% of the estimated correct anchor force when all values of residual nodal forces are less than 0.05 kips. The number of iterations required to obtain this value varies considerably depending on the magnitudes of various design parameters. In most cases, however, the convergence was reached in less than 200 iterations.

Comparison to Another Method

The following case was analyzed by Doyle and Chu with fixed end conditions (7)

Liner thickness = 0.375 in.
Anchor spacing in the direction of model = 21.0 in.
Anchor spacing in the width direction of model = 10.5
Size of anchors = 3/4 in. diameter studs
Liner yield stress = 43.0 ksi
Design strain = uniaxial yield strain
Weak panel has zero load carrying capacity
Number of anchors in model = 20

The results using the initial stress method are compared to Doyle and Chu's results as follows:
<table>
<thead>
<tr>
<th>Method</th>
<th>Maximum anchor force (kips)</th>
<th>Maximum anchor displacement (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Doyle and Chu</td>
<td>25.433</td>
<td>0.1103</td>
</tr>
<tr>
<td>Initial stress method</td>
<td>25.904</td>
<td>0.118</td>
</tr>
</tbody>
</table>

As can be seen from this data, the analysis results from the two methods agree very well. The differences are probably due to the differences in idealizing the anchor load-displacement characteristics.
CHAPTER IV.
ELASTO PLASTIC FINITE ELEMENT METHOD OF ANALYSIS

The stress analysis techniques developed to date, as noted before, are mostly one-dimensional, ignoring the effect of the biaxial stress field. It has been stated that this approach results in a conservative design (8, 16). The forces and displacements determined by one-dimensional analysis are substantial for some loading conditions and therefore, the designer is faced with the problem of revising the design (mainly, by providing additional anchors) which usually leads to increased costs.

It is also noted that in some sections of the liner such as those in the vicinity of internal component supports, representing the actual structure by a one-dimensional model is somewhat unrealistic.

In order to evaluate the adequacy of and conservatism associated with the one-dimensional analysis method, a two-dimensional analysis technique may be used. Such a method should be based on assumptions similar to those used in the one-dimensional methods. A new analysis technique which is based on the finite element method was developed during the course of this study. In this method a segment of the liner plate, separate from the backing concrete is considered. The liner plate is stressed by concrete-imposed strains and is restrained by flexible anchors at the nodes. Material nonlinearities are dealt with using the initial stress method as was also used in the one-dimensional method presented in Chapter III. The method, as presented
herein, is applicable to stud-supported liners. Application to liners with continuous anchors would require minor revisions in the method.

Assumptions

The analysis model consists of a two-dimensional liner segment together with flexible anchors at the nodes (Fig. 9). The following assumptions are made regarding the model and the behavior of its components:

a) The liner assembly is subject to design strains imposed by the backing concrete. The strain within a triangular plate element is constant.

b) The friction between the liner plate and the backing concrete is negligible.

c) The stud anchors provide resistance to displacements, only in the plane of the liner. Anchor characteristics are determined experimentally and represented by four linear segments as shown in Fig. 7.

d) At the boundaries of the section analyzed, the relative displacement between the liner and concrete is negligible.

e) The effect of liner curvature is neglected.

f) The liner panel behavior (both weak and strong) is elastic-perfectly plastic and von Mises yield criterion is applicable.

The first three assumptions listed above are the same as those used in one-dimensional analysis. Assumption d) implies that an adequate number of panels should be included in the model analyzed.
(see Chapter V). Experimental results and one-dimensional analysis show that ignoring the effect of liner curvature (assumption e) is conservative (2). Elastic-perfectly plastic liner behavior and von Mises yield criterion (assumption f) have also been assumed in previous studies (5, 8).

The problem as defined above is a plane stress plate problem with fixed edge conditions and flexible supports at discrete interior points. It is obvious that if all panel characteristics are the same and design strains are uniform, no stresses will exist in the anchors and the plate will be subject to uniform compressive stress field. On the other hand, if design strains and/or panel behavior are non-uniform, differential inplane forces will develop in adjacent panels which must be resisted by the intervening anchors.

Formulation of the Problem

It can be shown that the plane stress problem in finite element formulation is reduced to solving a set of simultaneous equations of the form

\[ [K_T] \{ \delta_T \} + \{ F_{CT} \} - \{ R_T \} = 0 \]  

(4.1)

where

\[ [K_T] = [K_p] + [K_s] \]

\[ [K_T] \text{ = total stiffness matrix} \]

\[ [K_p] \text{ = plate stiffness matrix} \]

\[ [K_s] \text{ = anchor stiffness matrix} \]

\[ ^1 \text{The principles of discrete element-matrix displacement method are well established and will not be repeated here.} \]
\[
\{\delta_T\} = \begin{bmatrix} \delta \end{bmatrix}
\]

\(
\{\delta_T\} = \text{total nodal displacement vector}
\)

\(
\{\delta\} = \text{unknown nodal displacement vector}
\)

\(
\{\delta_R\} = \{0\} = \text{nodal displacement vector corresponding to boundary restraints}
\)

\[
\{F_{CT}\} = \begin{bmatrix} F_{CD} \\ F_{CR} \end{bmatrix}
\]

\(
\{F_{CT}\} = \text{total initial nodal force vector}
\)

\(
\{F_{CD}\} = \text{initial nodal force vector corresponding to unknown nodal displacements}
\)

\(
\{F_{CR}\} = \text{initial nodal force vector corresponding to boundary restraints}
\)

\[
\{R_T\} = \begin{bmatrix} R \\ R \end{bmatrix}
\]

\(
\{R_T\} = \text{total external nodal force vector}
\)

\(
\{R\} = \{0\} = \text{external nodal force vector corresponding to unknown nodal displacements}
\)

\(
\{R\} = \text{external nodal force vector corresponding to boundary restraints.}
\)

Displacements and nodal forces are positive if they are in the positive \(x\) and \(y\) directions (see Fig. 9).

The above equation may be partitioned to obtain (33):

\[
\begin{bmatrix}
K & K_{DR} \\
K_{RD} & K_{RR}
\end{bmatrix}
\begin{bmatrix}
\delta \\
0
\end{bmatrix}
+ \begin{bmatrix}
F_{CD} \\
F_{CR}
\end{bmatrix}
- \begin{bmatrix}
0 \\
R
\end{bmatrix}
= 0
\]

\[\text{(4.2)}\]
where $[K]$ is a square, symmetric matrix which corresponds to unknown nodal displacements. Also, $[K_{RD}]$ is a rectangular submatrix that corresponds to external reactions due to unit values of unknown displacements, $[K_{DR}]$ represents forces corresponding to unknown displacements due to unit displacements of the support restraints and, finally, $[K_{RR}]$ is a square submatrix which contain forces corresponding to support restraints due to unit displacements at support restraints.

The external force vector corresponding to unknown displacements is zero because there are no external forces applied at the corresponding nodes. It is also noted that, in Eq. (4.2), the number of elements of vector $\delta$ represents the degrees of freedom and that of vector $R_R$ represents number of zero displacements.

In the problem at hand only the unknown nodal displacements, $\{\delta\}$, are of interest because the stresses in the plate elements and forces in the anchors can be determined using these displacements. The computation of vector $R_R$ is not necessary because these forces are not used in the nodal equilibrium equations.

The unknown nodal displacements may be obtained from [via Eq. (4.2)]

$$[K]\{\delta\} + \{F_{CD}\} = 0 \quad (4.3)$$

The elastic solution to the above equation is given by

$$\{\delta\} = - [K]^{-1}\{F_{CD}\} \quad (4.4)$$

Once the unknown nodal displacements are determined, the elastic stresses and forces may be obtained using the following equations:
\[
\{e\}^e = [B]^e\{\delta\}^e 
\]

\[
\{\sigma\}^e = [D]\{(e)^e - \{\varepsilon\}^e\} 
\]

\[
\{S\} = K_1\{\delta\} 
\]

where

\(\{e\}^e\) = element strains due to nodal displacements

\(\{\delta\}^e\) = element nodal displacements

\([B]^e\) = element strain displacement matrix

\(\{\sigma\}^e\) = element stresses

\(\{\varepsilon\}^e\) = element design strains

\([D]\) = elasticity matrix

\(\{S\}\) = anchor shear forces

\(K_1\) = initial anchor stiffness.

In the above formulation compressive design strains are treated as equivalent initial thermal strains and thus are positive. Stresses and strains due to nodal displacements are positive if tensile. The anchor shear forces (acting on the anchors) are positive if they act in the positive \(x\) and \(y\) directions.

**Plate elements**

Since the stud anchors are usually arranged in a rectangular pattern, it is natural to represent the liner plate with triangular elements between the anchors as shown in Fig. 9. Using linear nodal displacements with two degrees of freedom at each node (plane stress case), the vectors and matrices defined in the preceding paragraphs take the form (see Fig. 9)

\[
\{\delta\}^e = \{(\delta_x)_i \ (\delta_y)_i \ (\delta_x)_j \ (\delta_y)_j \ (\delta_x)_m \ (\delta_y)_m\}^T 
\]

(4.8)
where superscript "T" indicates transpose of a vector or matrix.

Also

$$[B]_e^T = \frac{1}{2\Delta} \begin{bmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix}$$

(4.9)

where \(i, j, m = \text{nodes}\)
\(x, y = \text{coordinates of the nodes}\)
\(\Delta = \text{area of the triangular element}\)
\(b_i = y_j - y_m\)
\(c_i = x_m - x_j\)

and the other coefficients are obtained by a cyclic permutation of subscripts in the order of \(i, j, m\).

The strains and stresses for the elements are

$$\{\varepsilon\}_e^T = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{bmatrix}^T$$

(4.10)

$$\{\sigma\}_e^T = \begin{bmatrix} \sigma_x & \sigma_y & \tau_{xy} \end{bmatrix}^T$$

(4.11)

where \(\varepsilon_x, \varepsilon_y = \text{normal strains in } x \text{ and } y \text{ directions, respectively}\)
\(\gamma_{xy} = \text{shearing strain}\)
\(\sigma_x, \sigma_y = \text{normal stresses in } x \text{ and } y \text{ directions, respectively}\)
\(\tau_{xy} = \text{shearing stress}\)

and the elasticity matrix is given by

$$[D] = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix}$$

(4.12)
where $E =$ modulus of elasticity

$\mu =$ Poisson's ratio.

The plate stiffness matrix is obtained from

$$[K_p] = \sum [k]^e \quad (4.13)$$

where $[k]^e$ is the element stiffness matrix and summation is taken over the entire region. The element contributions are determined from

$$[k]^e = \int_v [B]^e^T[D][B]^e_dv \quad (4.14)$$

### Anchors

As indicated earlier, experimentally obtained anchor characteristics are used in the method. In forming the initial stiffness matrix for the total assembly, initial anchor stiffnesses are added at the nodes where anchors exist. These stiffnesses are

$$K_x = K_y = K_1 \quad (4.15)$$

where $K_x$, $K_y$ are the stiffnesses in two orthogonal directions and $K_1$ is the initial, uniaxial anchor stiffness (Fig. 7).

### Initial nodal forces

The initial nodal force vector $\{F_{CD}\}$, is obtained from the assembly of element initial nodal forces which are given by

$$\{F_{CD}\}^e = - \int_v [B]^e^T[D] \{e\}_c^e_dv \quad (4.16)$$

It is to be noted that if all plate elements have the same properties and the design strains are uniform, the initial nodal force vector $\{F_{CD}\}$ (after assembly) will be identically zero resulting in
zero relative displacements. Then the stresses in the liner panels are
due to design strains only [Eq. (4.6)] and the anchor shear forces
are zero.

Notes on the formation of the matrices

As noted above, the stresses and forces in the liner components
may be obtained after the nodal displacements are determined from
Eq. (4.4). Thus, it is sufficient to form the reduced global stiffness
matrix \([K]\) rather than the total global stiffness matrix \([K_T]\).
Similarly, only the nodal force vector \(\{F_{CD}\}\) needs to be assembled.

In the computer program, when an element stiffness matrix is
formed those terms corresponding to unknown displacements are trans­
ferred to the global matrix and the remaining terms are disregarded.
A similar procedure is used in forming the vector \(\{F_{CD}\}\).

Solution Technique

In the analysis of PCRV liners an elastic solution as described
in the preceding section will not usually be correct if some liner
panels and anchors are stressed beyond the elastic limit. For such
cases a method of nonlinear analysis is needed.

In the analysis of structures where material nonlinearity is
involved and where a biaxial or triaxial stress field exists an
incremental approach is essential. The reason for this is that, in
the inelastic range, the stress strain relationship depends on the
state of total stress. This relationship is usually assumed to be
"piecewise linear" and is given by (31, 32, 36, 37, 38)
\[
\{\Delta \sigma\} = [D_p] \{\Delta \varepsilon\}
\]

where \(\{\Delta \sigma\}\) = incremental stresses

\([D_p]\) = elasto-plastic or plasticity matrix

\(\{\Delta \varepsilon\}\) = incremental strains.

An explicit expression for \([D_p]\), for a material obeying the von Mises yield law, is given in Appendix B.

As noted in Chapter III, the initial stress method of nonlinear analysis was adopted in this study. In this method the initial stiffness matrix is used throughout which results in savings in the computer time.

In the solution technique developed in this study, the loads (in the problem at hand the design strains) at the elastic limit are determined first. The remainder of the load is applied in increments, determining incremental displacements, strains and stresses. In computing the incremental stresses from the incremental strains the elasticity matrix \([D]\) is used for those elements which remain elastic. For those elements in a state of yield the plasticity matrix \([D_p]\) is used.

After determining the incremental and hence the total stresses, the equivalent nodal forces are computed for each plate element and transferred into a global equivalent nodal force vector. In addition the anchor forces are computed using the total nodal displacements and actual anchor stiffnesses. If one or more plate elements or anchors are in the inelastic range, there will be unbalanced forces at the corresponding nodes. For equilibrium, the existence of fictitious
"holding forces" must be assumed at these nodes. The next step is to remove these fictitious forces by allowing the structure (with unchanged elastic properties) to deform further. Thus, an additional set of displacements and corresponding strains and stresses are computed. This process is termed an "iteration." Once again there will be unbalanced forces at some nodes which must be removed. The process is continued until the holding forces required for equilibrium become negligible. When this happens, iteration within an increment is ended and a new increment is applied.

The incremental analysis method briefly discussed above will now be given in detail using a step-by-step approach. The procedure is similar to that given in Ref. (32).

Step 1. Establish the reduced elastic stiffness matrix, \([K]\), invert and store.

Step 2. Assuming all panels and anchors remain elastic, determine the initial nodal force vector, \(\{F_{CD}\}\).

Step 3. Using Eqs. (4.4) through (4.7) calculate

\[
\{\delta\}_e, \{\varepsilon\}_e, \{\sigma\}_e, \{S\}_e
\]

where subscript "e" indicates an elastic solution.

Step 4. a) Check each plate element for plastic flow (based on von Mises yield criterion, see Appendix B) by determining

\[
\phi = \left(\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2\right)^{1/2} - \bar{\sigma}
\]

(4.18)

where \(\bar{\sigma}\) = yield stress in uniaxial tension.
If $\phi \leq 0$, element is elastic (or, at most, has just yielded).
If $\phi > 0$, plastic flow occurs.
For those elements in the plastic range, determine
\[
\alpha = \frac{\bar{\sigma}}{\phi + \bar{\sigma}}
\] (4.19)
where $\alpha$ is a factor with which the magnitude of design strains at the start of panel yielding is determined (see Appendix B).

b) Similarly, for each anchor determine
\[
\beta = \frac{S_{A1}}{S_e}
\] (4.20)
where $\beta$ is a factor with which the magnitude of design strains at the start of anchor yielding (more properly, limit of initial anchor stiffness, Fig. 7) is determined and $S_{A1}$ is the anchor force at the elastic limit.

It is to be noted that if all $\alpha \geq 1.0$ and $\beta \geq 1.0$, the solution is elastic and no further computation is necessary.

Step 5. Let $\eta$ be the smallest $\alpha$ or $\beta$. Scale down all elastic values determining and storing
\[
\{\delta\}_{\text{lim}}, \{e\}_{\text{lim}}, \{\sigma\}_{\text{lim}}, \{S\}_{\text{lim}}
\]
where subscript "lim" indicates values at the yield of the weakest element. Typically,
\[
\{\delta\}_{\text{lim}} = \eta \{\delta\}_e
\] (4.21)
Step 6. Determine and store incremental nodal displacements from

\[ \Delta \delta^j_{1} = \frac{1 - \eta}{INC} \Delta \delta \]

(4.22)

where \( \Delta \delta^j_{1} \) are the incremental nodal displacements and "INC" indicates the number of increments to be used in plastic analysis. This process is equivalent to solving Eq. (4.3) under incremental design strains.

Step 7. Apply incremental displacements and, using Eq. (4.5) and (4.6) determine

\[ \{ \Delta \varepsilon \}_{1}, \{ \Delta \sigma \}_{1} \]

where subscript "1" indicates the first increment.

Step 8. Add \( \{ \Delta \varepsilon \}_{1} \) and \( \{ \Delta \sigma \}_{1} \) to existing strains and stresses, respectively, determining (for each element)

\[ \{ \varepsilon \}_{1} = \{ \varepsilon \}_{0} + \{ \Delta \varepsilon \}_{1} \]

\[ \{ \sigma \}_{1} = \{ \sigma \}_{0} + \{ \Delta \sigma \}_{1} \]

where \( \{ \varepsilon \}_{0} \) = strains at the beginning of increment

\( \{ \varepsilon \}_{1} \) = strains at the end of increment

\( \{ \sigma \}_{0} \) = stresses at the beginning of increment

\( \{ \sigma \}_{1} \) = stresses at the end of increment.

Determine \( \phi_{0} \) and \( \phi_{1} \) using stresses \( \{ \sigma \}_{0} \) and \( \{ \sigma \}_{1} \), respectively, and Eq. (4.18). Check the flow condition for each element:

a) if \( \phi_{1} \leq 0 \) and \( \phi_{0} < 0 \) the element is still in the elastic range (or, at most, has just yielded),

store \( \{ \varepsilon \}_{1} \) and \( \{ \sigma \}_{1} \), and continue.

b) If \( \phi_{1} < 0 \) and \( \phi_{0} \geq 0 \) the element was previously yielded, now unloading, store \( \{ \varepsilon \}_{1} \) and \( \{ \sigma \}_{1} \), and
continue (the significance of plate unloading is discussed in later sections),

c) If $\phi_1 > 0$ and $\phi_o > 0$ the element was previously in yield, go to step 9.

d) If $\phi_1 > 0$ and $\phi_o < 0$ the element was previously in the elastic range, now in the plastic range. Determine the intermediate stresses at which flow starts.

These stresses may be obtained using (see Appendix B)

$$\alpha = \frac{\bar{\sigma}}{\phi_1 + \sigma}$$

Reset

$$\{\varepsilon\}_0 = \alpha \{\varepsilon\}_1$$

$$\{\sigma\}_0 = \alpha \{\sigma\}_1$$

With this process, the total strains and stresses at the yield level are determined. Note that $\{\varepsilon\}_0$ and $\{\sigma\}_0$ thus computed are the "corrected" strains and stresses at the beginning of increment, respectively, and are stored as such, destroying the previous values. Also determine

$$\{\Delta \varepsilon\}_1 = \{\varepsilon\}_1 - \{\varepsilon\}_0$$

which is computed using the new values of $\{\varepsilon\}_0$.

The new $\{\Delta \varepsilon\}_1$ are the "corrected" incremental strains to be used in step 9.

Note that the elements satisfying the conditions in 8.a or 8.b do not go through step 9.
Step 9. The incremental strains \( \{ \Delta \varepsilon \} \) for those elements which go through steps 8.c or 8.d are in the plastic range and the corresponding stresses must be obtained using the plasticity matrix. With \( \{ \sigma \}_o \), determine \([D_p] \) matrix\(^1\) (see Appendix B). Then

\[
\{ \Delta \sigma \}'_1 = [D_p] \{ \Delta \varepsilon \}_1
\]

(4.23)

where \( \{ \Delta \sigma \}'_1 \) are the actual changes in stresses. The stresses at the end of the increment are redefined as

\[
\{ \sigma \}_1 = \{ \sigma \}_o + \{ \Delta \sigma \}'_1
\]

where \( \{ \sigma \}_1 \) are the "corrected" stresses at the end of the increment and are stored as such, destroying the previous values.

It should be noted here that the plastic stress-strain relationship given above is correct for infinitesimal values only (32). Since finite values are used, the resulting stresses \( \{ \sigma \}_1 \) may deviate from the yield surface. Therefore, these stresses must be brought back to the yield surface (31). This may be accomplished by simply multiplying the stresses by the factor \( \frac{\sigma}{\Phi_1 + \sigma} \) in which the value of \( \Phi_1 \) is computed with the corrected \( \{ \sigma \}_1 \).

Step 10. Determine \( \{ \delta \}_1 = \{ \delta \}_o + \{ \Delta \delta \}_1 \)

where \( \delta_o \) = node displacements at the beginning of increment

\( \delta_1 \) = node displacements at the end of increment.

Determine the actual anchor forces from

\[
\{ S \}_1 = \{ g(\delta) \}_1
\]

(4.24)

\(^1\)In this study the plasticity matrix is based on the stresses existing at the beginning of an increment, following Ref. (36). In Ref. (32) the stresses \( \{ \sigma \}_1 \) as determined in step 8 was used in computing \([D_p] \).
where \( g(b) \) is the nonlinear stiffness function for the anchors.

Step 11. If the stress field in any element is determined through step 9, equilibrium at the nodes associated with that element will be disturbed. The reason for this is that, in determining the incremental stresses the elasto-plastic matrix is used whereas Eq. (4.4) is based on elastic relationships. Similarly, if any resultant anchor displacement exceeds the elastic limit there will be unbalanced forces at that node.

The element equivalent nodal forces at the end of the increment are

\[
\{F\}_1^e = \int \{B\}_1^T \{\sigma\}_1^e \, dv
\]  

(4.25)

and the total equivalent nodal force matrix \( \{F\}_1 \) is obtained by assembling the element nodal forces. The anchor forces are \( \{S\}_1 \). For equilibrium at the nodes

\[
\{F\}_1 + \{S\}_1 = 0
\]  

(4.26)

However, as indicated above, there will be unbalanced forces at some nodes and Eq. (4.26) will not be satisfied. Therefore, there must be a set of "holding forces" (external forces) at these nodes which must be assumed to exist for equilibrium. The holding forces are given by

\[
\{HF\}_1 = \{F\}_1 + \{S\}_1
\]  

(4.27)

The external holding forces thus obtained are positive if they are in the positive x and y directions.
Step 12. Since the holding forces cannot exist, they must be removed by applying equal and opposite external forces. When this is done there will be additional displacements at the nodes and associated changes in strains and stresses. The additional displacements, \( \{ \Delta \delta \}_2 \), may be obtained using the general matrix-displacement equation

\[
[K] \{ \Delta \delta \}_2 = - \{ HF \}_1
\]

(4.28)

where the (-) sign is due to the fact that the "applied" forces are equal and opposite to the holding forces. In Eq. (4.28) the same reduced elastic stiffness matrix as in Eq. (4.4) has been used.

The solution to Eq. (4.28) results in the first "iterative" displacements, \( \{ \Delta \delta \}_2 \). The corresponding strains and stresses are given by (in terms of elements)

\[
\{ \Delta \varepsilon \}_2 = [B]^e \{ \Delta \delta \}_2^e
\]

(4.29)

\[
\{ \Delta \sigma \}_2 = [D] \{ \Delta \varepsilon \}_2^e
\]

(4.30)

It is to be noted that Eq. (4.29) is the same as Eq. (4.5). Equation (4.30) does not, of course, contain a design strain term and thus is different from Eq. (4.6).

Step 13. After determining the iterative values, steps 8 through 12 are repeated. The only difference is that the term "increment" is replaced by "iteration" and "incremental" by "iterative." The nodal displacements are continuously summed (step 10) and the current stresses and strains are computed and stored (steps 8 and 9).

This iterative process is stopped when the holding forces determined in step 11 are small. If
where $ER$ is a preset constant, the iterative displacements to be obtained in step 12 may be considered negligible. When this happens iteration is ended.

Step 14. After the iteration is ended another increment is applied and steps 7 through 13 are repeated. After all the increments are applied, the final strains and stresses in the plate elements and, forces, in the anchors are obtained.

General comments on the solution technique

The solution technique as described above is essentially a relaxation process. When an increment is applied, the yielded elements carry no additional load since perfect plasticity is assumed (however, changes occur in the stress field). The holding forces at the nodes maintain equilibrium. The removal of these forces during the iteration process cause additional displacements which in turn result in relaxation in the elastic elements and, increase the resisting forces in the anchors. The cumulative displacements converge to the correct solution as a series of approximations.

Criteria for Solution Accuracy

In the elasto-plastic finite element method the accuracy of the results depend on the magnitude of the holding forces when the iteration process is stopped and the number of increments applied (31). Due to the nature of the problem at hand, there is another problem associated
with the number of increments. This is the problem of unloading of strong panels. These subjects are discussed below.

**Magnitude of residual holding forces**

It was found in this study that, within the range of variables studied, the computed maximum anchor forces are within 1% of the estimated correct anchor forces when all values of residual holding forces are less than about 0.3 kips (per anchor). The estimated correct anchor force was determined by conducting several analyses of a problem, using different magnitudes of residual holding forces, and then extrapolating to find the anchor force corresponding to zero residual holding forces.

In the case of one-dimensional analysis, the residual holding force for the same accuracy was found to be about 0.05 kips (per unit width). Considering the fact that in all the problems analyzed the anchor spacing was 7.5 in. or greater, the residual nodal forces for one- and two-dimensional analysis are in the same order.

**Number of increments**

Since the plastic deformations are incremental, the loads (design strains) in the plastic range must be applied in increments. The number of increments required for accurate results naturally depend on the problem under consideration. In general, using smaller load increments (design strain increments) will result in greater accuracy (31). Greater design strains will result in greater incremental design strains for a given number of increments and a given weak panel yield point. Also, a lower elastic limit for any panel will result in greater
design strain increments for a given number of increments and design strains. Therefore, it may be concluded that greater design strains and/or lower elastic limit will require more increments for the same accuracy.

It was found in this study that, within the range of variables studied, using four increments gave sufficiently accurate results. This point is illustrated by an example in Chapter V. It may be pointed out that, for problems of similar size (with respect to degrees of freedom) but of different nature, the number of increments used has varied from 4 to 14 (32).

The sufficiency of using four increments was apparent from the computer results also. It was stated in the preceding section that, after step 9 in the iteration process, the yielded elements are checked to make sure that stresses are on the yield surface. In the computer program the flow value $\phi_1$ as determined from the corrected stresses $[\sigma]_1$ (at the end of step 9) was output for each yielded element. In the problems analyzed the maximum flow value was 0.088 ksi. Since the yield strength of steel ($\sigma$) in these problems was taken to be 36 or 60 ksi, the deviation from the yield surface as determined from the $\phi_1/\sigma$ ratio was less than 1%. Thus it was concluded that, such a small error will not affect the results appreciably.

**Unloading of strong panels**

When the design strains are applied to a liner section the assumed weak panel(s) yield first. The strains at which yielding starts are the strains at elastic limit for weak element(s) and are below the
yield level for the other elements. When the incremental strains are applied some of the strong panels may also yield. During the iteration process within one increment the yielded strong elements may unload because the strains tend to accumulate in the weak elements. If the unloading is such that a permanent set occurs, even though the final stresses are below the yield level, the results will be somewhat erroneous.

In order to prevent the unloading of strong panels in this manner a large number of increments should be used. This would of course be more critical under higher design strains. However, in the problems analyzed, four increments resulted in sufficient accuracy (see Chapter V).
CHAPTER V.

COMPARISON OF ONE- AND TWO-DIMENSIONAL ANALYSES

Purpose and Scope

It was stated in the introduction that one of the purposes of this study is to evaluate the adequacy of one-dimensional analysis. In this chapter such an evaluation is attempted by studying typical PCRV liner problems using one- and two-dimensional methods presented in Chapters III and IV and, comparing the analysis results.

Of course, any numerical or analytical method of analysis can be evaluated in the light of applicable test data. Unfortunately however, there does not appear to be any test data available which is applicable to the problem at hand (21). The reason for that appears to be the difficulty of simulating all geometric and loading conditions of the liner assembly and, local effects which must be postulated in design analysis. The tests conducted to date have been directed to either evaluating component characteristics or observing the behavior of a specimen which represents a segment of the liner assembly. The results of the first type of tests (e.g., weak panel characteristics, shear load-slip behavior of anchors) have been extensively used in design analysis (2, 7, 8, 15, 16). In the second type of tests the interest has mainly been in observing the elastic or inelastic buckling of liner panels between anchors (2, 13, 14). It should be added that some of the local variations postulated in design (such as yield point or thickness variation) have not been studied in these tests at all.
Design analysis

As noted earlier, in the design of liners a section of the liner is analyzed using the concrete-imposed strains as the loading. In this analysis the existence of a weak panel is postulated and the effects of local variations on liner components are investigated. If one-dimensional analysis is used for this purpose, the implication is that several weak panels exist perpendicular to the analysis model so that a unit width strip will accurately simulate the actual conditions. However, some of the variations listed in Chapter II are "local" (e.g., water pressure behind the liner) while others may extend over a large segment of the liner (e.g., thickness variation). Therefore, in design analysis, the effect of local variations on a single panel or several panels should be investigated.

In order to evaluate the adequacy of one-dimensional method two different problems were considered. In the first problem it was assumed that only one weak panel exists in the liner. In the second problem a row of weak panels were assumed. These two problems were intended to represent the two types of local effects mentioned in the preceding paragraph.

Variables

The behavior of a PCRV liner is influenced by the following variables:

1. Liner plate yield stress ($\sigma$)
2. Liner plate thickness (h)
3. Anchor spacing ($a_{\theta}$-circumferential, $a_z$-axial)
4. Anchor stiffness (K) and
5. Design strains ($\varepsilon_{c\theta}$-circumferential, $\varepsilon_{cz}$-axial).

In the problems analyzed in this chapter however, $\bar{\sigma}$, h, $a_\theta$, and K were considered to be constant, with the values taken from the Fort St. Vrain design data. Since the purpose was to compare the results of one- and two-dimensional analyses, using a constant value for these variables were considered adequate. The Fort St. Vrain design data gives

$$\bar{\sigma} = 60 \text{ ksi}$$

$$h = 0.75 \text{ in.}$$

$$a_\theta = 7.5 \text{ in.}$$

$$a_z = 7.5 \text{ in.}$$

$K =$ stiffness of $3/4$ in. diameter studs as determined from tests (Fig. 7).

The parameters selected for the purpose of comparing the two methods and the values of these parameters were

$$a_z/a_\theta = 1.0, 1.5, 2.0$$

$$\varepsilon_{c\theta} = 750, 1000, 1250, 1500 \mu\text{in./in.}$$

$$\varepsilon_{cz}/\varepsilon_{c\theta} = 0.5, 0.75, 1.0.$$  

The first parameter is used for investigating the validity of the one-dimensional method assumption that anchor stiffnesses may be uniformly distributed over the width of the model. The design strains for the Fort St. Vrain liner were from about 800 to 1500 $\mu\text{in./in.}$ in the circumferential direction and, from about 400 to 770 $\mu\text{in./in.}$ in the axial direction. The actual design strains and their ratios are within the ranges of parameters selected.
Local variations

The effect of local variations listed in Chapter II, in general, is to reduce the inplane load carrying capacity of an assumed weak panel. In one-dimensional analysis, it is customary to determine the weak panel stiffness from test results or using approximate methods. In the case of two-dimensional analysis however, there does not appear to be any analytical method available for this purpose. Applicable experimental results are rather limited also. For this reason, in the problems presented in this chapter, the effects of local variations were taken into account indirectly. This was done by assuming elastic-perfectly plastic weak panel behavior with a uniaxial yield strength less than that of strong panels. Two different yield points were assumed for the weak panels which were

\[ \bar{\sigma}_\omega = 0, \text{ and} \]
\[ \bar{\sigma}_\omega = 0.6 \bar{\sigma}. \]

It has been stated that zero inplane load capacity may come about if, due to a combination of adverse local effects, three full plastic hinges form in a panel [one-dimensional analysis, Ref. (2)]. Formation of three plastic hinges in a panel was reported in some experimental studies on models although the stress distribution was not determined (14). In an actual PWRV liner such a situation (i.e., zero average stress) is highly unlikely, but it may nevertheless be assumed so that stresses and strains in the liner components can be calculated under the worst possible conditions.
The case of \( \sigma_\omega = 0.6 \bar{\sigma} \) is used for a panel with 0.1875 in. initial deflection (Fig. 5) and was obtained experimentally (5). Although the test panel dimensions were different, it was used in one problem in order to evaluate the effect of inward deflection extending over a large segment of the liner.

Description of Study

Liner with one weak panel

The plan of the liner section analyzed is shown in Fig. 10. Description of one-dimensional and two-dimensional models are given in the following paragraphs.

One-dimensional model. The one-dimensional model is indicated in Fig. 10. As discussed before, the model consists of a unit strip of the liner plate together with stud anchors. The weak panel is assumed to have zero inplane load carrying capacity.

Since one of the main assumptions in this type of analysis is that the ends are fixed (i.e., the relative movement between the liner and backing concrete is zero), the number of panels to be included in the model is of primary importance. The criterion used by some authors (7, 21) was that the displacement of the end anchor should not exceed two percent of the maximum anchor displacement which occurs at the anchor(s) adjacent to the weak panel. Using this criterion, the number of panels required has been found to be about 21 panels for typical PCRV liner problems (7) and 31 panels for typical PCRV liner problems (29). Of course these numbers would depend on the magnitude
of design strains and inplane load carrying capacity of the weak panel in the inelastic range.

Since the inplane load capacity of the weak panel was assumed to be zero in this problem, the number of panels required to satisfy the above criterion would be about 31. However, using 31 panels each in two directions in the finite element program would have been too costly because of the number of degrees of freedom involved. For this reason a square model with 19 panels in each direction was used. This resulted in a problem with 180 degrees of freedom in the finite element method. Since the purpose was to compare the results of the two methods it was concluded that using 19 panels would be sufficient in this study, even though the two percent criterion mentioned in the preceding paragraph was not satisfied.

In order to evaluate the sufficiency of this model for the purpose of this study, a second model, consisting of seven panels in each direction was also analyzed for some combinations of the parameters. The results of this analysis (not given here) showed the same trend as those obtained from the larger model.

Two-dimensional model. Because of symmetry only a quarter of the structure shown in Fig. 10 needs to be analyzed. The resulting problem and the finite element mesh used is shown in Fig. 11. Nodal displacement conditions in x and y directions are also indicated in Fig. 11.

Problems analyzed. The 36 different problems which resulted from the parameter selection are listed in Table 2. Anchor spacing and design strains for both methods are given for each case. The other
Table 2. Variable values for liner with one weak panel$^a$

<table>
<thead>
<tr>
<th>Case</th>
<th>$a_2$</th>
<th>$\epsilon_c$</th>
<th>$\epsilon_{c\theta}$</th>
<th>$\epsilon_{cz}$</th>
<th>$\epsilon_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.5</td>
<td>750</td>
<td>375</td>
<td>863</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>7.5</td>
<td>750</td>
<td>563</td>
<td>919</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7.5</td>
<td>750</td>
<td>750</td>
<td>975</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>7.5</td>
<td>1000</td>
<td>500</td>
<td>1150</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>7.5</td>
<td>1000</td>
<td>750</td>
<td>1225</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>7.5</td>
<td>1000</td>
<td>1000</td>
<td>1300</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7.5</td>
<td>1250</td>
<td>625</td>
<td>1438</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>7.5</td>
<td>1250</td>
<td>938</td>
<td>1531</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>7.5</td>
<td>1250</td>
<td>1250</td>
<td>1625</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>7.5</td>
<td>1500</td>
<td>750</td>
<td>1725</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>7.5</td>
<td>1500</td>
<td>1125</td>
<td>1838</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>7.5</td>
<td>1500</td>
<td>1500</td>
<td>1950</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>11.25</td>
<td>750</td>
<td>375</td>
<td>863</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>11.25</td>
<td>750</td>
<td>562</td>
<td>919</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>11.25</td>
<td>750</td>
<td>750</td>
<td>975</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>11.25</td>
<td>1000</td>
<td>500</td>
<td>1150</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>11.25</td>
<td>1000</td>
<td>750</td>
<td>1225</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>11.25</td>
<td>1000</td>
<td>1000</td>
<td>1300</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>11.25</td>
<td>1250</td>
<td>625</td>
<td>1438</td>
<td></td>
</tr>
</tbody>
</table>

$^a$In addition the following are common to all cases: $\sigma = 60$ ksi, $h = 0.75$ in., $a_\theta = 7.5$ in., $\sigma_\omega = 0$, diameter of stud anchors = $3/4$ in.
<table>
<thead>
<tr>
<th>Case</th>
<th>$a_z$</th>
<th>Two-dimensional</th>
<th>One-dimensional</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\varepsilon_{c\theta}$</td>
<td>$\varepsilon_{cz}$</td>
</tr>
<tr>
<td>20</td>
<td>11.25</td>
<td>1250</td>
<td>938</td>
</tr>
<tr>
<td>21</td>
<td>11.25</td>
<td>1250</td>
<td>1250</td>
</tr>
<tr>
<td>22</td>
<td>11.25</td>
<td>1500</td>
<td>750</td>
</tr>
<tr>
<td>23</td>
<td>11.25</td>
<td>1500</td>
<td>1125</td>
</tr>
<tr>
<td>24</td>
<td>11.25</td>
<td>1500</td>
<td>1500</td>
</tr>
<tr>
<td>25</td>
<td>15.0</td>
<td>750</td>
<td>375</td>
</tr>
<tr>
<td>26</td>
<td>15.0</td>
<td>750</td>
<td>563</td>
</tr>
<tr>
<td>27</td>
<td>15.0</td>
<td>750</td>
<td>750</td>
</tr>
<tr>
<td>28</td>
<td>15.0</td>
<td>1000</td>
<td>500</td>
</tr>
<tr>
<td>29</td>
<td>15.0</td>
<td>1000</td>
<td>750</td>
</tr>
<tr>
<td>30</td>
<td>15.0</td>
<td>1000</td>
<td>1000</td>
</tr>
<tr>
<td>31</td>
<td>15.0</td>
<td>1250</td>
<td>625</td>
</tr>
<tr>
<td>32</td>
<td>15.0</td>
<td>1250</td>
<td>938</td>
</tr>
<tr>
<td>33</td>
<td>15.0</td>
<td>1250</td>
<td>1250</td>
</tr>
<tr>
<td>34</td>
<td>15.0</td>
<td>1500</td>
<td>750</td>
</tr>
<tr>
<td>35</td>
<td>15.0</td>
<td>1500</td>
<td>1125</td>
</tr>
<tr>
<td>36</td>
<td>15.0</td>
<td>1500</td>
<td>1500</td>
</tr>
</tbody>
</table>

variables which are common to all problems are given following the table.
Liner with more than one weak panel

Since some local variations may occur over a large segment of the liner while others are localized, a second type of problem was analyzed for this condition. The following data were used in this analysis:

\[ \bar{\sigma} = 60 \text{ ksi} \]
\[ h = 0.75 \text{ in.} \]
\[ a_\theta = 7.5 \text{ in.} \]
\[ a_z = 7.5 \text{ in.} \]
\[ \epsilon_{c\theta} = 1500 \mu \text{in./in.} \]
\[ \epsilon_{cz} = 750 \mu \text{in./in.} \]
\[ \sigma_\omega = 0 \text{ and } 0.6 \bar{\sigma} \]

\( K \) = stiffness of 3/4 in. diameter studs as determined from tests (Fig. 7).

The above values were again taken from the Fort St. Vrain design data (16).

It was again assumed that the center panel has zero load carrying capacity. In addition, the panels above and below the center panel and extending over the full length of the model were assumed to have a yield point of 0.6 \( \bar{\sigma} \).

One-dimensional models. Two one-dimensional models were analyzed. The first one included the center panel (\( \sigma_\omega = 0 \)) and the second one included one of the other weak panels (\( \sigma_\omega = 0.6 \bar{\sigma} \)). Both of these models are indicated in Fig. 12. The number of panels included was the same as before.
Two-dimensional model. The two-dimensional problem for this case is also shown in Fig. 12. The two different weak panels assumed in this analysis are indicated in the figure. Nodal displacement conditions are the same as before (Fig. 11).

Discussion of Results

Liner with one weak panel

The maximum anchor forces and displacements obtained from the two methods of analysis are shown in Table 3. The two-dimensional cases with an elastic solution are indicated with an asterisk in the table.

As was mentioned previously, all inelastic cases were analyzed using 4 load increments (in the plastic range) and 0.3 kips maximum residual holding forces. In order to verify the adequacy of using 4 increments and 0.3 kips residual force the last 12 cases were re-analyzed using 8 increments and 0.1 kips residual force. The anchor forces for the latter are shown in parenthesis. It is seen that maximum anchor forces in two cases differed by less than one percent.

The final stress and strain field in all panels were similar in both analyses, thus indicating that appreciable permanent set did not take place in strong panels.

The significance of analysis results are discussed in the following paragraphs.
Table 3. Maximum anchor forces and displacements for the liner problem with one weak panel

<table>
<thead>
<tr>
<th>Case</th>
<th>One-dimensional Displacements, in.</th>
<th>Two-dimensional Displacements, in.</th>
<th>One-dimensional Forces, kips</th>
<th>Two-dimensional Forces, kips</th>
</tr>
</thead>
<tbody>
<tr>
<td>1*</td>
<td>0.031805</td>
<td>0.002097</td>
<td>18.226</td>
<td>1.748</td>
</tr>
<tr>
<td>2*</td>
<td>0.034612</td>
<td>0.002387</td>
<td>18.765</td>
<td>1.990</td>
</tr>
<tr>
<td>3*</td>
<td>0.037385</td>
<td>0.002712</td>
<td>19.050</td>
<td>2.261</td>
</tr>
<tr>
<td>4*</td>
<td>0.046484</td>
<td>0.002796</td>
<td>19.995</td>
<td>2.331</td>
</tr>
<tr>
<td>5*</td>
<td>0.050481</td>
<td>0.003182</td>
<td>20.408</td>
<td>2.653</td>
</tr>
<tr>
<td>6*</td>
<td>0.054553</td>
<td>0.003616</td>
<td>20.828</td>
<td>3.015</td>
</tr>
<tr>
<td>7</td>
<td>0.062297</td>
<td>0.004093</td>
<td>21.630</td>
<td>4.825</td>
</tr>
<tr>
<td>8</td>
<td>0.067584</td>
<td>0.005362</td>
<td>22.178</td>
<td>6.557</td>
</tr>
<tr>
<td>9</td>
<td>0.072879</td>
<td>0.007274</td>
<td>22.725</td>
<td>8.576</td>
</tr>
<tr>
<td>10</td>
<td>0.078809</td>
<td>0.007775</td>
<td>23.333</td>
<td>9.166</td>
</tr>
<tr>
<td>11</td>
<td>0.085298</td>
<td>0.009141</td>
<td>24.008</td>
<td>10.776</td>
</tr>
<tr>
<td>12</td>
<td>0.091790</td>
<td>0.011373</td>
<td>24.675</td>
<td>11.790</td>
</tr>
<tr>
<td>13*</td>
<td>0.038437</td>
<td>0.002356</td>
<td>19.159</td>
<td>1.964</td>
</tr>
<tr>
<td>14*</td>
<td>0.041565</td>
<td>0.002726</td>
<td>19.485</td>
<td>2.272</td>
</tr>
<tr>
<td>15*</td>
<td>0.044697</td>
<td>0.003143</td>
<td>19.811</td>
<td>2.621</td>
</tr>
<tr>
<td>16*</td>
<td>0.054980</td>
<td>0.003141</td>
<td>20.869</td>
<td>2.619</td>
</tr>
<tr>
<td>17</td>
<td>0.059391</td>
<td>0.003634</td>
<td>21.330</td>
<td>3.030</td>
</tr>
<tr>
<td>18</td>
<td>0.063839</td>
<td>0.004258</td>
<td>21.791</td>
<td>5.020</td>
</tr>
<tr>
<td>19</td>
<td>0.072294</td>
<td>0.005617</td>
<td>22.658</td>
<td>6.622</td>
</tr>
<tr>
<td>20</td>
<td>0.077959</td>
<td>0.007608</td>
<td>23.254</td>
<td>8.969</td>
</tr>
<tr>
<td>21</td>
<td>0.083626</td>
<td>0.009338</td>
<td>23.839</td>
<td>11.009</td>
</tr>
<tr>
<td>22</td>
<td>0.089691</td>
<td>0.011132</td>
<td>24.458</td>
<td>11.714</td>
</tr>
<tr>
<td>23</td>
<td>0.096586</td>
<td>0.013275</td>
<td>24.908</td>
<td>12.389</td>
</tr>
<tr>
<td>24</td>
<td>0.103525</td>
<td>0.015501</td>
<td>25.234</td>
<td>13.090</td>
</tr>
<tr>
<td>25*</td>
<td>0.042623</td>
<td>0.002540</td>
<td>19.590</td>
<td>2.117 (2.117)</td>
</tr>
<tr>
<td>26*</td>
<td>0.049330</td>
<td>0.002825</td>
<td>19.935</td>
<td>2.355 (2.355)</td>
</tr>
<tr>
<td>27*</td>
<td>0.049285</td>
<td>0.003281</td>
<td>20.280</td>
<td>2.736 (2.736)</td>
</tr>
<tr>
<td>28*</td>
<td>0.059849</td>
<td>0.003387</td>
<td>21.375</td>
<td>2.823 (2.823)</td>
</tr>
<tr>
<td>29</td>
<td>0.064422</td>
<td>0.003766</td>
<td>21.855</td>
<td>3.140 (3.140)</td>
</tr>
<tr>
<td>30</td>
<td>0.069306</td>
<td>0.004718</td>
<td>22.350</td>
<td>5.563 (5.574)</td>
</tr>
<tr>
<td>31</td>
<td>0.077828</td>
<td>0.006834</td>
<td>23.235</td>
<td>8.056 (8.073)</td>
</tr>
<tr>
<td>32</td>
<td>0.083642</td>
<td>0.009268</td>
<td>23.835</td>
<td>10.927 (10.947)</td>
</tr>
<tr>
<td>33</td>
<td>0.089485</td>
<td>0.011832</td>
<td>24.435</td>
<td>11.832 (11.860)</td>
</tr>
</tbody>
</table>

*Cases with an * indicate elastic solution in two-dimensional analysis.

Numbers in parentheses for the last 12 cases are the anchor forces obtained using 8 increments and 0.1 kips residual holding forces. All other results are obtained using 4 increments and 0.3 kips residual forces.
Maximum anchor displacements. Since the nature of the problem is displacement limited and since the anchor behavior is nonlinear, the most significant results are the maximum anchor displacements.

The maximum anchor displacements for the case of $\alpha_2/\alpha_0$ are shown in Fig. 13. It is seen that one-dimensional analysis predicts much larger anchor movements. Considering all cases analyzed, displacements obtained from one-dimensional analysis are about 6 to 17 times greater than those obtained from the two-dimensional analysis.

Therefore, it may be concluded that assuming the existence of a weak panel due to local effects and then conducting a strip analysis is a highly conservative design practice.

The effect of magnitude of design strains may be observed from Fig. 13 and the values listed in Table 3. As the design strain increases, anchor displacements also increase. The trend of the data obtained from the two methods is similar.

The effect of anchor spacing in the axial direction is shown in Fig. 14. It is seen that displacements increase with the increase in anchor spacing and that rate of increase in the two-dimensional case is greater.
It was mentioned in the preceding section that adequate number of panels should be included in the section being analyzed. The criterion mentioned for this purpose was that the displacement at the last anchor should not exceed two percent of the maximum anchor displacements. Obviously this criterion is not satisfied in the case of one-dimensional analysis. The last anchor displacement in the most critical case (case 36) was about ten percent of the maximum anchor displacement.

In two-dimensional analysis the anchor displacement near the boundaries was about 5.2 percent of the maximum anchor displacement. This indicates that, for the same accuracy, fewer panels need to be included (in both directions) in two-dimensional analysis.

Anchor forces. The anchor forces for the case of $a_z/a_0 = 1.0$ are shown in Fig. 15. As expected, the maximum anchor forces obtained from the one-dimensional analysis are much larger. The trend of the data from the two methods are not similar because of nonlinear anchor behavior.

An important design consideration is the extent of the disturbance caused by the existence of a weak panel. This disturbance may be evaluated by considering the anchor forces in a row of anchors. Figure 16 shows the relative forces in the first row of anchors in x-direction as obtained from the two methods. It is seen that the forces obtained in anchors away from the weak panel, as a fraction of maximum anchor force, are considerable in the case of one-dimensional analysis. This means that when an anchor is highly stressed due to a local effect, the adjacent anchors will also be highly stressed. On the other
hand, the two-dimensional analysis shows that the effect of a local variation is quite "local" and that the disturbance dies down very rapidly.

Figure 16 also indicates that the relative anchor forces are practically independent of the design strains.

Weak panel strains. When a weak panel load carrying capacity (i.e., average normal stress) is reduced, the concrete-imposed strains tend to accumulate in this panel. The weak panel strains obtained from either analysis are not "strains" in the true sense, rather they indicate average displacements in a panel.

The maximum liner strain is another design consideration and must be limited to the code allowables. The maximum principal strain obtained from the two-dimensional analysis was about 6400 \( \mu \text{in.}/\text{in.} \) (case 12) whereas the corresponding strain from the one-dimensional analysis was found to be about 26,400 \( \mu \text{in.}/\text{in.} \). Again much higher values were obtained from one-dimensional analysis.

**Liner with more than one weak panel**

The three problems analyzed were

a) One-dimensional analysis which includes a weak panel with 
   \[ \sigma_w = 0.6 \bar{\sigma} \]

b) One-dimensional analysis which includes a weak panel with 
   \[ \sigma_w = 0 \]

c) Two-dimensional analysis which includes center weak panel with 
   \[ \sigma_w = 0 \] and all the other panels above and below with \( \sigma_w = 0.6 \bar{\sigma} \).
The maximum anchor displacements and forces for these three cases are given below:

<table>
<thead>
<tr>
<th>Case</th>
<th>Maximum anchor displacement, in.</th>
<th>Maximum anchor force, kips</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.019834</td>
<td>14.453</td>
</tr>
<tr>
<td>b</td>
<td>0.097900</td>
<td>24.675</td>
</tr>
<tr>
<td>c</td>
<td>0.025465</td>
<td>16.227</td>
</tr>
</tbody>
</table>

In liner design practice, the design is usually based on a one-dimensional analysis in which a panel with the lowest estimated load carrying capacity is included. The above values for anchor forces and displacements indicate that this is a conservative design practice.

The assumption of two different weak panels is a practical one. For example, there may be a weld seam offset (as permitted in a design specification) extending a considerable distance in the axial or circumferential direction. The panels which include the offset will carry smaller inplane loads than the adjacent panels. Furthermore, some of these panels may be subject to other local effects (such as water pressure behind the liner) thus carrying further reduced loads. If the design is based on one-dimensional analysis which includes all these effects the result will be conservative. On the other hand, the two-dimensional analysis shows that the effect of a local variation superimposed on a more general one will not be as critical.
CHAPTER VI.

STUDY OF THE EFFECT OF LOCAL VARIATIONS

Description of Study

As noted in preceding chapters, design analysis of PCRV liners involves consideration of postulated local effects. The effects of these local variations and the manner in which they are incorporated in the analysis is seldom discussed in the literature. Furthermore, studies on the problem of combining these effects are lacking.

The local effects are usually taken into account using a one-dimensional analysis method (5, 8). The use of a two-dimensional method for this purpose would require a method by which the inelastic stress-strain relationships for rectangular thin or medium-thick plates under edge displacements and lateral pressures can be determined. Since such a method does not appear to be available, study of the local effects was accomplished by using the one-dimensional initial stress method discussed in Chapter III.

In one-dimensional analysis the stiffness of a weak panel is needed. This can be determined using approximate methods (2, 3). For this purpose the "strut analogy" was used in this study (Appendix A). The strut analysis gives average stress-average strain relationships for columns with rectangular cross sections and subject to bending and axial loads. Thus, the effects of local variations can be readily studied using one-dimensional methods and strut analogy.
Among the nine different types of local effects listed in Chapter II all but one were studied and are discussed in this chapter. The effects of local loads was not included in this study.

In order to evaluate the effects of local variations on liner components a "basic" design was selected. The basic design is that of the Fort St. Vrain PCRV liner. Therefore, it is important to point out that the results given in this chapter are valid only for this specific design and may only be interpreted as indicating a trend as far as other designs are concerned.

It should also be pointed out that in this study the shear resistance of the cooling tubes is neglected. This assumption is conservative and has also been made by some British designers (30).

Method of Analysis

The computer program developed for one-dimensional analysis requires a separate input for each panel and each anchor. With this feature, any local variation of known location can be readily accommodated. In the design analysis, however, the local variations should be assumed to exist in the most unfavorable manner. Usually a weak panel is identified and local variations are assumed to occur in or next to this weak panel in the following manners:

1. Initial inward deflection

   Apply the assumed initial deflection to the weak panel and develop its characteristic by the strut analysis.
2. Variation in liner thickness

   Establish the weak panel characteristic based on the thickness of the thinnest panel. Establish the strong panel characteristic based on the maximum panel thickness.

3. Variation in liner yield stress

   Establish the weak panel characteristic based on the minimum yield stress. Establish the strong panel characteristic based on the maximum yield stress.

4. Variation in anchor spacing

   Establish the weak panel characteristic using the largest anchor spacing. Possibility of a loss of an anchor can be treated by doubling the regular anchor spacing at the weak panel (i.e., the weak panel length is twice that of other panels).

5. Variation in anchor stiffness

   Assume that one of the anchors adjacent to the weak panel has reduced stiffness. Use a different anchor characteristic to reflect the variation in anchor stiffness.

6. Water pressure behind the liner

   Develop the weak panel characteristic considering water pressure as the lateral load, using the strut analysis.

7. Concrete void behind the liner

   If the liner surface is not subjected to internal pressure, the existence of a concrete void has no effect in the analysis. If the internal pressure exists, the liner panel covering the concrete void should support the pressure by plate bending. For liner analysis, develop the weak panel characteristic by strut
analysis, considering the internal pressure as the lateral load. It is noted that the bending deflection of the panel in this case is opposite in direction to those due to initial inward deflection and water pressure behind the liner.

8. Local high temperature (hot-spots)

This effect is taken into account by adding the corresponding thermal strains to the panel design strains (5). It is likely that the increased local temperature distribution will be non-uniform. However for design purposes, one or more panels may be assumed to have uniform temperature increases. The thermal gradient in the thickness direction is usually ignored in the design of both PCRV and containment liners (5, 8, 16).

Combinations of various local variations can be accommodated by assuming that all local variations to be combined occur in or adjacent to the same weak panel. This is justified since strains tend to accumulate in a weak panel. A new weak panel characteristic is then developed for the combination on hand (using strut analysis, if plate bending is involved).

When the design strain is uniform and no structural discontinuity exists the location of the weak panel for design is immaterial. When the design strain is variable the weak panel should be selected at the location of maximum strain gradient and maximum strain (5). It may be necessary at times to try several possible locations in order to identify the critical weak panel for a given liner under a given design strain condition.
The design data for the Fort St. Vrain liner, in the circumferential direction, is as follows:

- **Liner thickness**  
  \( h = 0.75 \text{ in.} \)

- **Liner yield stress**\(^1\)  
  \( \overline{\sigma} = 60 \text{ ksi} \)

- **Initial lateral deflection of weak panel (assumed)**  
  \( w_{om} = 1/16 \text{ in.} \)

- **Anchor spacing (x- and y-directions)**  
  \( a = 7.5 \text{ in.} \)

- **Anchor size and type**  
  3/4 in. diameter headed studs

- **Design strains**\(^2\)  
  \( \varepsilon_c = 1731 \mu\text{in./in.} \)

The local variations introduced into the basic design (individually and in combinations) and related assumptions are shown in Table 4 as cases a to k. The design strains for case k are different as will be discussed later on. In all cases the weak panel is assumed to have 1/16 in. initial inward deflection, except as noted.

In the analyses of these cases, the model consisted of 31 panels, 30 intermediate nodes and two fixed ends. Weak panel characteristics were determined using fixed-ended strut analogy. For this purpose, stress ratio-average strain curves were plotted using the strut analysis computer output. The stress ratio is defined as the ratio of average stress to yield stress, \( \sigma_o/\overline{\sigma} \) (see Appendix A). Then these curves were idealized using three linear segments. One of these curves is shown

---

\(^1\)For SA-537 Gr. B steel. However, the specified minimum yield stress in the design was 56 ksi.

\(^2\)Uniform design strain in the circumferential direction. This value is computed from the data given in Ref. (17) and corresponds to a loading case: prestress + shrinkage + creep + thermal loads.
Table 4. Local variations studied

<table>
<thead>
<tr>
<th>Case</th>
<th>Design strain, ( \mu \text{in./in.} )</th>
<th>Initial defl., in.</th>
<th>Other local variations</th>
<th>Assumptions on local variations</th>
</tr>
</thead>
<tbody>
<tr>
<td>a-1</td>
<td>1731(^{a})</td>
<td>1/16</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>a-2</td>
<td>1731(^{a})</td>
<td>1/8</td>
<td>None</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>1731(^{a})</td>
<td>1/16</td>
<td>Thickness variation(^{b})</td>
<td>( h = 0.74 ) in., weak panel</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( h = 0.806 ) in., other panels</td>
</tr>
<tr>
<td>c</td>
<td>1731(^{a})</td>
<td>1/16</td>
<td>Anchor stiffness</td>
<td>One stud stiffness is one-half of the others</td>
</tr>
<tr>
<td>d</td>
<td>1731(^{a})</td>
<td>1/16</td>
<td>Loss of an anchor</td>
<td>Loss of an anchor doubles the spacing</td>
</tr>
<tr>
<td>e</td>
<td>1731(^{a})</td>
<td>1/16</td>
<td>Water pressure</td>
<td>( q = 125 ) psi, behind weak panel</td>
</tr>
<tr>
<td>f-1</td>
<td>1731(^{a})</td>
<td>1/16</td>
<td>Hot spots</td>
<td>( \Delta T = 100 ) °F adjacent to weak panel</td>
</tr>
<tr>
<td>f-2</td>
<td>1731(^{a})</td>
<td>1/16</td>
<td>Hot spots</td>
<td>( \Delta T = 100 ) °F, two panels adjacent to weak panel</td>
</tr>
<tr>
<td>f-3</td>
<td>1731(^{a})</td>
<td>1/16</td>
<td>Hot spots</td>
<td>( \Delta T = 200 ) °F, two panels adjacent to weak panel</td>
</tr>
</tbody>
</table>

\(^{a}\)From Refs. (16) and (17).

\(^{b}\)From ASTM Standards, Part 4, A20.
Table 4. Continued

<table>
<thead>
<tr>
<th>Case</th>
<th>Design strain, μin./in.</th>
<th>Initial defl., in.</th>
<th>Other local variations</th>
<th>Assumptions on local variations</th>
</tr>
</thead>
<tbody>
<tr>
<td>g</td>
<td>1731^a</td>
<td>1/16</td>
<td>Yield point variation^c</td>
<td>$\sigma_y = 48$ ksi, weak panel</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\sigma_y = 60$ ksi, other panels</td>
</tr>
<tr>
<td>h-1</td>
<td>1731^a</td>
<td>1/16</td>
<td>Combination of b, e, f-1</td>
<td>-</td>
</tr>
<tr>
<td>h-2</td>
<td>1731^a</td>
<td>1/8</td>
<td>Combination of b, e, f-1</td>
<td>-</td>
</tr>
<tr>
<td>i-1</td>
<td>1731^a</td>
<td>1/16</td>
<td>Combination of b, d, e, f-1, g</td>
<td>-</td>
</tr>
<tr>
<td>i-2</td>
<td>1731^a</td>
<td>1/8</td>
<td>Combination of b, d, e, f-1, g</td>
<td>-</td>
</tr>
<tr>
<td>j</td>
<td>1731^a</td>
<td>-</td>
<td>Loss of panel load carrying capacity</td>
<td>Weak panel carries no load</td>
</tr>
<tr>
<td>k-1</td>
<td>981^a</td>
<td>1/16</td>
<td>Concrete void</td>
<td>q = 750 psi on weak panel</td>
</tr>
<tr>
<td>k-2</td>
<td>981^a</td>
<td>-</td>
<td>Concrete void</td>
<td>Weak panel carries no load</td>
</tr>
</tbody>
</table>

^cStrong panels are assumed to have 25% higher yield point.
in Fig. 17. The average strains and average stresses corresponding to the three points which describe these three linear segments (the fourth point is the origin) are designated as \((e_{yw}, \bar{\sigma}_w), (e_{r1}, \sigma_{r1}), \) and \((e_{r2}, \sigma_{r2})\), as shown in Fig. 17.

The anchor stiffness characteristics were determined from the idealized load-displacement relationship, shown in Fig. 7.

The computed maximum anchor forces and displacements, weak panel strains, approximate additional lateral deflections, and locations of reference anchors are shown in the left half of Table 5. Weak panel average stress-average strain values are given in the remainder of Table 5.

The reference anchor is defined as the anchor with displacement no more than 10% of the maximum anchor displacement. Its location is given by counting the anchors sequentially starting with the maximum stressed anchor (also with maximum displacement) as anchor number 1. Since in most cases, the last anchor in a model has a displacement of approximately 2% of the maximum displacement, a rough indication on the extent of disturbance caused by a local variation is obtainable with the added information from the reference anchor.

Discussion of Results

Discussion of the results in Table 5 and their significance is given below in the order of cases in Table 5.

Case a. Initial inward deflection: If the initial lateral deflection of the weak panel is assumed to be 1/16 in., the resulting
Table 5. Effects of local variations

<table>
<thead>
<tr>
<th>Case</th>
<th>Maximum anchor force, kips</th>
<th>Maximum anchor displ., in.</th>
<th>Weak panel strain, in./in.</th>
<th>Approx. lat. defl.</th>
<th>Location of ref. anchor</th>
<th>Weak panel characteristics^a</th>
</tr>
</thead>
<tbody>
<tr>
<td>a-1</td>
<td>2.83</td>
<td>0.0024</td>
<td>0.0024</td>
<td>0.025</td>
<td>10</td>
<td>53.52 40.26 34.4 0.001739 0.0095 0.0150</td>
</tr>
<tr>
<td>a-2</td>
<td>11.20</td>
<td>0.0095</td>
<td>0.0043</td>
<td>0.098</td>
<td>11</td>
<td>48.72 37.74 32.8 0.001583 0.0100 0.0150</td>
</tr>
<tr>
<td>b</td>
<td>11.45</td>
<td>0.0103</td>
<td>0.0045</td>
<td>0.086</td>
<td>11</td>
<td>53.52 40.26 34.4 0.001739 0.0095 0.0150</td>
</tr>
<tr>
<td>c</td>
<td>2.95</td>
<td>0.0025</td>
<td>0.0024</td>
<td>0.025</td>
<td>10</td>
<td>53.52 40.26 34.4 0.001739 0.0095 0.0150</td>
</tr>
<tr>
<td>d</td>
<td>19.10</td>
<td>0.0387</td>
<td>0.0069</td>
<td>0.552</td>
<td>12</td>
<td>49.68 26.58 19.7 0.001615 0.0062 0.0150</td>
</tr>
<tr>
<td>e</td>
<td>11.32</td>
<td>0.0099</td>
<td>0.0044</td>
<td>0.122</td>
<td>11</td>
<td>50.70 35.28 30.5 0.001648 0.0087 0.0150</td>
</tr>
<tr>
<td>f-1</td>
<td>11.29</td>
<td>0.0098</td>
<td>0.0036</td>
<td>0.063</td>
<td>6</td>
<td>53.52 40.26 34.4 0.001739 0.0095 0.0150</td>
</tr>
<tr>
<td>f-2</td>
<td>13.69</td>
<td>0.0172</td>
<td>0.0049</td>
<td>0.098</td>
<td>4</td>
<td>53.52 40.26 34.4 0.001739 0.0095 0.0150</td>
</tr>
</tbody>
</table>

^a σ_r1, σ_r2, and ε_r1, ε_r2 are the reference stresses and strains, respectively, of the idealized stress-strain curve (Fig. 17).

^b Additional to initial deflection.

^c Reference anchor is defined as the anchor with displacement no more than 10% of the maximum anchor displacement. The number in this column gives the location of reference anchor, counting the maximum stressed anchor as number 1.

^d The above values of anchor forces and displacements are to be compared with ultimate anchor strength of 32.5 kips and ultimate displacement of 0.341 in. (29).
Table 5. Continued

<table>
<thead>
<tr>
<th>Case</th>
<th>Maximum anchor force, kips</th>
<th>Maximum anchor displ., in.</th>
<th>Weak panel strain, in./in.</th>
<th>Approx. lat. defl., in.</th>
<th>Location of ref. anchor</th>
<th>Weak panel characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\bar{\sigma}<em>w$ $\sigma</em>{r1}$ $\sigma_{r2}$ $\varepsilon_{yw}$ $\varepsilon_{r1}$ $\varepsilon_{r2}$</td>
</tr>
<tr>
<td>f-3</td>
<td>18.83</td>
<td>0.0361</td>
<td>0.0083</td>
<td>0.204</td>
<td>5</td>
<td>53.52 40.26 34.4 0.001739 0.0095 0.0150</td>
</tr>
<tr>
<td>g</td>
<td>16.70</td>
<td>0.0272</td>
<td>0.0098</td>
<td>0.232</td>
<td>11</td>
<td>43.20 36.00 26.4 0.001404 0.0063 0.0133</td>
</tr>
<tr>
<td>h-1</td>
<td>19.00</td>
<td>0.0378</td>
<td>0.0109</td>
<td>0.342</td>
<td>11</td>
<td>50.70 35.28 30.5 0.001648 0.0087 0.0150</td>
</tr>
<tr>
<td>h-2</td>
<td>19.11</td>
<td>0.0388</td>
<td>0.0113</td>
<td>0.311</td>
<td>11</td>
<td>46.68 36.00 29.5 0.001517 0.0079 0.0150</td>
</tr>
<tr>
<td>i-1</td>
<td>23.93</td>
<td>0.0847</td>
<td>0.0126</td>
<td>0.967</td>
<td>12</td>
<td>28.32 12.78 9.9 0.000920 0.0052 0.0150</td>
</tr>
<tr>
<td>i-2</td>
<td>24.04</td>
<td>0.0857</td>
<td>0.0128</td>
<td>0.936</td>
<td>12</td>
<td>26.06 11.81 9.8 0.000847 0.0053 0.0150</td>
</tr>
<tr>
<td>j</td>
<td>25.03</td>
<td>0.0993</td>
<td>0.0282</td>
<td>-</td>
<td>13</td>
<td>-   -   -   -   -   -</td>
</tr>
<tr>
<td>k-1</td>
<td>0.20</td>
<td>0.0002</td>
<td>0.0010</td>
<td>0.013</td>
<td>9</td>
<td>37.86 25.14 20.4 0.001188 0.0063 0.0150</td>
</tr>
<tr>
<td>k-2</td>
<td>19.59</td>
<td>0.0434</td>
<td>0.0133</td>
<td>-</td>
<td>12</td>
<td>-   -   -   -   -   -</td>
</tr>
</tbody>
</table>
forces and deformations are rather low. The reason for this result is that the design strains are lower than the yield strain of the steel used. If the design strains were assumed to be at yield strain level, the predicted maximum anchor force would be 11.76 kips which is much larger than the 2.83 kips shown in Table 5.

In the case of 1/8 in. initial inward deflection, anchor forces, and displacements are, predictably, higher. Although the increase would largely depend on the design strain as discussed above, it may be concluded that larger initial inward curvature will result in greater anchor forces and displacements. Thus, it is imperative that the maximum initial deflection permitted by the design specification be used in the analysis.

Case b. Effect of thickness variation: The thickness variation shown in Table 4 is based on tolerances for a 3/4 in. plate shown in ASTM Standards, Part 4, A20. The anchor force is seen to increase considerably, in fact, more than that due to 1/8 in. initial deflection. It may be concluded that such variation should be considered in the design analysis.

Case c. Effect of anchor stiffness variation: In this case it was assumed that one anchor adjacent to the weak panel had one-half of the stiffness of other anchors. The result shows that such a variation is not significant. Thus, it may be concluded that lower stiffness of a single anchor is not important as long as it is considered to have sufficient tensile strength to hold the panel in place (see Case d).

Case d. Effect of loss of an anchor: If the anchor is completely lost (i.e., no shear or tensile load capacity) the two adjacent panels
would become one panel which is assumed to be the weak panel. The resulting anchor force is larger than those cases in which only a single variation is considered. Therefore, it may be concluded that, for this specific design, effect of loss of an anchor is very important and such a loss should be postulated in design analyses.

Case e. Effect of water pressure: In this case it was assumed that the weak panel had a lateral pressure of 125 psi in addition to the assumed 1/16 in. initial lateral deflection. The magnitude of the lateral pressure is stated to be the maximum credible condition for the Fort St. Vrain vessel (16). The resulting anchor forces and displacements although large are not as great as those in Case d. However, it should be pointed out that the effect of water pressure would be greatly increased if the anchor spacing is larger.

Case f. Effect of hot spots: There are two possible critical locations for hot spots:

1. The weak panel is the hot spot: If the weak panel is already strained beyond yield, the effect of temperature increase will be to increase lateral deflection thus reducing its load carrying capacity. This in turn will cause increases in the anchor forces. However, there does not appear to be any data available on the behavior of initially yielded struts or plates. For this reason the case of weak panel being subject to temperature increase is not studied.

2. The hot spot is away from the weak panel: In this case the worst effect will come about if hot spot occurs adjacent to
the weak panel. Case f gives the results of analyses with this assumption.

In Case f-1 it is assumed that the panel adjacent to the weak panel will be subject to 100 °F temperature increase. In Case f-2 two consecutive panels adjacent to the weak panel are assumed to be hot. Finally, in Case f-3, two consecutive panels adjacent to the weak panel are assumed to have 200 °F higher temperature. The magnitudes of differential temperatures are taken from the proposed code (24) and reflect the normal and the emergency loading conditions. Since the area over which temperature increase occurs is not known one and two panels are considered.

The results show that anchor forces increase considerably under the assumed conditions. It is noted also that assuming hot spots on two consecutive panels is more critical than assuming hot spot on one panel. The possibility of hot spots should be considered in any design analysis.

One interesting observation is that the effect of hot spots seems to be more localized compared to other local variations. This is reflected by the low numbers for the location of reference anchors as given in Table 5.

Case g. Yield strength variation: Although the minimum yield strength of SA.537 Gr. B steel is 60 ksi (for 3/4 in. thickness) it was assumed in this case that the weak panel yield stress is 48 ksi. This is based on the assumption that yield stress variation may be as much as 25% of the minimum (8).
The anchor force obtained for this case is larger than other cases considered so far (except Cases d and f-3). This indicates the importance of local yield stress variation. Thus, in design analyses, such a variation should be postulated and the effect of radiation on yield strength of steel should be evaluated (34).

Cases h and i. Combined effect of local variations: The combination of more than one adverse variation should be based on a probabilistic study. Since such a study is not available, four different but arbitrary combinations were studied.

In Case h initial deflection, thickness variation, water pressure and hot spot were combined. The significant results obtained may be summarized as follows:

1. The effect of adverse variations are not additive. In fact, the anchor force for the combined case is much smaller than the sum of anchor forces due to individual variations.

2. It was indicated in Cases a that the magnitude of initial lateral deflection, without any other local variation, affects the anchor forces substantially. However, Cases h show that when initial inward deflection is considered together with other local variations, the magnitude of initial deflection does not appear to make much difference.

3. Although the condition represented by the combination may be considered extreme, resulting anchor forces are still less than two-thirds of the anchor ultimate strength in shear. The displacements are less than 25% of ultimate.
In Cases i, all adverse variations were combined (initial deflection, water pressure, hot spot, loss of an anchor, and yield strength variation). Such a condition is probably highly unlikely. The results show that anchor forces would be about 73% of the anchor ultimate strength. The displacements are about 24% of the ultimate. It is noted that, as in Cases h, the magnitude of initial lateral deflection makes little difference when it is combined with other local variations.

Case j. Effect of loss of load carrying capacity of a panel: As an extreme (but not probable) condition it was assumed in this case that the weak panel loses its load carrying capacity completely. The resulting maximum anchor force is about 77% of the anchor ultimate strength. However, the maximum anchor displacement is only about 27% of the ultimate displacement. Therefore, it may be concluded that even in an unlikely event of loss of a panel load carrying capacity, the liner would meet the functional requirements.

Case k. Effect of concrete void: Study of the effect of concrete void behind the liner will be meaningless under the design strain used in Cases a-j. This is because the design strain represents a loading condition in which internal pressure does not exist.

However, when there is internal pressure, a panel with concrete behind it would be subject to higher lateral pressure, thus losing much of its inplane load carrying capacity. It is noted that the design strains will be much smaller when internal pressure exists.

Case k gives the results of two analyses for which design strains were taken from Refs. (16) and (17).
It is seen that the resulting anchor forces and displacements are negligible if the weak panel load-displacement characteristic is based on 1/16 in. initial deflection. However, if the weak panel is assumed to carry no inplane load (by assuming the formation of plastic hinges at three points), anchor forces and displacements are considerable. The maximum displacement (12% of the ultimate) is still much less than the 25% to 50% allowable.
CHAPTER VII.
SUMMARY AND CONCLUSIONS

Summary

An overview of current PCRV liner design philosophy was presented in Chapter II. It was shown that the usual practice of design based on stress analysis rather than stability considerations is rational.

The stress analysis of liners is usually conducted by idealizing a unit width liner strip, together with anchors but separate from the backing concrete. An analysis technique using this approach was given in Chapter III. In addition, a two-dimensional elasto-plastic finite element method of liner analysis was presented in Chapter IV. In both methods, the initial stress method of nonlinear (due to material properties) analysis was employed.

Typical PCRV liner problems were analyzed using one- and two-dimensional methods (Chapter V). In Chapter VI procedures were presented for investigating the effect of local variations, using the one-dimensional method.

Finally, an approximate method with which elastic-plastic stiffness characteristics of struts may be determined was presented in Appendix A.

Conclusions

The main conclusions of this study may be summarized as follows:

1. The usual practice of one-dimensional analysis is a conservative approach to liner design problems. This conclusion is true whether
the local effects which must be considered in design occur over a small or large area of the liner plate.

2. The two-dimensional approach which accounts for the biaxial stress field in the liner plate results in less critical stresses and displacements in liner components.

3. Liner stress analysis problems may be reduced to solving nonlinear simultaneous equations. The nonlinearity is due to material properties.

4. One-dimensional analysis shows that local variations which may occur as a result of construction imperfections and component failures do have substantial effects on the predicted forces and displacements. Therefore, for safe design, the local effects should be considered as specified by the proposed code.

5. The so-called "weak panel" characteristics which are needed in one-dimensional analysis may be obtained, in lieu of testing, using an approximate method.

Recommendations for Future Research

Although the elasto-plastic finite element method presented in this dissertation is a practical one, additional work needs to be done on the following.

1. The method as developed considers fixed end boundary conditions. To be more general, flexible boundary conditions should be incorporated in the computer code.
2. Since the plane stress case was considered, there is need to develop methods with which inelastic stress-strain behavior of rectangular plates subject to edge forces as well as lateral pressures may be determined.

3. It is noted that local variations were arbitrarily applied in past designs. Therefore there is a need to determine, based on probability, the extent and magnitude of these effects. Furthermore, studies on the combinations of local effects which should be included in the normal and accident category loading conditions are needed.
REFERENCES


ACKNOWLEDGMENTS

The author is indebted to his adviser, Dr. Ti-ta Lee, for his support and guidance, during the investigation and writing of the dissertation. He is also indebted to Dr. H. A. Elleby for his valuable suggestions on computer programming and nonlinear analysis.

Dr. F. P. Schauer, Directorate of Licensing, USAEC, offered invaluable assistance in using the AEC computer facilities and Mr. K. L. Johnson wrote the program for one-dimensional analysis. Their help is gratefully acknowledged. The assistance of Mr. M. L. Porter, fellow graduate student, and of many persons of the Iowa State Computation Center in developing the computer programs is deeply appreciated.

Perhaps most important, the author wishes to thank his wife, Meral, for her patience and understanding during the preparation of this dissertation.

This study was conducted while the author was serving as co-investigator on a project entitled "Assessment of Behavior and Design of Steel Liners for Concrete Reactor Vessels" sponsored by the USAEC with additional funds provided by the Engineering Research Institute, Iowa State University, Ames, Iowa. The computer time made available by the USAEC and funds provided by the Department of Civil Engineering are appreciated.
APPENDIX A.

STRUT ANALYSIS

It has been suggested that load vs axial deformation characteristics of unit-width struts may be used in lieu of the load vs in-plane deformation characteristics of liner panels in the analysis of liner assembly (3).

The elastic-plastic analysis of struts, subjected to axial load and lateral pressure is summarized in this appendix. The purpose of the analysis is to establish the load vs axial deformation characteristics. The analysis is an extension of Jezek's work (25, 26) to include lateral pressure. Fixed-end and simply supported struts are considered separately.

The following assumptions are made in the analysis:

a. Initial and deflected shapes of the strut are sinusoidal,
b. Deflections are small,
c. The material is elastic-perfectly plastic,
d. Plane sections remain plane.

Fixed End Struts

Lateral displacement

Based on the first assumption, the lateral displacement is given by (Fig. 18):

\[ w_o = w_{om} \sin^2 \left( \frac{nx}{a} \right) \]
\[ w = w_m \sin^2 \left( \frac{nx}{a} \right) \]
where

\[ w_0 = \text{initial lateral displacement} \]
\[ w_{om} = \text{initial lateral displacement at midspan} \]
\[ w = \text{additional lateral displacement} \]
\[ w_m = \text{additional lateral displacement at midspan} \]
\[ a = \text{initial length of the strut} \]

Curvature

For small deflections, change in curvature due to additional lateral deflection is:

\[ \phi = \frac{d^2 w}{dx^2} \]  
(A.2)

The maximum change in curvature occurs at the fixed-ends (and at the midspan) and is given by:

\[ \phi_m = \frac{2\pi^2}{a^2} w_m \]  
(A.3)

Axial load and moments

The axial force on a unit-width strut is:

\[ P = \sigma_o h \]  
(A.4)

where \( \sigma_o \) = average stress
\[ h = \text{thickness of the strut} \]

The bending moment may be expressed by:

\[ M = \frac{P}{2} (w_{om} + w_m) \cos \frac{2\pi x}{a} + \frac{qa^2}{12} \left( 1 + 6 \frac{x^2}{a^2} - 6 \frac{x}{a} \right) \]  
(A.5)

where \( q \) = intensity of lateral pressure. The first term in Eq. (A.5) is due to axial load and the second term is due to lateral load.
The moment is considered positive when it induces tensile stresses in the top fibers.

The maximum moment occurs at the support and is:

\[ M_m = \frac{P}{2} (w_{om} + w_m) + \frac{qa^2}{12} \quad \text{(A.6)} \]

**Average stress vs lateral deflection relations**

With the assumed elastic-perfectly plastic stress-strain relation, three different stress distributions are possible.

**Case I. Maximum stress is below yield point.**

The stress and strain distributions for this case is shown in Fig. 19. From this figure, at the point of maximum moment,

\[ \frac{\sigma_2}{h_2} = \frac{\sigma_2 - \sigma_0}{h/2} = \frac{12M_m}{h^3} = \frac{12}{h^3} \left[ \frac{P}{2} (w_{om} + w_m) + \frac{qa^2}{12} \right] \]

\[ = \frac{1}{h^3} \left[ 6\sigma_0 h (w_{om} + w_m) + qa^2 \right] \]

and,

\[ \phi_m = \frac{\varepsilon_2}{h_2} = \frac{\sigma_2}{E h_2} = \frac{1}{E h^3} \left[ 6\sigma_0 h (w_{om} + w_m) + qa^2 \right] \]

Using this expression and Eq. (A.3),

\[ \frac{1}{E h^3} \left[ 6\sigma_0 h (w_{om} + w_m) + qa^2 \right] = \frac{2\pi^2}{a^2} w_m \]

and now dividing both sides by \( \overline{\sigma} \) and rearranging:

\[ \frac{\sigma_0}{\overline{\sigma}} = \frac{k \left( \frac{w_m}{h} \right) - \frac{9}{6\overline{\sigma}} \left( \frac{a}{h} \right)^2}{\frac{w_{om}}{h} + \frac{w_m}{h}} \quad \text{(A.7)} \]

where \( k = \frac{\pi^2 E}{3\overline{\sigma} \left( \frac{a}{h} \right)^2} \).
Equation (A.7) gives a nondimensionalized relationship between $\sigma_0$ and $w_m$. This equation is valid for $q < \frac{2h^2}{a^2} \sigma$ (maximum $q$ causing $\sigma_2 = \sigma$ without axial force), and as long as $\sigma_2 \leq \sigma$.

Since $v_2 = \sigma_0 + \frac{6M_m}{h} = \sigma_0 + \frac{6}{h} \left[ \frac{\sigma h}{2} (w_{om} + w_m) + \frac{qa^2}{12} \right]$

$$= \sigma_0 \left[ 1 + 3 \left( \frac{w_{om}}{h} + \frac{w_m}{h} \right) + \frac{qa^2}{2h^2} \right]$$

the condition of $\sigma_2 \leq \sigma_y$ is expressible as

$$\sigma_0 \left[ 1 + 3 \left( \frac{w_{om}}{h} + \frac{w_m}{h} \right) \right] + \frac{qa^2}{2h^2} \leq \sigma$$

Nondimensionalizing and rearranging:

$$\frac{\sigma_0}{\sigma} \leq \frac{1 - \frac{q}{2\sigma} \left( \frac{a}{h} \right)^2}{1 + 3 \left( \frac{w_{om}}{h} + \frac{w_m}{h} \right)}$$

when $\sigma_2 = \sigma$, the equal sign applies in the above expression.

Equating the $\frac{\sigma_0}{\sigma}$ expression obtained in this manner to that in Eq. (A.7), and solving for $w_m$, the limiting magnitude of additional deflection at which yielding of the extreme fibers commence is obtained as:

$$\left( \frac{w_m}{h} \right)_{\text{lim}} = \frac{1}{6k} \left[ -k - 3k \frac{w_{om}}{h} + 1 + \sqrt{(k + 3k \frac{w_{om}}{h} - 1)^2 + 12k \frac{w_{om}}{h}} \right] + \frac{q}{6\sigma} \left( \frac{a}{h} \right)^2$$

(A.8)

Case II. Compression side has yielded (Fig. 20).

When the compression side has yielded, the stress distribution may be as shown in Fig. 18. Using the conditions of equilibrium
\[ \int_{h_1}^{h_2} \sigma_2 \, dz = P \quad \text{and} \quad \int_{h_1}^{h_2} \sigma_2' \, dz = M \]

and noting that \( e_1 + c_1 + d_1 = h \), the following two equations are obtained (25, 26):

\[ \frac{e_1}{h} = \frac{3M}{(\sigma - \sigma_0)h^2} - \frac{1}{2} \quad \text{and} \quad \frac{c_1}{h} = \frac{9}{8} \left( \frac{\left( \frac{\sigma}{\sigma_0} - \frac{2M}{h^2} \right)^2}{(\sigma - \sigma_0)^3} \right) \quad (A.9) \]

At the point of maximum moment, the second expression takes the form of:

\[ \frac{c_1}{h} = \frac{9}{8} \left[ \frac{1 - \frac{\sigma_0}{\sigma}}{\left( 1 + \frac{w_m}{h} + \frac{w}{h} \right) - \frac{qa^2}{6\sigma} \frac{h}{2}} \right] \left( \frac{1}{1 - \frac{\sigma_0}{\sigma}} \right)^2 \]

With this value of \( c_1 \), and by equating the two curvature relationships

\[ \phi_m = \frac{2\pi^2}{a^2} w_m \quad \text{and} \quad \phi_m = \frac{\sigma}{E c_1} \]

and rearranging one obtains:

\[ [(2c_5 - 1)^2 - c_4] + [3c_4 + 2(c_3 + 1)(2c_5 - 1)] \left( \frac{\sigma_0}{\sigma} \right) \]

\[ + [(c_3 + 1)^2 - 3c_4] \left( \frac{\sigma_0}{\sigma} \right)^2 + c_4 \left( \frac{\sigma_0}{\sigma} \right)^3 = 0 \quad (A.10) \]

where

\[ c_3 = \frac{w_m}{h} + \frac{w}{h} \]

\[ c_4 = \frac{4}{27k} \frac{w}{h} \]

\[ c_5 = \frac{qa^2}{12\sigma h^2} \]
Equation (A.10) relates $\sigma_0$ and $w_m$ in Case II. Actual values of stress ratio $\frac{\sigma_0}{\sigma}$ may be obtained by solving Eq. (A.10) for a given $w_m/h$ ratio.

Case III. Both sides have yielded.

Proceeding as in Case II, the distances $c_2$ and $e_2$ (Fig. 21) can be determined as follows (25):

$$c_2 = \sqrt{\frac{3}{4} \left(1 - \frac{\sigma_0^2}{\sigma^2} - \frac{3M}{2h^2} \right)}$$
and
$$e_2 = \frac{1}{2} \left(1 + \frac{\sigma_0}{\sigma} \right) - \frac{c_2}{h}$$

At the point of maximum moment, the first expression takes the form of:

$$c_2 = \sqrt{\frac{3}{4} \left(1 - \frac{\sigma_0^2}{\sigma^2} \right) - \frac{3M}{2h^2}}$$

With this value of $c_2$, and by equating the two curvature relationships

$$\phi_m = \frac{2\pi^2}{a^2} w_m \quad \text{and} \quad \phi_m = \frac{\sigma}{E c_2}$$

the stress ratio for Case III is found to be

$$\frac{\sigma_0}{\sigma} = - \left(\frac{w_{om}}{h} + \frac{w_m}{h} \right) + \sqrt{\left(\frac{w_{om}}{h} + \frac{w_m}{h} \right)^2 + \left(1 - \frac{1}{27k\left(\frac{h}{w_m}\right)^2} \right) - \frac{q a^2}{35h^2}}$$ (A.11)

Apparent axial strain

The two ends of a strut displace toward each other by an amount equal to the sum of: a) contraction due to strain and b) shortening due to lateral deflection.

The total shortening of a strut due to lateral deflection, $l_\delta$, is given by the following approximate expression:
The total shortening of a strut due to strain, $\lambda_m$, is given by:

$$\lambda_m = \int_0^a \epsilon_m \, dx$$

The expression for $\epsilon_m$, strain at midheight, is different for the three different stress distribution cases. Thus:

Case I: 

$$\epsilon_m = \frac{\sigma_0}{E}$$

Case II: 

$$\epsilon_m = \frac{c_1}{h} + \frac{e_1}{h^2} - \frac{1}{2} \frac{1}{\epsilon}$$

Case III: 

$$\epsilon_m = \frac{c_2}{h} + \frac{e_2}{h^2} - \frac{1}{2} \frac{1}{\epsilon}$$

where $\epsilon = \text{yield strain}$.

For convenience, define an "apparent axial strain, $\epsilon_a$" as:

$$\epsilon_a = \frac{\lambda_s + \lambda_m}{a}$$

$$= \frac{\pi^2 \left[ \left( \frac{w_0}{h} + \frac{w_1}{h^2} \right)^2 - \left( \frac{w_0}{h} \right)^2 \right]}{4 \left( \frac{a}{h} \right)^2} + \frac{1}{a} \int_0^a \epsilon_m \, dx$$

(A.13)

The last term above may be replaced by:

$$\frac{1}{a} \int_0^a \epsilon_m \, dx = \frac{1}{n} \sum_{i=1}^n \epsilon_m$$

(A.14)
**Computational procedure**

The actual steps required in establishing load vs axial deformation (apparent average strain) curves of struts are given below. It is noted that since each of these curves consists of loading and unloading portions, lateral deflection is not unique for a given axial load. However, a unique stress ratio, $\sigma_o/\bar{\sigma}$ is obtained for a given lateral deflection $w_m/h$.

**Step 1:** For a given strut, define the following:

- $a/h$, $\bar{\sigma}$, $E$, $w_{om}$, $q$.

**Step 2:** Determine $(w_m/h)_\text{lim}$ by Eq. (A.8).

**Step 3:** Select a $(w_m/h) > (w_m/h)_\text{lim}$ and determine whether the stress distribution at the maximum moment point belongs to Case II or Case III.

**Step 4:** Compute $\sigma_o/\bar{\sigma}$ from Eqs. (A.10) or (A.11).

**Step 5:** Compute moment at several points along the strut.

**Step 6:** Determine stress distribution case and compute $\varepsilon_m$ at each point by Eq. (A.12).

**Step 7:** Compute $\varepsilon_a$ by Eqs. (A.13) and (A.14).

**Step 8:** Repeat Steps 3 through 7 for several increasingly higher $w_m/h$ values.

**Step 9:** Plot $w_m/h$ vs $\varepsilon_a$.

The determination of stress distribution case in Steps 3 and 6 is accomplished by computing $e_1$ and $c_1$ from Eq. (A.9) and then examining them to see if

$$\frac{e_1}{h} + \frac{2c_1}{h} \geq 1.0$$
and

\[ 0 < \frac{e_1}{h} < 1.0 \]

These inequalities are obtained from Eq. (A.3). Thus if any of these is not satisfied the stress distribution is Case III.

**Simply Supported Struts**

Similar derivation as in the case of fixed-end strut results in the following equations for simply-supported struts:

**Lateral displacement:** The lateral displacement is assumed to be:

\[ w_o = w_{om} \sin \left( \frac{\pi x}{a} \right) \]

\[ w = w_m \sin \left( \frac{\pi x}{a} \right) \]  \hspace{1cm} (A.15)

**Curvature:** The maximum change in curvature at mid-height is

\[ \phi_m = \frac{\pi^2}{a^2} w_m \]  \hspace{1cm} (A.16)

**Axial load and maximum moment:**

\[ P = c_h \]  \hspace{1cm} (A.17)

\[ M_m = P (w_{om} + w_m) + \frac{qa^2}{8} \]  \hspace{1cm} (A.18)

**Average stress vs lateral deflection relations:**

**Case I.** In the elastic range the stress ratio is given by

\[ \frac{\sigma_o}{\sigma} = \frac{k}{4} \left( \frac{w_m}{h} \right) + \frac{q}{8\sigma} \left( \frac{a}{h} \right)^2 \]  \hspace{1cm} (A.19)
where \( k = \frac{\pi^2 E}{3\overline{\sigma} (\frac{a}{h})^2} \) as defined before.

Equation (A.19) is valid for \( q < \frac{4h^2}{3a} \overline{\sigma} \) and \( \sigma_2 \leq \overline{\sigma} \).

The limiting additional deflection is:

\[
\left( \frac{w_m}{h} \right)_{\text{lim}} = \frac{1}{12k} \left[ -k - 6k \frac{w_{om}}{h} + 4 \sqrt{k + 6k \frac{w_{om}}{h} - 4} + 96k \left( \frac{w_{om}}{h} + \frac{q}{8\overline{\sigma}} \frac{a}{h} \right)^2 \right]
\]

\[\text{(A.20)}\]

**Case II.** The cubic equation for the solution of stress ratio when only one side has yielded becomes:

\[
[(3c_5 - 1)^2 - 2c_4] + [6c_4 + 2(2c_3 + 1)(3c_5 - 1)] \frac{\sigma_0}{\sigma} + (2c_3 + 1)^2 - 6c_4 \left( \frac{\sigma_0}{\sigma} \right)^2 + 2c_4 \left( \frac{\sigma_0}{\sigma} \right)^3 = 0
\]

\[\text{(A.21)}\]

where \( c_3, c_4, \) and \( c_5 \) are as defined for the case of fixed-end struts.

**Case III.** If both sides have yielded,

\[
\frac{\sigma_0}{\sigma} = -2 \left( \frac{w_{om}}{h} + \frac{w_m}{h} \right) + \sqrt{4 \left( \frac{w_{om}}{h} + \frac{w_m}{h} \right)^2 + \left( 1 - \frac{4}{27k^2 \left( \frac{w_m}{h} \right)^2} - \frac{q_a^2}{2\overline{\sigma}h^2} \right)}
\]

\[\text{(A.22)}\]

Apparent axial strain can be determined as in the case of fixed-end struts.
APPENDIX B.

YIELD CRITERION, ELASTIC LIMIT, PLASTICITY MATRIX

Yield Criterion

Among the yield criteria developed for steels the von Mises yield criterion is probably the most commonly used. In this study, the von Mises yield criterion (maximum distortion energy theory) was adopted since it can be readily computerized.

According to the maximum distortion energy theory yielding (plastic flow) begins when the distortion energy equals the distortion energy at yield in simple tension.

Distortion energy in a uniaxial tension test is given by (35)

\[ U_d = \frac{1 + \mu}{3E} \sigma_1^2 = \frac{1}{6G} \sigma_1^2 \]  

(B.1)

where

- \( U_d \) = distortion energy
- \( \mu \) = Poisson's ratio
- \( E \) = modulus of elasticity
- \( G \) = shear modulus
- \( \sigma_1 \) = tensile stress.

In the more general case

\[ U_d = \frac{1}{2G} J_2 \]  

(B.2)

where

\[ J_2 = \frac{1}{3} (I_1^2 + 3I_2) \]

\( I_1, I_2 \) = first and second stress invariants.

For plastic flow to occur, then (with \( \sigma_1 = \sigma \))

\[ J_2 \geq \frac{1}{3} \sigma^2 \]  

(B.3)
or, explicitly,

\[
\frac{1}{6} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right] \geq \frac{1}{3} \sigma^2
\]

(B.4)

where \(\sigma_x, \sigma_y, \sigma_z\) = normal stresses

\(\tau_{xy}, \tau_{yz}, \tau_{zx}\) = shearing stresses.

In plane stress case \((\sigma_z = \tau_{yz} = \tau_{zx} = 0)\)

\[
\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 - \frac{\sigma^2}{\sigma} \geq 0
\]

(B.5)

Defining

\[
\phi = (\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2)^{1/2} - \bar{\sigma}
\]

(B.6)

if \(\phi \leq 0\), the case is elastic (or, at most, has just yielded), and

\(\phi > 0\), the case is plastic.

Elastic Limit

In plastic analysis it is necessary to determine the loads (or displacements) at which plastic flow commences. The following procedure may be used for this purpose (plane stress case):

Let \((\sigma_x)^e, (\sigma_y)^e, (\tau_{xy})^e\) be the elastic solution to a given loading.

Assume that

\[
\phi = ((\sigma_x)^e)^2 + (\sigma_y)^e^2 - (\sigma_x)^e (\sigma_y)^e + 3(\tau_{xy})^e^2)^{1/2} - \bar{\sigma} > 0
\]

(B.7)

Let \((\sigma_x)^{lim}, (\sigma_y)^{lim}, (\tau_{xy})^{lim}\) be the stresses at elastic limit so that

\[
\phi_{lim} = [(\sigma_x)^{lim}^2 + (\sigma_y)^{lim}^2 - (\sigma_x)^{lim} (\sigma_y)^{lim} + 3(\tau_{xy})^{lim}^2)^{1/2} - \bar{\sigma} = 0
\]

(B.8)
Since Hooke's law is valid in the elastic range, stresses $\{\sigma\}_e$ and $\{\sigma\}_\text{lim}$ must be proportional. Let

$$\alpha = \frac{(\sigma_x)_\text{lim}}{(\sigma_x)_e} = \frac{(\sigma_y)_\text{lim}}{(\sigma_y)_e} = \frac{(\tau_{xy})_\text{lim}}{(\tau_{xy})_e}$$

(B.9)

where $\alpha$ is a constant. Substituting Eqs. (B.9) in Eq. (B.8)

$$\alpha((\sigma_x)_e^2 + (\sigma_y)_e^2 - (\sigma_x)_e(\sigma_y)_e + 3(\tau_{xy})_e^2)^{1/2} - \bar{\sigma} = 0$$

(B.10)

Substituting Eq. (B.7) in Eq. (B.10) and rearranging

$$\alpha[\phi + \bar{\sigma}] - \bar{\sigma} = 0$$

or

$$\alpha = \frac{\bar{\sigma}}{\phi + \bar{\sigma}}$$

(B.11)

In order to find the value of loads or displacements at the start of yielding, these loads and displacements need to be multiplied by the factor $\alpha$.

It was noted in Chapter IV that after step 9 is executed the resulting stresses are checked and, if necessary, brought back to the yield surface by taking proportionate values. In finding the stresses on the yield surface, the $\alpha$ factor as determined from Eq. (B.11) is used.

**Plasticity Matrix**

In nonlinear problems the elastic relationship given by

$$\{\sigma\} = [D]\{\varepsilon\}$$

(B.12)
is no longer valid. However, similar stress-strain relationships can be obtained for nonlinear cases (31, 32, 36, 37, 38).

It has been well established that the plastic deformations are incremental (36). Using the more generally accepted Prandtl-Reuss equations and von Mises yield criterion, the incremental stress-strain relationship may be expressed by (36)

$$\{d\sigma\} = [D_p]\{d\varepsilon\}$$

(B.13)

where $[D_p]$ is termed the plasticity matrix. Therefore, in finite element analysis, the only requisite for yielded elements is to replace the elasticity matrix $[D]$ by the plasticity matrix $[D_p]$ and to use sufficiently small strain increments to determine the incremental stress because Eq. (B.13) is valid only for small stresses and strains.

The plasticity matrix for plane stress case takes the form (36)

$$[D_p] = \frac{E}{Q} \begin{bmatrix}
\frac{\sigma_y'}{2} + p \\
\frac{\sigma_x'}{2} + p \\
\frac{\tau_{xy}}{1 + \mu} \\
\frac{\tau_{xy}}{1 + \mu} \\
\frac{R}{2(1 + \mu)} + \frac{2H'}{9E} (1 - \mu) \sigma'\end{bmatrix}$$

(B.14)

where $\sigma_x', \sigma_y' =$ deviatoric stresses

$\tau_{xy} =$ shear stress

$\sigma =$ uniaxial yield stress

$H' =$ slope of the uniaxial stress-strain curve in the plastic region
\[ P = \frac{4H'}{9E} \frac{\sigma^2}{\sigma_x} + \frac{2\tau^2_{xy}}{1 + \mu} \]

\[ R = \sigma_x^2 + 2\mu \sigma_x \sigma_y + \sigma_y^2 \]

\[ Q = R + (1 - \mu^2)P. \]

If perfect plasticity is assumed the plasticity matrix becomes

\[
[D_p] = \frac{E}{Z} \left[ \begin{array}{ccc}
\sigma_y^2 + \frac{2\tau^2_{xy}}{1 + \mu} & \frac{\sigma_y^2}{\sigma_y} + \frac{2\mu}{1 + \mu} \tau_{xy} & \frac{\sigma_y^2}{\sigma_y} + \frac{2\tau^2_{xy}}{1 + \mu} \\
\frac{\sigma_y^2 + \mu \sigma_y}{\tau_{xy}} & \frac{\sigma_y^2}{\sigma_y} + \frac{\mu \sigma_y}{\tau_{xy}} & \frac{\sigma_y^2}{\sigma_y} + \frac{2\tau^2_{xy}}{1 + \mu} \\
\frac{\sigma_y^2 + \mu \sigma_y}{\tau_{xy}} & \frac{\sigma_y^2 + \mu \sigma_y}{\tau_{xy}} & \frac{\sigma_y^2}{\sigma_y} + \frac{2\tau^2_{xy}}{1 + \mu} \\
\end{array} \right]_{\text{SYM.}}
\]

\[ (B.15) \]

where \[ Z = \sigma_x^2 + 2\mu \sigma_x \sigma_y + \sigma_y^2 + 2(1 - \mu)\tau_{xy}^2 \]
APPENDIX C.

FIGURES
Fig. 1. PCRV general configuration [from Ref. (16)].
Fig. 2. Variation of uniform minimum buckling strains with anchor spacing.
Fig. 3. One-dimensional models used by Parker (5) and Doyle and Chu (7).
Fig. 4. Model used by Bechtel [from Ref. (8)].
Fig. 5. Comparison of panel (experimental) and strut (theoretical) characteristics.
Fig. 6. Idealized panel characteristics.
Fig. 7. Load-displacement characteristics of 3/4 in. diameter stud anchors [from Ref. (29)].
Fig. 8. One-dimensional stress analysis model used in this study.
Fig. 9. Liner assembly and plate element in finite element method.
Fig. 10. Liner section analyzed (showing one weak panel in the center of the section).
Weak elements

One-dimensional model

<table>
<thead>
<tr>
<th>Nodes</th>
<th>Displacement Conditions $(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(free, fixed)</td>
</tr>
<tr>
<td>2</td>
<td>(fixed, free)</td>
</tr>
<tr>
<td>3</td>
<td>(fixed, fixed)</td>
</tr>
<tr>
<td>interior</td>
<td>(free, free)</td>
</tr>
</tbody>
</table>

Fig. 11. Finite element problem with one weak panel.
Weak elements with $\bar{\sigma}_w = 0$

Weak elements with $\bar{\sigma}_w = 0.6 \bar{\sigma}$

--- One-dimensional models

Fig. 12. Finite element problem with more than one weak panel.
\( \bar{\sigma} = 60 \text{ ksi} \)
\( \bar{\sigma}_w = 0 \)
\( h = 0.75 \text{ in.} \)
\( a_\theta = 7.5 \text{ in.} \)
\( a_z = 7.5 \text{ in.} \)

---

Two-dimensional analysis

One-dimensional analysis

---

Fig. 13. Maximum anchor displacements as influenced by design strains.
Fig. 14. Maximum anchor displacements as influenced by anchor spacing.
Fig. 15. Maximum anchor forces as influenced by design strains.
Fig. 16. Forces in anchors relative to the most-stressed anchor. (Anchor 1 is the most stressed anchor, adjacent to the weak panel.)
Fig. 17. Computed and idealized stress ratio-average strain relationships for a fixed-end strut.
Fig. 18. Fixed-end strut.

Fig. 19. Strain and stress distribution for strut analysis — Case I.
Fig. 20. Strain and stress distribution for strut analysis — Case II.

Fig. 21. Strain and stress distribution for strut analysis — Case III.