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Aspects of bivariate CDF iteration

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ASPECTS OF BIVARIATE CDF ITERATION

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Aspects of bivariate CDF iteration

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Soetarto Sastrosoewignjo

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Classical theory of iteration is due to the works of Schroeder (1871) and Koenigs (1884), and a recent significant contribution to the theory, called "regular iteration," is due to Szekeres (1958). An essential ingredient in the theory of iteration is the existence of a fixed "attractive" point \(z_0\), real or complex, fixed under iteration \(\phi^n(\cdot)\), such that \(\phi^n(z_0) = z_0\) for all \(n\). A natural example of iteration occurs in stochastic processes, namely in branching processes, in which case \(\phi^n(\cdot)\) represents the probability generating function of the number of offsprings in the \(n\)th generation; see e.g. in Feller (1950), Harris (1963), Ross (1970), Athreya and Ney (1972), and Winger (1972). Two examples of stochastic iteration of the form \(G_n(\cdot) = \phi^n G(\cdot)\), referred to as cdf iteration, can be seen in (1) Gnedenko (1943) and (2) Thomas and David (1967). The first example concerns a subsequence of geometrically growing size of a sequence of extreme value cdf's. The second example concerns the cdf of the value of certain stylized stochastic zero-sum two-person games of perfect information (i.e., SSZSTPGPI) of length \(n\). The limit laws of \(G_n\) in these two cases are given by Gnedenko (1943) and Thomas (1967), respectively.

More functional equations involving Thomas’ limit laws are given by Winger (1972) and Chung (1975); in the former case based on "easily iterated maps," stemming from the works of Schroeder (1871), Koenigs (1884), and Szekeres (1958), and in the latter case based on amplification of Winger's results, using the theory of regularly varying
monotone functions. An example of bivariate stochastic iteration also is given by Winger (1972), as the bivariate Galton-Watson branching processes. With bivariate iteration of this type, one considers a sequence of two-valued two-variable real functions. This is not the case with bivariate cdf iteration as studied in this thesis, in which the author treats a sequence of single-valued two-variable real functions. Contributory results concerning bivariate cdf's that will prove useful in this research include the bounds of Fréchet (1951), the concept "concordance" of Gini and Tchen, as cited by Marshall and Olkin (1976), in their "theory of majorization," the concept "positive or negative quadrant dependence" of Lehmann (1966), and the "Marshall-Olkin bivariate exponential cdf" of Marshall and Olkin (1967).

This thesis primarily deals with bivariate cdf iteration

$$F_n(x,y) = \lambda^{(n)}(G(x),H(y),P(x,y)) = \lambda(G_{n-1}(x),H_{n-1}(y),P_{n-1}(x,y)),$$

where

$$G_n(x) \equiv \phi_1^{(n)}G(x) \quad \text{and} \quad H_n(y) \equiv \phi_2^{(n)}H(y),$$

the marginals of $$F_n(x,y),$$ are two (univariate) cdf iterations. A natural example of this model is given by the bivariate extension of the SSZSTPGPI of Thomas and David (1967) and Thomas (1967). Weak convergence of $$F_n$$ and conditions for asymptotic independence and dependence are studied for this problem. A second problem dealt with in this thesis is a related one in extreme value theory,
i.e., the study of extreme values of iid essentially bounded bivariate r.v.'s, for which conditions for asymptotic independence and dependence also are studied.

Specifically, a review of univariate results, is given in Sections 2.1, 2.2, and 2.3, including the following topics.
1. Simple and monotone iteration. Partial results of Schroeder (1871), Koenigs (1884) and Szekeres (1958), presented in a similar method as in Ross (1970).
2. SSZSTPGPI of Thomas and David (1967), and characterization of the limit laws of the maximin cdf's of Thomas (1967).
3. "Iteration-related problem" in extreme values, and limit laws for maxima of iid bounded above bivariate r.v.'s given in Gnedenko (1943).

General considerations concerning bivariate cdf's, are given in Chapter 3, covering the following topics. The class $\mathcal{C}(G,H)$ of bivariate cdf's with marginals $G$ and $H$, its Fréchet bounds, Fréchet (1951), its convexity, concordance concept of Gini and Tchen (Marshall and Olkin, 1976), quadrant dependence of Lehmann (1966), the Marshall-Olkin distribution of Marshall and Olkin (1967), and basic definitions.

Formulation of bivariate maximin iteration is given in Chapter 4, yielding the joint distribution of the values of two correlated SSZSTPGPI's. Conditions for asymptotic independence and dependence of such values are established in Theorems 4.3.3, 4.4.1, and 4.5.1.

Bivariate maxima of iid bounded (above) bivariate r.v.'s are studied in Chapter 5. Conditions for asymptotic independence and
dependence are given in Lemmas 5.2.3 and 5.2.4, for the case when the
marginals are regular. More specific similar conditions are given for
both bivariate extremes, in Lemma 5.3.1(d), (e) and Corollaries 5.3.1
and 5.3.2, when the marginals are uniform.

A summary of findings is given in Chapter 6, which, in brief, 
may be expressed as follows: In both cases, i.e., (maximin and mini-
max) bivariate cdf iteration and the extreme value case, the upper
Fréchet bound (i.e., UFB) plays a dominant role in establishing asymp-
totic dependence. In the former case, the UFB provides essentially the
only instance of dependence, while, in the latter case, mixtures in-
volving the UFB provide such instances. In the former case, asymptotic
independence holds, essentially, unless the UFB is achieved at a certain
critical point, while, in the latter case, asymptotic independence is
established for the case \(F(x,y) \leq G(x)H(y)\) at \((x,y)\) near a certain
critical point.
2. UNIVARIATE FUNCTIONAL ITERATION

2.1. Simple and Monotone Iteration

In this section an elementary account is given of certain aspects of functional iteration for use in the later development. Although the details of the presentation are the author's, the account in spirit is essentially covered in Schroeder (1871), Koenigs (1884), and Szekeres (1958), while the mode of presentation is motivated by Ross (1970).

**Definition 2.1.1.** A function \( \phi(x) \) is said to be a simple increasing iterand on an interval \( [\alpha, \beta] \), \( \alpha < \beta \), if \( \phi(x) \) is non-decreasing on \( [\alpha, \beta] \) with \( \phi(\alpha) > \alpha \) and \( \phi(\beta) \leq \beta \).

**Definition 2.1.2.** A simple increasing iterand \( \phi(x) \) on \( [\alpha, \beta] \) is said to be right-attracted if \( \phi(x-) \equiv \sup_{\varepsilon>0}(\phi(x-\varepsilon)) > x \) for \( x \in (\alpha, \beta) \).

**Definition 2.1.3.** A simple increasing iterand \( \phi(x) \) on \( [\alpha, \beta] \) is said to be left-attracted if \( \phi(x+) \equiv \inf_{\varepsilon>0}(\phi(x+\varepsilon)) < x \) for \( x \in (\alpha, \beta) \).

**Definition 2.1.4.** A function \( \phi(x) \) is said to be an increasing iterand on \( [\alpha, \beta] \) if \( \phi(x) \) is either a simple increasing iterand on \( [\alpha, \beta] \), or \( \phi(x) \) consists of a finite set of simple increasing iterands, each defined on an element of a finite interval-partition of \( [\alpha, \beta] \), and is monotone on \( [\alpha, \beta] \).

**Definition 2.1.5.** An increasing iterand \( \phi(x) \) on \( [\alpha, \beta] \) is said to be essentially separated at an interior point \( \pi \in (\alpha, \beta) \) if \( \phi(x) \) is left-attracted on \( (\alpha, \pi) \) and is right-attracted on \( (\pi, \beta) \).

**Definition 2.1.6.** An increasing iterand \( \phi(x) \) on \( [\alpha, \beta] \) is said to be essentially connected at an interior point \( \pi \in (\alpha, \beta) \) if \( \phi(x) \)
is right-attracted on \((\alpha, \pi)\) and is left-attracted on \((\pi, \beta)\).

The relevance of an iterand being left- and right-attracted, or essentially-separated, or connected, will be seen in Lemmas 2.1.1, 2.1.2, 2.1.3, and 2.1.4.

Given an increasing iterand \(\varphi(x)\) on \([\alpha, \beta]\), in order for the following definition to be meaningful, it is necessary to establish the convention that by the associated first iterand \(\varphi^{(1)}(x)\) is meant the increasing iterand \(\varphi(x)\) itself on \([\alpha, \beta]\).

**Definition 2.1.7.** Let \(\varphi(x)\) be an increasing iterand on \([\alpha, \beta]\). Then, for \(n = 1, 2, 3, \ldots\), the associated \((n+1)\)st iterand \(\varphi^{(n+1)}(x)\) on \([\alpha, \beta]\) is defined by

\[
\varphi^{(n+1)}(x) = \varphi(\varphi^{(n)}(x)) \quad \text{for} \quad x \in [\alpha, \beta]. \quad (2.1.1)
\]

Using Definitions 2.1.1 and 2.1.4 one obtains easily from relation (2.1.1) that, for each \(n\), if \(\varphi(x)\) is a (simple) increasing iterand on \([\alpha, \beta]\), then \(\varphi^{(n+1)}(x)\) is also a (simple) increasing iterand on \([\alpha, \beta]\). Also, using Definitions 2.1.2 and 2.1.3, if \(\varphi(x)\) is a simple increasing iterand on \([\alpha, \beta]\), relation (2.1.1) implies, for each \(n\), that, if \(\varphi(x)\) is right- or left-attracted, then \(\varphi^{(n+1)}(x)\) is also right- or left-attracted. Relation (2.1.1) also underlies:

**Fact 2.1.1.** If \(\varphi^{(s)}(x), s \geq 1\) denotes the \(s\)th iterand associated with an increasing iterand \(\varphi(x)\) on \([\alpha, \beta]\), then, for \(n \geq 2\) and \(k: 1 \leq k < n\),

\[
\varphi^{(k)}(\varphi^{(n-k)}(x)) = \varphi^{(n-k)}(\varphi^{(k)}(x)) \quad (2.1.2)
\]
The following two facts can also be verified using the previously given definitions.

**Fact 2.1.2.** If $\phi(x)$ is a left-attracted simple increasing iterand on $[a, \beta]$, then $\phi(x)$ is right-continuous at $x = a$ and $\phi(a^+) = \phi(a) = a$.

**Fact 2.1.3.** If $\psi(x)$ is a right-attracted simple increasing iterand on $[a, \beta]$, then $\psi(x)$ is left-continuous at $x = \beta$ and $\psi(\beta^-) = \psi(\beta) = \beta$.

Fact 2.1.2 is verified as follows: By assumption, for each $\varepsilon > 0$ such that $a + \varepsilon < \beta$,\
\[\phi(a + \varepsilon) > \phi(a) > a, \quad (2.1.3)\]

using Definition 2.1.1. In addition, by Definition 2.1.3, one also gets\
\[a + \varepsilon > \phi(a + \varepsilon) > \phi(a + \varepsilon). \quad (2.1.4)\]

With inequalities (2.1.3) and (2.1.4) holding for arbitrary $\varepsilon > 0$ provided $a + \varepsilon < \beta$, one concludes that $\phi(x)$ is right-continuous at $x = a$, and\
\[a = \phi(a^+) = \phi(a) = a. \quad (2.1.5)\]

Fact 2.1.3 is verified in a similar fashion, to get the reversed inequalities\
\[\beta - \varepsilon < \phi(\beta - \varepsilon) \leq \phi(\beta) \leq \beta, \quad (2.1.6)\]
which hold for each $\epsilon > 0$ such that $\beta - \epsilon > \alpha$, and one concludes that

$$\beta = \phi(\beta-) = \phi(\beta) = \beta. \tag{2.1.7}$$

The following four Lemmas provide basic results concerning the limiting iterand $\phi_L(x)$ of the associated $n$th iterand $\phi^{(n)}(x)$ of a (simple) increasing iterand $\phi(x)$ on $[\alpha, \beta]$.

**Lemma 2.1.1.** If $\phi(x)$ is a left-attracted simple increasing iterand on $[\alpha, \beta], \alpha < \beta$, then the limit, as $n \to \infty$, of the associated $n$th iterand $\phi^{(n)}(x)$ is equal to $\phi_L(x) = \lim_{n \to \infty} \phi^{(n)}(x)$, given by

$$\phi_L(x) = \begin{cases} 
\alpha & \text{for } x \in [\alpha, \beta) \\
\beta & \text{for } x = \beta \{ \text{if } \phi(\beta) < \beta \\
\beta & \text{if } \phi(\beta) = \beta 
\end{cases}.$$  

**Lemma 2.1.2.** If $\phi(x)$ is a right-attracted simple increasing iterand on $[\alpha, \beta], \alpha < \beta$, then the limit, as $n \to \infty$, of the associated $n$th iterand $\phi^{(n)}(x)$ is equal to $\phi_L(x) = \lim_{n \to \infty} \phi^{(n)}(x)$, given by

$$\phi_L(x) = \begin{cases} 
\beta & \text{for } x \in (\alpha, \beta] \\
\beta & \text{for } x = \alpha \{ \text{if } \phi(\alpha) > \alpha \\
\beta & \text{if } \phi(\alpha) = \alpha 
\end{cases}.$$  

**Proof of Lemma 2.1.1:** Using Definitions 2.1.1 and 2.1.2, we have for $x \in [\alpha, \beta)$,

$$\alpha \leq \phi(2)(x) \leq \phi(\phi(x)+) \leq \phi(x) \leq \phi(x+) < \infty,$$

with the third inequality being strict unless $\phi(x) = \alpha$; inductively,
we have
\[ \alpha \leq \phi^{(n+1)}(x) \leq \phi^{(n)}(x^+), \quad \phi^{(n)}(x) \leq \phi^{(n)}(x^+) < x, \quad (2.1.8) \]
with the third inequality being strict as long as \( \phi^{(n)}(x) > \alpha \). This fact can easily be seen by means of a sketch, as in Figure 2.1(a).

Now, monotonicity of \( \phi^{(n)}(x) \in [\alpha, \beta] \) in (2.1.8) implies that
\[ \phi^{(n)}(x) \rightarrow x_0, \text{ as } n \to \infty, \text{ some } x_0 \in [\alpha, \beta], \quad (2.1.9) \]
which in turn implies
\[ \phi(x_0) = \phi(x_0^+) = x_0, \text{ some } x_0 \in [\alpha, \beta]. \quad (2.1.10) \]
Assuming \( \alpha < x_0 < \beta \), we have \( \phi(x_0) \leq \phi(x_0^+) < x_0 \), contradicting (2.1.10), so that
\[ \lim_{n \to \infty} \phi^{(n)}(x) = \phi(\alpha^+) = \alpha \text{ for each } x \in [\alpha, \beta]. \quad (2.1.11) \]
The result in (2.1.11) also follows for \( x = \beta \) whenever \( \phi(\beta) < \beta \).

Finally, if \( \phi(\beta) = \beta \), then \( \phi^{(n)}(\beta) = \beta \) for all \( n \). The assertion of the lemma is therefore proven.

Figure 2.1(a). Graphical demonstration of the relation \( \phi^{(n+1)}(x) \leq \phi^{(n)}(x) \).
The proof of Lemma 2.1.2 is very similar to that of Lemma 2.1.1, as suggested by the sketch in Figure 2.1(b). An explicit proof will not be given here.

One may notice that Definitions 2.1.2 and 2.1.3 are justified by Lemmas 2.1.1 and 2.1.2. Namely, in the limit, as \( n \to \infty \), nth iterands, at any interior point \( x_0 \) of \( [\alpha, \beta] \), are either "attracted to the left," \( \alpha \), or "attracted to the right," \( \beta \).

**Lemma 2.1.3.** If an increasing iterand \( \phi(x) \) on \( [\alpha, \beta] \) is essentially separated at \( \pi \in (\alpha, \beta) \), then the limit, as \( n \to \infty \), of the associated nth iterand \( \phi^{(n)}(x) \) is equal to \( \phi_L(x) = \lim_{n \to \infty} \phi^{(n)}(x) \), where

\[
\phi_L(x) = \begin{cases} 
\beta & \text{for } x \in (\pi, \beta] \\
\pi & \text{for } x = \pi \left\{ \begin{array}{ll}
\text{if } \phi(\pi) > \pi \\
\text{if } \phi(\pi) = \pi \\
\text{if } \phi(\pi) < \pi 
\end{array} \right. \\
\alpha & \text{for } x \in [\alpha, \pi)
\end{cases} \tag{2.1.12}
\]
Proof: \( \phi(x) \) on \([\alpha,\beta]\) is left-attracted simple increasing, and \( \phi(x) \) on \([\beta,\pi]\) is right-attracted simple increasing. Thus relation (2.1.12) follows immediately from Lemmas 2.1.1 and 2.1.2.

**Lemma 2.1.4.** If an increasing iterand \( \phi(x) \) on \([\alpha,\beta]\) is essentially connected at \( \pi \in (\alpha,\beta) \), then the associated nth iterand \( \phi^{(n)}(x) \) is continuous at \( \pi \) with \( \phi^{(n)}(\pi) = \pi \) for each \( n \), and its limit, as \( n \to \infty \), is equal to

\[
\phi_L(x) = \begin{cases} 
\beta & \text{for } x = \beta \text{ if } \phi(\beta) = \beta \\
\pi & \text{for } x \in (\alpha,\beta) \\
\alpha & \text{for } x = \alpha \text{ if } \phi(\alpha) > \alpha \text{ if } \phi(\alpha) = \alpha
\end{cases}
\]

(2.1.13)

Proof: Let \( \phi(x) \) be an increasing iterand on \([\alpha,\beta]\), essentially connected at \( \pi \in (\alpha,\beta) \). Then, \( \phi(x) \) on \([\alpha,\pi]\) is simple increasing, and is right-attracted on \((\alpha,\pi)\). Similar properties hold for \( \phi^{(n)}(x) \) for each \( n \), by the remark after Definition 2.1.7, so that, by Fact 2.1.3, \( \phi^{(n)}(x) \) is left-continuous at \( \pi \) for each \( n \). On the other hand, \( \phi(x) \) on \([\pi,\beta]\), and hence also \( \phi^{(n)}(x) \) on \([\pi,\beta]\) for each \( n \), is left-attracted simple increasing, so that Fact 2.1.2 implies that \( \phi^{(n)}(x) \) is right-continuous at \( \pi \) for each \( n \). Both one-sided continuity facts concerning \( \phi^{(n)}(x) \) imply that \( \phi^{(n)}(x) \) is in fact continuous at \( \pi \). Now, applying Lemma 2.1.1 to \( \phi(x) \) on \([\pi,\beta]\) and Lemma 2.1.2 to \( \phi(x) \) on \([\alpha,\pi]\), assertion (2.1.13) follows.

Again, it can be noticed, from Lemmas 2.1.3 and 2.1.4, that, for an increasing iterand \( \phi(x) \) on \([\alpha,\beta]\) to be essentially separated
or essentially connected, it is necessary that the limit as $n \to \infty$ of the $n$th iterand at an interior point $x_0 \in (\alpha, \beta)$, $\phi^{(n)}(x_0)$, is "separated" from $\pi$ or "connected" to $\pi$, respectively, for some $\pi \in (\alpha, \beta)$.

Decreasing iterands are not of main concern for this thesis. They are included, however, for the sake of completeness.

**Definition 2.1.8.** A function $\phi(x)$ is said to be a decreasing iterand on $[\alpha, \beta]$, $\alpha < \beta$, if

(a) $\phi(\cdot)$ is a decreasing function from $[\alpha, \beta]$ onto $[\gamma, \delta] \subset [\alpha, \beta]$, $\gamma < \delta$, with an inverse $\phi^{-1}(\cdot)$ on $[\gamma, \delta]$.

(b) There is a fixed point $\pi \in (\gamma, \delta)$ such that $\phi(\pi) = \pi$, $\phi(x) < \phi^{-1}(x)$ for $x \in [\gamma, \pi)$ and $\phi^{-1}(x) < \phi(x)$ for $x \in (\pi, \delta]$. (2.1.14)

Now, given a decreasing iterand $\phi(x)$ on $[\alpha, \beta]$, the associated $n$th iterand $\phi^{(n)}(x)$ on $[\alpha, \beta]$ for $n \geq 1$ is defined similarly as in Definition 2.1.7. But, while Fact 2.1.1 pertains equally to decreasing iterands, as it does to increasing ones, it does not analogously follow that the associated $n$th iterand $\phi^{(n)}(x)$ is itself a decreasing iterand, as indicated in the following.

**Fact 2.1.4.** If $\phi(x)$ is a decreasing iterand on $[\alpha, \beta]$, $\alpha < \beta$, then, for each $k \geq 1$, the $2k$th iterand $\phi^{(2k)}(x)$ on $[\alpha, \beta]$ is an increasing essentially connected iterand on $[\alpha, \beta]$, and the associated $(2k+1)$st iterand $\phi^{(2k+1)}(x)$ on $[\alpha, \beta]$ is a decreasing essentially connected iterand on $[\alpha, \beta]$, where this last term is defined analogous to Definition 2.1.6.

Fact 2.1.4 may be verified as follows:
(i) \( \phi^{(2)}(x) = \phi(\phi(x)) \) is an increasing function from \([\alpha, \beta]\) onto 
\([\gamma, \delta]\) \subseteq [\alpha, \beta] \), because it is a composition of two decreasing func-
tions. Clearly it also follows that

\[
\phi^{(2)}(\alpha) \geq \alpha, \quad \phi^{(2)}(\beta) \leq \beta.
\] (2.1.15)

(ii) \( \phi(\pi) = \pi \) implies \( \phi^{(2)}(\pi) = \pi \). (2.1.16)

(iii) If \( x_0 \in [\alpha, \pi] \), then monotonicity of \( \phi(\cdot) \) and (2.1.14) imply
\( \phi(x_0) \in [\pi, \delta] \), and (2.1.14) implies

\[
x_0 = \phi^{-1}(\phi(x_0)) < \phi(\phi(x_0)) = \phi^{(2)}(x_0).
\] (2.1.17)

If \( x_\ast \in [\pi, \beta] \), then \( \phi(x_\ast) \in [\alpha, \pi] \), and (2.1.14) implies

\[
x_\ast = \phi^{-1}(\phi(x_\ast)) > \phi(\phi(x_\ast)) = \phi^{(2)}(x_\ast).
\] (2.1.18)

(2.1.15)-(2.1.18) verify that \( \phi^{(2)}(x) \) is an increasing essentially
cconnected iterand on \([\alpha, \beta]\). 

Now, for each \( k \geq 1 \), \( \phi^{(2k)}(x) \) is simply the kth associated
iterand for \( \phi^{(2)}(x) \) on \([\alpha, \beta]\), and hence it is also an increasing
essentially connected iterand on \([\alpha, \beta]\); on the other hand, \( \phi^{(2k+1)}(x) \)
is again nonincreasing on \([\alpha, \beta]\), which is clear from the expression
\( \phi(\phi^{(2k)}(x)) \) where \( \phi(\cdot) \) is decreasing and \( \phi^{(2k)}(\cdot) \) is increasing.

The above-described oscillating behavior of decreasing iterands
underlines the fact that the iteration of increasing iterands is indeed
a monotone operation, in the sense of Figures 2.1(a) and 2.1(b).
Thus, increasing iterands might also be called monotone iterands.
Lemma 2.1.5. If \( \phi(x) \) is a decreasing iterand on \([\alpha, \beta]\), with the fixed point \( \Pi = \phi(\Pi) \in (\alpha, \beta) \), then, as \( n \to \infty \), the associated \( n \)th iterand \( \phi^{(n)}(x) \) on \([\alpha, \beta]\) converges to \( \Pi \).

Proof: From Fact 2.1.4, \( \phi^{(2)}(x) \) is an increasing essentially connected iterand on \([\alpha, \beta]\) with its fixed point \( \Pi \in (\alpha, \beta) \) such that \( \phi^{(2)}(\Pi) = \Pi \). Noting also that \( \phi^{(2)}(\alpha) > \alpha \) and \( \phi^{(2)}(\beta) < \beta \), one gets, using Lemma 2.1.4, that

\[
\lim_{k \to \infty} \phi^{(2k)}(x) = \Pi \quad \text{for all } x \in [\alpha, \beta].
\] (2.1.19)

It suffices to show that

\[
\lim_{k \to \infty} \phi^{(2k+1)}(x) = \Pi \quad \text{for } x \in [\alpha, \beta].
\] (2.1.20)

From (2.1.17) one gets, for \( x_0 \in [\alpha, \Pi] \),

\[
x_0 < \phi^{(2)}(x_0) \leq \Pi,
\]

so that

\[
\phi(x_0) > \phi^{(3)}(x_0) \geq \Pi,
\]

and, inductively,

\[
\phi^{(2k+1)}(x_0) > \phi^{(2k+3)}(x_0) \geq \Pi.
\] (2.1.21)

Analogously, for \( x_* \in (\Pi, \beta] \),
\[ \phi^{(2k+1)}(x^*_k) < \phi^{(2k+3)}(x^*_k) \leq \pi. \] (2.1.22)

In addition, \( \phi(x) \) is continuous at \( \pi \), because, assuming \( \phi(\pi-) > \pi \), one gets from (2.1.14) that \( \pi = \phi^{-1}(\phi(\pi-)) < \pi \), which leads to a contradiction. Similarly, assuming \( \phi(\pi+) > \pi \) will lead to a contradiction. Therefore, (2.1.19), (2.1.21), (2.1.22) and the continuity of \( \phi(*) \) at \( \pi \) imply

\[ \lim_{k \to \infty} \phi^{(2k+1)}(x) = \phi(\lim_{k \to \infty} \phi^{(2k)}(x)) = \phi(\pi) = \pi, \]

and (2.1.20) follows.

Examples of increasing iterand, as given below, conclude this section.

**Example 2.1.1.** The maximin function \( \phi(x) \) is defined (see Thomas and David (1967)) to be

\[ \phi(x) = (1 - (1-x)^2)^2, \text{ for } x \in [0,1]. \] (2.1.23)

Among the properties of \( \phi(*) \) detailed in Section 2.2 are the fact that \( \phi(*) \) is continuous on \( [0,1] \), with one interior fixed point \( a \in (0,1) \) such that \( \phi(a) = a \), where

\[ a^2 - 3a + 1 = 0, \quad 0 < a < 1. \]

Also,

\[ 0 < \phi(x) < x \text{ for } 0 < x < a, \]

\[ 1 > \phi(x) > x \text{ for } a < x < 1, \]
\[ \phi(0) = 0 \quad \text{and} \quad \phi(1) = 1. \]

The maximin function \( \phi(x) \) is, therefore, a monotone essentially separated iterand on \([0,1]\). The limit, as \( n \to \infty \), of the associated \( n \)-th iterand \( \phi^{(n)}(x) \) on \([0,1]\), applying Lemma 2.1.3, is equal to

\[
\phi_L(x) = \begin{cases} 
0, & \text{for } x \in [0,a) \\
a, & \text{for } x = a \\
1, & \text{for } x \in (a,1].
\end{cases} \tag{2.1.24}
\]

**Example 2.1.2.** Consider a probability generating function \( \phi(t) \) defined as follows

\[
\phi(t) = \sum_{i=0}^{\infty} p_i t^i \quad \text{for } t \in [0,1]
\]

and

\[ \phi(0) = p_0, \quad \text{where} \quad p_i \geq 0, \quad i \geq 0 \]

and

\[
\sum_{i=0}^{\infty} p_i = 1. \tag{2.1.25}
\]

One is chiefly interested in the case where not all \( p_i, \quad i \geq 2 \), are equal to zero, in which case \( \phi(t) \) has strictly positive first and second derivatives:

\[
\phi'(t) = \sum_{i=1}^{\infty} i p_i t^{i-1} > 0 \quad \text{for } t \in [0,1]
\]
and

\[ \varphi''(t) = \sum_{i=2}^{\infty} (i-1) \beta_i \mathbf{1}^{-i-2} > 0 \text{ for } t \in [0,1]. \]

Therefore, \( \varphi(t) \) is strictly increasing and strictly convex on \((0,1)\)
with

\[ \varphi(0) = p_0 \geq 0 \quad \text{and} \quad \varphi(1) = 1. \quad (2.1.26) \]

Now there are three possible cases.

Case 1: \( p_0 > 0 \) and \( 0 < \varphi'(1) < 1 \) \quad (2.1.27)
Case 2: \( p_0 > 0 \) and \( \varphi'(1) > 1 \) \quad (2.1.28)
Case 3: \( p_0 = 0 \) and \( \varphi'(1) > 1 \) \quad (2.1.29)

In Case 1, convexity, (2.1.26) and (2.1.27) imply \( \varphi(t) > t \) for \( t \in [0,1] \), and \( \varphi(t) \) is therefore a simple increasing right-attracted iterand on \([0,1]\).

In Case 2, convexity, (2.1.26) and (2.1.28) imply that there is a unique fixed point \( \pi \in (0,1) \) such that \( \varphi(\pi) = \pi \), \( \varphi(t) > t \) for \( t \in [0,\pi] \), and \( \varphi(t) < t \) for \( t \in (\pi,1) \). In this case, \( \varphi(t) \) is therefore a monotone essentially connected iterand on \([0,1]\).

In Case 3, convexity, (2.1.26) and (2.1.29) imply that \( \varphi(t) < t \) for \( t \in (0,1) \), so that \( \varphi(t) \) is a simple increasing left-attracted iterand on \([0,1]\).

The limit, as \( n \to \infty \), of the associated nth iterands \( \varphi^{(n)}_1(t) \), \( \varphi^{(n)}_2(t) \) and \( \varphi^{(n)}_3(t) \) for these three cases are, respectively,

\[ \varphi^{(n)}_L(t) = 1 \text{ for } t \in [0,1], \quad (2.1.30) \]
Example 2.1.3. Consider the Cantor function \( \phi(\cdot) \) defined as in Chung (1974), pp. 12-13, as follows. From the closed interval \([0,1]\), the "middle third" interval \( \left( \frac{1}{3}, \frac{2}{3} \right) \) is removed; from each of the two remaining disjoint closed intervals the middle third, \( \left( \frac{1}{9}, \frac{2}{9} \right) \) and \( \left( \frac{7}{9}, \frac{8}{9} \right) \), respectively, are removed and so on. After \( n \) steps, the number of disjoint open intervals that have been removed is 

\[ 1 + 2 + \ldots + 2^{n-1} = 2^n - 1, \]

and the total number of remaining disjoint closed intervals left is \( 2^n \). Denote the removed intervals in order of position from left to right by \( J_{n,k} \), \( 1 \leq k \leq 2^n - 1 \), and their union \( U_n = \bigcup_{k=1}^{2^n-1} J_{n,k} \). As \( n \to \infty \), \( U_n \) increases to an open set \( U \); the complement \( C \) of \( U \) with respect to \([0,1]\) is a perfect set, called the Cantor ternary set. \( C \) is of measure zero. Now, for each \( n \) and \( k \), \( n \geq 1 \), \( 1 \leq k \leq 2^n - 1 \), let \( C_{n,k} = \frac{k}{2^n} \); and define a function \( \phi_0 \) on \( U \) as follows:

\[ \phi_0(x) = C_{n,k} \quad \text{for} \quad x \in J_{n,k}. \]

Completing the definition of \( \phi(\cdot) \) on \([0,1]\) by

\[ \phi(x) = \inf_{x \in U} \phi_0(t), \quad \text{for} \quad x \notin C, \]

and

\[ \phi_L^2(t) = \begin{cases} \Pi & \text{for } t \in [0,1), \\ 1 & \text{for } t = 1 \end{cases}, \quad (2.1,31) \]

\[ \phi_L^3(t) = \begin{cases} 0 & \text{for } t \in [0,1), \\ 1 & \text{for } t = 1 \end{cases}, \quad (2.1,32) \]
\( \phi(x) \) is increasing and continuous on \([0,1]\) with \( \phi(0) = 0, \phi(1) = 1 \) (see Chung (1974, pp. 12-14) including Exercise 5, p. 6). \( \phi(x) \) is flat on each \( J_{n,k} \) but strictly increasing on \( C \). It has a fixed point \( x = \frac{1}{2} \) such that \( \phi\left(\frac{1}{2}\right) = \frac{1}{2} \), and \( \phi(x) > x \) for \( x \in (0,\frac{1}{2}) \) and \( \phi(x) < x \) for \( x \in (\frac{1}{2},1) \). Therefore, \( \phi(\cdot) \) is an essentially connected increasing iterand on \([0,1]\). The associated nth iterand \( \phi^{(n)}(\cdot) \) has the limit

\[
\phi^*_L(x) = \begin{cases} 
0, & x = 0 \\
\frac{1}{2}, & x \in (0,1) \\
1, & x = 1 
\end{cases} \quad (2.1.33)
\]
as \( n \to \infty \).

In certain "stochastic iteration processes," one considers a sequence \( \{X_n\} \) of random variables with the distribution functions

\[
\Pr[X_n \leq x] = \phi^{(n)}(x), \quad x \in [0,1],
\]
where \( \phi(\cdot) \) is the Cantor function and \( \phi^{(n)}(\cdot) \) is the associated nth iterand. Looking at the limit \( \phi^*_L(x) = \lim_{n} \phi^{(n)}(x) \) given in (2.1.33), one may conclude that the sequence of random variable \( X_n \) converges almost surely to a "Bernoullian type" random variable \( X \), where \( \Pr[X = 0+] = \frac{1}{2} \) and \( \Pr[X = 1] = \frac{1}{2} \).

2.2. Stylized Stochastic Games

Certain "stylized" stochastic zero-sum two person games of perfect information are treated in this work, where, as in Blackwell and
Girshick (1954), "two-person" indicates that there are two-players playing against each other, "zero-sum" denotes the fact that the total amount of payoffs received by both players from their opponent always is zero, while the word "stylized" indicates that there always are two alternatives available at each step or move of each player. The word "stochastic" then indicates that the payoffs are random variables, which we assume to be iid. Values of such games are computed as "iterated maximins" of the random payoffs. Another analogous iteration involves "iterated maximums" which may be interpreted as the maxima of random samples of geometrically growing size. For the "iterated maximins", the main results of Thomas and David (1967) and Thomas (1967) are reviewed in the first half of this section, and, for the "iterated maximums," the main results of Gnedenko (1943), restricted to extreme values of iid bounded random variables, are reviewed in the second half of the section.

2.2.1. The Iterated Maximin Operation

Consider two players, player I and player II, alternately choosing one of two alternative moves with \( n \) choices to be made in all by each. Corresponding to each of the \( 4^n \) possible sequences of moves, there are \( 4^n \) payoffs \( X_1, X_2, \ldots, X_{4^n} \) for player I and \( 4^n \) payoffs \( -X_1, -X_2, \ldots, -X_{4^n} \) for player II. The value \( U_n \) of such a game is computed by backward induction as follows:

For \( n = 1 \), \( U_1 = \max(\min(X_1, X_2), \min(X_3, X_4)) \).

For \( n \geq 2 \), \( U_n \) is obtained through intermediate values \( Z_k^i \);
\( k = 1,2,\ldots,n-1; \ i = 1,2,\ldots,4^{n-k}; \) as follows:

\[
\begin{align*}
Z_1^i &= \max(\min(X_{4i-3},X_{4i-2}), \min(X_{4i-1},X_{4i})), \ i = 1,2,\ldots,4^{n-1} \\
Z_2^i &= \max(\min(Z_{1,4i-3},Z_{1,4i-2}), \min(Z_{1,4i-1},Z_{1,4i})), \ i = 1,2,\ldots,4^{n-2} \\
Z_{n-1}^i &= \max(\min(Z_{n-2,4i-3},Z_{n-2,4i-2}), \min(Z_{n-2,4i-1},Z_{n-2,4i})), \ i = 1,2,3,4 \\
U_n &= \max(\min(Z_{n-1,1},Z_{n-1,2}), \min(Z_{n-1,3},Z_{n-1,4})). \quad (2.2.1)
\end{align*}
\]

The value \( U_n \) sometimes will be written also as \( \text{Mm}^{(n)}(X_k's) \).

Suppose now that the payoffs \( X_1,X_2,\ldots,X_n \) are iid r.v.'s with the common cdf \( G \). The cdf \( G_n \) of the value \( U_n \) is given in the following theorem.

**Theorem 2.2.1.**

(a) For \( n = 1 \), the cdf \( G_1 \) of \( U_1 \) is of the form

\[
G_1(x) = \phi(G(x)),
\]

where \( \phi \) is the maximin function \( \phi(t) = [1-(1-t)^2]^2, \ t \in [0,1] \) as given in Example 2.1.1.

(b) For \( n \geq 2 \), the cdf \( G_n \) of \( U_n \) is computed iteratively as follows:

\[
\begin{align*}
G_1(x) &= \phi(G(x)), \\
G_2(x) &= \phi(G_1(x)) \equiv \phi(\phi(G(x))) \equiv \phi^{(2)}(G(x)), \\
& \vdots \\
G_n(x) &= \phi(G_{n-1}(x)) \equiv \phi(\phi^{(n-1)}(G(x))) \equiv \phi^{(n)}(G(x)). \quad (2.2.2)
\end{align*}
\]
where \( \phi \) is defined as in (a).

Proof:
(a) For \( n = 1 \),

\[
G_1(x) = \Pr[U_1 \leq x]
\]

\[
= \Pr[\max(\min(X_1, X_2), \min(X_3, X_4)) \leq x]
\]

\[
= \Pr[\min(X_1, X_2) \leq x, \min(X_3, X_4) \leq x]
\]

\[
= [\Pr[\min(X_1, X_2) \leq x]]^2
\]

\[
= [1 - \Pr[\min(X_1, X_2) > x]]^2
\]

\[
= [1 - \Pr[X_1 > x, X_2 > x]]^2
\]

\[
= [1 - (\Pr[X_1 > x])^2]^2
\]

\[
= [1 - (1-G(x))^2]^2
\]

\[\equiv \phi(G(x)), \ x \in \mathbb{R}\]

with the fourth and seventh equations following by iid arguments,

(b) For \( n \geq 2 \), the result follows by induction:

Assume it is true that

\[G_{n-1}(x) = \phi(G_{n-2}(x)).\]

Then,
\[ G_n(x) = \Pr[U_n \leq x] \]
\[ = \Pr[\max(\min(Z_{n-1}^1, Z_{n-1}^2), \min(Z_{n-1}^3, Z_{n-1}^4)) \leq x] \]
\[ = \Pr[\min(Z_{n-1}^1, Z_{n-1}^2) \leq x, \min(Z_{n-1}^3, Z_{n-1}^4) \leq x] \]
\[ = [\Pr[\min(Z_{n-1}^1, Z_{n-1}^2) \leq x]]^2 \]
\[ = [1 - \Pr[\min(Z_{n-1}^1, Z_{n-1}^2) > x]]^2 \]
\[ = [1 - (\Pr[Z_{n-1}^1 > x])^2]^2 \]
\[ = [1 - (1 - G_{n-1}(x))^2]^2 \]
\[ = \varphi(G_{n-1}(x)) \equiv \varphi^{(n)}(G(x)) \]

The second equation follows from (2.2,1), the fourth and sixth equations follow by iid arguments and the seventh equation follows by the fact that \( Z_{n-1}^1 \equiv U_{n-1} \), which was assumed to have the cdf \( G_{n-1}(x) = \varphi(G_{n-2}(x)) \).

The asymptotic distribution of \( U_n \) was studied by Thomas and David (1967) and Thomas (1967). On the way to the main results, we first characterize the properties of the maximin function \( \varphi \) as follows.

**Fact 2.2.1.** The equation \( x = (1-x)^2, 0 < x < 1 \) has a unique solution, to be called \( a \).

**Fact 2.2.2.** The maximin function \( \varphi(x) = [1 - (1-x)^2]^2 \) is
a monotone iterand on \([0,1]\). Furthermore,

(i) 0, a and 1 are the only fixed points of \(\phi\) on \([0,1]\) such that \(\phi(x) = x\);

(ii) \(0 < \phi(x) < x\), \(0 < x < a\), \(x < \phi(x) < 1\), \(a < x < 1\);

(iii) \(\phi\) is continuous on \([0,1]\) and strictly increasing on \((0,1)\).

Fact 2.2.3. \(\phi^{(n)}\) satisfies (i)-(iii) of Fact 2.2.2.

Fact 2.2.4. The monotone iterand \(\phi\) is essentially separated at \(a\), so that, using Lemma 2.1.3, \(\phi^{(n)}\) has the following limit

\[
\phi_L(x) = \lim_{n} \phi^{(n)}(x) = \begin{cases} 
0, & 0 \leq x < a \\
a, & x = a \\
1, & a < x \leq 1 
\end{cases}
\]  

(2.2.3)

Fact 2.2.5. (Thomas and David, 1967): \(\phi^{(n)}\), \(n \geq 1\) is twice differentiable, with the first derivative at the interior fixed point \(a\) equal to

\[
\phi^{(n)'}(a) = b^n, \text{ where } b = 4a > 1.
\]

Fact 2.2.6. (Thomas and David, 1967): The number \(m\) in \((a,1)\) satisfying \(3m^2 - 6m + 2 = 0\) is such that

(i) \(\phi''(x) > 0\), \(0 \leq x < m\),

(ii) \(\phi''(x) < 0\), \(m < x \leq 1\),

(iii) \(\phi''(m) = 0\).

We now consider characterizing the limit cdf (see Definition 2.2.2.) of the cdf \(G_n\) of the game value \(U_n = M_{m^{(n)}}(X_k)\) when the \(\{X_k\}\)s
are iid with the common uniform cdf $G(x) = x$, $0 \leq x \leq 1$. The presentation represents a slight condensation of the argument in Thomas and David (1967). Using Theorem 2.2.1, we have $G_n(x) = \phi^{(n)}(x)$, $x \in [0,1]$. We need a sequence $\{a_n, b_n\}$, $b_n > 0$ such that

$$G_n(a_n + b_n u) \longrightarrow \$ (u), u \in \mathbb{R}$$

for some nondegenerate cdf $\$.

To this end, define

$$L_n(u) = \Pr[b_n(U_n - a) \leq u] = \Pr[u_n \leq a + \frac{u}{b_n}]$$

$$= G_n(a + \frac{u}{b_n}) = \phi^{(n)}(a + \frac{u}{b_n}), u \in [-ab^n, (1-a)b^n]. \quad (2.2.4)$$

Then, we have the following.

**Lemma 2.2.1.** For each $n \geq 1$, we have

(i) $L_n$ is a continuous cdf on $[-ab^n, (1-a)b^n]$ and is strictly increasing on $(-ab^n, (1-a)b^n)$.

(ii) $L_n$ is differentiable, and $L_n'(0) = 1$.

(iii) $L_n(0) = a$; $L_n(-ab^n) = 0$; $L_n((1-a)b^n) = 1$; $0 < L_n(u) < 1$, $-ab^n < u < (1-a)b^n$.

**Proof:** $L_n(u) = \phi^{(n)}(a + \frac{u}{b_n})$, $u \in [-ab^n, (1-a)b^n]$,

(i) follows from Fact 2.2.3 (iii),

(ii) follows from Fact 2.2.5, and

(iii) follows from Facts 2.2.2 (i) and (ii).

**Lemma 2.2.2.** $L_n$ satisfies the following functional equation:
\[ L_{m+n}(u) = \phi^{(m)}(L_n\left(\frac{u}{b^m}\right)), \quad u \in [-ab^{m+n}, (1-a)b^{m+n}], \]

\[ m,n = 1, 2, \ldots \tag{2.2.5} \]

Proof: Using (2.2.4),

\[ L_{m+n}(u) = \phi^{(m+n)}(a + \frac{u}{b^{m+n}}) = \phi^{(m)}(a + \frac{u/b^m}{b^n}) \]

\[ = \phi^{(m)}(L_n\left(\frac{u}{b^m}\right)), \quad u \in [-ab^{m+n}, (1-a)b^{m+n}] \]

The domains of \( L_{m+n} \) and \( L_n \) are consistent since

\[ u \in [-ab^{m+n}, a+b^{m+n}] \iff \frac{u}{b^m} \in [-ab^n, (1-a)b^n]. \]

Lemma 2.2.3. (Thomas and David, 1967): For any interval \( I = [u_0, u_1], \) there exists \( n_0 \) such that

\[ L_{n+1}(u) \geq L_n(u), \quad u \in I, \quad n \geq n_0. \tag{2.2.6} \]

Proof: Given \( I = [u_0, u_1], \) find \( n_0 \) large enough so that

\[ -ab^n \leq u \leq (m-a)b^n, \quad n \geq n_0. \tag{2.2.7} \]

Since \( L_1(0) = \phi(a) = a \) and \( L_1'(0) = \frac{1}{b}, \phi'(a) = 1 \) by Fact 2.2.5, and \( L_1''(z) \geq 0, -a \leq z \leq m-a, \) by Fact 2.2.6, we have

\[ L_1(z) \geq a + z, \quad -a \leq z \leq m-a. \tag{2.2.8} \]
But, for \( u \) satisfying (2.2.7), we have

\[-a < \frac{u}{b^n} < m-a, \quad n \geq n_0, \tag{2.2.9}\]

so that (2.2.8) and (2.2.9) imply

\[L_1\left(\frac{u}{b^n}\right) \geq a + \frac{u}{b^n}, \quad \forall \epsilon > 0, \quad n \geq n_0, \tag{2.2.10}\]

which leads to

\[\phi^{(n)}\left(L_1\left(\frac{u}{b^n}\right)\right) \geq \phi^{(n)}\left(a + \frac{u}{b^n}\right), \quad \forall \epsilon > 0, \quad n \geq n_0, \tag{2.2.11}\]

by monotonicity of \( \phi^{(n)} \). Now, using (2.2.5) and the definition of \( L_n \), (2.2.11) is equivalent to (2.2.6).

**Lemma 2.2.4.** (Thomas and David, 1967): \( L_n(u) \) converges pointwise to a nondecreasing function \( \phi(u), \quad -\infty < u < \infty \), with \( \phi(0) = a \).

**Proof:** For each \( u \in \mathbb{R} \), \( L_n(u) \) is bounded and, in view of Lemma 2.2.3, is eventually nondecreasing in \( n \), so that it has a limit \( \phi(u) \). The limit is nondecreasing on \( \mathbb{R} \) since \( L_n \) is nondecreasing on \( \mathbb{R} \) for each \( n \). \( \phi(0) = a \), since \( L_n(0) = a \) for all \( n \).

**Corollary 2.2.1.** \( \phi(u) = \lim_{n} L_n(u) \) satisfies the following functional equation:

\[\phi(u) = \phi^{(k)}\left(\frac{\phi(u)}{b^k}\right), \quad \forall u \in \mathbb{R}, \quad k = 1, 2, \ldots \tag{2.2.12}\]

**Proof:** Taking limits in (2.2.5) the result follows.
Define Thomas and David's bound functions $\lambda$ and $\mu$ as follows:

$$\lambda(x) = 1 - (1-a) \cdot \left(\frac{1-x}{l-a}\right)^b$$

$$\mu(x) = a \left(\frac{x}{a}\right)^b, \quad x \in [0,1]$$

and define $\lambda^{(n)}$ and $\mu^{(n)}$ of $\lambda$ and $\mu$ on $[0,1]$ analogously to $\phi^{(n)}$ of $\phi$.

Fact 2.2.7. (Thomas and David, 1967):

(i) $\lambda(x) \leq \phi(x) \leq \mu(x), \quad 0 \leq x \leq 1$.

(ii) $\lambda$ and $\mu$ are monotone increasing on $[0,1]$.

(iii) $\lambda^{(n)}(x) = 1 - (1-a) \cdot \left(\frac{1-x}{l-a}\right)^b$

$$\mu^{(n)}(x) = a \cdot \left(\frac{x}{a}\right)^b$$

Fact 2.2.8. (Thomas and David, 1967):

(i) $\lambda^{(n)}(a + \frac{u}{b^n}) \longrightarrow \alpha(u) = 1 - (1-a) \exp(-u/(1-a))$.

(ii) $\mu^{(n)}(a + \frac{u}{b^n}) \longrightarrow \beta(u) = a \exp(u/a)$.

(iii) Both $\alpha(u)$ and $\beta(u)$ are increasing and differentiable on a neighborhood of 0, with $\alpha'(0) = \beta'(0) = 1$.

Lemma 2.2.5. (Thomas and David, 1967): There is a neighborhood $J$ of 0 in which

$$\alpha(u) \leq \lambda(u) \leq \beta(u). \quad (2.2.13)$$

The proof is done by showing the existence of an interval $J = (u^-, u^+)$ around 0 such that eventually,
for $u \in J$, and then taking the limit as $n \to \infty$ and using Lemma 2.2.4 and Fact 2.2.8 to get (2.2.13).

**Corollary 2.2.2.** There is a neighborhood $J$ of 0 such that

$$\mathcal{L}(u) < m < 1, u \in J. \quad (2.2.14)$$

This corollary follows from (2.2.13) and using the fact that $\beta$ is an easily computed continuous increasing function with $\beta(0) = \mathcal{L}(0) = a < m$.

**Lemma 2.2.6.** $\mathcal{L}(u)$ is a continuous strictly increasing cdf on $R$.

**Proof:** In view of Lemma 2.2.6, since $\mathcal{L}(u)$ is bounded below and above by $\alpha(u)$ and $\beta(u)$, $\alpha(0) = \mathcal{L}(0) = \beta(0) = a$ and $\alpha(u)$ and $\beta(u)$ are strictly increasing and continuous, there is a neighborhood $J$ of 0 such that

$$0 < \mathcal{L}(u) < 1, u \in J.$$ 

Using the relation (2.2.12),

$\mathcal{L}(u) < 1, u < 0, u \in J$ implies $\mathcal{L}(0-) = a$

and

$\mathcal{L}(u) < 1, u > 0, u \in J$ implies $\mathcal{L}(0+) = a$.

Therefore, $\mathcal{L}(u)$ is continuous at $a$. Also, $\mathcal{L}(-\infty) = \lim_{n \to \infty} L_n(-ab^n) = 0$.
and \( F(\pm \infty) = \lim_{n \to \infty} \frac{(1-a)b^n}{L_n} = 1 \), since \( L_n(ab^n) = 0 \),
\( L_n((1-a)b^n) = 1, \forall n \).

Supposing \( f(u) = a \) on an interval \( I \) containing 0, (2.2.12) implies

\[
\phi(k) (f(u)) = u \xi_i, \quad k = 1, 2, \ldots ;
\]

and, taking the limit as \( k \to \infty \),

\[
\phi(-\infty) = \phi(\infty) = 1,
\]

which is impossible. Hence, \( f(u) \) is continuous and strictly increasing on an interval containing 0. Thus, it has been shown that \( f(u) \) is a cdf with values in \((0,1)\) on \((-\infty, +\infty)\). A similar argument (Thomas and David, 1967) verifies that \( f \) is in fact strictly increasing on \((-\infty, +\infty)\).

We will now consider the limit cdf of the game values

\[
U_n = \text{Mm(}n\{X_k\}, \quad \text{when the} \ X_k \ 's \ \text{are iid with right continuous common cdf} \ G. \ \text{Define} \ x_a \in \mathbb{R}, \ \text{such that}
\]

\[
G(x_a - \varepsilon) < a \leq G(x_a + \varepsilon) \quad \forall \varepsilon > 0 , \quad (2.2.15)
\]

that is,

\[
x_a = \inf\{x: G(x) \geq a\} . \quad (2.2.16)
\]

Consider a sequence \( \{b_n\}, b_n > 0 \), and define
\[ Q_n(u) = \Pr\left[ \frac{U_n - x}{b_n} \leq u \right] = \Pr\left[ U_n \leq x_a + b_n u \right] \]

\[ = G_n(x_a + b_n u) = \phi(n)(G(x_a + b_n u)) . \quad (2.2.17) \]

**Lemma 2.2.7.** Let \( \{b_n\}, b_n > 0, \) be given and \( Q_n \) be defined as in (2.2.17). The following relation holds for \( u \in \mathbb{R}: \)

\[ Q_{n+m}(u) = \phi^{(m)}(Q_n(b_n \frac{b_{n+m}}{b_n} u)) , \quad n, m = 1, 2, \ldots . \quad (2.2.18) \]

**Proof:** For any \( n \geq 1 \) and \( m = 1, \) we have

\[ Q_{n+1}(u) = \phi(\phi(n)(G(x_a + b_{n+1} u))) \]

\[ = \phi(\phi(n)(G(x_a + b_{n+1} u))) \]

\[ = \phi(Q_n(b_n \frac{b_{n+1}}{b_n} u)) . \quad (2.2.19) \]

Therefore (2.2.18) holds for each \( n \geq 1, m = 1. \) Now assume that (2.2.18) holds for \( n \geq 1, m \geq 1. \) Then,

\[ Q_{m+n+1}(u) = \phi(Q_{m+n}(b_{m+n+1} \frac{b_{m+n+1}}{b_{m+n}} u)) \]

\[ = \phi(\phi^{(m)}(Q_n(b_n \frac{b_{m+n}}{b_{m+n}} \cdot \frac{b_{m+n+1}}{b_{m+n}} u))) \]

\[ = \phi^{(m+1)}(Q_n(-\frac{m+n+1}{b_n} u)) , \]

\[ = \phi^{(m+1)}(Q_n(-\frac{m+n+1}{b_n} u)) , \]

\[ = \phi^{(m+1)}(Q_n(-\frac{m+n+1}{b_n} u)) , \]

\[ = \phi^{(m+1)}(Q_n(-\frac{m+n+1}{b_n} u)) . \]
where the first equation follows from (2.2.1a) upon replacing \( n, b_n \) by \( m+n, b_{m+n} \), and the second equation follows from the assumption that (2.2.18) holds at \( u \). The lemma is proved by induction.

Corollary 2.2.3. Let \( 0 < \beta \leq 1 \), and define

\[
\tilde{Q}_n(u) = \Pr\left[ \frac{U_n - x}{\beta^n} \leq u \right] = \phi^{(n)}(G(x + \beta^n u)), \quad (2.2.20)
\]

Then, we have

\[
\tilde{Q}_{m+n}(u) = \phi^{(m)}(\tilde{Q}_n(\beta^m u)), \quad u \in \mathbb{R}, \quad m, n = 1, 2, \ldots. \quad (2.2.21)
\]

Proof: The result follows directly from Lemma 2.2.7, upon replacing \( b_n \) by \( \beta^n \), \( n = 1, 2, \ldots \).

Remark 2.2.1. It is easy to see that since \( \phi^{(n)} \) is increasing and continuous at \( a \) with \( \phi^{(n)}(a) = a \) and \( G \) is nondecreasing with \( G(x^-) \leq a \leq G(x^+) \); then \( Q_n \) and \( \tilde{Q}_n \), defined as in (2.2.17) and (2.2.20), are continuous at \( 0 \) for each \( n \) whenever \( G(x^-) = a = G(x^+) \).

Definitions 2.2.1 and 2.2.2 and Theorem 2.2.2 given below hold in general for any (sequence and limit of) cdf's.

Definition 2.2.1. (Gnedenko, 1943): Two nondegenerate cdf's \( Q_1 \) and \( Q_2 \) are said to be of the same type if there are real constants \( \beta > 0, \alpha \in \mathbb{R} \) such that
\[ Q_1(x) = Q_2(\alpha + \beta x), \quad x \in \mathbb{R}. \quad (2.2.22) \]

**Definition 2.2.2.** A sequence \( \{ G_n \} \) of cdf's is said to converge in distribution to a nondegenerate cdf \( Q \), if there is a sequence \( \{ a_n, b_n \}, \quad b_n > 0 \) such that

\[ G_n(a_n + b_n x) \longrightarrow Q(x) \quad (2.2.23) \]

for all continuity points \( x \) of \( Q \).

**Theorem 2.2.2.** (Gnedenko, 1943 and Feller, 1966): Suppose (2.2.23) holds with a nondegenerate cdf \( Q \). Then,

\[ G_n(c_n + d_n x) \longrightarrow \mathcal{K}(x) \quad (2.2.24) \]

for some nondegenerate cdf \( \mathcal{K} \) and a sequence \( \{ c_n, d_n \}, \quad d_n > 0 \), if and only if there exist real constants \( b > 0 \) and \( a \) such that

\[ \lim_{n \to \infty} \frac{d_n}{b_n} = b \quad \text{and} \quad \lim_{n \to \infty} \frac{c_n - a_n}{b_n} = a, \quad (2.2.25) \]

and

\[ \mathcal{K}(x) = Q(a + bx), \quad x \in \mathbb{R}. \]

Now we reconsider the sequences \( \{ G_n \} \) and \( \{ Q_n \} \) for \( U_n \) and \( \frac{U_n - a}{b_n} \), as defined in (2.2.2) and (2.2.17).

**Definition 2.2.3.** (Thomas, 1967): A cdf \( G \) is said to belong to the domain of attraction of a nondegenerate cdf \( Q \) (denoted by
if the cdf $G_n$ of $U_n$ converges in distribution to $Q$.

**Definition 2.2.4.** (Thomas, 1967): The nondegenerate cdf $Q$ is said to be a limit cdf if $\mathcal{D}(Q)$ is not empty.

Let $\mathcal{L}$ be the class of nondegenerate limit cdf's $Q$ that satisfy

$$Q(-\varepsilon) < a < Q(\varepsilon) \text{ for all } \varepsilon > 0,$$

(2.2.26)

and contains exactly one member of every possible type.

Clearly, we may also view $\mathcal{L}$ as a class of location-scale equivalence classes of cdf's, each member of $\mathcal{L}$ being a class of cdf's of the same type representable by one representative cdf.

**Lemma 2.2.8.** Suppose $Q \in \mathcal{D}(Q)$, with location-scale norming constants $(x_{a}, b_{n})$, $b_{n} > 0$, where $Q \in \mathcal{L}$. The following hold

(a1) $G(x_{a} -) < a \implies Q(0-) = 0$

(a2) $G(x_{a} -) = a \iff Q(0-) = a$

(b1) $G(x_{a}) = a \iff Q(0) = a$

(b2) $G(x_{a}) > a \iff Q(0) = 1$.

Proof: $G$ and $Q$ are nondecreasing and right continuous; $G$ satisfies (2.2.15), so that

$$G(x_{a} -) \leq a \leq G(x_{a}),$$

(2.2.27)

satisfies (2.2.26), so that $Q(0-) \leq a \leq Q(0)$. $\phi^{(n)}$ is continuous increasing with $\phi^{(n)}(0) = 0$; $0 < \phi^{(n)}(x) < x$, $x \in (0,a)$; $\phi^{(n)}(a) = a$;
and

\[ x < \phi^{(n)}(x) < 1, \quad x \in (a, 1), \quad \forall n, \quad (2.2.28) \]

and

\[
\begin{align*}
0, & \quad 0 \leq x < a \\
\lim_{n} \phi^{(n)}(x) = a, & \quad x = a \\
1, & \quad a < x \leq 1,
\end{align*}
\]

as given in Fact 2.2.4.

Since \( G \in \mathcal{A}(Q) \), for some \( [x_{a}, b_{n}], b_{n} > 0 \), we have

\[
Q_{n}(u) \equiv \phi^{(n)}(G(x_{a} + b_{n}u)) \rightarrow Q(u)
\]

at all continuity points \( u \) of \( Q \). Therefore,

1. \( Q_{n}(0-) \leq \phi^{(n)}(G(x_{a})). \quad (2.2.29) \)

Taking the limit of (2.2.29) as \( n \to \infty \), and assuming \( G(x_{a}) < a \) and using (2.2.28), we have

\[
0 \leq Q(0-) \leq \lim_{n} \phi^{(n)}(G(x_{a})) = 0,
\]

so that (a1) follows.

2. \( Q_{n}(0) = \phi^{(n)}(G(x_{a})). \quad (2.2.30) \)

Taking the limit of (2.2.30) as \( n \to \infty \) and using (2.2.28) again,
The negation of (al) is of the form \( Q(0-) = 0 \implies G(x^-) \geq a \). However, since \( G(x^-) \neq a \), (a2) follows.

The converse of (al) and (a2) of Lemma 2.2.8 are not necessarily true as shown in the following.

**Example 2.2.1.** Let

\[
\begin{align*}
x, & \quad 0 \leq x < a \\
G(x) &= a, \quad a \leq x < 1+a \\
1, & \quad x \geq 1+a.
\end{align*}
\]

With the location-scale constants \((a,b) = (a,l)\), \(\forall n\); we have

\[
L_n(u) = G_n(a+u) = \phi^n(a+u)
\]

\[
\begin{align*}
0, & \quad u < 0 \\
Q_0(u) &= a, \quad 0 \leq u < 1 \\
1, & \quad u \geq 1.
\end{align*}
\]

Clearly, \( Q_0 \in \mathcal{F} \subseteq \mathcal{G} \), and

\[
x_a = a, \quad G(x^-) = a, \quad Q(0-) = 0.
\]

This shows that

(i) \( G(x^-) = a \implies Q(0-) = a \)

(ii) \( Q(0-) = 0 \implies G(x^-) < a \).

**Theorem 2.2.3.** (Thomas, 1967): The cdf \( Q \in \mathcal{G} \) if and only if there exists a constant \( \beta, 0 < \beta \leq 1 \), such that
\( \phi^{(k)}(Q(u^k)) = Q(u) \), \( u \in \mathbb{R} \), \( k = 1, 2, \ldots \) \hspace{1cm} (2.2.31)

**Proof:** First suppose that \( Q \in \mathcal{E}_{\phi} \). By definition, there is a cdf \( G \) and a sequence \( \{a_n, b_n\}, b_n > 0 \), for which

\[
G_{n}(a_n + b_n u) = \phi^{(n)}(G(a_n + b_n u)) \rightarrow Q(u), \quad u \in \mathbb{R}. \tag{2.2.32}
\]

Define the function

\[
Q_1(u) = \phi^{-1}(Q(u)). \tag{2.2.33}
\]

Since \( \phi \) is a continuous strictly increasing function from \( [0, 1] \) onto \( [0, 1] \), and \( Q \in \mathcal{E}_{\phi} \), then \( Q_1 \) is a nondegenerate cdf. It also follows from (2.2.32) and (2.2.33) that

\[
G_{n}(a_{n+1} + b_{n+1} u) = \phi^{(n)}(G(a_{n+1} + b_{n+1} u)) \rightarrow Q_1(u). \tag{2.2.34}
\]

From (2.2.32), (2.2.34) and Theorem 2.2.2, it follows that \( Q \) and \( Q_1 \) are of the same type, i.e., there are constants \( \beta > 0, \alpha \), with

\[
\beta = \lim_{n} \frac{b_{n+1}}{b_n}, \quad \alpha = \lim_{n} \frac{a_{n+1} - a_n}{b_n} \tag{2.2.35}
\]

such that

\[
Q_1(u) = Q(\alpha + \beta u), \quad u \in \mathbb{R}. \tag{2.2.36}
\]

From (2.2.33) and (2.2.36) we have
\[ Q(u) = \phi(Q(\alpha + \beta u)), \quad u \in \mathbb{R}. \quad (2.2.37) \]

We need to show that \( \alpha = 0 \) and \( 0 < \beta \leq 1 \). First, note that \( \beta \geq 0 \), since \( \beta = \lim_{n \to \infty} \frac{b_{n+1}}{b_n} \) with \( b_n > 0 \) for all \( n \). Also, \( \beta \neq 0 \) since, if \( \beta = 0 \), (2.2.37) implies

\[ Q(u) = \phi(Q(\alpha)) = \text{constant}, \quad u \in \mathbb{R}, \]

which is impossible, since \( Q \) is a nondegenerate cdf. Therefore, \( \beta > 0 \). Then, to show that \( \alpha = 0 \); put \( u = 0 \) in (2.2.37) to get

\[ \phi(Q(\alpha)) = Q(0) \geq \alpha, \quad (2.2.38) \]

since \( Q \) satisfies (2.2.26), and this implies

\[ Q(\alpha) \geq \alpha, \quad \text{and hence} \quad \alpha \geq 0. \quad (2.2.39) \]

Put \( u = -\alpha/\beta \) in (2.2.37) to get

\[ Q(-\frac{\alpha}{\beta}) = \phi(Q(0)) \geq \alpha, \]

since \( Q(0) \geq \alpha \), which implies \( -\alpha/\beta \geq 0 \), so that

\[ \alpha \leq 0, \quad (2.2.40) \]

since \( \beta > 0 \).

Then (2.2.39) and (2.2.40) give

\[ \alpha = 0. \quad (2.2.41) \]

To show that \( \beta \neq 1 \), suppose that (2.2.37) holds for \( \alpha = 0 \) and
some $\beta > 1$. Then for $u < 0$,  

$$0 \leq \varphi(Q(\beta u)) \leq Q(\beta u) \leq Q(u) < a,$$

using Fact 2.2.2 of $\varphi$, so that (2.2.37) can hold only if $Q(u) = 0$, $u < 0$ which is impossible for $Q \in \mathcal{E}_I \cup \mathcal{E}_{II}$. Also, for $u > 0$,  

$$\varphi(Q(\beta u)) > Q(\beta u) > Q(u) > a,$$

using Fact 2.2.2, so that (2.2.37) can hold only if $Q(u) = a$, $u > 0$, which is impossible for $Q \in \mathcal{E}_I \cup \mathcal{E}_{III} \cup \mathcal{E}_0$.

Hence, we have  

$$Q(u) = \varphi(Q(\beta u)) , \ u \in \mathbb{R}, \ 0 < \beta \leq 1 \quad (2.2.42)$$

and, finally, it is easy to show that (2.2.42) is equivalent to (2.2.31).

Conversely, suppose (2.2.31) holds for some $Q \in \mathcal{E}$ and $0 < \beta \leq 1$. Then, taking $G = Q$ and $b_n = \beta^n$, $a_n = 0$, we have  

$$G_k(a_k + b_k u) = \varphi^{(k)}(Q(\beta^k u)) \equiv Q(u) , \ \forall k,$$

so that $G \in \mathcal{D}(Q)$.

**Lemma 2.2.9.** (Thomas, 1967): The only limit cdf $Q \in \mathcal{E}$ that satisfies  

$$Q(u) = \varphi^{(k)}(Q(u)) , \ k = 1,2,3,... \quad (2.2.43)$$

is of the form:
0, u < 0
\[ Q_0(u) = a, \quad 0 \leq u < k \quad (2.2.44) \]
1, 1, u ≥ k,

where k > 0. The class \( S_0 \) of limit cdf's of this type can therefore be represented by (2.2.44) with k = 1.

Proof: The proof is immediate, since 0, a and 1 are the only fixed points of \( \phi^{(n)}(x) \) on \([0,1]\), for all n. \( Q_0 \) is a limit cdf since we can choose \( G = Q_0 \), and \( G \in \mathcal{S}(Q_0) \).

As in Thomas (1967), define the classes \( S_1, S_{II} \) and \( S_{III} \) of cdf's in \( \mathcal{S} \phi \) as follows:

\[
S_1 = \{ Q \in \mathcal{S}: 0 < Q(u) < a \text{ for } -\infty < u < 0, a < Q(u) < 1 \text{ for } 0 < u < \infty, Q(0-) = a = Q(0) \};
\]
\[
S_{II} = \{ Q \in \mathcal{S}: 0 < Q(u) < a \text{ for } -\infty < u < 0, Q(0-) = a, Q(0) = 1 \};
\]
\[
S_{III} = \{ Q \in \mathcal{S}: a < Q(u) < 1 \text{ for } 0 < u < \infty, Q(0-) = 0, Q(0) = a \}. \quad (2.2.45)
\]

Lemma 2.2.10. (Thomas, 1967): \( \mathcal{S} \phi = S_0 \cup S_1 \cup S_{II} \cup S_{III} \).

Proof: As in Thomas (1967), let \( \mathcal{S}' = \mathcal{S} \phi - S_0 \). Then, we need to show that

\[
\mathcal{S}' = S_1 \cup S_{II} \cup S_{III}. \quad (2.2.46)
\]

Using Theorem 2.2.3 and Lemma 2.2.9, we have that a nondegenerate cdf \( Q \) is in \( \mathcal{S}' \) if and only if
\( L(u) = \phi^{(k)}(L(\beta^k u)), u \in \mathbb{R}, k = 1,2,... \)
for some \( \beta, 0 < \beta < 1 \). \hfill (2.2.47)

Fact 2.2.3 then leads to

\[
\begin{align*}
Q(u) = 1, \text{ some } u > 0 & \implies Q(\beta^k u) = 1 \quad \forall k \\
& \implies Q(0^+) = 1 \implies Q(0) = 1 ; \\
Q(u) = 0, \text{ some } u < 0 & \implies Q(\beta^k u) = 0 \quad \forall k \\
& \implies Q(0^-) = 0 .
\end{align*}
\hfill (2.2.48)
\]

We also have

\( Q(u) > a \) for \( u > 0 \) , \hfill (2.2.49)

for, if not, the functional equation

\[
\phi^{(k)}(Q(u)) = Q(u/\beta^k) \quad \text{for } 0 < \beta < 1; k = 1,2,...
\hfill (2.2.50)
\]

implies \( Q(\frac{u}{\beta^k}) = a , 0 < \beta < 1, \quad \forall k \implies Q(+\infty) = a < 1 \) which is impossible.

Also, Fact 2.2.4 applied to the functional equation (2.2.50) leads to

\[
\begin{align*}
Q(u) < 1, \text{ some } u > 0 & \implies Q(0^+) = a \\
Q(u) > 0, \text{ some } u < 0 & \implies Q(0^-) = a .
\end{align*}
\hfill (2.2.51)
\]

Since \( \mathbb{S} \) contains only nondegenerate cdf's, relations (2.2.48), (2.2.49), and (2.2.51) establish the lemma.
Lemma 2.2.11. Given a cdf $G$, suppose there is a real constant $x_a$ and a sequence $\{b_n\}, b_n > 0$ such that

$$Q_n(u) \equiv \phi^{(n)}(G(x_a + b_n u)) \to Q(u), \ u \in \mathbb{R} \quad (2.2.52)$$

for some $Q \in \mathcal{Q}$. Then,

$$x_a = \inf \{x: G(x) \geq a\} \quad (2.2.53)$$

Proof: First, assume that $Q \in \mathcal{Q}_0$ and (2.2.52) holds, so that we may assume

$$0, \ u < 0$$

$$\phi^{(n)}(G(x_a + b_n u)) \to a, \ 0 \leq u < 1 \quad (2.2.54)$$

$$1, \ u \geq 1.$$

Hence, $\phi^{(n)}(G(x_a)) \to a$, and by Fact 2.2.4 this implies

$$G(x_a) = a \quad (2.2.55)$$

Also, given $\epsilon > 0$,

$$\phi^{(n)}(G(x_a - \epsilon)) \leq \sup_{\delta > 0} \phi^{(n)}(G(x_a - b_n \delta)) \quad (2.2.56)$$

since $G$ is monotone. The RHS of (2.2.56) converges to 0 by (2.2.54), so that, in view of Fact 2.2.4, this implies

$$G(x_a - \epsilon) < a \quad (2.2.57)$$
Since $\epsilon$ is arbitrary, (2.2.55) and (2.2.57) imply that $x_a$ satisfies (2.2.53).

Next, assume that $Q \in L_0 = L_1 \cup L_2 \cup L_3$ and (2.2.52) holds. Hence,

$$Q(-\epsilon) < a < Q(\epsilon) \quad \forall \epsilon > 0,$$

but the RHS of (2.2.56) converging to $Q(-\epsilon) < a$, so that, by Fact 2.2.4,

$$G(x_a - \epsilon) < a. \quad (2.2.59)$$

Also,

$$\phi^{(n)}(G(x_a + \epsilon)) \geq \inf_{\delta > 0} \phi^{(n)}(G(x_a + b \delta)). \quad (2.2.60)$$

The RHS of (2.2.60) converging to $Q(\epsilon) > a$, so that this and Fact 2.2.4 imply that

$$G(x_a + \epsilon) > a. \quad (2.2.61)$$

Now, since $\epsilon > 0$ was arbitrary, and $G$ is monotone and right continuous, (2.2.59) and (2.2.61) imply that $x_a$ satisfies (2.2.53).

**Corollary 2.2.4.** Suppose that $G \in \mathcal{B}(Q)$ for some $Q \in L_0$.

Then the following hold:

1. If $Q \in L_0$, then $G(x_a) = a; G(x_a - \epsilon) < a, \forall \epsilon > 0$.

2. If $Q \in L_1$, then $G(x_a) = a = G(x_a - \epsilon); G(x_a - \epsilon) < a < G(x_a + \epsilon), \forall \epsilon > 0$. 

Lemma 2.2.12. Suppose a sequence of real constants \( \{a_n, b_n\} \), \( b_n > 0 \) is such that

\[
\frac{b_{n+1}}{b_n} \to \beta, \quad 0 < \beta < 1
\]  
(2.2.62)

\[
\frac{a_{n+1} - a_n}{b_n} \to 0.
\]  
(2.2.63)

Then the following hold

\[
a_n \to x_a, \quad \text{for some real constant } x_a
\]  
(2.2.64)

and

\[
\frac{x_a - a_n}{b_n} \to 0.
\]  
(2.2.65)

Proof: Let \( \epsilon \) and \( \delta \) be such that \( 0 < \epsilon < \delta = \frac{1}{3} \min(\beta, 1-\beta) \).

Clearly, \( 0 < \beta + \delta < 1 \). Let

\[
K = \frac{1}{1-(\beta+\delta)},
\]  
(2.2.66)

(2.2.62) and (2.2.63) imply that for some \( N \),

\[
0 < b_n < 1, \quad n \geq N
\]  
(2.2.67)
\[ 0 < \frac{b_{n+1}}{b_n} < \beta + \varepsilon, \quad n \geq N \quad (2.2.68) \]

\[ 0 \leq \left| \frac{a_{n+1} - a_n}{b_n} \right| < \frac{\varepsilon}{k\beta}, \quad n \geq N \quad (2.2.69) \]

(2.2.68) implies

\[ 0 < \frac{b_{n+k}}{b_n} = \prod_{j=1}^{k} \frac{b_{n+j-1}}{b_{n+j-1}} < (\beta + \varepsilon)^k, \quad k \geq 1, \quad n \geq N \quad (2.2.70) \]

(2.2.66), (2.2.69), and (2.2.70) imply

\[ 0 \leq \left| \frac{a_{n+k} - a_n}{b_n} \right| \leq \sum_{j=1}^{k} \left| \frac{a_{n+j} - a_{n+j-1}}{b_{n+j-1}} \right| \cdot \frac{b_{n+j-1}}{b_n} \]

\[ < \sum_{j=1}^{k} \frac{\varepsilon}{k\beta} (\beta + \delta)^j < \sum_{j=1}^{k} \frac{\varepsilon}{k\beta} (\beta + \delta)^j \leq \varepsilon. \quad (2.2.71) \]

Hence,

\[ 0 \leq |a_{N+k} - a_N| < b_N \varepsilon < \varepsilon, \quad \forall k \geq 1, \quad (2.2.72) \]

and this implies that \( \{a_n\} \) is a Cauchy sequence as e.g., in Rudin (1976), so that (2.2.64) follows.

Now, from (2.2.71),

\[ 0 \leq \left| \frac{a_{n+k} - a_n}{b_n} \right| < \varepsilon, \quad k \geq 1, \quad n \geq N \quad (2.2.73) \]

(2.2.73) and (2.2.64) imply that taking the limit in (2.2.73) as
so that (2.2.65) follows.

**Theorem 2.2.4.** (Thomas, 1967). Suppose that \( G \in \mathcal{B}(\mathbb{Q}) \) for some \( Q \in \mathcal{L}' \). Then there is a sequence \( \{b_n\} \) of positive constants such that

\[
C^j(u) = \varphi^j(G(x^n b_n u)) > Q(u) \quad (2.2.75)
\]

at all continuity points \( u \) of \( Q' \), where

\[
x_a = \inf\{x : G(x) > x\}. \quad (2.2.76)
\]

**Proof:** (See Lemma 4 of Thomas (1967) for an alternative proof.) Since \( G \in \mathcal{B}(\mathbb{Q}) \), \( Q \in \mathcal{L}' \), by definition there is a sequence of real constants \( \{a_n,b_n\}, b_n > 0 \) such that

\[
Q^a_n(u) = \varphi^j(G(a^+_n b_n u)) \longrightarrow Q(u) \quad (2.2.77)
\]

at all continuity points \( u \) of \( Q' \).

Now we can write

\[
Q^a_{n+1}(u) = \varphi^j(G(a^+_{n+1} b_{n+1} u)) = \varphi^j(G(a^+_n b_n \left[ \frac{a^+_{n+1} - a_n}{b_n} + \frac{b_{n+1}}{b_n} u \right]))
\]
or,

\[ Q^{a}_{n+1}(u) = \phi(Q^n[-\frac{a+1-a}{b_n} + \frac{b+1}{b_n} u]) \quad (2.2.78) \]

Convergence of \( Q^n \) to \( Q \) as in (2.2.77) and strict monotonicity and boundedness of \( \phi \) imply that

\[ Q(u) = \phi(Q(a + \beta u)) \quad (2.2.79) \]

where

\[ \beta = \lim \frac{b_{n+1}}{b_n}, \quad \alpha = \lim \frac{a_{n+1}-a}{b_n} \quad (2.2.80) \]

The proof of Theorem 2 in Thomas (1967) (our Lemma 2.2.9) applies here to deduce from (2.2.79) that in fact,

\[ 0 < \beta \leq 1, \quad \alpha = 0 \quad (2.2.81) \]

and, since \( Q \notin S_0, \beta \neq 1 \) in view of Lemma 2.2.9. Now (2.2.62) and (2.2.63) imply, in view of Lemma 2.2.12, that

\[ a_n \rightarrow x_a, \text{ some real constant } x_a \quad (2.2.82) \]

and

\[ \frac{x-a}{b_n} \rightarrow 0. \quad (2.2.83) \]

Now, let
\[ Q_n(u) = \varphi^{(n)}(G(x + b_n u)). \tag{2.2.84} \]

Then, we can write

\[ Q_n(u) = \varphi^{(n)}(G(a + b_n [\frac{-a_n}{b_n} + u])) = Q_n^a(\frac{x-a_n}{b_n} + u). \tag{2.2.85} \]

We will show that

\[ Q_n(u) \rightarrow Q(u) \tag{2.2.86} \]

at all continuity points \( u \) of \( Q \). Using the expression in (2.2.85) we have

\[
|Q_n(u) - Q(u)| \leq |Q_n^a(\frac{x-a_n}{b_n} + u) - Q(\frac{x-a_n}{b_n} + u)| + |Q(\frac{x-a_n}{b_n} + u) - Q(u)| \tag{2.2.87}
\]

with the second term converging to 0 since (2.2.83) holds and by continuity of \( Q \) at its continuity points, while the first term of (2.2.87) is no bigger than

\[
\sup_{u}|Q_n^a(u) - Q(u)|
\]

that tends to zero, since monotonicity and boundedness of \( Q_n^a \) and
continuity of \( Q \) imply uniform convergence in (2.2.77) at continuity points of \( Q \).

Therefore, (2.2.75) follows, and, in view of Lemma 2.2.11, (2.2.76) holds, so that the theorem is established.

**Theorem 2.2.5.** (Thomas, 1967): Suppose that \( G \) is a cdf with a positive continuous derivative \( g(x) = G'(x) \) on an interval \( I \) containing \( x_a \). Then \( G \in \mathcal{L}(\mathcal{L}) \), where

\[
\mathcal{L}(u) = \lim_{n \to \infty} \mathcal{L}_n(u) = \lim_{n \to \infty} \phi^{(n)}(a + \frac{u}{b^n}).
\]

A sufficient condition for \( G \) to be in the domain of attraction of a limit cdf \( Q \) in the class \( \mathcal{L}_I, \mathcal{L}_{II} \) or \( \mathcal{L}_{III} \) is given by Thomas (1967). Three particular families of limit cdf's, one in each class, are given in the form of \( Q_{I,\gamma,\tau}, Q_{II,\gamma}, \) and \( Q_{III,\gamma} \) defined as follows:

\[
\mathcal{L}_I: Q_{I,\gamma,\tau}(u) = \begin{cases} 
\mathcal{L}(-|u|^\gamma), & -\infty < u < 0 \\
\mathcal{L}(\tau u^\gamma), & 0 \leq u < \infty 
\end{cases}
\]

\[
\mathcal{L}_{II}: Q_{II,\gamma}(u) = \begin{cases} 
\mathcal{L}(-|u|^\gamma), & -\infty < u < 0 \\
1, & 0 \leq u < \infty 
\end{cases}
\]

\[
\mathcal{L}_{III}: Q_{III,\gamma}(u) = \begin{cases} 
0, & -\infty < u < 0 \\
\mathcal{L}(u^\gamma), & 0 \leq u < \infty 
\end{cases}
\]

where \( \tau > 0, \gamma > 0 \).

**Lemma 2.2.13.** (Thomas, 1967): The distributions \( Q_{I,\gamma,\tau} \)
\( Q_{II, \gamma} \) and \( Q_{III, \gamma} \) belong respectively to classes \( S_I \), \( S_{II} \), and \( S_{III} \).

The proof is given by Thomas (1967) by letting

\[
\beta = \left( \frac{1}{b} \right)^{1/\gamma}, \quad \gamma > 0
\]  

(2.2.88)

and

\[
-|u|^{\gamma}, \quad -\infty < u < 0
\]

\[
z = u^{\gamma}, \quad 0 < u < \infty, \quad (2.2.89)
\]

and using the functional relation

\[
\phi^{(k)} \left[ \frac{z}{b^k} \right] = \Phi(z), \quad -\infty < z < \infty, \quad k = 1, 2, \ldots
\]

to show that \( Q_{I, \gamma, \tau}, Q_{II, \gamma}, \) and \( Q_{III, \gamma} \) satisfy

\[
\phi^{(k)}[Q(\beta^k u)] = Q(u), \quad -\infty < u < \infty, \quad k = 1, 2, \ldots, \quad 0 < \alpha < 1.
\]

\( \Delta \)

**Theorem 2.2.6.** (Thomas, 1967): A nondegenerate cdf \( Q \in \mathcal{F}_g \) if and only if there exists a constant \( \beta, \quad 0 < \beta \leq 1 \), for which

\[
b \cdot C(\beta u) = C(u), \quad -\infty < u < \infty
\]  

(2.2.90)

where

\[
C(u) = F^{-1}(Q(u)), \quad -\infty < u < \infty.
\]  

(2.2.91)

In fact, Lemma 2.2.12 gives a subclass of (one or two parameter)
solutions to Thomas' equation (2.2.90). The complete set of solutions to Thomas' equation is given in Theorem 5.5.4 of Chung (1975).

2.2.2. The Iterated Maximum Operation

An operation analogous to the "iterated maximin" is the "iterated maximum," defined as follows. Suppose an iid sequence \( \{X_k\} \)'s of r.v.'s is given and define \( U_1 = \max\{X_1, X_2\} \), \( U_2 = \max\{\max(X_1, X_2), \max(X_3, X_4)\} \), etc., and hence \( U_n = M^{(n)} \{X_k\} \)'s, the nth iteratively computed maximum involving the first \( 2^n \) of the \( X_k \)'s. Note that, if \( W_m = \max\{X_1, \ldots, X_m\} \), it is easy to see that \( U_n = W_{2^n} \), \( n \geq 1 \), so that \( \{U_n\} \) is a subsequence of \( \{W_m\} \). The cdf of \( U_n \) is of the form

\[
G_n(x) = (G(x))^{2^n} = \phi^{(n)}(G(x)) ,
\]

where \( G \) is the common cdf of the \( X_k \)'s, and where \( \phi(x) = x^2 \),

\( 0 \leq x \leq 1 \), has the following properties:

(a) 0 and 1 are the only fixed points of \( \phi \) on \([0,1]\). (2.2.93)

(b) \( \phi \) is strictly increasing and \( 0 < \phi(x) < x \), \( 0 < x < 1 \). (2.2.94)

(c) \( \phi_n(x) = \lim_{n \to \infty} \phi^{(n)}(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases} \) (2.2.95)

(d) \( \phi^{(n)}(1) = 2^n \), \( n \geq 1 \) (2.2.96)

(e) \( \phi^{(n)}(1 + \frac{u}{2^n}) = (1 + \frac{u}{2^n})^{2^n} \to e^u \), \( u \leq 0 \). (2.2.97)

Now, if \( G(x) = x \), \( 0 \leq x \leq 1 \), then from (2.2.92) and (2.2.97), we see that \( G_n \), the cdf of \( U_n \), converges in distribution to \( M \), where
M is the exponential cdf of the form \( M(u) = e^u, \ u \leq 0 \).

**Corollary 2.2.5.** The cdf \( G^m \) of \( W_m \) converges in distribution to \( M \).

**Proof:** \( G^m(x) = (G(x))^m = x^m \). Replacing the norming constants \((1, \frac{1}{2^n}) \) in (2.3.6) by \((1, \frac{1}{m}) \), we have

\[
G^m(1 + \frac{u}{m}) = (1 + \frac{u}{m})^m ,
\]

(2.2.98)

and, taking the limit as \( m \to \infty \), (2.2.98) tends to the limit of (2.2.97).

We will now consider the case when the \( X^k \)'s are bounded r.v.'s with the common cdf \( G \). Assume for some real constant \( x_0 \)

\[
G(x_0) = 1 , \ G(x_0 - \epsilon) < 1 \ \forall \epsilon > 0 .
\]

(2.2.99)

Consider the class \( M_1 \) of possible limit law of \( G_n \), where \( Q \in M_1 \) is of the form

\[
0 < Q(u) < 1, \ -\infty < u < 0 , \ Q(0-) = Q(0) = 1 .
\]

(2.2.100)

The following lemmas, similar in spirit to the development in Gnedenko (1943), follow analogously to Theorems 2.2.3 and 2.2.4, where \( G \in D(Q) \) is defined analogously as Definition 2.2.4.

**Lemma 2.2.14.** \( Q \in M_1 \) if and only if the following functional equation holds

\[
Q(u) = \phi(Q(\beta u)) = [Q(\beta u)]^2
\]

(2.2.101)
for some $\beta$, $0 < \beta < 1$.

Lemma 2.2.15. Suppose $G \in \mathcal{B}(q)$ for some $Q \in M_1$. Then, for some real constants $x_0$ and $\{b_n\}$, $b_n > 0$,

$$G_n(x_0 + b_nu) \equiv \{G(x_0 + b_nu)\}^{2^n} \rightarrow Q(u), \ u \leq 0, \quad (2.2.102)$$
and $x_0$ satisfies (2.2.99).

As in Gnedenko (1943), we will now consider three types of limit laws of extreme values, denoted by $\phi^\alpha, \psi^\alpha$ and $\Lambda$, as follows:

$$\phi^\alpha(u) = \begin{cases} 0, \ u \leq 0 \\ e^{-u^\alpha}, \ u > 0 \end{cases} \quad (2.2.103)$$

$$\psi^\alpha(u) = \begin{cases} e^{-(u^\alpha)}, \ u \leq 0 \\ 1, \ u > 0 \end{cases} \quad (2.2.104)$$

and

$$\Lambda(u) = e^{-e^{-u}} \quad (2.2.105)$$

Definition 2.2.5. A cdf $G$ is said to belong to the domain of attraction of a nondegenerate cdf $Q$ (in the sense of the maximum), written $G \in \mathcal{B}(Q)$, if (2.2.106) holds for some sequence $\{a_n, b_n\}$, $b_n > 0$.

Lemma 2.2.16. (Gnedenko, 1943): A necessary and sufficient condition for a cdf $G$ and a sequence $\{a_n, b_n\}$, $b_n > 0$, to satisfy

$$G_n(a_n + b_nu) \rightarrow Q(u) \quad (2.2.106)$$
for all real \( u \) as \( n \to \infty \), is that

\[
\lim_{n \to \infty} n G(a_n + b_n u) = -\log Q(u),
\]

for all real values of \( u \) for which \( Q(u) \neq 0 \).

**Theorem 2.2.7.** (Gnedenko, 1943): A necessary and sufficient condition for \( G \in \mathcal{B}(\Psi_\alpha) \) is as follows:

1. for some real constant \( x_0 \)
   \[
   G(x_0) = 1 \quad \text{and} \quad G(x_0 - \varepsilon) < 1, \quad \forall \varepsilon > 0.
   \]

2. \[
   \lim_{u \to 0+} \frac{G(x_0 + ku)}{G(x_0 + u)} = k^\alpha \quad \forall k > 0.
   \]

3. \( a_n \) of (2.2.106) may be set equal to \( x_0 \).

**Corollary 2.2.6.** If a cdf \( G \) satisfies the three conditions in Theorem 2.2.7, then \( b_n \) of (2.2.106) satisfies

\[
b_n \to 0 \quad \text{as} \quad n \to \infty,
\]

and \( G \) is left continuous at \( x_0 \) with

\[
G(x_0^-) = 1.
\]

Note that the symbol \( \overline{G} \) has been used to denote an upper cdf, i.e.,

\[
\overline{G}(x) = 1 - G(x), \quad x \in \mathbb{R}.
\]
3. GENERAL CONSIDERATIONS CONCERNING BIVARIATE CDF'S

3.1. Introductory

In this chapter, we review some basic bivariate results required in the sequel. Section 3.2 deals with the class \( C(G,H) \) of bivariate cdf's \( F \) with fixed marginals \( G \) and \( H \) and its general properties, such as convexity of \( C(G,H) \), Fréchet bounds (Fréchet, 1951) for \( C(G,H) \), the concept quadrant dependence in \( C(G,H) \) (Lehmann, 1966) and certain of its ramifications, and the concordance concept of Gini and Tchen (Marshall and Olkin, 1976). Section 3.3 deals with basic definitions of weak convergence and asymptotic independence in bivariate cdf iteration.

3.2. Bivariate cdf's with Fixed Marginals

Suppose two cdf's \( G \) and \( H \) defined on the reals are given. Consider the class \( C(G,H) \), each member of which is a bivariate cdf on \( \mathbb{R}^2 \) having the marginals \( G \) and \( H \). Note that the class \( C(G,H) \) is not empty, since a cdf \( I \) defined by \( I(x,y) = G(x) \cdot H(y) \), \((x,y) \in \mathbb{R}^2\), belongs to \( C(G,H) \); the cdf \( I \) is the joint cdf of two stochastically independent random variables, with cdf's \( G \) and \( H \), respectively.

In most cases, the class \( C(G,H) \) has uncountably many members, as we shall see. Certain bounds for \( C(G,H) \) are given by Fréchet (1951), in the following theorem.

**Theorem 3.2.1.** (Fréchet Bounds-Fréchet, 1951): The class \( C(G,H) \), where \( G \) and \( H \) are cdf's on the reals, has the lower-
bound $F^0$ and the upper-bound $F^*$ given by

$$F^0(x,y) = [G(x) + H(y) - 1]^+, \quad (x,y) \in \mathbb{R}^2 \quad (3.2.1)$$

and

$$F^*(x,y) = G(x) \wedge H(y), \quad (x,y) \in \mathbb{R}^2, \quad (3.2.2)$$

where $s^+ = \max(s,0)$ and $p \wedge q = \min(p,q)$, in the sense that

$$F \in \mathcal{C}(G,H) \text{ implies } F^0(x,y) \leq F(x,y) \leq F^*(x,y), \quad (x,y) \in \mathbb{R}^2. \quad (3.2.3)$$

Furthermore, $F^0$ and $F^*$ are cdf's which belong to $\mathcal{C}(G,H)$.

Proof: Let $F$ be any member of $\mathcal{C}(G,H)$, and consider a bivariate $(X,Y)$ with the cdf $F$. First, since $\Pr\{X > x, Y > y\} = 1 - G(x) - H(y) + F(x,y) \geq 0$, we have $F(x,y) \geq G(x) + H(y) - 1$, and, since also $F(x,y) \geq 0$,

$$F(x,y) \geq [G(x) + H(y) - 1]^+ \equiv F^0(x,y), \quad (x,y) \in \mathbb{R}^2. \quad (3.2.4)$$

Now,

$$F(x,y) \leq F(x, +\infty) = G(x), \quad F(x,y) \leq F(+\infty, y) = H(y),$$

so that

$$F(x,y) \leq G(x) \wedge H(y) = F^*(x,y) \text{ for } (x,y) \in \mathbb{R}^2. \quad (3.2.5)$$

Relations (3.2.4) and (3.2.5) verify (3.2.3).

It remains to show that both $F^0$ and $F^*$ are indeed legitimate members of the class $\mathcal{C}(G,H)$. Note that, in order for a function $F$
to form a bivariate cdf, we need to verify that the following conditions (Cramér, 1945):

(i) \( F(-\infty, y) = F(x, -\infty) = 0 \),

(ii) \( F(+\infty, +\infty) = 1 \),

(iii) \( F(x+, y) = F(x, y+) = F(x, y) \), and

(iv) \( F(x+h, y+k) + F(x,y) - F(x+h,y) - F(x,y+k) \geq 0 \)

hold for all \( x \in \mathbb{R}, y \in \mathbb{R}, h \geq 0 \) and \( k \geq 0 \).

It is easy to show that \( F^0 \) and \( F^* \) satisfy (i) - (iii). To show that (iv) also holds for \( F^0 \) and \( F^* \), we can argue as follows. Suppose \( (x, y) \in \mathbb{R}^2 \) and \( h, k \geq 0 \).

(a) \( \Delta_{h,k}^2 F^*(x, y) = F^*(x+h, y+k) + F^*(x, y) - F^*(x+h, y) - F^*(x, y+k) \).

Assume that \( G(x) < H(y) \), then \( G(x) = F^*(x, y) = F^*(x, y+k) \implies F^*(x, y) - F^*(x, y+k) = 0 \), so that

\[
\Delta_{h,k}^2 F^*(x, y) = F^*(x+h, y+k) - F^*(x+h, y)
\]

\[= \begin{cases} 
G(x+h) - G(x+h) = 0 & \text{if } G(x+h) \leq H(y) \\
G(x+h) \land H(y+k) - H(y) \geq 0 & \text{if } G(x+h) > H(y).
\end{cases}
\]

Hence \( \Delta_{h,k}^2 F^*(x, y) \geq 0 \) if \( G(x) \leq H(y) \). Similar result also holds if \( G(x) \geq H(y) \).

(b) \( \Delta_{h,k}^2 F^0(x, y) = F^0(x+h, y+k) + F^0(x, y) - F^0(x+h, y) - F^0(x, y+k) \).

Let \( s(x, y) = G(x) + H(y) - 1 \). By monotonicity of \( G \) and \( H \), \( s \) is nondecreasing in \( x \) for given \( y \) and nondecreasing in
y for given x. We have the following possibilities:

1) \( s(x,y) \geq 0 \)

2) \( s(x,y) < 0, s(x+h,y) \geq 0, s(x,y+k) \geq 0 \)

3a) \( s(x+h,y) < 0, s(x,y+k) \geq 0 \)

3b) \( s(x+h,y) \geq 0, s(x,y+k) < 0 \)

4) \( s(x+h,y) < 0, s(x,y+k) < 0, s(x+h,y+k) \geq 0 \)

5) \( s(x+h,y+k) < 0 \).

Now, writing \( F^0(\cdot,\cdot) = s(\cdot,\cdot) \) when \( s(\cdot,\cdot) \geq 0 \) and \( F^0(\cdot,\cdot) = 0 \) when \( s(\cdot,\cdot) < 0 \), all cases 1) - 5) will result in

\[
\Delta^2_{h,k} F^0(x,y) \geq 0;
\]

e.g., for the case 3a), we have

\[
s(x+h,y) < 0 \implies s(x,y) < 0
\]

and

\[
x(x,y+k) \geq 0 \implies s(x+h,y+k) \geq 0,
\]

by monotonicity of \( G \), so that, in this case,

\[
\Delta^2_{h,k} F^0(x,y) = (G(x+h) + H(y+k) - 1) + 0
\]

\[
- (G(x) + H(y+k) - 1) - 0
\]

\[
= G(x+h) - G(x) \geq 0.
\]

The sketch of the proof is the author's. For another mode of proof for this theorem, see Mardia (1970). The bound cdf's \( F^0 \) and \( F^* \) of \( C(G,H) \) are commonly called the Fréchet bounds, even though Hoeffding, 1940, introduced these bounds earlier, as cited by Marshall and Olkin (1976).
Now, assume that \((X,Y)\) has the cdf \(F \in \mathcal{C}(G,H)\), and define
\[
\bar{G}(x) = 1 - G(x) = \Pr\{X > x\},
\]
\[
\bar{H}(y) = 1 - H(y) = \Pr\{Y > y\},
\]
\[
\bar{F}(x,y) = 1 - G(x) - H(y) + F(x,y) = \Pr\{X > x, Y > y\}. \tag{3.2.6}
\]
We call \(\bar{G}, \bar{H}\) and \(\bar{F}\) the upper cdf's associated with \(G, H\) and \(F\).

Consider the following partition of the probability mass on the plane:

\[
F^{(1)}(x,y) = \Pr\{X > x, Y > y\} = \bar{F}(x,y)
\]
\[
F^{(2)}(x,y) = \Pr\{X \leq x, Y > y\}
\]
\[
F^{(3)}(x,y) = \Pr\{X \leq x, Y \leq y\} = F(x,y)
\]
\[
F^{(4)}(x,y) = \Pr\{X > x, Y \leq y\}, (x,y) \in \mathbb{R}^2. \tag{3.2.7}
\]

Each one of the four possible alternative expressions for the distribution of \((X,Y)\) in (3.2.7) is called a pseudo-cdf of \((X,Y)\). Note that only two of these expressions are monotone, i.e., \(F^{(1)}\) (non-increasing), and \(F^{(3)}\) (nondecreasing). It is understood, e.g., that the "pseudo-marginals" of \(F^{(2)}\) are given by 
\[
G(x) = \sup_{y} F^{(2)}(x,y) = \lim_{y \to -\infty} F^{(2)}(x,y) \quad \text{and} \quad \bar{H}(y) = \sup_{x} F^{(2)}(x,y) = \lim_{x \to +\infty} F^{(2)}(x,y).
\]
Other marginals are obtained analogously.

In fact, in view of (3.2.6) and (3.2.7), we can easily verify that the class \(\mathcal{C}(G,H) \equiv \mathcal{C}^3(G,H)\) of cdf's \(F \equiv F^3\), may be seen
equivalently as the classes

\[ C^2(G,H), C^4(G,H) \text{ and } C^1(G,H) \]  

(3.2.8)
of \( F^{(2)}, F^{(4)}, \) and \( F^{(1)} \), in the sense that the triples

\[ \{G,H,F\}, \{G,H,F^{(2)}\}, \{G,H,F^{(4)}\} \]

and \( \{G,H,F^{(1)}\} \) are in equivalence relations; e.g., define the relation "\( \equiv \)" as \( \{G,H,F\} \equiv \{G,H,F^{(2)}\} \), or "\( \{G,H,F\} \) is in relation \( \equiv \) with \( \{G,H,F^{(2)}\} \)" if and only if

\[ H(x) = 1 - H(x) \]

and

\[ F(x,y) = G(x) - F^{(2)}(x,y), \ (x,y) \in \mathbb{R}^2, \]

and note that the relation "\( \equiv \)" is indeed reflexive, symmetric and transitive.

An immediate property of the class \( C(G,H) \) is given in the following theorem.

Theorem 3.2.2. Let \( F \) be a bivariate cdf with the marginals \( G \) and \( H \). Then \( F \) is continuous if and only if \( G \) and \( H \) are continuous.

Proof: Assume, (see Cramér, 1945) that \( G \) and \( H \) are continuous cdf's on \( \mathbb{R}^1 \). Given \( \epsilon > 0 \), and \( x_0, y_0 \in \mathbb{R} \), we can find \( \delta_\epsilon(x_0,y_0) \) such that, if \( x,y \in \mathbb{R} \) with \( |x-x_0| < \delta_\epsilon(x_0,y_0) \) and \( |y-y_0| < \delta_\epsilon(x_0,y_0) \), then

\[ |G(x) - G(x_0)| < \epsilon/2 \text{ and } |H(y) - H(y_0)| < \epsilon/2. \]  

(3.2.9)
Now, for any \( h, k \in \mathbb{R}^1 \) with \(|h|, |k| < \frac{1}{2} \delta \), then

\[
|F(x_0 + h, y_0 + k) - F(x_0, y_0)| \\
\leq |F(x_0 + |h|, y_0 + |k|) - F(x_0 - |h|, y_0 - |k|)| \\
\leq |G(x_0 + |h|) - G(x_0 - |h|)| + |H(y_0 + |k|) - H(y - |k|)| \\
< \varepsilon/2 + \varepsilon/2 = \varepsilon, \text{ by (3.2.9)}.
\]

Therefore, since \( \varepsilon \) is arbitrary, \( F \) is continuous at \( (x_0, y_0) \), and hence also on \( \mathbb{R}^2 \) since \( (x_0, y_0) \) was arbitrary.

Conversely, suppose \( G \) is discontinuous at \( x_0 \) with

\[
G(x_0) - G(x_0-) = \alpha > 0. \tag{3.2.10}
\]

Since \( G(x_0) = \sup_y F(x_0, y) \), there is \( y_0 \) such that

\[
G(x_0) - F(x_0, y_0) < \alpha/2. \tag{3.2.11}
\]

Note also that

\[
G(x_0-) - F(x_0-, y_0) \geq 0. \tag{3.2.12}
\]

Subtracting (3.2.11) from the sum of (3.2.10) and (3.2.12) yields

\[
F(x_0, y_0) - F(x_0-, y_0) > \alpha/2,
\]

which implies that \( F \) is discontinuous at \( (x_0, y_0) \).
The class $C(G,H)$, if both $G$ and $H$ are nondegenerate, has uncountably many members, as is seen from the convexity of the class, as established in the following lemma.

**Lemma 3.2.1.** The class $C(G,H)$ is convex, i.e., for $F^1$ and $F^2$ in $C(G,H)$, a mixture $F^\lambda$ of $F^1$ and $F^2$ given by

$$F^\lambda(x,y) = \lambda F^1(x,y) + (1-\lambda)F^2(x,y), \quad (x,y)\in \mathbb{R}^2, \quad \lambda \in (0,1) \quad (3.2.13)$$

belongs to $C(G,H)$.

**Proof:** $F^\lambda$ is clearly nondecreasing with

$$\inf F(x,y) = \inf F(x,y) = 0, \quad \sup \sup F(x,y) = 1,$$

and also, writing $\Delta^2_{h,k} F(x,y)$ for the "double difference"

$$F(x+h,y+k) + F(x,y) - F(x+h,y) - F(x,y+k), \quad (x,y)\in \mathbb{R}^2,$$

$$h \geq 0, \quad k \geq 0,$$

it is easily seen that

$$\Delta^2_{h,k} F^\lambda(x,y) = \lambda \Delta^2_{h,k} F^1(x,y) + (1-\lambda) \Delta^2_{h,k} F^2(x,y) \geq 0,$$

since both $F^1$ and $F^2$ are cdf's. Therefore, $F^\lambda$ is a cdf on $\mathbb{R}^2$, and indeed, $F^\lambda$ is also in $C(G,H)$, since

$$\sup F^\lambda(x,y) = \lambda \sup F^1(x,y) + (1-\lambda) \sup F^2(x,y)$$

$$= \lambda G(x) + (1-\lambda) G(x) = G(x),$$
and similarly,

$$\sup_{x} F^\lambda(x,y) = H(y),$$

so that $F^\lambda$ has the marginals $G$ and $H$.

**Corollary 3.2.1.** The classes $C^2(G,H)$, $C^4(G,H)$ and $C^1(G,H)$ given in (3.2.8) are convex.

Another property of Fréchet Bounds is, that (Feller, 1966, p. 166)

$F^0$ is singular on the set $S_0 = \{(x,y): G(x) + H(y) = 1\}$

and

$F^*$ is singular on the set $S_* = \{(x,y): G(x) = H(y)\}$.

**Example 3.2.1.** Consider, as in (3.2.15), the uniform cdf: $\mathcal{U}(x) = x, x \in [0,1]$, and the class $\mathcal{C}(\mathcal{U}, \mathcal{U})$ containing all bivariate cdf's with marginals $\mathcal{U}$. The Fréchet bounds for $\mathcal{C}(\mathcal{U}, \mathcal{U})$ are

$$F^0(x,y) = [x+y-1]^+, \ (x,y) \in [0,1]^2$$

and

$$F^*(x,y) = x\wedge y, \quad (x,y) \in [0,1]^2.$$ 

$F^0$ is singular on the line $x + y = 1$, and $F^*$ on the line $x = y$, with respect to Lebesgue, noticing that $F^0$ is flat and zero-valued on $S_{00} = \{(x,y) \in [0,1]^2: x + y \leq 1\}$ and strictly increasing on $S_{01} = \{(x,y) \in (0,1)^2: x + y > 1\}$, while for each point $a \in (0,1)$, $F^*(x,a)$ and $F^*(a,x)$ are strictly increasing on $\{x \in (0,1): x < a\}$, and flat on $\{x: x \geq a\}$.

**Lemma 3.2.2.** Suppose the lower and upper Fréchet bounds
and $F^*$ of a class $C(G,H)$ are the same. Then $C(G,H)$ contains only a single member

$$F(x,y) = G(x) H(y), \quad (x,y) \in \mathbb{R}^2$$

(3.2.14)

and, furthermore, $F$ is degenerate (or improper).

Proof: It is clear that $C(G,H)$ has only a single member $F = F^0 = F^*$. Now we write

$$[G(x) + H(y) - 1] = G(x) \land H(y)$$

(3.2.15)

Assume that $G(x) \leq H(y)$, then (3.2.15) reduces to

$$[G(x) + H(y) - 1] = G(x)$$

(3.2.16)

1. if $G(x) + H(y) - 1 < 0$, then (3.2.16) implies $G(x) = 0$, so that $G(x) H(y) = 0 = G(x) \land H(y)$,

(3.2.17)

2. if $G(x) + H(y) - 1 \geq 0$, then (3.2.16) implies $G(x) + H(y) - 1 = G(x) \implies H(y) = 1$ so that

$$G(x) H(y) = G(x) = G(x) \land H(y)$$

(3.2.18)

(3.2.15) and (3.2.18) imply that $F(x,y) \in C(G,H)$ has the form (3.2.14).

Assumption of $G(x) \geq H(y)$ leads to the same conclusion; in particular, to the cases

$$H(y) = 0 \quad \text{and} \quad G(x) = 1.$$

Hence, the only possibility is that, for some $(x_0, y_0)$,

$$G(x) = \begin{cases} 0, & x < x_0 \\ 1, & x \geq x_0 \end{cases} \quad \text{and} \quad H(y) = \begin{cases} 0, & y < y_0 \\ 1, & y \geq y_0 \end{cases}$$
The next issue of this section is to exploit the possibility of transforming members of \( C(G,H) \), \( G \) and \( H \) continuous, by \( G \) and \( H \).

**Lemma 3.2.3.** (Whitt, 1976): Suppose a r.v. \( X \) has the cdf \( G \) and a r.v. \( U \) has the uniform cdf \( \mathcal{U} \) on \([0,1]\). Let \( G^{-1}(y) = \inf\{x; G(x) > y\} \). Then the following hold:

(i) \( G^{-1}(U) \) has the cdf \( G \).

(ii) if \( G \) is continuous then \( G(X) \) has the cdf \( \mathcal{U} \).

**Lemma 3.2.4.** Suppose r.v.'s \( X \) and \( Y \) have continuous cdf's \( G \) and \( H \) respectively, and have the joint cdf \( F \). Define r.v.'s \( U \) and \( V \) by \( U = G(X) \) and \( V = H(Y) \). Then the joint cdf \( F^1 \) of \( U \) and \( V \) belongs to the class \( C(\mathcal{U},\mathcal{U}) \). Furthermore, if \( X \) and \( Y \) are stochastically independent, then \( U \) and \( V \) are stochastically independent.

**Proof:** \( U \) and \( V \) have the common cdf \( \mathcal{U} \) in view of Lemma 3.2.3. So, \( F^1 \in C(\mathcal{U},\mathcal{U}) \), with

\[
F^1(u,v) = \Pr[U \leq u, V \leq v] = \Pr[G(X) \leq u, H(Y) \leq v] = \Pr[X \leq G^{-1}(u), Y \leq H^{-1}(v)] = F(G^{-1}(u), H^{-1}(v)) \tag{3.2.19}
\]

where we have \( x = G^{-1}(x), y = H^{-1}(y) \), since \( G \) and \( H \) are continuous nondecreasing. If \( X \) and \( Y \) are stochastically independent, the expression in (3.2.19) becomes
\[ F^{1}(u,v) = G(G^{-1}(u)) \cdot H(H^{-1}(v)) \]
\[ = u \cdot v, \quad (u,v) \in [0,1]^2, \]

so that \( U \) and \( V \) are stochastically independent.

**Example 3.2.2.** Consider Gumbel's bivariate exponential distribution given by (Gumbel, 1960):

\[ F_\theta(x,y) = 1 - e^{-x} - e^{-y} + e^{-(x+y+\theta xy)}, \quad x > 0, \ y > 0, \]
\[ 0 \leq \theta \leq 1. \quad (3.2.20) \]

The marginals \( G \) and \( H \) of \( F \) are identical, i.e.,

\[ G(t) = H(t) = 1 - e^{-t}, \quad t > 0, \ \exists \ G_0(t). \]

Notice that

\[ F_\theta(x,y) = e^{-(x+y+\theta xy)}, \quad x > 0, \ y > 0 \]

and

\[ \overline{G}(t) = \overline{H}(t) = e^{-t}, \quad t > 0, = \overline{G}_0(t). \]

Then

\[ F_\theta \in C(G_0, G_0), \quad \overline{F}_\theta \in \overline{C}(\overline{G}_0, \overline{G}_0) = \overline{C}(\overline{G}_0, \overline{G}_0). \]

Let \( U = G(X) \) and \( V = H(Y) \), as in Lemma 3.2.1, where \( G \) and \( H \) are continuous on \( \mathbb{R}^+ \). Both \( U \) and \( V \) have the uniform cdf \( \mathcal{U} \), and the joint cdf of \( U \) and \( V \) is given by
\[ F^1_\theta(u,v) = \Pr[U \leq u, V \leq v] \]
\[ = 1 - (1-u) - (1-v) + (1-u)(1-v)e^{-\theta \log(1-u) \cdot \log(1-v)}, \]
\[(u,v) \in (0,1)^2.\]

Clearly,

\[ F^1_\theta \in C(U,U). \]

We can see that \(X\) and \(Y\) are stochastically independent if and only if \(\theta = 0\). The same condition holds for the independence of \(U\) and \(V\).

The previous example illustrates that any subclass \(C_\Theta(G,H)\), \(\Theta\) a suitable parameter space, of bivariate cdfs in \(C(G,H)\), \(G\) and \(H\) absolutely continuous with support \((\lambda_G, \omega_G)\) and \((\lambda_H, \omega_H)\), is equivalent to some subclass \(C_\Theta(U,U)\) of \(C(U,U)\), where \(C_\Theta(U,U)\), contains \(F^1_\theta\), the cdf of \((U,V)\), \(U = G(X), V = H(Y)\), where \((X,Y)\) has the cdf \(F_\theta \in C_\Theta(G,H)\). Furthermore, dependence or independence are common to both \(F_\theta\) and \(F^1_\theta\), for each \(\theta \in \Theta\). This observation will be useful later when we discuss the issue of asymptotic independence.

It should be noted that the subclass \(C_\Theta(G,H)\) does not necessarily have the general properties of \(C(G,H)\), such as achievable Fréchet bounds and convexity. Subclasses with nonachievable Fréchet bounds are found in Marshall and Olkin (1967) and Feller (1966), while a nonconvex subclass is given in Feller (1966).

The subclass of cdfs given in Example 3.2.2 involves exponential
marginals. Another one-parameter class is due to Morgenstern (1956), and has the form:

\[ F(x,y) = G(x)H(y)[1 + \alpha[1-G(x)][1-H(y)]], \quad |\alpha| \leq 1. \]

Another view of dependence or independence \( C(G,H) \) is given in terms of the concordance measure of Gini and Tchen, as cited by Marshall and Olkin (1976); another such view is given by Lehmann (1966) in terms of negative quadrant dependence. A property analogous to the latter will be given below, in Definition 3.2.3.

**Definition 3.2.1.** (Marshall and Olkin, 1976): Assume \( F^1 \) and \( F^2 \) are in \( C(G,H) \). \( F^1 \) is said to be of less concordance than \( F^2 \) if

\[ F^1(x,y) \leq F^2(x,y), \quad (x,y) \in \mathbb{R}^2. \quad (3.2.21) \]

**Definition 3.2.2.** (Lehmann, 1966): Assume \( F \in C(G,H) \). \( F \) is said to be of negative quadrant dependence in \( C(G,H) \) if

\[ F(x,y) \leq G(x)H(y), \quad (x,y) \in \mathbb{R}^2. \quad (3.2.22) \]

**Remark 3.2.1.** It follows from Definitions 3.2.1 and 3.2.2 that \( F \) in \( C(G,H) \) is of negative quadrant dependence if and only if \( F \) is of less concordance than \( I \), where \( I(x,y) = G(x)H(y), \quad (x,y) \in \mathbb{R}^2 \), is also in \( C(G,H) \).

**Remark 3.2.2.** The author does not use exactly the same definition as Lehmann, as appears in the next definition.

Before getting to the next definition, consider the pseudo cdf's
F(1), F(2), F(3), and F(4) associated with F ∈ C(G,H) given in $(3,2,7)$. Notice that when evaluated at $(x,y) = (0,0)$, F(1), F(2), F(3), and F(4) represent the probability mass of the first-, second-, third-, and fourth-quadrant, respectively. Now we have the following.

**Definition 3.2.3.** Suppose $F \in C(G,H)$ and $F^{(1)}$, $F^{(2)}$, $F^{(3)}$, and $F^{(4)}$ are defined as in $(3.2.7)$, $F$ is said to be of first-, second-, third- or fourth-quadrant dependence at $(x_0,y_0)$ if and only if

$$
F^{(1)}(x_0,y_0) \leq G(x_0) H(y_0),
$$

$$
F^{(2)}(x_0,y_0) \leq G(x_0) H(y_0),
$$

$$
F^{(3)}(x_0,y_0) \leq G(x_0) H(y_0),
$$

or

$$
F^{(4)}(x_0,y_0) \leq \bar{G}(x_0) H(y_0),
$$

respectively.

It turns out in the following lemma that the four types of quadrant-dependence represent two almost complementary pairs of equivalent properties.

**Lemma 3.2.5.** (Lehmann, 1966):

(a) First-quadrant-dependence and third-quadrant-dependence are equivalent, and are called odd-quadrant-dependence.

(b) Second-quadrant-dependence and fourth-quadrant-dependence are equivalent, and are called even-quadrant-dependence.

(c) Odd-quadrant-dependence and even-quadrant-dependence are two "almost" complementary properties.
Proof: Assume \((x,y) \in \mathbb{R}^2\).

(a) \[ F^{(1)}(x,y) \leq \overline{G(x)} \cdot \overline{H(y)} \iff \\
1 - \overline{G(x)} - \overline{H(y)} + F^{(1)}(x,y) \iff 1 - \overline{G(x)} - \overline{H(y)} + \overline{G(x)H(y)} \]
\[ \iff F(x,y) \leq (1 - \overline{G(x)}) (1 - \overline{H(y)}) \]
\[ \iff F^{(3)}(x,y) \leq G(x) \cdot H(y). \]

(b) \[ F^{(2)}(x,y) \leq G(x) \cdot H(y) \iff \\
1 - G(x) - H(y) + F^{(2)}(x,y) \leq 1 - G(x) - H(y) + G(x)H(y) \]
\[ \iff F^{(4)}(x,y) \leq (1 - G(x)) (1 - H(y)) = G(x)H(y). \]

(c) Non first-quadrant dependence at \((x,y)\) may be written as

\[ F^{(1)}(x,y) > \overline{G(x)} \cdot \overline{H(y)} \iff \\
\overline{H(y)} - F(x,y) < \overline{H(y)} - \overline{G(x)H(y)} \]
\[ \iff F^{(2)}(x,y) < [1 - \overline{G(x)}]H(y) = G(x)\overline{H(y)}. \]

Notice that quadrant dependence equivalence occurs only between classes of monotone pseudo-cdf's (i.e., odd-quadrant dependence), and between classes of nonmonotone pseudo-cdf's (i.e., even-quadrant dependence).

**Example 3.2.3.** More examples of classes of bivariate cdf's with prescribed marginals given in Feller (1966), pp 99-100 and 165, involve classes of bivariate cdf's with normal marginals.
1. (Feller cites the following example, due to E. Nelson): Assume
\( g(\cdot) = h(\cdot) \) is a normal density function. Let \( u(\cdot) = v(\cdot) \) be
an odd continuous function on the reals, vanishing outside
\([-1,1]\). If \( |u| < (2\pi e)^{-\frac{1}{2}} \), then
\[
 f(x,y) = g(x)h(y) + u(x)v(y)
\]
represents a bivariate density which is not normal, but whose
marginal densities are both normal. Clearly, the class of bi­
variate cdf's with normal marginals is bigger than the class of
bivariate normal cdf's, given the marginals. Either class does
not contain the Fréchet bounds.

2. Consider the class of bivariate normal cdf's with unit variance
but different correlation coefficients. When \( \phi_1 \) and \( \phi_2 \)
are two members of this class then \( \phi_\lambda = \lambda \phi_1 + (1-\lambda) \phi_2 \), \( \lambda \in (0,1) \),
does not belong to the class, so that the class is not convex.

3.3. Weak Convergence and Asymptotic Independence

Having defined the class \( \mathcal{C}(G,H) \), where \( G \) and \( H \) are given
cdf's, it is natural to consider sequences \( \mathbf{s}_n, \mathbf{s}_n \subseteq \mathcal{C}(G_n,H_n) \).

In a certain stochastic bivariate iteration, \( G_n \) and \( H_n \) are the
nth iterands for given cdf's \( G \) and \( H \), while \( \mathbf{s}_n \) consists of all
iterands \( F_n \) for \( F \in \mathcal{C}(G,H) \). Then, if \( G_n \) and \( H_n \) converge in
distribution to some cdf's \( G \) and \( H \), we "hope" to be able to find a
necessary and sufficient condition for the elements \( F_n \) of \( \mathbf{s}_n \) to
converge in distribution, and, furthermore, as in Chapters 4 and 5,
to characterize the bivariate limit laws involved.
Bivariate extreme value theory, as for example in the works of Gnedenko (1943), Villasenor (1976), and Galambos (1978), provides a more usual illustration of these phenomena.

Preliminaries in the general theory of bivariate convergence in distribution are as follows.

**Definition 3.3.1.** A bivariate cdf \( F \) is said to belong to the class \( \Gamma(x_0, y_0) \) if and only if for some nondegenerate cdf's \( Q \) and \( \mathcal{K} \), each having at most one discontinuity, i.e., at \( x_0 \) and \( y_0 \), respectively, it follows that

\[
F \in \mathcal{C}(Q, \mathcal{K}) = \Gamma(x_0, y_0).
\]

**Definition 3.3.2.** A sequence \( \{F_n\} \) of bivariate cdf's is said to converge weakly or in distribution to a cdf \( F \in \Gamma(x_0, y_0) \), denoted by \( F_n \xrightarrow{W} F \) or \( F_n \xrightarrow{D} F \), if and only if there are sequences of constants \( \{a_n\}, \{c_n\}, \{b_n\} \) and \( \{d_n\} \), \( a_n, c_n \in \mathbb{R}, b_n, d_n > 0 \), such that, as \( x \neq x_0 \) or \( y \neq y_0 \),

\[
\lim_{n} F_n(a_n + b_n x, c_n + d_n y) = F(x, y).
\]

**Corollary 3.3.1.** If \( F_n \in \mathcal{C}(Q_n, H_n) \) is such that \( F_n \xrightarrow{D} F \), where \( F \in \mathcal{C}(Q, \mathcal{K}) \subseteq \Gamma(x_0, y_0) \), then \( Q_n \xrightarrow{D} Q \) and \( H_n \xrightarrow{D} \mathcal{K} \).

**Definition 3.3.3.** A sequence of bivariate cdf's \( F_n \in \mathcal{C}(Q_n, H_n) \) is said to be asymptotically independent if there are nondegenerate cdf's \( Q, \mathcal{K} \) such that

\[
F_n \xrightarrow{D} F, \text{ where } \quad F(x, y) = Q(x)\mathcal{K}(y), (x, y) \in \mathbb{R}^2.
\]
Considerations specific to bivariate iterations are as follows.

**Definition 3.3.4.** A function $F_1$ is said to be a bivariate monotone iterand of a cdf $F \in C(G,H)$, if $F_1$ is a bivariate cdf and for some functions $\phi_1(\cdot), \phi_2(\cdot)$ and $\lambda(\cdot, \cdot, \cdot)$, $F_1$ can be expressed as

$$F_1(x,y) = \lambda(G(x), H(y), F(x,y))$$

(3.3.1)

where, the marginals of $F_1$,

$$G_1(x) = \sup_y F_1(x,y) = \phi_1(G(x))$$

and

$$H_1(y) = \sup_x F_1(x,y) = \phi_2(H(y))$$

are monotone cdf iterands of $G$ and $H$, respectively.

Note that the function $F_1$ in Definition 3.3.4 may, for example, have the specialized forms

$$F_1(x,y) = \lambda(F(x,y))$$

(3.3.2)

or

$$F_1(x,y) = \lambda(G(x), H(y))$$

(3.3.3)

Defining the zeroth iterand $F_0$ of $F$ as $F$, then, for $n \geq 1$, the $n$th monotone iterand of $F$ (in its general form, (3.3.1)) is given by

$$F_n(x,y) = \lambda(G_{n-1}(x), H_{n-1}(y), F_{n-1}(x,y)),$$

(3.3.4)
where $G_{n-1}$, $H_{n-1}$ and $F_{n-1}$ are the $(n-1)$st monotone iterands of $G$, $H$, and $F$. It is clear from (3.3.4) and Definition 3.3.4 that $F_n$ belongs to a subclass $G_n$ of $G(G_n, H_n)$.

To conclude this section, an observation will be made about the relationship between a bivariate monotone iterand and its limit under iteration, when a certain simple normalizing constant exists in the form of $(a, \frac{1}{b^n})$, $(c, \frac{1}{d^n})$ for some $b > 1$, $d > 1$.

**Lemma 3.3.1.** Suppose that the $n$th bivariate monotone iterand $\lambda^{(n)}$ for a certain monotone iterand $\lambda$ converges weakly to a non-degenerate limit cdf $\mathcal{F} \in \Gamma(0,0)$ with the marginals $Q$ and $\mathcal{X}$, in the sense that

$$
\mathcal{F}_n(u,v) \equiv \lambda^{(n)}(G(a + \frac{u}{b^n}), H(c + \frac{v}{d^n}), F(a + \frac{u}{b^n}, c + \frac{v}{d^n})) \\
\xrightarrow{D} \mathcal{F}(u,v),
$$

(3.3.5)

for some $a, c \in \mathbb{R}$, $b, d > 1$; then the following holds:

(a) $Q_n(u) \equiv Q_n(a + \frac{u}{b^n}) \equiv \varphi_1^{(n)}(G(a + \frac{u}{b^n})) \xrightarrow{D} Q(u)$

$$
\mathcal{X}_n(v) \equiv \mathcal{X}_n(c + \frac{v}{d^n}) \equiv \varphi_2^{(n)}(H(c + \frac{v}{d^n})) \xrightarrow{D} \mathcal{X}(v). 
$$

(3.3.6)

(b) $\varphi_1^{(k)}(Q(-\frac{u}{b^k})) = Q(u)$

$$
\varphi_2^{(k)}(\mathcal{X}(-\frac{v}{d^k})) = \mathcal{X}(v), \ k = 1,2,3,\ldots, u, v \neq 0.
$$

(3.3.7)
(c) \( \lambda^{(k)}(Q, (\frac{u}{b^k}, \frac{v}{b^k}), \mathcal{H}(\frac{u}{b^k}, \frac{v}{b^k})) = \mathcal{F}(u, v) \),

\[ u, v \neq 0, \quad k = 1, 2, \ldots \quad (3.3,8) \]

Proof: The proof is given for one marginal only, since the other marginal may be treated analogously.

(a) The results follow from marginal weak convergence as in Corollary 3.3.1.

(b) \( Q_{n+k}^{(n+1)}(u) \equiv \phi^{(n+1)}_1(G(a + \frac{u}{b^{n+k}})) \equiv \phi^{(n+1)}_1(G(a + \frac{u}{b^{n+k}})) \)

\[ \equiv \phi^{(n+1)}_1(Q_n^{(\frac{u}{b^k})}) . \]

Taking the limit of both sides, as \( n \to \infty \), using the result in (a) and the fact that taking the limit inside \( \phi^{(n)}_1 \) is justified by continuity of \( \phi^{(n+1)}_1 \), we have, as \( u \neq 0 \),

\[ Q(u) = \phi^{(n+1)}_1(Q_n^{(\frac{u}{b^k})}), \quad k = 1, 2, \ldots . \]

(c) \( \mathcal{F}_{n+k}(u, v) \equiv \lambda^{(k)}(Q_n^{(\frac{u}{b^k})}, \mathcal{H}_n^{(\frac{v}{d^k})}, \mathcal{F}_n^{(\frac{u}{b^k}, \frac{v}{d^k})}) . \)

Taking the limit of both sides, using (3.3.5) and (3.3.7) and continuity of \( \lambda^{(k)} \), we have (3.3.8).
4. BIVARIATE MAXIMIN ITERATION

4.1. Introductory

This chapter deals with the bivariate extension of the iterated maximin operations of Thomas and David (1967) and Thomas (1967) given in Chapter 2. It is concerned with the joint distribution of the values of two similarly structured zero-sum two-person games of perfect information (i.e., ZSTPGPI's), each with mutually independent terminal payoff's (i.e., MITP's), in which:

(i) the pairs of payoffs at corresponding terminal nodes have specified not-necessarily independent joint distributions, and
(ii) payoffs are mutually independent otherwise.

An important special case comes about when a second ZSTPGPI with MITP's is constructed from an initial ZSTPGPI with MITP's by the addition of iid measurement error terms mutually independent of the terminal payoffs of the initial game. In this special case, the joint distribution of the two game values yields the distribution of their ratio, and thus a global measure of the effect of measurement error on families of ZSTPGPI's adequately modeled by the assumption that terminal payoffs are mutually independent.

As in Thomas and David (1967), the discussion is focused on the case of constant number of alternative moves available to a player.

Consider thus two pairs of players (Ia, IIa) and (Ib, IIb) alternately choosing one of several alternative moves, with p and q alternatives available respectively to I and II at each move, and with n choices to be made in all by each player. Corresponding to
each of the \((pq)^n\) possible sequences of moves in the two games, there are \((pq)^n\) pairs of terminal payoffs: \((x,y)_{i_1,i_2,...,i_{2n}}\) to players \((Ia, Ib)\), where the odd-valued indices range from 1 to \(p\), and the even-valued ones from 1 to \(q\).

The joint value

\[
(u_n, v_n) = (u(x_{i_1,...,i_{2n}}), v(y_{i_1,...,i_{2n}}))
\]

of such a pair of games is

\[
M_{\max}^{(n)}(x_{i_1,...,i_{2n}}, y_{i_1,...,i_{2n}})
\]

\[
\equiv \{M_{\max}^{(n)}(x_{i_1,...,i_{2n}}), M_{\max}^{(n)}(y_{i_1,...,i_{2n}})\}
\]

\[
\equiv \{\max_{i_1} \min_{i_2} \cdots \max_{i_{2n-1}} \min_{i_{2n}} (x_{i_1,i_2,...,i_{2n-1},i_{2n}}), \max_{i_1} \min_{i_2} \cdots \max_{i_{2n-1}} \min_{i_{2n}} (y_{i_1,i_2,...,i_{2n-1},i_{2n}})\}
\]

\[
\equiv \{M_{\max}^{(n)}(x), M_{\max}^{(n)}(y)\}, \tag{4.1.1}
\]

to be computed under the joint distribution:

\[
\mathcal{J}(x_1,y_1; x_2,y_2; \ldots; x_{(pq)^n}, y_{(pq)^n})
\]

\[
= \prod_{j=1}^{(pq)^n} F(x_j,y_j), \text{ where } F \in \mathcal{C}(G,H). \quad \tag{4.1.2}
\]
The joint distribution $F_n$ of the pair of values $(U_n, V_n)$ is studied in Section 4.2. Weak convergence of $F_n$ is studied in Section 4.3, where asymptotic independence of

$$(U_n, V_n) = \{M_n^{(n)}(X), M_n^{(n)}(Y)\}$$

is characterized when both the distribution $G_n$ of $M_n^{(n)}(X)$ and the distribution $H_n$ of $M_n^{(n)}(Y)$ converge in law.

4.2. The Bivariate Maximin Operation

This section, as well as the following, deals with the joint distribution of the pair $(U_n, V_n)$ of values of games given in Section 4.1, for the case $p = q = 2$.

Definition 4.2.1. A function $F(x, y)$ is said to be a bivariate maximin iterand of $F \in \mathcal{C}(G, H)$ if it has the form

$$\lambda(G(x), H(y), F(x, y)), (x, y) \in \mathbb{R}^2,$$  \hspace{1cm} (4.2.1)

with

$$\lambda(s, t, z) = [1 - (1-s)^2 - (1-t)^2 + (1-s-t + z)^2]^2,$$

$$(s, t, z) \in [0, 1]^3, \hspace{1cm} (4.2.2)$$

or, equivalently,

$$\lambda(s, t, z) = [1 - \overline{s}^2 - \overline{t}^2 + \overline{z}^2]^2,$$

where

$$\overline{s} = 1-s, \hspace{0.2cm} \overline{t} = 1-t, \hspace{0.2cm} \overline{z} = 1-s-t+z, \hspace{0.2cm} (s, t, z) \in [0, 1]^3.$$  \hspace{1cm} (4.2.3)
We will see in the next theorem that the function $\lambda(G(x), H(y), F(x,y))$ is a bivariate monotone iterand.

**Theorem 4.2.1.** If $F \in \mathcal{C}(G, H)$ is a nondegenerate bivariate cdf, then $F_1(x,y) = \lambda(G(x), H(y), F(x,y))$ is the cdf of $(U_1, V_1)$; furthermore, $F_1$ is a bivariate monotone iterand of $F$ with the marginals $G_1(x) = \phi(G(x))$ and $H_1(y) = \phi(H(y))$, $(x, y) \in \mathbb{R}^2$, where $\phi(t) = [1 - (1-t)^2]^2$, $t \in [0, 1]$.

*Proof:* Let

\[
\begin{pmatrix}
X_1 \\
Y_1 \\
X_2 \\
Y_2 \\
X_3 \\
Y_3 \\
X_4 \\
Y_4
\end{pmatrix}
\]

be iid bivariate r.v.'s with the common cdf's $F \in \mathcal{C}(G, H)$. Consider the bivariate r.v.

\[
\begin{pmatrix}
U_1 \\
V_1
\end{pmatrix} = \begin{pmatrix}
\max[\min(X_1, X_2), \min(X_3, X_4)] \\
\max[\min(Y_1, Y_2), \min(Y_3, Y_4)]
\end{pmatrix}.
\]

Suppose $F_1$ is the cdf of $\begin{pmatrix} U_1 \\ V_1 \end{pmatrix}$. Then,

\[
F_1(x,y) = \Pr \left\{ \begin{array}{l}
U_1 \leq x \\
V_1 \leq y
\end{array} \right\}
\]

\[
= \Pr \left\{ \begin{array}{l}
\max[\min(X_1, X_2), \min(X_3, X_4)] \leq x \\
\max[\min(Y_1, Y_2), \min(Y_3, Y_4)] \leq y
\end{array} \right\}
\]
\[
\begin{align*}
&= \left\{ \Pr \left( \min(X_1, X_2) \leq x \right) \right\}^2 \\
&= \left\{ 1 - \Pr[\min(X_1, X_2) > x] - \Pr[\min(Y_1, Y_2) > y] \\
&\quad + \Pr \left( \min(X_1, X_2) > x \right) \right\}^2 \\
&= \left\{ 1 - \left( \Pr(X_1 > x) \right)^2 - \left( \Pr(Y_1 > y) \right)^2 \\
&\quad + \left( \Pr \left( \begin{array}{c}
X_1 > x \\
Y_1 > y
\end{array} \right) \right)^2 \right\},
\end{align*}
\]

where the third and last equations follow by iid argument. The last expression can then be written as

\[
F_1(x, y) = \left[ 1 - (1 - G(x))^2 - (1 - H(y))^2 + (1 - G(x) - H(y) + F(x, y))^2 \right]^2
\]

\[
= \lambda(G(x), H(y), F(x, y)).
\]

Furthermore, it is easy to see that

\[
G_1(x) = \sup_y F_1(x, y) = [1 - (1 - G(x))^2]^2 = \phi(G(x))
\]

and

\[
H_1(y) = \sup_x F_1(x, y) = [1 - (1 - H(y))^2]^2 = \phi(H(y)),
\]

are the marginals of $F_1$, are the maximin iterands discussed in Section 2.2.1. Therefore, $F_1(x, y)$ satisfies Definition 3.3.4 of a bivariate
monotone iterand.

Facts about the bivariate maximin iterand $F_1$ listed below, holding under certain conditions on $F \in \mathcal{C}(G,H)$, may be easily verified from its expression in (4.2.1)-(4.2.2), Theorem 4.2.1 and monotonicity of cdf's. As always, $\overline{G}$, $\overline{H}$, and $\overline{F}$ are to denote the upper cdf's associated with $G$, $H$, and $F$, respectively.

Fact 4.2.1. (Independence): If $F(x,y) = G(x)H(y)$, then $F_1(x,y) = G_1(x)H_1(y)$.

Fact 4.2.2. (Concordance): If $F^1, F^2 \in \mathcal{C}(G,H)$, with $F^1(x,y) \leq F^2(x,y)$, then $F^1_1(x,y) \leq F^2_1(x,y)$. In other words, if $F^1$ is of less concordance than $F^2$ in $\mathcal{C}(G,H)$, then $F^1_1$ is of less concordance than $F^2_1$, in the subclass $\mathcal{G}_1$ of $\mathcal{C}(G_1,H_1)$.

This fact may be verified by deriving from (4.2.1)-(4.2.2), the following equation:

$$
F^2_1(x,y) - F^1_1(x,y) = \left( \sqrt{F^2_1(x,y)} + \sqrt{F^1_1(x,y)} \right) \cdot

(F^2(x,y) - F^1(x,y))
$$

and noticing that the first two factors of the RHS of (4.2.4) are non-negative.

Fact 4.2.3. (Quadrant dependence): If $F$ is of even/odd quadrant dependence, then $F_1$ is also of even/odd quadrant dependence.

Lemma 4.2.1. If $F \in \mathcal{C}(G,H)$, then, for $n \geq 1$,

$$
F_n(x,y) \equiv \lambda^{(n)}(G(x), H(y), F(x,y))

\equiv \lambda(G_{n-1}(x), H_{n-1}(y), F_{n-1}(x,y))
$$

(4.2.5)
the \( n \)th maximin iterand of \( F \), is the cdf of the joint value \( (U_n, V_n) \) defined in (4.1.1) and (4.1.3), when \( F \) is the common cdf of the \((pq)^n\) iid pairs of r.v.'s in (4.1.2).

Proof: When \( n = 1 \), \( F_1 \) is the cdf of \((U_1, V_1)\) as given in Theorem 4.2.1, so that by the convention \( G_0 = G, H_0 = H \) and \( F_0 = F \), the result follows. When \( n \geq 2 \), note that \((U_n, V_n)\) may be written as

\[
\{\max[\min(U_{n-1}^1, U_{n-1}^2), \min(U_{n-1}^3, U_{n-1}^4)], \\
\max[\min(V_{n-1}^1, V_{n-1}^2), \min(V_{n-1}^3, V_{n-1}^4)]\}
\]

where \((U_{n-1}^i, V_{n-1}^i)\) are iid bivariate r.v.'s having the same distribution as \((U_{n-1}, V_{n-1})\). Assuming the distribution of \((U_{n-1}, V_{n-1})\) is

\[
F_{n-1}(x,y) = \lambda^{(n-1)}(G(x), H(y), F(x,y))
\]

\[
\equiv \lambda(G_{n-2}(x), H_{n-2}(y), F_{n-2}(x,y)),
\]

then, applying Theorem 4.2.1 again, the cdf of \((U_n, V_n)\) is given by (4.2.5).

**Lemma 4.2.2.** \( F_n \) belongs to a subclass \( \mathcal{G}_n \) of \( \mathcal{C}(G_n, H_n) \), \( n \geq 1 \).

Proof: The result follows from Theorem 4.2.1 and Lemma 4.2.1.

**Lemma 4.2.3.** (Monotonicity concordance): Suppose \( F^1 \), \( F^2 \in \mathcal{C}(G,H) \) with \( F^1(x,y) \leq F^2(x,y) \). Then for each \( n \geq 1 \),
\[ F_{n}^{1}(x, y) \leq F_{n}^{2}(x, y) . \]  
\[ (4.2.6) \]

Proof: For \( n = 1 \), equation (4.2.4) of Fact 4.2.2 follows. For \( n \geq 2 \), the same procedure may be extended inductively by successive decomposition to get the following expression:

\[
F_{n}^{2}(x, y) - F_{n}^{1}(x, y) = \sum_{k=1}^{n} \left[ \sqrt{F_{n-k+1}^{2}(x, y)} + \sqrt{F_{n-k+1}^{1}(x, y)} \right] .
\]

\[ (4.2.7) \]

All the first \( 2n \) factors of the RHS of (4.2.7) are nonnegative, so that \( F_{n}^{2}(x, y) - F_{n}^{1}(x, y) \geq 0 \) implies \( F_{n}^{2}(x, y) - F_{n}^{1}(x, y) \geq 0 \) and (4.2.6) follows.

It is also easy to verify from Theorem 4.2.1 and Lemmas 4.2.1 and 4.2.2 that the following holds.

Fact 4.2.4. For \( F \in \mathcal{G}(G, H) \) and \( m, n \geq 1 \), the following equation holds:

\[
F_{m+n}^{(m)}(x, y) = \lambda^{(m)}(G_{n}(x), H_{n}(y), F_{n}(x, y))
\]

\[ = \lambda^{(m)}(\varphi^{(n)}(G(x)), \varphi^{(n)}(H(y))) \]

\[ + \lambda^{(n)}(G(x), H(y), F(x, y)) . \]

(4.2.8)

Weak convergence of \( F_{n}^{1} \) and conditions for asymptotic independence of \((U_{n}, V_{n})\) will be studied in the next section. As a preliminary,
we will conclude this section by establishing some iterative functional equations involving the cdf $F_n$ of the normalized values of $(U_n, V_n)$.

Let

$$x_a = \inf \{ x : G(x) \geq a \}$$

and

$$y_a = \inf \{ y : H(y) \geq a \},$$

and define, for some sequences $\{b_n\}, b_n > 0$ and $\{d_n\}, d_n > 0$, the following cdf's:

$$Q_n(u) = \Pr \left( \frac{U - x_a}{b_n} \leq u \right) = G_n \left( x_a + b_n u \right),$$

$$K_n(v) = \Pr \left( \frac{V - y_a}{d_n} \leq v \right) = H_n \left( y_a + d_n v \right),$$

and

$$F_n(u, v) = \Pr \left( \frac{U - x_a}{b_n} \leq u, \frac{V - y_a}{d_n} \leq v \right)$$

$$= \Pr \left( x_a + b_n u, y_a + d_n v \right), \quad (4.2.9)$$

The following fact follows.

**Fact 4.2.5.** For $m \geq 1, n \geq 1$, $F_{m+n}$ defined as in (4.2.9) satisfies the following functional equation:
To verify Fact 4.2.5, we may write

\[
\mathcal{F}_{m+n}(u,v) = F_{m+n}(x + b_{m+n} u, y + d_{m+n} v)
\]

\[
= \lambda^m (G_n (x + b_{m+n} u), H_n (y + d_{m+n} v), F_n (x + b_{m+n} u, y + d_{m+n} v))
\]

\[
= \lambda^m [G_n (x + b_{m+n} u), H_n (y + d_{m+n} v)]
\]

\[
F_n (x + b_{m+n} u), y + d_{m+n} v)
\]

\[
= \lambda^m (G_n (b_{m+n} u), H_n (d_{m+n} v), F_n (b_{m+n} u, d_{m+n} v)),
\]

which follows using (4.2.9) and (4.2.8).

**Lemma 4.2.4.** Suppose \( F^i, F^j \in \mathcal{C}(G,H) \) and \( G_n, H_n, F^i_n \), \( i = 1,2 \) are defined as in (4.2.9), then for \( m, n \geq 1 \), the following functional equation holds:

\[
\mathcal{F}_{m+n}^2(u,v) - \mathcal{F}_{m+n}^1(u,v)
\]

\[
= \prod_{k=1}^{n} \left[ \sqrt{\mathcal{F}_{m+n-k+1}^2 \left( \frac{b_{m+n}}{b_{m+n-k+1}} u, \frac{d_{m+n}}{d_{m+n-k+1}} v \right)} \right.
\]

\[
+ \sqrt{\mathcal{F}_{m+n-k+1}^1 \left( \frac{b_{m+n}}{b_{m+n-k+1}} u, \frac{d_{m+n}}{d_{m+n-k+1}} v \right)} \right]
\]
\[
\left\{ \frac{2}{m+n-k} \left( \frac{b_{m+n}}{b_{m+n-k}} u, \frac{d_{m+n}}{d_{m+n-k}} v \right) \right\} + \frac{1}{m} \left( \frac{b_{m+n}}{b_{m+n-k}} u, \frac{d_{m+n}}{d_{m+n-k}} v \right) \}
\]
\[
\left\{ \frac{2}{m} \left( \frac{b_{m+n}}{b_{m}} u, \frac{d_{m+n}}{d_{m}} v \right) \right\} - \frac{1}{m} \left( \frac{b_{m+n}}{b_{m}} u, \frac{d_{m+n}}{d_{m}} v \right) \}
\]

(4.2.11)

Proof: First, by (4.2.9), write

\[
\mathcal{F}_{m+n}^2 (u, v) = \mathcal{F}_{m+n}^{-1} (u, v)
\]

\[
= \mathcal{F}_{m+n}^2 (x \begin{pmatrix} y & + t \end{pmatrix} a + \begin{pmatrix} d \end{pmatrix} m+n, v) - \mathcal{F}_{m+n}^1 (x \begin{pmatrix} y & + t \end{pmatrix} a + \begin{pmatrix} d \end{pmatrix} m+n, v)
\]

\[
= \prod_{k=1}^{n} \left\{ \frac{\mathcal{F}_{m+n-k+1}^2 (x \begin{pmatrix} y & + t \end{pmatrix} a + \begin{pmatrix} d \end{pmatrix} m+n, v)}{\mathcal{F}_{m+n-k}^1 (x \begin{pmatrix} y & + t \end{pmatrix} a + \begin{pmatrix} d \end{pmatrix} m+n, v)} \right\}.
\]

using similar procedure as in obtaining (4.2.7).

Then, writing

\[
(b_{m+n}, d_{m+n}) = \begin{pmatrix} b_{m+n-k+1} \end{pmatrix} \begin{pmatrix} b_{m+n} \end{pmatrix} \begin{pmatrix} d_{m+n-k+1} \end{pmatrix}, \begin{pmatrix} d_{m+n} \end{pmatrix}, k = 0, 1, \ldots, n+1,
\]

and using
(4.2.9) again, (4.2.11) follows.

Weak convergence of $F_n$ (and hence $P_n$) is considered in the next section.

Before proceeding, we will first consider the following functional equation involving $\lambda$, $Q$, $\mathcal{N}$ and $\mathcal{F}$ with $Q$, $\mathcal{N} \in \mathcal{L}_0$, assuming weak convergence $F_n \rightarrow F$ takes place.

**Lemma 4.2.5.** Suppose that $G \in \mathcal{F}(Q)$ and $H \in\mathcal{F}(\mathcal{N})$ with $Q$, $\mathcal{N} \in \mathcal{L}_0$, and suppose that, for some $b_n > 0$, $d_n > 0$, $n = 1, 2, \ldots,$

$$F_n(u,v) \equiv \lambda^{(n)}(G(x + b_n u), H(y + d_n v), F(x + b_n u, y + d_n v))$$

$$\overset{D}{\rightarrow} \mathcal{F}(u,v) \quad (4.2.12)$$

for some nondegenerate cdf $\mathcal{F}$. Then the following functional equation holds:

$$\mathcal{F}(u,v) \equiv \lambda^{(k)}(Q^{k_u}, \mathcal{N}^{k_v}, \mathcal{F}(\beta u, \delta v)),$$

$$(u,v) \in \mathbb{R}^2, \ k = 1, 2, \ldots \quad (4.2.13)$$

for some reals $\beta$ and $\delta$ such that $0 < \beta, \delta < 1$ (with $\beta = 1$ or $\delta = 1$ if and only if $Q \in \mathcal{L}_0$ or $\mathcal{N} \in \mathcal{L}_0$).

**Proof:** $(4.2.12)$ and $G \in \mathcal{F}(Q), H \in\mathcal{F}(\mathcal{N})$ imply, by marginal weak convergence, that we have

$$(a) \quad G_n(u) \equiv G_n(x + b_n u) \overset{D}{\rightarrow} G(u),$$

$$\mathcal{N}_n(v) \equiv H_n(y + d_n u) \overset{D}{\rightarrow} \mathcal{N}(v),$$
and, by Theorem 2.2.3 and (2.2.35), that

\[(b) \lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \beta, \text{ some } \beta: 0 < \beta \leq 1\]

\[\lim_{n \to \infty} \frac{d_{n+1}}{d_n} = \delta, \text{ some } \delta: 0 < \delta \leq 1\]

and

\[Q(u) = \phi^{(k)}(Q(\beta^k u)), k = 1, 2, \ldots\]

\[M(v) = \phi^{(k)}(M(\delta^k v)), k = 1, 2, \ldots\]

(Note, in view of Lemma 2.2.9, \(\beta = 1\) or \(\delta = 1\) if and only if \(Q \in \mathcal{S}_0\) or \(M \in \mathcal{Y}_0\).) Now, using (4.2.10),

\[(c) \quad \lambda^{(k)}(n_{k+1}u, v) = \lambda^{(k)}(Q_n(b_{n+k}u, M^{d_{n+k}}v), Q_n(b_{n+k}u, d_{n+k}v))\]

and taking the limit as \(n \to \infty\) and using (a) and (b) the result (4.2.13) follows.

4.3. Weak Convergence and Asymptotic Independence in a Special Case

In this section we assume that \(F \in \mathcal{C}(G, H)\) and \(G \in \mathcal{B}(Q), H \in \mathcal{B}(N)\)

for some \(Q, N \in \mathcal{S}_0\). Therefore, in view of Theorem 2.2.4 and Lemma 2.2.11, we assume that

\[x_a = \inf\{x: G(x) \geq a\}\]

and

\[y_a = \inf\{y: H(y) \geq a\}\]
are finite, and for some reals $b_n, d_n > 0$, $n = 1, 2, \ldots$, the following hold:

\[
Q_n(u) \equiv \varphi^{(n)}(G(x_{\alpha_n} + b_n)) \rightarrow Q(u) \\
K_n(v) \equiv \varphi^{(n)}(H(y_{\alpha_n} + d_v)) \rightarrow K(v) ,
\]

\[(4.3.2)\]

**Definition 4.3.1.** A cdf $F \in C(G,H)$ is said to belong to the domain of attraction of a nondegenerate cdf $\mathcal{F} \in \mathcal{C}(Q,K)$, written $F \in \mathcal{B}(\mathcal{F})$, if, for some $Q, K \in \mathcal{L}'$, (4.3.1) and (4.3.2) hold and

\[
\mathcal{F}_n(u,v) \equiv \lambda^{(n)}(G(x_{\alpha_n} + b_n), H(y_{\alpha_n} + d_v), F(x_{\alpha_n} + b_n, y_{\alpha_n} + d_v))
\]

\[
\rightarrow \mathcal{F}(u,v).
\]

\[(4.3.3)\]

Assume that $\{X_k, Y_k\}, k = 1, 2, \ldots$ are an iid sequence of bivariate r.v.'s with the common cdf $F \in C(G,H)$, and assume that

\[(U_n, V_n) = \{M_m^{(n)}(X), M_m^{(n)}(Y)\}
\]

\[(4.3.4)\]

as in Lemma 4.2.1.

**Definition 4.3.2.** The values $U_n$ and $V_n$ are said to be asymptotically independent if $F \in \mathcal{B}(\mathcal{F})$, where

\[\mathcal{F}(u,v) = Q(u) K(v)\]

\[(4.3.5)\]

for some $Q, K \in \mathcal{L}'$.

It may also be of interest to see how big the class $\mathcal{S} \subset C(Q,K)$
of limit cdf's of \( F_n \in \mathcal{G}_n \subseteq \mathcal{C}(R^n) \) in fact is for \( F \in \mathcal{C}(G,H) \).

In this regard, (Concordance-Monotonicity) Lemma 4.2.1 might suggest that iteration on both Fréchet bounds of \( \mathcal{C}(G,H) \) be considered. The following is the result on the corresponding limit cdf for the upper Fréchet bound.

**Theorem 4.3.1.** Suppose that

\[
G \in \mathcal{L}(\mathcal{Q}), \ H \in \mathcal{L}(\mathcal{K}),
\]

with \( \mathcal{Q}, \mathcal{K} \in \mathcal{F} \). Let \( F^\ast(x,y) = G(x)\Lambda_H(y), (x,y) \in \mathbb{R}^2 \) be the upper Fréchet bound of \( \mathcal{C}(G,H) \). Then for each \( n \geq 1 \),

\[
F^\ast_n(x,y) = G_n(x)\Lambda_{H_n}(y), \tag{4.3.7}
\]

and furthermore,

\[
W^\ast_n(u,v) \equiv \lambda^{(n)}(G(a^n u + b^n u), H(a^n v + d^n v), F^\ast(a^n u + b^n u, a^n v + d^n v))
\]

\[
\lim_{n \to \infty} W^\ast_n(u,v) \Rightarrow Q(u) \Lambda_N(v), \tag{4.3.8}
\]

where \((a^n, b^n), (y^n, d^n)\) are appropriate norming constants for \( G \) and \( H \).

**Proof:** Consider the first iterand \( F^\ast_1 \).

\[
F^\ast_1(x,y) = \lambda(G(x),H(y), G(x)\Lambda_H(y))
\]

\[
= [1 - (1-G(x))^2 - (1-H(y))^2 + (1-G(x)\Lambda_H(y))/2]^2
\]

\[
= [1 - (1-G(x)\Lambda_H(y))^2 - (1-G(x)\Lambda_H(y))^2 + (1-G(x)\Lambda_H(y))^2] \cdot (1-G(x)\Lambda_H(y))^2.
\]
= [1 - (1 - G(x) ∧ H(y))]²

= φ(G(x) ∧ H(y))

= φ(G(x)) ∧ φ(H(y)), by monotonicity of φ

= G₀(x) ∧ H₀(y).

By induction, we get the nth iterand

\[ \mathbb{F}_n^*(x, y) = G_n(x) \land H_n(y). \]  \hspace{1cm} (4.3.9)

Now, by (4.3.6), we have

\[ Q_n(u) = G_n(x + b_n u) \xrightarrow{D} Q(u), \]  \hspace{1cm} (4.3.10)

\[ H_n(v) = H_n(y + d_n v) \xrightarrow{D} H(v), \]  \hspace{1cm} (4.3.11)

for some constants \( b_n > 0, d_n > 0 \) and \( x, y \), where \( Q, H \in \mathcal{F} \).

Therefore,

\[ \mathbb{F}_n(u, v) = \lambda^{(n)}(G_n(x + b_n u), H_n(y + d_n v), F_n(x + b_n u, y + d_n v)) \]

\[ = G_n(x + b_n u) \land H_n(y + d_n v), \]  \hspace{1cm} by (4.3.9)

\[ = Q_n(u) \land H_n(v) \]

\[ \xrightarrow{D} Q(u) \land H(v), \]  \hspace{1cm} (4.3.12)

by (4.3.10) and (4.3.11).
Theorem 4.3.1 in effect states that the upper-bounding operation commutes with (1) the iteration operation, and (2) the limit operation. The following example shows that not even (1) holds for the lower bound.

**Example 4.3.1.** Suppose \( f^{0} \in C(U,U) \), where \( U \) is to denote the uniform cdf \( U(t) = t \), \( 0 \leq t \leq 1 \), and \( f^{0} \) is the lower Fréchet bound of \( C(U,U) \). We know in Chapter 2 that the marginal cdf's \( \Phi_{n} \) and \( \Psi_{n} \) converge in distribution to \( \Phi \), i.e.,

\[
\Phi_{n}(u) = \varphi^{(n)}(a + \frac{u}{b}) \quad \rightarrow \quad \Phi(u)
\]

\[
\Psi_{n}(v) = \varphi^{(n)}(a + \frac{v}{b}) \quad \rightarrow \quad \Psi(v)
\]

Note that since \( f^{0} \) has the form

\[
f^{0}(x,y) = [x+y-1]^{+}, \quad (x,y) \in [0,1]^{2}
\]

it is easy to see that there is a neighborhood \( N_{a} \) of \((a,a)\) such that \( f^{0}(x,y) = 0 \), \((x,y) \in N_{a}\) since \( 0 < a < \frac{1}{2} \).

The first iterand \( f^{0}_{1} \) at \((a,a)\) is as follows:

\[
f^{0}_{1}(0,0) = f^{0}_{1}(a,a) = \lambda(a,a,0)
\]

\[
= [1 - 2(1-a)^{2} + (1-2a+a)^{2}]^{2}
\]

\[
= 4a^{4}
\]

The marginals \( \Phi_{1}(u) \) and \( \Psi_{1}(v) \) evaluated at \( u = v = 0 \) are as
follows:

\[ Q_1(0) = \mathcal{X}_1(0) = \phi(a) = a, \]

so that

\[ [Q_1(0) + \mathcal{X}_1(0) - 1]^+ = 0 \neq g_1(0,0). \]

Therefore, we can conclude that the lower Fréchet bound of the class \((Q, \mathcal{X})\) is not the same as \(g_n\), the nth maximin iterand of the lower Fréchet bound of the original class \(C(\mathcal{U}, \mathcal{U})\).

**Corollary 4.3.1.** Suppose \(G \in \mathcal{G}(\mathcal{Q})\), \(H \in \mathcal{G}(\mathcal{V})\) with \(Q, \mathcal{X} \in \mathcal{Q}_\mathcal{X}\). The upper Fréchet bound \(F^\ast\) of the class \(C(\mathcal{Q}, \mathcal{X})\) is achievable as the limit cdf of

\[ F_n^\ast(u,v) \equiv \lambda^{(n)}(G(x_{a+b}^u), H(y_{a+d}^v), F^\ast(x_{a+b}^u, y_{a+d}^v)), \]

the nth maximin iterand of the upper Fréchet bound \(F^\ast\) of \(C(G,H)\). The lower Fréchet bound \(F^0\) of \(C(\mathcal{Q}, \mathcal{X})\), however, is not a limit cdf in a maximin iteration (see Theorem 4.3.2).

**Corollary 4.3.2.** If the common cdf of the iid \(\{X_k, Y_k\}\)'s is \(F^\ast\), the upper Fréchet bound of \(C(G,H)\), then \(U_n\) and \(V_n\) are asymptotically dependent.

This is succinctly expressed, in the case \(Q = \mathcal{X}\), by

\[ F^\ast(u,v) = Q(u) \land Q(v) \]

\[ = Q(u \land v), \quad (u,v) \in \mathbb{R}^2. \quad (4.3.13) \]
The following theorem indicates that iteration on \( F(x_a, y_a) \)
alone will provide valuable information on asymptotic dependence.

**Theorem 4.3.2.** Suppose that \( G(x_a) = H(y_a) = a, G \in \mathcal{B}(Q), \)
\( H \in \mathcal{B}(\mathcal{K}) \) and \( Q, \mathcal{K} \in \mathcal{L}_1 \). Let \( F \in \mathcal{C}(G,H) \). Then

\[
\lim_{n} F_n(x_a, y_a) = \begin{cases} 
a & \text{if } F(x_a, y_a) = a \\
a^2 & \text{if } F(x_a, y_a) < a \end{cases}
\]  
(4.3.14)

where \( F_n \) is the nth maximin iterand of \( F \).

**Proof:** Since \( G(x_a) = H(y_a) = a \), where \( 0 < a < \frac{1}{2} \), and
\( F \in \mathcal{C}(G,H) \), it is easy to see by looking at the Fréchet bounds
\( F^0(x_a, y_a) \) and \( F^*(x_a, y_a) \), that

\[
F(x_a, y_a) \in [0,a] .
\]  
(4.3.15)

In view of Fact 2.2.3, it is also easy to see that

\[
G_n(x_a) = \phi^n(G(x_a)) = a
\]

and

\[
H_n(y_a) = \phi^n(H(y_a)) = a
\]  
(4.3.16)

for all \( n \).

The nth maximin iterand \( F_n \) of \( F \) can therefore be written as

\[
F_n(x_a, y_a) = \lambda^n(G(x_a), H(y_a), F(x_a, y_a))
\]
Since \( a \) is fixed, it is now seen that the expression in (4.3.17) forms a univariate iteration. We may therefore view (4.3.17) in the form:

\[
\phi^{(n)}(f) = \lambda(a, a, \phi^{(n-1)}(f))
\]  

(4.3.18)

where

\[
\phi(f) = \lambda(a, a, f)
\]

\[
= [1 - 2(1-a)^2 + (1-2a+f)^2]^2, \ f \in [0,a].
\]  

(4.3.19)

The function \( \phi(\cdot) \) has the following properties:

(a) \( \phi(\cdot) \) has positive first and second derivative on \((0,a)\), so that \( \phi(\cdot) \) is strictly increasing and convex on \((0,a)\).

(b) \( \phi(0) = 4a^4 > 0 \) and \( \phi(a) = a \).

(c) \( \phi'(a) = 4a > 1 \).

(a) - (b) imply that \( \phi(\cdot) \) has a unique interior fixed point on \((0,a)\) such that

\[
\phi(f) = f, \ 0 < f < a.
\]  

(4.3.20)

The solution of (4.3.20) is

\[
f = a^2.
\]  

(4.3.21)
Hence,

\( \phi(f) > f, \ 0 \leq f < a^2, \ \phi(a^2) = a^2 \) and

\( \phi(f) < f, \ a^2 < f < a, \ \phi(a) = a. \)

From (a) and (d), we have \( \phi(f) \) an essentially connected increasing iterand on \([0,a]\), and, by Lemma 2.1.4,

\[
\lim_{n \to \infty} \phi^{(n)}(f) = \begin{cases} 
    a, & \text{for } f = a \\
    a^2, & \text{for } 0 \leq f < a.
\end{cases}
\] (4.3,22)

Since the variable \( f \) was used to denote \( F(x_a, y_a) \) and

\( \phi^{(n)}(f) = F_n(x_a, y_a) \) in (4.3.17), the assertion (4.3.14) follows. \( \square \)

Characterization of asymptotic independent of \((U_n, V_n)\) is given next, when both \( \mathcal{Q} \) and \( \mathcal{M} \) are in \( \mathcal{L}_1 \).

**Theorem 4.3.3.** Suppose that \( G \in \mathcal{B}(\mathcal{Q}) \), \( H \in \mathcal{B}(\mathcal{M}) \), with \( \mathcal{Q}, \ \mathcal{M} \in \mathcal{L}_1 \). A necessary and sufficient condition for \((U_n, V_n)\) to be asymptotically independent is \( F(x_a, y_a) < a \).

**Proof:** Since \( G \in \mathcal{B}(\mathcal{Q}) \), \( H \in \mathcal{B}(\mathcal{M}) \), with \( \mathcal{Q}, \ \mathcal{M} \in \mathcal{L}_1 \), in view of Theorem 2.2.4 and Lemma 2.2.9, there are marginal norming constants \([x_a, b_n], b_n > 0\) and \([y_a, d_n], d_n > 0\) with \( G(x_a) = H(y_a) = a \) such that

\[
\mathcal{Q}_n(u) \equiv G_n(x_a + b_n u) \longrightarrow \mathcal{Q}(u), \ u \in \mathcal{R} \] (4.3,23)

and

\[
\mathcal{M}_n(v) \equiv H_n(y_a + d_n v) \longrightarrow \mathcal{M}(v), \ v \in \mathcal{R} \] (4.3,24)

with
\[
\frac{b_{n+1}}{b_n} \to \beta, \quad 0 < \beta < 1, \quad \frac{d_{n+1}}{d_n} \to \delta, \quad 0 < \delta < 1. \quad (4.3.25)
\]

Let
\[
\mathcal{J}_n(u,v) = \mathcal{Q}_n(u) \cdot \mathcal{K}_n(v)
\]
and
\[
\mathcal{J}(u,v) = \mathcal{Q}(u) \cdot \mathcal{K}(v). \quad (4.3.26)
\]

First, assume that \( F(x_a, y_a) < a \). Then we will show that \( F \in \mathcal{B}(\mathcal{J}) \) by showing that
\[
|\mathcal{J}_n(u,v) - \mathcal{J}(u,v)| \to 0 , \quad (4.3.27)
\]
since from (4.3.23), (4.3.24) and (4.3.26) we know that \( \mathcal{J}_n \to \mathcal{J} \).

Since we are assuming \( F(x_a, y_a) < a \), in view of Theorem 4.3.2, we have
\[
\mathcal{J}_n(0,0) = F_n(x_a, y_a) \to a^2. \quad (4.3.28)
\]

Using (4.3.23), (4.3.24), (4.3.28), and the functional equations (4.2.14) and (4.2.13) we will establish (4.3.27) as follows:

1. Continuity and boundedness of \( \mathcal{Q}, \mathcal{K} \in \mathcal{F}_1 \) establish the existence of a neighborhood \( N_0 = I_0 \times J_0 \) of \((0,0)\) such that
\[
|\mathcal{Q}(u) - \mathcal{Q}(0)| < \frac{a^2}{8}, \quad |\mathcal{K}(v) - \mathcal{K}(0)| < \frac{a^2}{8}, \quad (u,v) \in N_0. \quad (4.3.29)
\]

2. The convergence in (4.3.23)-(4.3.25) and (4.3.28) implies the existence of an integer \( N \) such that the following hold:
(a) \(|Q_n(u) - Q(u)| < \frac{a^2}{8}, \quad u \in \Omega, \; n \geq N \) \hspace{1cm} (4.3.30)

(b) \(|\mathcal{M}_n(v) - \mathcal{M}(v)| < \frac{a^2}{8}, \quad v \in \Omega, \; n \geq N \) \hspace{1cm} (4.3.31)

(c) \(0 < \frac{b_{n+k}}{b_n} < 1, \quad 0 < \frac{d_{n+k}}{d_n} < 1, \quad k \geq 1, \; n \geq N \) \hspace{1cm} (4.3.32)

(d) \(|J_n(0,0) - a^2| < \frac{a^2}{8}, \quad n \geq N \) \hspace{1cm} (4.3.33)

for which (a) and (b) are justified since \(Q_n\) and \(\mathcal{M}_n\) are monotone and bounded while \(Q\) and \(\mathcal{M}\) are continuous so that (4.3.23)-(4.3.24) indicates uniform convergence.

3. Note that \(Q, \mathcal{M} \in L^1\), \(G(x_a) = H(y_a) = a\), and Fact 2.2.3 imply that 
\(Q(0) = \mathcal{M}(0) = a, \; Q_n(0) = \mathcal{M}_n(0) = a \; \forall \; n\), and using 1., 2. (a) and (b),

\[ |J_n(u,v) - J_n(0,0)| \leq |Q_n(u) - Q_n(0)| + |\mathcal{M}_n(v) - \mathcal{M}_n(0)|, \]

\[ < \frac{a^2}{4} + \frac{a^2}{4} = \frac{a^2}{2}, \; (u,v) \in \Omega, \; n \geq N. \] \hspace{1cm} (4.3.34)

If we define

\[ r_n(u, v) = \sqrt{J_n(u, v)} + \sqrt{J_n(u, v)} \]

\[ s_n(u, v) = \overline{J}_n(u, v) + \overline{J}_n(u, v) \]

\[ t_n(u, v) = J_n(u, v) - J_n(u, v) \] \hspace{1cm} (4.3.35)

where \(\overline{J}_n = 1 - Q_n(u) - \mathcal{M}_n(v) + J_n(u, v)\), we have from (4.3.29)-
(4.3.31) and (4.3.33)-(4.3.35) that, with \( a = 0.382 \),

\[
\left| r_n(u,v) \right| < \rho = a \sqrt{\frac{13}{8}} + a + \frac{a^2}{4} < 0.92
\]

\[
\left| s_n(u,v) \right| < \zeta = 1 - 2(a - \frac{a^2}{4}) + \frac{13}{8} a^2 + (1-a+\frac{a^2}{4})^2 < 0.985
\]

\[
\left| t_n(u,v) \right| < \tau = \frac{7}{8} a^2, \quad (u,v) \in N_0, \; n \geq N.
\] (4.3.36)

4. From (4.3.32), \((u,v) \in N_0\) implies that \( \left( \frac{b_{k+n}}{b_n} u, \frac{d_{k+n}}{b_n} v \right) \in N_0, \; k \geq 1, \)

\( n \geq N, \) since \( N_0 \) is a neighborhood of \((0,0)\). Now, using (4.3.35), and (4.2.14) with

\[
\sum_{k=1}^{n} \left| \begin{array}{c}
 r_{m+n-k+1} \left( \frac{b_{k+n}}{b_n} u, \frac{d_{k+n}}{b_n} v \right) \\
 s_{m+n-k} \left( \frac{b_{m+n-k}}{b_n} u, \frac{d_{m+n-k}}{d_m} v \right) \\
 t_{m+n-k} \left( \frac{b_{m+n}}{b_m} u, \frac{d_{m+n}}{d_m} v \right)
\end{array} \right| < (\rho \zeta)^n \tau
\]

\((u,v) \in N_0, \; n \geq 1, \; m \geq N,\) (4.3.37)

by the results in (4.3.36). Taking the limit as \( n \to \infty \), (4.3.27) follows for \((u,v) \in N_0\), since \( 0 < \rho \zeta < 1 \).

5. To complete the proof of (4.3.27) for any \((u,v) \in \mathbb{R}^2\), but \((u,v) \in N_0\), find an integer \( k \) such that
(β^u, δ^v) ∈ N_0 ,

and

\[
\left( \frac{b^{n+k}}{b^n} u, \frac{d^{n+k}}{d^n} v \right) \in N_0, \quad n \geq N .
\]

In view of (4.3,25), we have

\[
b^{n+k} = \prod_{j=0}^{k-1} \frac{b^{n+j+1}}{b^{n+j}} \xrightarrow{n \to \infty} \beta^k , \quad 0 < \beta < 1
\]

and

\[
\frac{d^{n+k}}{d^n} \xrightarrow{n \to \infty} \delta^k , \quad 0 < \delta < 1
\]

so that by uniform convergence in (4.3,23) and (4.3,24),

\[
\mathcal{Q}_n \left( \frac{b^{n+k}}{b^n} u \right) \xrightarrow{n \to \infty} \mathcal{Q}(\beta^k u)
\]

\[
\mathcal{N}_n \left( \frac{d^{n+k}}{d^n} v \right) \xrightarrow{n \to \infty} \mathcal{N}(\delta^k v) \quad (4.3,38)
\]

and, by the result in 4.,

\[
\mathcal{F}_n \left( \frac{b^{n+k}}{b^n} u, \frac{d^{n+k}}{d^n} v \right) \xrightarrow{n \to \infty} \mathcal{Q}(\beta^k u) \mathcal{N}(\delta^k v) . \quad (4.3,39)
\]

Now, using the functional equations (4.2.10) and (4.2.13),
\[ \mathcal{F}_{n+k}(u,v) = \lambda^{(k)}(q_n\frac{b_{n+k}}{b_n}u, n\frac{d_{n+k}}{d_n}v), \mathcal{K}_{n}\left(\frac{b_{n+k}}{b_n}u, \frac{d_{n+k}}{d_n}v\right) \]

(4.3.40)

and, taking the limit as \( n \to \infty \), the RHS of (4.3.40) gives

\[ \lambda^{(k)}(q(\beta^k u), n(\delta^k v), q(\beta^k u) n(\delta^k v)) \]

\[ = \varphi^{(k)}(q(\beta^k u)) \varphi^{(k)}(n(\delta^k v)) \]

\[ = q(u) n(v) , \]

in view of Theorem 2.2.3. Therefore, \( F \in \mathcal{B}(\mathcal{J}) \).

For the converse, we will complete the proof by showing that, if

\[ F(x_a, y_a) = a, \]

then \( F \notin \mathcal{B}(\mathcal{J}) \). But this is immediate, since in view of

Theorem 4.3.2,

\[ \mathcal{F}_n(0,0) \equiv F_n(x_a, y_a) \longrightarrow a \text{ if } F(x_a, y_a) = a \]

i.e.,

\[ \lim_{n \to \infty} \mathcal{F}_n(0,0) \neq a^2 = q(0)n(0), \]

whatever \( b_n \) and \( d_n \).

4.4. Extension to the General Case

**Theorem 4.4.1.** Suppose that \( G \in \mathcal{K}(\mathcal{Q}), H \in \mathcal{B}(\mathcal{K}), \mathcal{K}, \mathcal{N} \in \mathcal{L} = \mathcal{L}_I \cup \mathcal{L}_{II} \cup \mathcal{L}_{III} \) and \( F \in C(G,H) \). Necessary and sufficient conditions for the asymptotic independence of \( U_n \) and \( V_n \) are given in the following table:
Table 4.1. Conditions for asymptotic independence

<table>
<thead>
<tr>
<th>( Q_{\text{in}} )</th>
<th>( S_{\text{I}} )</th>
<th>( S_{\text{II}} )</th>
<th>( S_{\text{III}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_{\text{I}} )</td>
<td>( \Phi(x,y_{a}) &lt; a )</td>
<td>( \Phi(x,y_{a}) &lt; a )</td>
<td>( \Phi(x,y_{a}) &lt; a )</td>
</tr>
<tr>
<td>( S_{\text{II}} )</td>
<td>( \Phi(x_{a},y_{a}) &lt; a )</td>
<td>( \Phi(x_{a},y_{a}) &lt; a )</td>
<td>( \Phi(x_{a},y_{a}) &lt; a )</td>
</tr>
<tr>
<td>( S_{\text{III}} )</td>
<td>( \Phi(x_{a},y_{a}) &lt; a )</td>
<td>( \Phi(x_{a},y_{a}) &lt; a )</td>
<td>( \Phi(x_{a},y_{a}) &lt; a )</td>
</tr>
</tbody>
</table>

Proof: For \( Q, \mathcal{K} \in S_{\text{I}}, \) the proof has been given in Theorem 4.3.3.

We note that the result and proof involved first \((u,v)\) in a neighborhood of \((0,0)\), and then an extension to all of \(\mathbb{R}^2\). For the other five cases, the appropriate neighborhoods are restricted to pertinent quadrants, (possibly excluding relevant bounding axes) to which the extensions apply. For example, for the case \( Q, \mathcal{K} \in S_{\text{II}}, \) the relevant neighborhood is excluded and restricted to the third quadrant. This case is now looked at in some detail.

\( G \in \mathfrak{B}(Q), \ H \in \mathfrak{B}(\mathcal{K}), \ Q, \mathcal{K} \in S_{\text{II}} \) imply, in view of Theorem 2.2.4 and Lemma 2.2.9 that, for some sequences \( \{b_n\}, \ b_n > 0, \ \{d_n\}, \ d_n > 0 \),

\[
Q_n(u) \equiv \phi^{(n)}(G(x_{a} + b_n u)) \xrightarrow{D} Q(u) \quad (4.4.1)
\]

\[
\frac{b_{n+1}}{b_n} \rightarrow \beta, \ \text{some} \ \beta: \ 0 < \beta < 1 \quad (4.4.2)
\]

\[
\mathcal{K}_n(v) \equiv \phi^{(n)}(H(y_{a} + d_n v)) \xrightarrow{D} \mathcal{K}(v) \quad (4.4.3)
\]
with

\[ Q(0-) \equiv \sup_{u<0} Q(u) = a \]  \hspace{1cm} (4.4.5)

\[ M(0-) \equiv \sup_{v<0} M(v) = a \]  \hspace{1cm} (4.4.6)

(4.4.1), (4.4.3), (4.4.5), (4.4.6), and monotonicity of \( G, H \) and \( g^{(n)} \) imply that

\[ G(x-a) \equiv \sup_{u<0} G(x-a+bu) = a \]  \hspace{1cm} (4.4.7)

\[ H(y-a) \equiv \sup_{v<0} H(y-a+dv) = a \]  \hspace{1cm} (4.4.8)

\[ Q^{(n)}(0-) \equiv \sup_{u<0} \phi^{(n)}(G(x-a+bu)) = \phi^{(n)}(G(x-a)) = a \forall n \]  \hspace{1cm} (4.4.9)

\[ M^{(n)}(0-) \equiv \sup_{v<0} \phi^{(n)}(G(y-a+dv)) = \phi^{(n)}(H(y-a)) = a \forall n \]  \hspace{1cm} (4.4.10)

since any violation of (4.4.7), (4.4.9) or (4.4.8), (4.4.10) will contradict (4.4.5) or (4.4.6) in view of Fact 2.2.3.

Now, writing

\[ F(x-a,y-a) = \sup_{u<0,v<0} F(x-a-a+u,y-a-a+dv) \]  \hspace{1cm} (4.4.11)

we have
\[ \mathcal{F}_n(0-,0-) \equiv \sup_{u<0,v<0} \lambda^{(n)}(G(x+\frac{b_n}{b_{n-1}}u), H(y+\frac{d_n}{d_{n-1}}v), F(x+\frac{b_n}{b_{n-1}}u, y+\frac{d_n}{d_{n-1}}v)) \]

\[ = \sup_{u<0,v<0} \lambda(Q_{n-1}(\frac{b_n}{b_{n-1}}u), \mathcal{W}_{n-1}(\frac{d_n}{d_{n-1}}v)) \]

\[ = \lambda(Q_{n-1}(0-), \mathcal{W}_{n-1}(0-), \mathcal{F}_{n-1}(0-,0-)) \]

\[ = \lambda(a, a, \mathcal{F}_{n-1}(0-,0-)) \forall n. \quad (4.4.12) \]

Assume first that \( F(x_a-, y_a-) < a \). Then (4.4.12) implies

\[ \mathcal{F}_n(0-,0-) \rightarrow a^2 = Q(0-)W(0-) \quad (4.4.13) \]

since the unique solution to

\[ f = \lambda(a, a, f), \quad 0 \leq f < a \]

is \( f = a^2 \), as in Theorem 4.3.2.

1. Continuity and boundedness of \( Q(u), u < 0 \) and \( W(v), v < 0 \) imply that for some deleted southwest neighborhood \( SW(0-,0-) = I_{0-} \times J_{0-} \) of \((0,0)\),

\[ |Q(u) - a| < \frac{a^2}{8}, \quad u \in I_{0-} \quad (4.4.14) \]

\[ |W(v) - a| < \frac{a^2}{8}, \quad v \in J_{0-} \quad (4.4.15) \]
2. Monotonicity and boundedness of $Q_n(u)$, $V_n(v)$ and continuity of $Q(u)$, $V(v)$, $u \in I_0^-$, $v \in J_0^-$ imply that for some $N_1$,

$$|Q_n(u) - Q(u)| < \frac{a^2}{8}, \quad u \in I_0^-, \quad n \geq N_1$$

(4.4.16)

$$|V_n(v) - V(v)| < \frac{a^2}{8}, \quad v \in J_0^-, \quad n \geq N_1.$$  

(4.4.17)

3. (4.4.2), (4.4.4), and (4.4.13) imply for some $N_2$,

$$0 < b_n < 1, \quad 0 < \frac{b_{n+1}}{b_n} < 1, \quad n \geq N_2$$

(4.4.18)

$$0 < d_n < 1, \quad 0 < \frac{d_{n+1}}{d_n} < 1, \quad n \geq N_2$$

(4.4.19)

$$|\bar{F}_n(0-,0-) - a^2| < \frac{a^2}{8}, \quad n \geq N_2.$$  

(4.4.20)

Let $N = \max(N_1, N_2)$.

4. From (4.4.9), (4.4.10), (4.4.14)-(4.4.17), and (4.4.13), we have

$$t_n(u,v) = |\bar{F}_n(u,v) - Q_n(u)V_n(v)|$$

$$\leq |\bar{F}_n(u,v) - \bar{F}_n(0-,v)| + |\bar{F}_n(0-,v) - \bar{F}_n(0-,0-)|$$

$$+ |\bar{F}_n(0-,0-) - a^2| + |a^2 - Q_n(u)V_n(v)|$$

$$\leq |Q_n(u) - Q_n(0-)| + |V_n(v) - V_n(0-)|$$
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\[ + \frac{a}{8} + \left( \frac{a}{4} + \frac{a}{64} \right) \]

\[ < 2 \cdot \frac{a}{8} + 2 \cdot \frac{a}{8} + \frac{a}{8} \]

\[ + \frac{a}{4} + \frac{a}{16} \]

\[ < \frac{3}{4} a^2 = \tau, \quad (u,v) \in \text{SW}(0-,0-) \]  \( n \geq N \).  \( \text{(4.4.21)} \)

5. Let

\[ x_n(u,v) = \sqrt{F_n(u,v) + Q_n(u)M_n(v)} \]

\[ s_n(u,v) = \overline{F_n(u,v)} + Q_n(u)\overline{M_n(v)} . \]

Then, we have, using (4.4.9), (4.4.10), (4.4.14)-(4.4.17), and (4.4.13) again

\[ |x_n(u,v)| < \rho = a \sqrt{\frac{13}{8}} + a + \frac{a^2}{4} < .92 \]  \( \text{(4.4.22)} \)

\[ |s_n(u,v)| < \zeta = 1 - 2(a - \frac{a}{4}) + \frac{13}{8} a^2 + (1-a+\frac{a^2}{4})^2 \]

\[ < .985. \]  \( \text{(4.4.23)} \)

for \( (u,v) \in \text{SW}(0-,0-) \), \( n \geq N \).

6. Using (4.2.14), with \( F_n^2(u,v) = F_n(u,v) \) and \( \overline{F_n^2(u,v)} = Q_n(u)\overline{M_n(v)}, \)

we have, using notations in 5.,

\[ |t_{n+k}(u,v)| = \left| \sum_{j=1}^{\pi} x_{n+k-j+1} \left( \frac{b_{n+k}}{b_{n+k-j+1}} u, \frac{d_{n+k}}{d_{n+k-j+1}} v \right) \right| \]

and, by (4.4.22), (4.4.23), and (4.4.21),

\[ |t_{n+k}(u,v)| < (\rho \xi)^{k} \tau, \quad (u,v) \in \text{SW}(0-,0-), \ n \geq N, \ k \geq 1. \]

Taking the limit as \( k \to \infty \), we have \( t_{n}(u,v) \to 0 \), \( (u,v) \in \text{SW}(0-,0-) \), which implies, since (4.4.1) and (4.4.3) hold, that

\[ \mathcal{F}_{n}(u,v) \longrightarrow Q(u)K(v), \ (u,v) \in \text{SW}(0-,0-) \ . \quad (4.4.24) \]

Now, for \( u < 0, v < 0 \) but \( (u,v) \not\in \text{SW}(0-,0-) \), find \( k \) such that

\( (\beta^{\ell}u, \delta^{\ell}v) \in \text{SW}(0-,0-), \ \ell \geq k \)

and

\[ \left( \frac{b_{n+k}}{b_{n}} u, \frac{d_{n+k}}{d_{n}} v \right) \in \text{SW}(0-,0-), \ n \geq N \]

and using (4.2.13),

\[ \mathcal{F}_{n}(u,v) = \lambda^{(k)} Q_{n} \left( \frac{b_{n+k}}{b_{n}} u, \frac{d_{n+k}}{d_{n}} v \right), \quad \mathcal{F}_{n} \left( \frac{b_{n+k}}{b_{n}} u, \frac{d_{n+k}}{d_{n}} v \right) \longrightarrow \lambda^{(k)} Q^{k}(\beta^{k}u), K^{k}(\delta^{k}v), \ Q^{k}(\beta^{k}u), K^{k}(\delta^{k}v) \]
= \phi^{(k)}[Q(\beta^k u)] \phi^{(k)}[N(\delta^k v)]

= Q(u) N(v) , \quad (4.4.25)

in view of Theorem 2.2.3. Therefore, (4.4.24) and (4.4.25) imply that

\[ 3^{(u,v)} > Q(u) N(v) , \quad u < 0 , \quad v < 0 . \quad (4.4.26) \]

For \((u,v) \in \mathbb{R}^2 - \{(u,v); u < 0, v < 0\}\), it is easy to see that both Fréchet bounds \(\mathcal{F}^0\) and \(\mathcal{F}^*\) of \(C(Q,N)\), i.e.,

\[ \mathcal{F}^0(u,v) = [Q(u) + N(v) - 1]^+ = \lim_{n} [Q_n(u) + N_n(v) - 1]^+ \]

and

\[ \mathcal{F}^*(u,v) = Q(u) \land N(v) = \lim_{n} Q_n(u) \land N_n(v) \]

are identical, i.e.,

\[ \mathcal{F}^0(u,v) = \mathcal{F}^*(u,v) = Q(u) N(v), \quad (u,v) \in \mathbb{R}^2 \]

Therefore,

\[ \mathcal{F}_n(u,v) \rightarrow Q(u) N(v), \quad (u,v) \in \mathbb{R}^2 \quad (4.4.27) \]

(4.4.26) and (4.4.27) imply that \(U_n\) and \(V_n\) are asymptotically independent.

For the converse, assume that \(F(x_a, y_a) = a\). Then, (4.4.12) implies that
\[ f_n(0-,0-) = a \forall n \text{ so that} \]
\[ f_n(0-,0-) \rightarrow a \neq a^2 = \mathcal{Q}(0-) \mathcal{X}(0-) \]

and therefore \( U_n \) and \( V_n \) are asymptotically dependent.

4.5. Uniform Marginals

The maximin iteration on a cdf \( F \in \mathcal{C}(G_0,G_0) \) when \( G_0 \) is the uniform cdf on \([0,1]\) is interesting for the following reasons:

(a) Correct location-scale marginal norming constants \((a, \frac{1}{b^n})\) are available (Thomas and David, 1967).

(b) Certain mixtures \( F^\lambda \) have simple forms, so that steps of the iteration become relatively easy to control.

(c) If \( F \) is not easily computed, one can find two bounding mixtures \( F^1 \) and \( F^2 \), both having simple forms, such that \( F^1(x,y) \leq F(x,y) \leq F^2(x,y) \) on some neighborhood \( N_a \) of \((a,a)\).

In essentially all cases, in view of Theorem 4.3.3., \( F^1, F^2 \in \mathcal{C}(x^2) \), where \( x^2(u,v) = x(u)x(v) \), so that by the monotonicity property of the iteration, as given in Theorem 4.2.3, the bounds remain valid, and become increasingly tight, under iteration.

The convexity property of \( \mathcal{C}(G_0,G_0) \) has been assumed in considering mixtures as in (b) and (c). But note that some parametric subclasses of \( \mathcal{C}(G_0,G_0) \) are not convex; for example, the class \( \mathcal{C}(G_0,G_0) \) consisting of bivariate normal uniform cdf's. A cdf \( F_\theta \in \mathcal{C}(G_0,G_0) \) is not easily computed as is seen, e.g., in Barnett (1980). However, if one is interested in considering intermediate steps of the maximin
iteration on $F_\theta \in \mathcal{C}(G_0, G_0)$, (b) and (c) might suggest that, by taking two bounding mixtures $F_1^\lambda$ and $F_2^\lambda$ in $\mathcal{C}(G_0, G_0)$, not necessarily in $\mathcal{C}(G_0', G_0')$, such that both are easily computed, maximin iterands $\mathcal{J}_n^\lambda_1$ and $\mathcal{J}_n^\lambda_2$ might still provide reasonably good approximates of $\mathcal{J}_n^\theta_0$. 

A detailed consideration of $\mathcal{C}(G_0, G_0)$ will not be elaborated in this section, since $\mathcal{C}(G_0, G_0)$ is covered by Theorem 4.3.3. Only a few points will be made here, in an attempt to help ease the computational aspects, such as (i) examples of easily computed mixtures, and (ii) an alternative approach to asymptotic independence for $\mathcal{C}(G_0, G_0)$.

1. Two examples of easily computed mixtures are given as follows:

(a) $F_1^\lambda(x,y) = \lambda F^*(x,y) + (1-\lambda)F^0(x,y)$, $0 < \lambda < 1$. Note that $F^0(x,y) = 0$, $(x,y) \in [0,\frac{1}{2}]^2$. Since $[0,\frac{1}{2}]^2$ contains a neighborhood $N_a$ of $(a,a)$, maximin iterands on $F_1^\lambda(x,y)$ tend to maximin iterands on $\lambda F^*(x,y) = \lambda x\wedge y, (x,y) \in [0,1]^2$ in view of the proof of Theorem 4.3.3.

(b) $F_2^\lambda(x,y) = \lambda F^*(x,y) + (1-\lambda)G_0(x) G_0(y)$

\[ = \lambda x\wedge y + (1-\lambda)xy \]

\[ = xy + \lambda(x\wedge y - xy), (x,y) \in [0,1]^2 \]

is also easily computed for maximin iteration.

2. An alternative approach to proving asymptotic independence: Suppose an easily computed mixture has been chosen, say $F_\lambda$, and assume that $F_\lambda(a,a) < a$. We can always write $F_\lambda$ as follows:

\[ F_\lambda(x,y) = xy + h_0(x,y), \quad (4.5.1) \]
hence, \(-a^2 \leq h_0(a, a) < a(1-a)\).

The \(n\)th maximin iterand \(F_n^{\lambda}(x, y) \equiv \lambda^{(n)}(x, y, F^{\lambda}(x, y))\) has the following form:

\[
F_n(x, y) = \phi^{(n)}(x)\phi^{(n)}(y) + h_n(x, y),
\]

where

\[
h_n(x, y) = h_{n-1}(x, y)\left[2(1-\phi^{(n-1)}(x))(1-\phi^{(n-1)}(y)) + h_{n-1}(x, y)\right]
\]

\[
+ \left[2\phi^{(n)}(x)\phi^{(n)}(y) + h_{n-1}(x, y)[2(1-\phi^{(n-1)}(x))(1-\phi^{(n-1)}(y))
\]

\]

Hence, proving asymptotic independence of \((U_n, V_n)\) may be achieved by showing that

\[
h_n(a + \frac{u}{b^n}, a + \frac{v}{b^n}) \to 0,
\]

first for \((u, v)\) on a neighborhood \(N_0\) of \((0, 0)\), and then for \((u, v) \in \mathbb{R}^2 - \bar{N_0}\) the result follows by applying Corollary 2.2.1. While the complete proof in this case may be more lengthy than that given in Theorem 4.3.3, it will be easier to trace intermediate iteration using (4.5.3).

Additional remarks concern the relation of maximin to minimax iteration. First consider the minimax function \(\psi\) defined as follows:
\[ \psi(x) = 1 - (1-x^2)^2, \quad 0 \leq x \leq 1. \quad (4.5.4) \]

It is easy to show that the associated nth iterands \( \psi^{(n)} \) and \( \varphi^{(n)} \) have the following relationship:

\[ \psi^{(n)}(x) = 1 - \varphi^{(n)}(1-x), \quad n \geq 1 \quad (4.5.5) \]

or, equivalently,

\[ \varphi^{(n)}(x) = 1 - \psi^{(n)}(1-x), \quad n \geq 1. \quad (4.5.6) \]

Putting \( x = a + \frac{u}{b^2} \) in (4.5.6), we have

\[ \varphi^{(n)}(a + \frac{u}{b^2}) = 1 - \psi^{(n)}(1 - a - \frac{u}{n}), \quad (4.5.7) \]

and taking the limit in (4.5.7) as \( n \to -\infty \),

\[ \mathcal{M}(u) = 1 - \mathcal{M}(-u), \quad (4.5.8) \]

where

\[ \mathcal{M}(u) = \lim_{n \to -\infty} \psi^{(n)}(1 - a + \frac{u}{n}), \quad (4.5.9) \]

is the uniform limit law under \( \psi \).

Note that, as is easily checked, \( 1 - a \) is the unique interior fixed point of \( \psi \) on \( (0,1) \). Now, with \( (W_n, Z_n) = [M^{(n)}(X), m^{(n)}(Y)] \) defined analogously as \( (U_n, V_n) = [M^{(n)}(X), m^{(n)}(Y)] \) in (4.1.3), we have the following two conditions for asymptotic independence which may be proved in a manner specializing and parallelizing
the argument in Theorem 4.3.3.

**Lemma 4.5.1.** For $F \in C(G_0, G_0)$, necessary and sufficient conditions for asymptotic independence of $(U_n, V_n)$ and of $(W_n, Z_n)$ are given by $F(a, a) < a$ and $F(1-a, 1-a) < 1-a$, respectively.
5. BIVARIATE EXTREMES

5.1. Introductory

Consider a sequence \( \{X_k, Y_k\} \), \( k = 1, 2, \ldots \) of iid bivariate r.v.'s with the common cdf \( F_{G,H}(x,y) \). Let \( (U_n, V_n) = (\max(X_1, X_2, \ldots, X_n), \max(Y_1, Y_2, \ldots, Y_n)) \). The cdf \( F_n \) of \( (U_n, V_n) \) is given by

\[
F_n(x,y) = F^n(x,y), \quad (x,y) \in \mathbb{R}^2,
\]

with the marginals \( G_n(x) = G^n(x) \) and \( H_n(y) = H^n(y) \) respectively.

An operation analogous to bivariate maximin iteration is the bivariate maximum iteration, that may be viewed, as in Section 5.2.2, as generating a subsequence of geometrically growing size of \( \{U_n, V_n\} \), i.e., \( \{W_k, Z_k\} = \{U_{2^k}, V_{2^k}\} \). It is clear that, if \( F_n \) converges in distribution to a nondegenerate limit law \( \mathbb{F} \), then also does \( F_{2^k} \). Hence, if \( U_n \) and \( V_n \) are asymptotically independent, then \( W_k \) and \( Z_k \) are also asymptotically independent.

In Chapter 4, conditions for asymptotic independence in bivariate maximin iteration are established based on the value of \( F \) at or around \( (x_a, y_a) \), where \( G(x_a) \) or \( G(x_a^-) \), and \( H(y_a) \) or \( H(y_a^-) \), are the marginal fixed points \( a \) of the iteration. In order to find an analogous condition of asymptotic dependence or independence for extreme values, we therefore restrict our attention now to the case where the \( \{X_k, Y_k\}'s \) are essentially bounded above, so that there are finite numbers \( x_0 \) and \( y_0 \) for which
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\[ G(x_0) = 1, \ G(x_0-\varepsilon) < 1 \ \forall \varepsilon > 0 \]

and

\[ H(y_0) = 1, \ H(y_0-\varepsilon) < 1 \ \forall \varepsilon > 0 \ . \]

In this manner, \( G(x_0) \) and \( H(y_0) \) are identified as the analogous marginal fixed points in the maximum iteration, so that a condition on the value of \( F \) near \((x_0, y_0)\) can be expected to lead to a condition for asymptotic dependence or independence for bivariate maxima. The case with some "regularity conditions" is considered in Section 5.2, and the more special case when both marginals are uniform is considered in Section 5.3.

5.2. Regular Marginals

As always, a "super bar" symbol such as in \( \bar{G}, \bar{H} \) or \( \bar{F} \) is to denote an upper cdf, while \( F^0, F^* \) and \( I_G(C,G,H) \) are defined as in Section 3.2.

**Definition 5.2.1.** A cdf \( F \in C(G,H) \) is said to have regular marginals (for maximum iteration) if the following hold:

1. For some \( x_0, y_0: -\infty < x_0, y_0 < +\infty \)

\[ G(x_0) = 1, \ G(x_0-\varepsilon) < 1 \ \forall \varepsilon > 0 \]

\[ H(y_0) = 1, \ H(y_0-\varepsilon) < 1 \ \forall \varepsilon > 0 \] \hspace{1cm} \hspace{1cm} (5.2.1)

2. For some \( \alpha, \beta: 0 < \alpha, \beta < +\infty \),
\[
\lim_{u \to 0} \frac{G(x_0 + ku)}{G(x_0 + u)} = k^x \quad \forall k > 0
\]

\[
\lim_{v \to 0} \frac{H(y_0 + kv)}{H(y_0 + v)} = k^y \quad \forall k > 0
\]  

(5.2.2)

If \( G \) and \( H \) satisfy (5.2.1) and (5.2.2), then, in view of Lemma 2.3.3 and Theorem 2.3.1, there are sequences \( \{b_n\}, b_n > 0 \) and \( \{d_n\}, d_n > 0 \) such that

\[
G^n(x_0 + b_n u) \rightarrow \psi_\alpha(u)
\]  

(5.2.3)

or, equivalently,

\[
-\log \psi_\alpha(u) = \begin{cases} (-u)^\alpha, & u \leq 0 \\ 0, & u > 0 \end{cases}
\]  

(5.2.4)

and

\[
H^n(y_0 + d_n v) \rightarrow \psi_\beta(u)
\]  

(5.2.5)

or, equivalently,

\[
-\log \psi_\beta(v) = \begin{cases} (-v)^\beta, & v \leq 0 \\ 0, & v > 0 \end{cases}
\]  

(5.2.6)

Suppose now that, for a cdf \( F \) satisfying Definition 5.2.1, there is a nondegenerate cdf \( \bar{F} \) such that, for some \( \{b_n\}, b_n > 0 \) and \( \{d_n\}, d_n > 0 \),
\( F_n(x_0 + b_n, y_0 + d_n) \longrightarrow \mathcal{F}(u,v) \) \hfill (5.2.7)

holds for \((u,v) \in \mathbb{R}^2\). Then, writing \( F(x,y) = 1 - G(x) - H(y) + \bar{F}(x,y) \), we have

\[
\mathcal{F}(u,v) = \lim_{n \to \infty} \left[ 1 + \frac{-nG(x_0 + b_n) - nH(y_0 + d_n) + nF(x_0 + b_n, y_0 + d_n)}{n} \right]^n
\]

\[
= e^{\log \psi_\alpha(u) + \log \psi_\beta(v) + w(u,v)}
\]

\[
= \psi_\alpha(u) \psi_\beta(v) e^{w(u,v)} \hfill (5.2.8)
\]

by (5.2.4) and (5.2.6), where

\[
w(u,v) = \lim_{n \to \infty} nF(x_0 + b_n, y_0 + d_n) \hfill (5.2.9)
\]

Hence, we have

**Lemma 5.2.1.** If \( F \in \mathcal{C}(G,H) \) has regular marginals (for maximum iteration), then \( F^n \) converges to a nondegenerate limit law if and only if for some \( x_0 \) and \( y_0 \) satisfying (5.2.1) and for some \( \{b_n\} \) and \( \{d_n\} \), \( b_n, d_n > 0 \ \forall \ n \), \( w(u,v) \) defined as in (5.2.9) exists and is bounded.

A condition for asymptotic independence is given in

**Lemma 5.2.2.** (Galambos, 1978): If \( F \in \mathcal{C}(G,H) \) has regular marginals for maximum iteration, then \( U_n \) and \( V_n \) are asymptotically independent if and only if
\[
\lim_{n \to \infty} n F(x_0 + b_n u, y_0 + d_n v) = 0 \quad (5.2.10)
\]

where \( x_0, y_0, \{b_n\} \) and \( \{d_n\} \) are such that (5.2.1), (5.2.2), (5.2.4) and (5.2.6) hold.

Proof: The result follows from (5.2.8) and (5.2.9), under which

\[
F(u,v) = \lim_{n \to \infty} F^n(x_0 + b_n u, y_0 + d_n v)
\]

if and only if (5.2.10) holds.

The following fact is easy to verify:

**Fact 5.2.1.** Suppose \( G \) and \( H \) satisfy (5.2.3) and (5.2.5) for some \( (x_0, b_n) \), \( (y_0, d_n) \). Let \( F^* \) be the upper Fréchet bound of \( C(G,H) \). Then

\[
F^*(x_0 + b_n u, y_0 + d_n v) \to \psi_\alpha(u) \psi_\beta(v), \quad (5.2.11)
\]
as \( n \to \infty \).

We will now consider a sufficient condition for asymptotic independence of \( (U_n, V_n) \) as follows:

**Lemma 5.2.3.** Suppose that \( F \in C(G,H) \) has regular marginals. A sufficient condition for \( U_n \) and \( V_n \) to be asymptotically independent is that

\[
F(x,y) \leq G(x) \cdot H(y) \quad (5.2.12)
\]

for \( (x,y) \) on a southwest neighborhood \( SW(x_0,Y_0) \) of \( (x_0,Y_0) \) where
(x_0, y_0) is defined as in (5.2.1).

Proof: Assume that α, β (> 0) and positive sequences \{b_n\}
and \{d_n\} are such that (5.2.4) and (5.2.6) hold.

Given any (u,v) ≤ 0, since b_n, d_n → 0 by Corollary 2.3.2,
there is an integer N_{uv} such that

\[ (x_0 + b_n u, y_0 + d_n v) \in SW(x_0, y_0), \quad n \geq N_{uv}. \]

Hence, if (5.2.12) holds,

\[ F(x_0 + b_n u, y_0 + d_n v) \leq G(x_0 + b_n u) H(y_0 + d_n v), \quad n \geq N_{uv} \quad (5.2.13) \]

and, by equivalence of odd-quadrant dependence (Lemma 3.2.5), (5.2.13)
is equivalent to

\[ 0 \leq n F(x_0 + b_n u, y_0 + d_n v) \leq \frac{nG(x_0 + b_n u) \cdot nH(y_0 + d_n v)}{n}. \quad (5.2.14) \]

Since (5.2.4) and (5.2.6) hold, the RHS of (5.2.14) converges to 0
as \( n \to \infty \), so that (5.2.10) follows, and hence, in view of Lemma
5.2.1, since \( u, v \leq 0 \) were arbitrary, \( U_n \) and \( V_n \) are asymptotically
independent.

Remark 5.2.1. Lemma 5.2.3 is slightly stronger than a similar
sufficient condition for asymptotic independence given in Villasenor
(1976), who requires inequality (5.2.12) to hold for \( (x,y) \in \mathbb{R}^2 \).

A sufficient condition for asymptotic dependence of \((U_n, V_n)\) is
given as follows:
Lemma 5.2.4. If \( F \in \mathcal{C}(G,H) \) has regular marginals (for maximum iteration) and is a mixture involving the upper Fréchet bound \( F^* \) of \( \mathcal{C}(G,H) \), then \( U_n \) and \( V_n \) are asymptotically dependent.

Proof: Assume that, for some

\[
\lambda: 0 < \lambda < 1, \quad F(x,y) = \lambda F^*(x,y) + (1-\lambda) F^1(x,y), \tag{5.2.15}
\]

where \( F^*(x,y) = G(x)H(y) \) and \( F^1 \in \mathcal{C}(G,H) \). Then, with \( F^0(x,y) = [G(x)+H(y)-1]^+ \), let

\[
F^\lambda(x,y) = \lambda F^*(x,y) + (1-\lambda) F^0(x,y) \tag{5.2.16}
\]

for \( \lambda \) given in (5.2.15).

Clearly, in view of the Fréchet bound – Theorem 3.2.1, (5.2.15) and (5.2.16) imply

\[
F^\lambda(x,y) \leq F(x,y) \quad \forall (x,y) \in \mathbb{R}^2, \tag{5.2.17}
\]

and this, in turn, in view of equivalence of odd-quadrant dependence (Lemma 3.2.5), implies that

\[
nF^\lambda(x,y) \leq nF(x,y) \quad \forall (x,y) \in \mathbb{R}^2. \tag{5.2.18}
\]

Since \( F \) has regular marginals, (5.2.3) and (5.2.5) hold for \( x_0, y_0 \) satisfying (5.2.1) and (5.2.2), and for some positive sequences \( \{b_n\} \) and \( \{d_n\} \). Then (5.2.18) implies

\[
nF^\lambda(x_0+b_n u, y_0+d_n v) \leq nF(x_0+b_n u, y_0+d_n v) \quad \forall (u,v) \in \mathbb{R}^2. \tag{5.2.19}
\]
Now, the LHS of (5.2.19) may be written as

\[ n\mathbb{F}(x_0+b_nu, y_0+d_nv) = n\{\lambda[1 - G(x_0+b_nu) - H(y_0+d_nv) + G(x_0+b_nu)\Lambda H(y_0+d_nv)] \]
\[ + (1-\lambda)[1 - G(x_0+b_nu) - H(y_0+d_nv) + [G(x_0+b_nu) + H(y_0+d_nv) - 1]^+]\}. \]  

(5.2.20)

Given any \( u, v < 0 \), since \( b_n, d_n \to 0 \) and \( G(x_0^-) = H(y_0^-) = 1 \) in view of Corollary 2.3.2, (5.2.1) and monotonicity of \( G \) and \( H \) imply that there is an integer \( N_{uv} \) such that

\[ G(x_0+b_nu) + H(y_0+d_nv) - 1 \geq 0, \quad n \geq N_{uv} \]  

(5.2.21)

(5.2.20) and (5.2.21) then reduce to

\[ n\mathbb{F}(x_0+b_nu, y_0+d_nv) = n\{\lambda[1 - G(x_0+b_nu) - H(y_0+d_nv) + G(x_0+b_nu)\Lambda H(y_0+d_nv)] \]
\[ + (1-\lambda)[1 - G(x_0+b_nu) - H(y_0+d_nv)] \]
\[ = \lambda(nG(x_0+b_nu))\Lambda(nH(y_0+d_nv)), \quad n \geq N_{uv}, \]  

(5.2.22)

and (5.2.19), (5.2.22), (5.2.25) and (5.2.27) imply that

\[ \lim_{n \to \infty} n\mathbb{F}(x_0+b_nu, y_0+d_nv) \geq \lim_{n \to \infty} n\mathbb{F}(x_0+b_nu, y_0+d_nv) \]
\[ = \lambda((-u)^q \land (-v)^q), \]  

(5.2.23)
with the limit in (5.2.23) positive for \( u, v < 0 \). Therefore, \( U_n \) and \( V_n \) are asymptotically dependent by Lemma 5.2.2.

**Corollary 5.2.1.** Suppose \( F \in \mathcal{C}(G,H) \) has regular marginals (for maximum iteration), and is a mixture of the form given in (5.2.15), involving \( F^* \) and \( F^1 \) satisfying \( F^0(x,y) \leq F^1(x,y) \leq G(x)H(y) \). Then the limit law of \( F^n \) is a generalized Marshall-Olkin type cdf of the form

\[
\Phi(u,v) = \psi_A(u) \psi_B(v) \cdot \lambda^(-u)^a \land (-v)^b, \quad u, v < 0 \quad (5.2.24)
\]

**Proof:** Consider \( F^\lambda \) defined in (5.2.16) and

\[
F^\lambda(x,y) = \lambda F^*(x,y) + (1-\lambda) G(x)H(y) \quad (5.2.25)
\]

Then

\[
n\Phi^\lambda(x,y) = n[1 - G(x)H(y) + \lambda F^*(x,y) + (1-\lambda) G(x)H(y)]
\]

\[
= n[(1-G(x))(1-H(y)) + \lambda F^*(x,y) - G(x)H(y))]
\]

\[
= \frac{nG(x) \cdot nH(y)}{n} + \lambda n[G(x)\land H(y) - G(x)H(y)] \quad (5.2.26)
\]

Given any \( u, v < 0 \), (5.2.26) implies for \( x_0, y_0 \) satisfying (5.2.1) and (5.2.2) and for positive sequences \( \{b_n\} \) and \( \{d_n\} \) such that (5.2.4) and (5.2.6) hold,

\[
n\Phi^\lambda(x_0 + b_n u, y_0 + d_n v) = \frac{nG(x_0 + b_n u) \cdot nH(y_0 + d_n v)}{n} + \lambda n[G(x_0 + b_n u)\land H(y_0 + d_n v) - G(x_0 + b_n u)H(y_0 + d_n v)] \quad (5.2.27)
\]
the first term of (5.2.26) tending to 0 as \( n \to \infty \), since the numerator tends to values bounded by 0 and 1 by (5.2.4) and (5.2.6), while the denominator \( n \to \infty \); the second term of (5.2.27) may be written as

\[
\lambda [nH(y_0 + d_n v) \cdot G(x_0 + b_n u) \wedge nG(x_0 + b_n u) \cdot H(y_0 + d_n v)]
\]

\[\to \lambda ((-u)\alpha \wedge (-v)\beta), \text{ as } n \to \infty, \quad (5.2.28)\]

using (5.2.4), (5.2.6) and Corollary 2.3.2. Therefore, the limit of (5.2.27), as \( n \to \infty \), becomes

\[
\overline{nF}(x_0 + b_n u, y_0 + d_n v) \to \lambda ((-u)\alpha \wedge (-v)\beta). \quad (5.2.29)
\]

Hence, the limits in (5.2.23) and (5.2.29) are the same, so that, for any \( F^1 \in \mathcal{C}(G, H) \) such that \( F^0(x, y) \leq F^1(x, y) \leq G(x)H(y) \), \( F \in \mathcal{C}(G, H) \) defined by

\[
F(x, y) = \lambda F^*(x, y) + (1-\lambda)F^1(x, y),
\]

will also lead to, in view of Lemma 3.2.4 again, the same limit

\[
\overline{nF}(x_0 + b_n u, y_0 + d_n v) \to \lambda ((-u)\alpha \wedge (-v)\beta), \quad u, v \leq 0, \quad (5.2.30)
\]

so that (5.2.8) and (5.2.30) imply that (5.2.24) holds.

5.3. Uniform Marginals

A special class of bivariate cdf's with regular marginals is given by \( \mathcal{C}(G_0, G_0) \) where \( G_0 \) is uniform on \([0,1]\) (\( \alpha = \beta = 1 \)), in
which case the previous regularity conditions hold (for both maximum and minimum iteration).

Let

\[ B(u) = e^u, \quad u \leq 0 \]  \hspace{1cm} \text{(5.3.1)}

\[ \Gamma(u) = 1 - e^{-u}, \quad u > 0 \]  \hspace{1cm} \text{(5.3.2)}

and

\[ B_n(u) = G_0^n(1 + \frac{u}{n}), \quad -n \leq u \leq 0 \]  \hspace{1cm} \text{(5.3.3)}

\[ \Gamma_n(u) = 1 - [1 - G_0^n(\frac{u}{n})]^n, \quad 0 \leq u \leq n \]  \hspace{1cm} \text{(5.3.4)}

Clearly \( B_n \) and \( \Gamma_n \) denote the cdf's of the normalized maxima and normalized minima, respectively, of iid r.v.'s with the common cdf \( G_0 \). Note also that \( B_n(u) \rightarrow B(u) \) and \( \Gamma_n(u) \rightarrow \Gamma(u) \) as \( n \rightarrow \infty \).

For \( P \in \mathcal{C}(G_0, G_0) \), define

\[ \mathcal{F}_n(u, v) = F^n(1 + \frac{u}{n}, 1 + \frac{v}{n}), \quad -n \leq u, v \leq 0 \]  \hspace{1cm} \text{(5.3.5)}

\[ \mathcal{K}_n(u, v) = 1 - \mathcal{F}_0^n(\frac{u}{n}) - \mathcal{F}_0^n(\frac{v}{n}) + \mathcal{F}_0^n(\frac{u}{n}, \frac{v}{n}), \quad 0 \leq u, v \leq n \]  \hspace{1cm} \text{(5.3.6)}

Hence, \( \mathcal{F}_n \) and \( \mathcal{K}_n \) are the cdfs of the normalized bivariate maxima and minima, respectively, of iid bivariate r.v.'s with the common cdf \( F \).

The following holds analogously as in Section 5.2.
Lemma 5.3.1. Let $F \in C(G_0, G_0)$ and $B, \Gamma, B_n, \Gamma_n, F_n$ and $K_n$ be defined as in (5.3.1)-(5.3.6). Then, assuming that convergence in distributions take place, we have:

(a) $\mathcal{H}_n(u, v) \rightarrow B(u)B(v) e^{w(u, v)}, \ u, v \leq 0$ as $n \rightarrow \infty$, where

$$w(u, v) = \lim_{n \rightarrow \infty} nF(1 + \frac{u}{n}, 1 + \frac{v}{n}).$$

(b) $K_n(u, v) \rightarrow \Gamma(u) \Gamma(v) + e^{u-v} (e^{z(u, v)} - 1), \ u, v > 0$ as $n \rightarrow \infty$, where

$$z(u, v) = \lim_{n \rightarrow \infty} nF\left(1 + \frac{u}{n}, 1 + \frac{v}{n}\right).$$

(c) Asymptotic independence of bivariate maxima, and of bivariate minima, of iid bivariate r.v.'s obtain, respectively, if and only if

$$w(u, v) = \lim_{n \rightarrow \infty} nF\left(1 + \frac{u}{n}, 1 + \frac{v}{n}\right) = 0$$

and

$$z(u, v) = \lim_{n \rightarrow \infty} nF\left(\frac{u}{n}, \frac{v}{n}\right) = 0.$$

(d) If $F(x, y) \leq G_0(x)G_0(y)$ for $(x, y)$ on a southwest neighborhood $SW(1, 1)$ of $(1, 1)$, then $w(u, v) = 0$ and hence the bivariate maxima is asymptotically independent.

(e) If $F(x, y) \leq G_0(x)G_0(y)$ for $(x, y)$ on a northeast neighborhood $NE(0, 0)$ of $(0, 0)$, then $z(u, v) = 0$ and hence the bivariate
minima is asymptotically independent.

(f) If \( F(x,y) = \lambda F^*(x,y) + (1-\lambda)F^1(x,y) \), \( 0 < \lambda < 1 \) with 
\[ F^0(x,y) \leq F^1(x,y) \leq G_0(x)G_0(y) \] then

1. \( \mathcal{F}_n(u,v) \rightarrow \mathcal{F}(u,v) \) as \( n \rightarrow \infty \),

\[ \mathcal{F}(u,v) = e^{u+v} - \lambda u v , \quad u, v \leq 0 \] (5.3.7)

2. \( \mathcal{K}_n(u,v) \rightarrow \mathcal{K}(u,v) \) as \( n \rightarrow \infty \), where

\[ \mathcal{K}(u,v) = 1 - e^{-u} - e^{-u} + e^{-u-v} + \lambda u v , \quad u, v > 0. \] (5.3.8)

(g) If \( F \) is a mixture involving the upper Fréchet bound, both bivariate extremes are asymptotically dependent.

Definition 5.3.1. (Marshall and Olkin, 1967): A bivariate cdf \( \mathcal{F} \) is said to be a Marshall-Olkin cdf if it has either one of the following two forms:

(a) \[ \mathcal{F}(u,v) = e^{\lambda_1 u + \lambda_2 v + \lambda_3 u v} , \quad u, v \leq 0 \] (5.3.9)

(b) \[ \overline{\mathcal{F}}(u,v) = e^{-\lambda_1 u - \lambda_2 v - \lambda_3 u v} , \quad u, v > 0. \] (5.3.10)

The (upper) cdf \( \overline{\mathcal{F}} \) in (5.3.10) is given by Marshall and Olkin (1967).

Corollary 5.3.1. If \( F \) is a mixture involving the upper Fréchet bound \( F^* \) and \( F^1 \in C(G_0, G_0) \) such that \( F^1(x,y) \leq G_0(x)G_0(y) \), then both bivariate extremes converge in distribution to the Marshall-Olkin limit law.
Proof: Writing \( uW = u + v - uA \) in (5.3.7), we have

\[
\mathcal{F}(u,v) = e^{(1-\lambda)u + (1-\lambda)v + \lambda uv}, \quad u, v \leq 0 ,
\]

which has the form (5.3.9) with parameters \( \lambda_1 = \lambda_2 = 1-\lambda \) and \( \lambda_{12} = \lambda \).

Similarly, writing \( uA = u + v - u\mathcal{W} \) in (5.3.8), we can write the corresponding upper cdf \( \overline{\mathcal{F}} \) as follows:

\[
\overline{\mathcal{F}}(u,v) = e^{-(1-\lambda)u - (1-\lambda)v - \lambda uv}, \quad u, v > 0 ,
\]

which has the form (5.3.10) with parameters \( \lambda_1 = \lambda_2 = 1-\lambda \), \( \lambda_{12} = \lambda \).

As a corollary to 5.3.1(g) and (h), sufficient conditions for asymptotic dependence for both extremes are given in the following.

**Corollary 5.3.2.** Suppose \( F \in \mathcal{C}(G_0, G_0) \). Sufficient conditions for (a) bivariate maxima and (b) bivariate minima to be asymptotically dependent are given, respectively, by the following:

(a) \( F(x,x) > x^2 \) on some deleted left-neighborhood

\[
L(l-) \text{ of } l ,
\]

and

\[
\lim_{x \downarrow l-} \frac{dF(x,x)}{dx} \neq 2 \quad \text{(5.3.14)}
\]

(b) \( F(x,x) > x^2 \) on some deleted right-neighborhood

\[
R(0+) \text{ of } 0 ,
\]

and

\[
\lim_{x \uparrow 0+} \frac{dF(x,x)}{dx} \neq 2 . \quad \text{(5.3.16)}
\]
Proof: The proof is given for part (a), while part (b) can be proved analogously.

Assume (5.3.13) and (5.3.14) hold, then

\[
\lim_{x \to 1^-} \frac{dF(x,x)}{dx} = 2 - 2\lambda, \text{ some } \lambda: 0 < \lambda \leq \frac{1}{2},
\]  

(5.3.17)

since \( x^2 < F(x,x) \leq x \), (5.3.13) and differentiability imply

\[
2 > \lim_{x \to 1^-} \frac{dF(x,x)}{dx} \geq 1. \text{ Now, let }
\]

\[
F^\lambda(x,y) = \lambda x \Lambda y + (1-\lambda)xy, (x,y) \in [0,1]^2.
\]  

(5.3.18)

Hence,

\[
\lim_{x \to 1^-} \frac{dF^\lambda(x,x)}{dx} = 2 - \lambda > 2 - 2\lambda
\]  

(5.3.19)

so that (5.3.13) and (5.3.17)-(5.3.19) imply that

\[
F(x,x) \geq F^\lambda(x,x) > x^2
\]  

(5.3.20)

on some deleted left-neighborhood \( L^\prime (l^-) \) of 1. Given any \( u < 0 \), (5.3.20), continuity and monotonicity of the cdf's in \( \mathcal{C}(G_0,G_0) \) imply that

\[
F^n(l + \frac{u}{n}, l + \frac{u}{n}) \geq F^\lambda^n(l + \frac{u}{n}, l + \frac{u}{n}) > (l + \frac{u}{n})^{2n}
\]  

(5.3.21)

for large \( n \).

Taking the limit of (5.3.21), as \( n \to \infty \), we have in view of (5.3.7),
\[ \lim_{n \to \infty} F^n(1 + \frac{u}{n}, 1 + \frac{u}{n}) \geq e^{2u-\lambda u} > e^{2u} = [B(u)]^{2}, \]

so that the bivariate maxima are asymptotically dependent.
6. CONCLUDING REMARKS

1. Marginal fixed points in bivariate (maximin and minimax) cdf iteration and its analog, i.e., the marginal essential maxima and minima in bivariate extremes are considered as critical points, near which cdf values determine asymptotic dependence/independence.

2. For both the cdf iteration and extreme value problems, the upper Fréchet bound (i.e., UFB) plays a dominant role in establishing asymptotic dependence. In the former case the UFB provides essentially the only instance of dependence; while, in the latter case, mixtures involving the UFB provide such instances.

3. For the case of uniform marginals, conditions (on the bivariate cdf $F$) for asymptotic independence (A.I.) and asymptotic dependence (A.D.) are given in the following table for both bivariate (maximin and minimax) iteration and bivariate extremes, for comparison.

<table>
<thead>
<tr>
<th>R.V.'s</th>
<th>A.I.</th>
<th>A.D.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximin</td>
<td>$F(a,a) &lt; a$ (N.S)$^a$</td>
<td>$F(a,a) = a$ (N.S)</td>
</tr>
<tr>
<td>Minimax</td>
<td>$F(1-a,1-a) &lt; 1-a$ (N.S)</td>
<td>$F(1-a,1-a) = 1-a$ (N.S)</td>
</tr>
<tr>
<td>Maxima</td>
<td>$F(x,y) \leq xy$, $(x,y) \in SW(1-,1-)$ (S)$^b$</td>
<td>$F(x,x) &gt; x^2$, $x \in L(1-)$ (S) $\lim_{x \uparrow 1-} \frac{dF(x,x)}{dx} \neq 2$ (S)</td>
</tr>
<tr>
<td>Minima</td>
<td>$F(x,y) \leq xy$, $(x,y) \in NE(0-,0-)$ (S)</td>
<td>$F(x,x) &gt; x^2$, $x \in R(0+)$ (S) $\lim_{x \uparrow 0+} \frac{dF(x,x)}{dx} \neq -2$ (S)</td>
</tr>
</tbody>
</table>

$^a$N.S = necessary and sufficient.

$^b$S = sufficient.
4. The convexity property of the class $C(G,H)$ has proved to be useful for establishing the above conditions for asymptotic independence or dependence.
7. REFERENCES


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