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The flow of a large scale general vortex near a solid boundary normal to the axis of the vortex

Hailezghi Tesfamariam
Iowa State University

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The flow of a large scale general vortex near a solid boundary normal to the axis of the vortex

by

Hailezghi Tesfamariam

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LIST OF SYMBOLS

\( \delta(r) \) the boundary layer thickness function
\( \delta_0^* \) a reference distance in the axial direction
\( \varepsilon \) the turbulence energy dissipation rate
\( K \) turbulence diffusivity
\( \mu \) molecular viscosity
\( p \) pressure
\( p_a \) the ambient pressure at large \( r \)
\( P \) the mean value of \( p - p_a \)
\( p' \) the fluctuating component of \( p - p_a \)
\( q \) the mean value of the turbulence energy, \( \frac{(u'^2 + v'^2 + w'^2)}{2} \)
\( q' \) the turbulence energy, \( \frac{(u'^2 + v'^2 + w'^2)}{2} \)
\( r \) the radial coordinate
\( \rho \) density
\( t \) the time coordinate
\( \theta \) the azimuthal coordinate
\( u \) radial velocity
\( U \) the mean value of the radial velocity, \( u \)
\( u' \) the fluctuating component of the radial velocity, \( u \)
\( v \) tangential velocity
\( V \) the mean value of the tangential velocity, \( v \)
\( v' \) the fluctuating component of the tangential velocity, \( v \)
\( v_m^* \) a reference velocity
\( w \) axial velocity

\( W \) the mean value of the axial velocity, \( w \)

\( w' \) the fluctuating component of the axial velocity, \( w \)

\( z \) the axial coordinate
I. INTRODUCTION

A. The General Vortex

When a cylindrical container filled with a fluid is rotated about its cylindrical axis, a rotating fluid flow is created inside the container in which the rotational velocity of the fluid varies from zero at the axis of the cylinder to the rotational velocity of the wall of the container at the inside wall of the container. When a cylindrical solid body is rotated about its axis in a fluid which is at rest, a rotating fluid flow is created in which the rotational velocity of the fluid varies from the rotational velocity of the outer edge of the body at the wall of the body to zero at large distances away from the body. When a small rotor is rotated inside a large stationary container filled with a fluid, a rotating fluid flow is created in which the rotational velocity of the fluid increases from zero at the rotational axis of the rotor to some maximum value at some distance from the rotational axis of the rotor and then decreases from the maximum value to zero at the inside wall of the container. (See Fig. 1.)

These three flows are representative examples of the three types of vortex flows that can exist. When they are considered in a cylindrical coordinate system, in the first type, the tangential velocity is monotonously increasing with
Fig. 1. Three types of rotating flows
increasing radius; in the second type, the tangential velocity
is monotonously decreasing with increasing radius; and in the
third type, the tangential velocity is increasing in some
radial positions and decreasing in some other radial posi­
tions. The third type is a combination of the first and
second types. For this reason it is referred to as a general
vortex.

The general vortex is the most prevalent vortex. It is
encountered more frequently than either one of the vortex
flows with monotonously varying tangential velocity. While
the latter are encountered mostly in man made flows, the
general vortex is encountered in both natural and man made
flows. The general vortex is formed in many different ways
and occurs in many different circumstances. Most general
vortices, though, occur in or are associated with any one or
a combination of the following five circumstances.

i) Flow past solid bodies. When a fluid is flowing
past a solid body, on the lee side of the body a general
vortex is created. Examples of this type of general vortex
are trailing vortices of aircrafts and vortices created near
buildings as wind blows over the earth's surface.

ii) Flow associated with rotating solid bodies. Any
solid body rotating in a fluid creates a vortex near it.
Examples of this type of general vortex are flows associated
with helicopter rotors and airplane propellers.
iii) **Angular inflow into a confined area.** When a fluid is made to enter a cylindrical type confined area at a small angle to the inner walls of the confinement a general vortex is created. Examples of this type of general vortex are flows associated with industrial processes such as flows in Hilsh tubes and cyclone separators.

iv) **Sink flow from a confined area.** When a fluid flows out from a confined area through a narrow exit a general vortex is formed. Examples of this type of general vortex are whirlpools and whirlwinds.

v) **Vortex flow associated with very large flow fields.** In very large flow fields such as planetary atmospheres and oceans density and temperature stratification of the flow field occurs. From the interaction of the different flows at different strata, flows traversing the strata and the flow field's boundaries large scale general vortices are created. Examples of this type of general vortex are hurricanes and tornadoes.

General vortex flow fields are of interest for many reasons. Large scale atmospheric vortices such as hurricanes and tornadoes produce much damage to human life and property in many parts of the world. Low speed flight thrives on flows created by propellers and rotors. Trailing vortices associated with large aircrafts are hazardous to small aircrafts. In industrial processes use is made of general vortex
flow fields in separation processes. For these reasons among many the study of general vortex flow fields is important. A good knowledge of the flow field of a general vortex would be the biggest asset in any endeavor to alleviate the danger from large scale atmospheric vortices or the danger from trailing vortices of large aircrafts. The knowledge would also be essential in any attempt to optimize the use of vortex flow fields in industrial processes.

B. General Vortex Flow Near a Solid Surface

A very important feature of vortex flow fields is the existence of a centrifugal force field due to the rotation of the fluid. In any vortex flow, there is a strong interaction between the centrifugal force and the pressure gradient in the radial direction. Near a solid boundary normal to the axis of the vortex, the centrifugal force decreases to zero rapidly while there is little axial change of the pressure. This gives rise to fluid motion along the radial direction which in turn gives rise to motion along the axial direction. The motion created along the axial direction may persist up to the next boundary normal to the axis of the vortex. This axial motion would also modify the rotational motion thus by completing a circle of modification and change. So, a solid boundary normal to the axis of a vortex affects the whole vortex flow field very significantly. In most of the cases
where a vortex flow is of interest the vortex flow interacts with a solid surface normal to the axis of the vortex. For this reason the flow of a vortex near a solid boundary normal to the axis of the vortex is of particular importance.

A vortex flow is governed by the Navier Stokes equations, the energy equation and the continuity equation. For an incompressible flow, with constant molecular viscosity and negligible temperature variations, the equations can be written in a cylindrical coordinate system as:

\[ \rho \{ u_r + uu_r + \frac{v}{r} u_\theta + wu_z - \frac{v^2}{r} \} = - p_r \]

\[ + \mu \{ u_{rr} + \frac{u_r}{r} + \frac{u_\theta^2}{r^2} + u_{zz} - \frac{u}{r^2} - 2 \frac{u}{r^2} v_\theta \} \]  \hspace{1cm} (1) \]

\[ \rho \{ v_r + uv_r + \frac{v}{r} v_\theta + wv_z + \frac{uv}{r} \} = \frac{p_\theta}{r} \]

\[ + \mu \{ v_{rr} + \frac{v_r}{r} + \frac{v_\theta^2}{r^2} + v_{zz} - \frac{v}{r^2} + 2 \frac{v}{r^2} u_\theta \} \]  \hspace{1cm} (2) \]

\[ \rho \{ w_r + uw_r + \frac{w}{r} w_\theta + ww_z \} = - p_z - \rho g \]

\[ + \mu \{ w_{rr} + \frac{w_r}{r} + \frac{w_\theta^2}{r^2} + w_{zz} \} \]  \hspace{1cm} (3) \]

\[ u_r + \frac{u}{r} + \frac{v_\theta}{r} + w_z = 0 \]  \hspace{1cm} (4)
In these equations, $u$, $v$ and $w$ are the radial, tangential and axial velocities, respectively; a subscript denotes differentiation and $t$, $r$, $\theta$ and $z$ are the time, radial, tangential and axial coordinate positions, respectively; and $\mu$, $\rho$, $p$ and $g$ are the molecular viscosity, density, pressure and gravitational acceleration, respectively. The solution of these equations in a region near a solid boundary with boundary conditions appropriate for a vortex flow used at appropriate boundaries would explain the flow of a vortex near a solid boundary normal to the axis of the boundary. The equations are coupled and nonlinear and their solutions are obtained by methods of the following three groups of methods.

i) **Perturbation methods.** Briefly, in these methods, the variables $u$, $v$, $w$ and $p$ are expanded in a series in terms of one or two parameters $\varepsilon < 1$. The expansions of the variables are then inserted into the equations (1, 2, 3 and 4). By grouping terms of the same power of $\varepsilon$ together, a series of easily solvable equations is obtained. An account of perturbation methods is given in a book by Van Dyke (1964).

ii) **Methods of reduction to ordinary differential equations.** In these methods, by integrating in one or two directions suitably or by making use of a suitable transformation, the partial differential equations are reduced to solvable ordinary differential equations. For a two
coordinate problem (which is what the solution of Equations 1, 2, 3 and 4 becomes when axisymmetric and steady flow assumptions are made) the most widely used methods of this type are the similarity and momentum integral methods. Accounts of similarity and momentum integral methods are given in many books, for example, in a book by Walz (1969).


Many solutions to the governing equations for a vortex flow near a solid boundary have been obtained by many investigators by methods of groups (i), (ii) and (iii). An example of a solution by a perturbation method is a solution reported by Smith and Smith (1965). In this solution, steady and axisymmetric flow is assumed and the variables are expanded in a series in terms of a parameter \( e = \delta_0^*/r_m^* \), where, \( \delta_0^* \) is a characteristic distance in the axial direction and \( r_m^* \) is a characteristic distance in the radial direction. An example of a solution by a method of group (ii) is a solution reported by Serrin (1972). Serrin uses a spherical coordinate system in his solution and by assuming a certain general form for the solution of the variables he finds a transformation which reduces the four partial differential equations to two solvable integro-differential equations. An example of a
solution by a finite difference method for partial differential equations is a solution reported by Chi and Jih (1974). In this solution, a steady and axisymmetric flow is assumed, equations for the tangential vorticity and stream function are derived from Equations 1, 3 and 4 and those two equations and Equation 2 are solved by the Gauss-Seidel successive iteration method.

Finite difference methods for partial differential equations are the simplest and most widely applicable of the groups of methods. Perturbation methods are restricted in application to cases where the expansion parameters that is used is small. Also, to get accurate solutions, one has to solve a large number of equations. The methods of group (ii) are more applicable than perturbation methods and they are cheaper to use than finite difference methods for partial differential equations for some problems. But, the methods of group (ii) are restricted by assumptions that are made in order to reduce the governing partial differential equations to ordinary differential equations. Moreover, obtaining solutions to the ordinary differential equations may not be easy. Finite difference methods for partial differential equations, on the other hand, do not have any restrictions from the point of view of the type of solution that is obtained. Their application is straightforward and general. There is a central problem with finite difference methods, the
problem of numerical instability, but, varying in degree from one finite difference method to another, this problem can be solved for a wide scope of applications.

Sometimes, for example, when solutions very close to a surface are sought, Equations 1-4 are not solved in full. They are reduced to a simpler set of equations. This reduction is done by either making an order of magnitude analysis or by expanding the variables in terms of a parameter which is a ratio of reference distances in the axial and radial directions and the reduced equations are obtained as the zero order equations. These equations are:

\[ \rho \left( u_t + uu_r + \frac{v}{r} u_\theta + wu_z - \frac{v^2}{r} \right) = p_r + u_{zz} \]  \hspace{1cm} (5)

\[ \rho \left( v_t + uv_r + \frac{v}{r} v_\theta + wu_z + \frac{uv}{r} \right) = \mu u_{zz} \]  \hspace{1cm} (6)

\[ p_z = 0 \]  \hspace{1cm} (7)

\[ u_r + \frac{u}{r} + \frac{v_\theta}{r} + w_z = 0 \]  \hspace{1cm} (8)

These equations are referred to as the boundary layer equations for a vortex. Sometimes, this term can be misleading. The term boundary layer implies that the equations are applicable only in regions very close to the surface, but for some vortex flows these "boundary layer equations" are a good
approximation to the whole equations at distances far from the surface. In such cases they should be looked at as approximate equations which are a good approximation to the whole equations when diffusion in the axial direction is much larger than diffusion in the radial direction and when the axial variation of the pressure is very small.

Equations 5, 6, 7 and 8 have been solved for vortex flows by many investigators. An example of a solution is that reported by Kuo (1971). In this solution, axisymmetric and steady state flow is assumed, and the equations are solved by a similarity and momentum integral method.

Vortex flows may be laminar or turbulent. When the flow is turbulent, in addition to the variation of the flow variables in time and space, there is a small scale fluctuation of the values of the variables at all points in time and space. So, when a turbulent flow is being analyzed, it is the mean values of the flow variables that is sought. The sets of equations 1-4 and 5-8 govern both laminar and turbulent flows. But, when the mean values of the flow variables of a turbulent flow are sought the equations have to be modified. From the Reynolds' decomposition approach to the problem, what this entails is the addition of Reynolds stress terms to Equations 1, 2, 3, 5 and 6. During the solution of the resulting equations, the Reynolds stress terms are modeled or solved for in some way so that the set of governing equations
containing the Reynolds stress terms are solvable. The way
this is done is an area of continuing research. A summary of
the approaches to the problem is provided in a book by
Launder and Spalding (1972). Most vortex flows of interest
are turbulent and so in a complete solution of a vortex flow
turbulence is an important aspect of the solution problem.

Many investigators have obtained theoretical solutions
to turbulent vortex flows near a solid boundary by using many
different turbulence models. An example of a solution is
that reported by Hsu and the present author (1976 and 1977).
In this solution, an axisymmetric flow is assumed, a
two equation model of turbulence is used and the boundary
layer equations are solved by a finite difference method.
The solution is the result of an exploratory analysis made
at the beginning of the research that is being reported in
this thesis and it is discussed in detail in the following
section.

C. An Exploratory Analysis of the Turbulent
Flow of a Vortex Near a Solid Boundary

In the exploratory analysis, the governing equations
were taken to be:

\[ U_t + U U_x + \frac{W}{\delta(x)} \cdot U_y + \frac{U^2 - V^2}{x} = \frac{1}{\delta^2(x)} \cdot \{(K + \mu)U_y\}_{\eta} \]  

(9)
\[ V_t + UV_r + \frac{W}{\delta(r)} \cdot V_\eta + \frac{UV}{r} = \frac{1}{\delta^2(r)} \cdot \{(K + \mu)V_\eta\}_\eta \] (10)

\[ \xi_t + U\xi_r + \frac{W}{\delta(r)} \xi_\eta = \frac{1}{\xi \delta^2(r)} \cdot \{(K + \mu)\xi_\eta\}_\eta 
+ \frac{K}{2\xi \delta^2(r)} \cdot (U^2_\eta + V^2_\eta) - \frac{\mu \xi_\eta^2}{2\xi \delta^2(r)} - \frac{C_D \xi^3}{2K} + \frac{C_D^4 \xi^4}{2\xi K^\infty(r)} \] (11)

\[ \zeta_t + U\zeta_r + \frac{W}{\delta(r)} \zeta_\eta = \frac{1}{\zeta \delta^2(r)} \cdot \{(K + \mu)\zeta_\eta\}_\eta 
+ \frac{C_D K \zeta}{2\delta^2(r) \zeta^2} \cdot (U^2_\eta + V^2_\eta) - \frac{C_D^2 \xi^2 \zeta}{2K} + \frac{C_D^2 \xi^2 (r) \zeta^2 \eta}{2\xi K^\infty(r)} \] (12)

\[ U_r + \frac{U}{r} + \frac{W}{\delta(r)} = 0 \] (13)

\[ K = \frac{\xi^4}{\zeta^2} \] (14)

\[ \delta_r^2 = \left[ \frac{r}{\nu^2} \{(K + \mu)U_\eta\}_\eta \right]_{\eta=0} \] (15)

In these equations, \( \xi, \zeta, K \) and \( \delta(r) \) are the square root of the kinetic energy of turbulence, the square root of the dissipation rate of turbulence, the turbulence diffusivity and the boundary layer thickness function, respectively; \( \eta = z/\delta(r) \) and the subscript \( \infty \) designates a value outside of the boundary layer. The equations are in nondimensional form. \( U, V \) and \( \xi, W, \zeta, K \) and \( \mu, \delta(r) \) and \( z, r \) and \( t \) are in terms of \( v_m^*, v_m^* \delta^*/r_m^*, \rho v_m^*, (v_m^* / r_m^*)^2, \rho v_m^* \delta^* / r_m^* \), \( \delta^* \), \( r_m^* \) and \( r_m^* / v_m^* \).
respectively; and \( v^*_{m} \), \( r^*_m \) and \( \delta^*_0 \) are a characteristic velocity, a characteristic length in the radial direction and a characteristic length in the axial direction, respectively. \( C_p \), \( C_1 \) and \( C_2 \) are nondimensional constants and their values were taken to be 0.1, 1.5 and 0.1, respectively.

The equations were solved by a finite difference method developed by Rubin and Lin (1972). The method was applied in the exploratory analysis as follows.

Any variable \( X \) is differenced as:

\[
x_{i,j,k}^{m+1} = \frac{(x_{i,j,k}^m + x_{i,j,k}^m)}{2}
\]

(16)

\[
(x_t)_{i,j,k}^{m+1} = \frac{(x_{i+1,j,k}^{m+1} - x_{i,j,k}^{m+1})}{\Delta t}
\]

(17)

\[
(x_r)_{i,j,k}^{m+1} = \frac{(x_{i+1,j,k}^{m} - x_{i,j,k}^{m} + x_{i,j+1,k}^{m} - x_{i,j,k}^{m} - x_{i-1,j,k}^{m} + x_{i+1,j,k}^{m})}{4\Delta r}
\]

(18)

\[
(x_n)_{i,j,k}^{m+1} = \frac{(x_{i+1,j+1,k}^{m+1} - x_{i,j+1,k}^{m+1} + x_{i,j,k}^{m+1} - x_{i,j-1,k}^{m+1})}{4\Delta n}
\]

(19)

\[
(x_{nn})_{i,j,k}^{m+1} = \frac{(x_{i+1,j+1,k}^{m} - 2x_{i,j,k}^{m} + x_{i,j-1,k}^{m} + x_{i,j+1,k}^{m} - 2x_{i,j,k}^{m}}{2\Delta n^2}
\]

(20)
Here, \( i, j, k \) and \( m \) represent the radial grid point number, the axial grid point number, the time grid point number and the iteration number, respectively; and \( \Delta r \), \( \Delta n \) and \( \Delta t \) are the radial, axial and temporal grid point interval sizes, respectively. When this differencing is introduced into Equations 9-12, linear coupled algebraic equations are obtained and they are solved simultaneously. This gives the solution for \( u \), \( v \), \( \xi \) and \( \zeta \). Then, \( K \) is obtained from 14 and \( w \) and \( \delta(r) \) are obtained by integrating 13 and 15 numerically by the trapezoidal method. At each time point \( k + \frac{1}{2} \), three iterations were used and the zeroth iteration value was obtained by a second order Taylor expansion from the previous two time points.

\( \mu \) was given the value of 0.001 and the following boundary conditions were used.

At \( r = 0 \), \( U(0,n), V(0,n), \xi(0,n), \zeta(0,n) = 0 \)
At \( n = 0 \), \( U(r,0), V(r,0), \xi(r,0), \zeta(r,0), W(r,0) = 0 \)
At \( n = 1 \), \( U(r,1) = 0 \), \( V(r,1) = U_\infty(r) = 1.5(1 - e^{-1.25r^2})/r \)
\( \xi(r,1) = \xi_\infty(r) = 0.01V_\infty(r) \), \( \zeta(r,1) = \zeta_\infty(r) = 0.1V_\infty(r) \)
The equations were integrated for a long time in the region $0 \leq \eta \leq 1$ and $0 \leq r \leq 4$ with $\Delta \eta = 0.05$ and $\Delta r = 0.2$ and the solution that was obtained at the point where the integration was stopped is shown in Figs. 2-6. In the figures, an asterisk designates a dimensional quantity. As it is seen in Fig. 2, the solution that was obtained is a 5-cell vortex. The term cell is being used here to describe an upflow region in between two downflow regions in the part of the flow outside of the boundary layer. The values that were obtained for the radial, tangential and axial velocities (Figs. 3-5) are similar to what have been obtained for one cell and two cell vortices by other investigators such as Kuo (1971) and Chi and Jih (1974). There are no solutions for the turbulence diffusivity by other investigators that the solution shown in Fig. 6 can be compared to. As it is seen in Fig. 6, very large variations of the values of the turbulence diffusivity were obtained.

D. The Problems of the Analysis of the Flow of a Vortex Near a Solid Boundary

In the paper by Rubin and Lin (1972), numerical results are presented which support the authors' claim that the method they have developed is better than many other finite difference methods. In the paper, a linearized consistency and stability analysis is provided. This is the case with most existing finite difference methods, only linearized
Fig. 2. The meridional flow pattern for a five-cell vortex
Fig. 3. The variation of the radial velocity for a five-cell vortex
Fig. 4. The variation of the tangential velocity for a five-cell vortex
Fig. 5. The variation of the axial velocity for a five-cell vortex.
Fig. 6. The variation of the turbulent viscosity for a five-cell vortex
numerical stability analysis is made. When the method by Rubin and Lin was used in the exploratory analysis of a vortex flow near a solid boundary, the linearized stability analysis that they provide was found to be inadequate. In order to keep the method stable, it was necessary to do numerical experiments during the solution of the governing equations. This can be both inefficient and frustrating.

A two equation model of turbulence was chosen in the exploratory analysis because it is half way between the most complicated and the most elementary models of turbulence. In Prandtl's mixing length approach, the turbulence diffusivity is proportional to a characteristic turbulence velocity and a characteristic turbulence length. In a two equation model of turbulence, basically, differential equations are provided for the turbulence characteristic velocity and length. In the model that was used the equations are for the turbulence kinetic energy and the turbulence dissipation rate. The model is relatively simple and easy to derive, but, like all 2 equation models, it has an imperfection. The imperfection of the model is in the values of the constants \( C_1, C_2 \) and \( C_p \). In the first place, there is no experimental evidence that they should be constants and secondly the universality of the constants is questionable. The problem which this imperfection creates is that when the values of the constants are physically incorrect, during the solution
of the governing equations over a long time, the physical error in the solution obtained for turbulence variables can grow. This can then lead to errors in the final solutions obtained for the velocities and pressure, depending in degree on the importance of turbulence in the flow that is being analyzed.

The boundary conditions that were used at \( \eta = 1 \) during the exploratory analysis is given in Equation 22. The conditions that were used are a combination of solutions to simplified forms of the governing equations and estimates. For a physically correct solution to the governing equations, the boundary conditions that are used at \( \eta = 1 \) must be physically correct solutions to the governing equations. Such a solution at \( \eta = 1 \) is hard to obtain because the governing equations are nonlinear and thus difficult to solve. The solutions that have existed up to now are simplified and thus restricted solutions.

So, in the analysis of the vortex flow near a solid boundary, there are three primary problems. They are the problem of numerical instability in the finite difference solution of the governing equations, the problem of finding a good turbulence model and the problem of determining boundary conditions at \( \eta = 1 \) which are solutions to the governing equations. In this thesis solutions are provided to these three problems. In Chapter II, the governing equations are
derived. In Chapter III, general solutions to the governing equations are obtained for use as the boundary conditions at $\eta = 1$. In Chapter IV, numerical methods which do not have any stability problem are developed for the solution of the governing equations. In Chapter V, as an example solution, an unsteady state flow of a vortex near a solid boundary is examined and the results are presented.

Some of the results presented in this thesis are of a much wider applicability than just vortex flows. The turbulence model that is presented in Chapter II is applicable to all turbulent flows. Also, the convergence proof method for some finite difference methods for parabolic equations that is developed in Chapter IV can probably be applied to any finite difference method for parabolic equations.
II. GOVERNING EQUATIONS FOR THE TURBULENT FLOW OF A VORTEX

A. The Equations for the Mean Values of the Flow Variables

From Equations 1, 2, 3 and 4, by decomposing the flow variables into their mean and fluctuating components, the equations for a turbulent flow are obtained as:

\[
\rho \left\{ (U + u')_t + (U + u') (U + u')_r + (W + w') (U + u')_z \\
+ \frac{(V + v')}{r} \cdot (U + u')_\theta - \frac{(V + v')^2}{r} \right\} = -(P + p')_r \\
+ \mu \left\{ (U + u')_{rr} + \frac{1}{r}(U + u')_r + \frac{1}{r^2}(U + u')_{\theta\theta} + (U + u')_{zz} \\
- \frac{(U + u')}{r^2} - \frac{2}{r^2} (V + v')_\theta \right\} 
\]

\[
(23)
\]

\[
\rho \left\{ (V + v')_t + (U + u') (V + v')_r + (W + w') (V + v')_z \\
+ \frac{(V + v')}{r} (V + v')_\theta + \frac{1}{r}(U + u') (V + v') = - \frac{1}{r} (P + p')_\theta \\
+ \mu \left\{ (V + v')_{rr} + \frac{1}{r}(V + v')_r + \frac{1}{r^2}(V + v')_{\theta\theta} + (V + v')_{zz} \\
+ \frac{2}{r^2}(U + u')_\theta - \frac{(V + v')}{r^2} \right\} 
\]

\[
(24)
\]
\[ \rho \{ (W + w')_t + (U + u')(W + w')_x + (W + w')(W + w')_z \} \]
\[ = -(P + p')_z + \mu \{(W + w')_{xx} + \frac{1}{r}(W + w')_r \}
+ \frac{1}{r^2}(W + w') \theta \theta + (W + w')_{zz} \]  \tag{25}
\[ (U + u')_r + \frac{1}{r}(U + u')_x + (W + w')_z = 0 \]  \tag{26}

In these equations \( U, V, W \) and \( P \) are the mean values of \( u, v, w \) and \( p - p_a \); and \( u', v', w' \) and \( p' \) are the fluctuation values of \( u, v, w \) and \( p - p_a \). Here, \( p_a \) is the ambient pressure at large \( r \).

By taking the time average of Equations 23-26, the equations for the mean values of the flow variables are obtained as:

\[ \rho \{(U_t + UU_r + WU_z + \frac{V}{r} \cdot U_\theta - \frac{V^2}{r}) = -P_r \]
\[ + \mu \{U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_\theta \theta + U_{zz} - \frac{2}{r^2} V_\theta - \frac{U}{r^2} \}
- \rho \{(u'u')_r + (u'w')_z + \frac{u'u'}{r} + \frac{1}{r}(u'v')_\theta - \frac{v'v'}{r} \} \]  \tag{27}
\[
\rho \{ V_t + UV_r + WV_z + \frac{V}{r} V_\theta + \frac{UV}{r} \} = - \frac{1}{r} p_\theta \\
+ \mu \{ V_{rr} + \frac{1}{r} V_r + \frac{1}{r^2} V_\theta \theta + V_{zz} + \frac{2}{r^2} U_\theta - \frac{V}{r^2} \} \\
- \rho \{ (u'v')_r + \frac{1}{r} (v'v')_\theta + (v'w')_z + \frac{2}{r} u'v' \} \\
(28)
\]

\[
\rho \{ W_t + UW_r + \frac{1}{r} VW_\theta + WW_z \} = - p_z \\
+ \mu \{ W_{rr} + \frac{1}{r} W_r + \frac{1}{r^2} W_\theta \theta + W_{zz} \} \\
- \rho \{ (u'w')_r + \frac{1}{r} (v'w') + (w'w')_z + \frac{u'w'}{r} \} \\
(29)
\]

\[
U_r + \frac{U}{r} + \frac{1}{r} V_\theta + W_z = 0 \\
(30)
\]

In these four equations, there are ten unknowns, \( U, V, W, P \), and the Reynolds' stress terms, \( u'u', u'v', u'w', v'v', v'w', w'w' \). So, in order to have a solvable problem, the Reynolds' stress terms have to be modeled or solved for in some way. As it was mentioned in Chapter I, there are several ways of handling this problem that are being practiced and studied. Basically, the methods of handling this problem are of two types. In the first type, the stress terms are expressed in terms of a single turbulence diffusivity function and appropriate strain rates. In the second type the stress terms are treated separately.
So far, the approach in the methods of the second type has been to derive equations for the Reynolds' stress terms from Equations 23-26. The new unknowns that arise in the derived equations are then solved for from experimental data or from further equations that are derived. But, at some point, experimental data or observation has to be used in order to close the problem. In the approaches of methods of the second type that exist, the number of equations that have to be solved can be very large. For example in an approach suggested by Davidov (1960), the number of equations is 23.

In the methods of the first type the additional equations due to turbulence modeling is not more than two. In the methods of the first type, the complexity of the turbulence models vary from the constant turbulence diffusivity model to the two equation models of turbulence. From the Prandtl's mixing length approach to turbulence point of view, conceptually, the two equation models of turbulence are the most complete models. But, as it was indicated in Chapter I, for universal applicability, the oversimplified assumptions and estimations that exist in the models have to be corrected.

In this thesis, the closure problem of Equations 27-30 is going to be solved by a new model of turbulence. The turbulence model is of the same degree of conceptual completeness as two equation models of turbulence, but it is simpler and more universal.
B. Modeling of the Turbulent Stress Terms

To begin with, the Reynolds' stress terms of Equations 27-30 are expressed in terms of a single turbulence diffusivity function and strain rates as:

\begin{align*}
\overline{u'u'} &= -2Ku_r, \quad \overline{u'v'} = -K\left\{ \frac{1}{r} U_\theta + V_r - \frac{V}{r} \right\}, \\
\overline{u'w'} &= -K(U_z + W_r), \quad \overline{v'v'} = -2K\left\{ \frac{1}{r} V_\theta + \frac{U}{r} \right\} \\
\overline{v'w'} &= -K\left\{ \frac{1}{r} W_\theta + V_z \right\}, \quad \overline{w'w'} = -2K W_z.
\end{align*}

Here, the diffusivity function K should be regarded as the average of six diffusivity functions. With the relations of Equation 31, the ten unknowns of Equations 27-30 are reduced to five unknowns, U, V, W, P and K. In the following, a way of determining K with one differential equation and three empirical relations is developed.

By multiplying Equations 23, 24 and 25 by \( u' \), \( v' \) and \( w' \) respectively, adding the resulting three equations, using Equations 26 and 30 to simplify some terms and taking the time average of the resulting equation, the equation for the kinetic energy of turbulence is obtained as:

\[ \text{See also discussion on the modeling in the appendix.} \]

\[ \text{Usually, } (2/3)q \text{ is added to the present modeling of the normal stresses, i.e. } \overline{u'u'} = (2/3)q - 2Ku_r \text{ etc.} \]
\[
\rho \{ q_r + u q_r + \frac{1}{r} v q_{\theta} + w q_z \} = - \rho \Lambda - \rho D_T
\]

\[
+ \mu \{ q_{rr} + \frac{1}{r} q_r + \frac{1}{r^2} q_{\theta\theta} + q_{zz} \} - \varepsilon
\]

where,

\[
\Lambda = \overline{u' u'} u_r + \overline{u' v'} \left\{ \frac{1}{r} u_{\theta} + v_r - \frac{v}{r} \right\} + \overline{u' w'} (u_z + w_r)
\]

\[
+ \overline{v' v'} \left\{ \frac{1}{r} v_{\theta} + u_r \right\} + \overline{v' w'} \left\{ \frac{1}{r} w_{\theta} + v_z \right\} + \overline{w' w'} w_z',
\]

\[
D_T = (\overline{u' q'} + \overline{u' p'})_r + \frac{1}{r} (\overline{u' q'} + \overline{u' p'})
\]

\[
+ \frac{1}{r} (\overline{v' q'} + \overline{v' p'})_\theta + (\overline{w' q'} + \overline{w' p'})_z,
\]

\[
\varepsilon = \mu \{ \overline{u' v'} \cdot \overline{u' v'} + \overline{v' v'} \cdot \overline{v' v'} + \overline{w' w'} \cdot \overline{w' w'}
\]

\[
+ \frac{u'^2}{r} + \frac{v'^2}{r} - \frac{2}{r^2} v' u'_{\theta} + \frac{2}{r^2} u' v'_{\theta} \},
\]

\[
q' = \frac{1}{2} (u'^2 + v'^2 + w'^2)
\]

and \( q = |q'| \)

In Equation 32, \( |q'| \) is the kinetic energy of turbulence, \( \rho \Lambda \) is the rate of production of turbulence energy, \( \rho D_T \) is the rate of diffusion of turbulence energy and \( \varepsilon \) is the rate of
viscous dissipation of turbulence energy.

The equation is general and applies to all turbulent flows. For several turbulent flows other than vortex flows, the turbulence terms in the equation have been examined by some investigators and so experimental data on the terms exist. In the turbulence model of this thesis these data are used to solve the turbulence closure problem. For experimental data, those gathered by Townsend (1949 and 1951), Laufer (1954) and Klebanoff (1955) are used. These data sets are chosen because they are for three different types of turbulent flows, turbulent flow in the wake of a cylinder, turbulent pipe flow and turbulent boundary layer flow. So, if a relation between turbulence terms satisfies all these different data sets, it has a very good likelihood of being widely applicable.

Examining the data of Townsend (1949), Laufer (1954) and Klebanoff (1955), a relationship is observed between the ratio of the production and dissipation rates of turbulence and the nondimensional quantity

\[ L = |\nabla \alpha| \frac{q^{3/2}}{\epsilon} \]  \hspace{1cm} (38)

where \( \alpha = 100(q/Q) \) and \( Q = \frac{1}{2}(U^2 + V^2 + W^2) \) \hspace{1cm} (39)

The relationship is shown in Fig. 7. As it is shown in the figure, the relationship can be approximated well by:

\[ \frac{\Lambda}{\epsilon} = 0.9 L^{0.33} \]  \hspace{1cm} (40)
Fig. 7. An empirical relation for the ratio of the production and dissipation rates of turbulence energy.
From Equations 31 and Equation 33 one obtains the relation:

\[ \Lambda = K \phi \]  \hspace{2cm} (41)

where,

\[ \phi = \left[ 2U_r^2 + \left( \frac{1}{r} U_\theta + V_r - \frac{V}{r} \right)^2 + (U_z + W_r)^2 \right. \]

\[ + 2 \left( \frac{1}{r} V_\theta + U \right)^2 + \left( \frac{1}{r} W_\theta + V_z \right)^2 + 2W_z^2 \]  \hspace{2cm} (42)

So, from this, a relation for \( K \) is obtained as:

\[ K = 0.9 \epsilon L^{0.33}/\phi \]  \hspace{2cm} (43)

Examining the data sets due to Laufer (1954) and Townsend (1949), a simple relation is observed between the ratio of turbulence energy diffusion and \( Q^{3/2} \) and the nondimensional quantity:

\[ M = a_x \cdot q^{3/2}/\epsilon \]  \hspace{2cm} (44)

Here an \( x \) subscript designates a differentiation in the direction of the fluctuating velocity component which is causing the diffusion. The relation is shown in Fig. 8, and it is an approximately linear relation. From this, for the diffusion terms of Equation 34, one obtains the empirical relations:
Fig. 8. An empirical relation for the diffusion of turbulence energy
\[ D_u = u'q' + u'p' = -0.171 \times 10^{-4} \alpha_r q^{3/2} \cdot Q^{3/2}/\varepsilon \]

\[ D_v = v'q' + v'p' = -0.171 \times 10^{-4} \frac{1}{r} \alpha_\theta q^{3/2} \cdot Q^{3/2}/\varepsilon \quad (45) \]

\[ D_w = w'q' + w'p' = -0.171 \times 10^{-4} \alpha_z q^{3/2} \cdot Q^{3/2}/\varepsilon \]

The relations of Equations 43 and 45 and Equation 32 contain \( \varepsilon \); so, a determination of \( \varepsilon \) is necessary.

By taking the dot product of the gradients of Equations 23, 24 and 25 and 2\( \mu \nu u' \), 2\( \mu \nu v' \) and 2\( \mu \nu w' \), respectively; by multiplying Equation 23 by 2\( \mu u'/r \) and 2\( \mu v'/r^2 \) and multiplying Equation 24 by 2\( \mu v'/r \) and -2\( \mu u'/r^2 \); by adding the resulting seven equations and taking the time average of the final equation, a differential equation for \( \varepsilon \) can be derived. In the equation, correlations of first derivatives and second derivatives of the fluctuation velocities appear. To have a closed mathematical problem, these correlation terms have to be modeled using experimental data. However, at this point in time, there is no adequate data on the correlations of first and second derivatives of fluctuating velocities for one to be able to do this. So, in this thesis, instead of a differential equation for \( \varepsilon \), an empirical relation for \( \varepsilon \) will be used.

Examining the experimental data of Townsend (1949), Laufer (1954) and Klebanof (1955), a relation common to the
three sets of data is observed between $\alpha$ and the nondimensional quantity:

$$N = \frac{\varepsilon}{\mu|V^2_Q| + q^{1/2}|VQ| \cdot (q/Q)}$$

(46)

The relation is shown in Fig. 9 and it can be expressed to a good degree of approximation by:

$$2N = S(\alpha) = \frac{0.69}{\alpha} (1 - e^{-43.5\alpha^2}) + \frac{0.38}{\alpha} e^{-0.6/\alpha}$$

(47)

From Equations 46 and 47 a relation for $\varepsilon$ is obtained as:

$$\varepsilon = \frac{1}{2} S(\alpha) \{\mu|V^2_Q| + q^{1/2}|VQ| \cdot (q/Q)\}$$

(48)

With this the turbulence closure problem is solved and the governing equations for the turbulent flow of a vortex can be written as:

$$\rho\left\{U_t + UU_r + \frac{V}{r} U_\theta + WU_z - \frac{V^2}{r}\right\} = -P_r$$

$$+ 2\{(K + \mu)U_r\}_r + \frac{1}{r}\{(K + \mu)(\frac{1}{r} U_\theta + V_r - \frac{V}{r})\}_\theta$$

$$+ \{(K + \mu)(U_z + W_r)\}_z + \frac{2}{r} (K + \mu)\{U_r - \frac{1}{r} V_\theta - \frac{U}{r}\}$$

(49)

---

1 The turbulence model, therefore, consists of a differential equation for $q$ and generalized expressions for $Du$, $\varepsilon$ and $\lambda/\varepsilon$ from which $K$ is determined.
Fig. 9. An empirical relation for the dissipation rate of turbulence energy
\[ \rho \{ v_t + uv_r + \frac{V}{r} v_\theta + wv_z + \frac{uv_z}{r} \} = - \frac{1}{r} p_\theta \]
\[ + \frac{2}{r} \cdot \{(K + \mu) \cdot v_\theta \}_\theta + \{(K + \mu) (\frac{1}{r} w_\theta + w_z) \}_z \]
\[ + \{(K + \mu) (\frac{1}{r} u_\theta + v_r - \frac{V}{r}) \}_r + \frac{2}{r} (K + \mu) (\frac{1}{r} u_\theta + v_r - \frac{V}{r}) \]
\[ (50) \]
\[ \rho \{ w_t + uw_r + \frac{V}{r} w_\theta + w w_z \} = - p_z + 2 \{(K + \mu) w_z \}_z \]
\[ + \frac{1}{r} \{(K + \mu) r (u_z + w_z) \}_r + \frac{1}{r} \{(K + \mu) (\frac{1}{r} w_\theta + v_z) \}_\theta \]
\[ (51) \]
\[ \rho \{ q_t + u q_r + \frac{V}{r} q_\theta + w q_z \} = \mu \{ q_{rr} + \frac{1}{r} q_r + \frac{1}{r^2} q_{\theta \theta} + q_{zz} \} \]
\[ + (Du)_r + \frac{1}{r} Du + \frac{1}{r} (Dv)_\theta + (Dw)_z + (0.9 L^{0.33} - 1) \epsilon \]
\[ (52) \]
\[ U_r + \frac{U}{r} + \frac{1}{r} V_\theta + W_z = 0 \]
\[ (30) \]

where \( K, \epsilon, Du, Dv, Dw \) and \( L \) are given by Equations 43, 48, 45 and 38, respectively.

C. The Governing Equations for the Axisymmetric Turbulent Flow of a Vortex

When a turbulent vortex flow is axisymmetric or when a solution that is averaged in the \( \theta \) coordinate is sought, Equations 49-52 and Equation 30 reduce to a simpler set of
equations. In nondimensional form, the reduced equations can be written as:

\[ U_t + UU_r + WU_z - \frac{V^2}{r} = - P_r + 2 \{(K + \mu)U_r\}_r \]

\[ + \{(K + \mu)(U_z + W_r)\}_z + \frac{2}{r}(K + \mu)(U_r - \frac{U}{r}) \]  \hspace{1cm} (53)

\[ V_t + UV_r + WV_z + \frac{UV}{r} = \{(K + \mu)V_z\}_z \]

\[ + \{(K + \mu)(V_r - \frac{V}{r})\}_r + \frac{2}{r}(K + \mu)(V_r - \frac{V}{r}) \]  \hspace{1cm} (54)

\[ W_t + UW_r + WW_z = - P_z + 2 \{(K + \mu)W_z\}_z \]

\[ + \frac{1}{r}\{r(K + \mu)(U_z + W_r)\}_r \]  \hspace{1cm} (55)

\[ U_r + U + W_z = 0 \]  \hspace{1cm} (56)

\[ q_t + Uq_r + Wq_z = \mu\{q_{rr} + \frac{1}{r}q_r + q_{zz}\} \]

\[ + (Du)_r + \frac{1}{r}Du + (Dw)_z + (0.9 L^{0.33} - 1)e \]  \hspace{1cm} (57)

Here, the velocities are in units of a reference velocity \( v^*_m \), \( r \) and \( z \) are in units of a reference distance \( r^*_m \), \( q \) is in units of \( v^*_m^2 \), \( P \) is in units of \( \rho v^*_m^2 \), \( \mu \) and \( K \) are in units of \( \rho v^*_m^2 r^*_m \) and \( e \) is in units of \( v^*_m^3/r^*_m \).
Approximations to Equations 53-57 or "boundary layer equations" can be obtained by neglecting the radial diffusion of flow quantities and the axial variation of the pressure. The equations are:

\[ U_t + UU_r + WU_z - \frac{V^2}{r} = -p_r + \{(K + \mu)U_z\}_z \]  

(58)

\[ V_t + UV_r + WV_z + \frac{UV}{r} = \{(K + \mu)V_z\}_z \]  

(59)

\[ p_z = 0 \]  

(60)

\[ U_r + \frac{U}{r} + W_z = 0 \]  

(61)

\[ q_t + Uq_r + Wq_z = \mu q_{zz} + (Dw)_z + (0.9 L^{0.33} - 1)\varepsilon \]  

(62)

In Equations 53-57, there are five unknowns and so there are enough equations for the five unknowns. However, during the solution of the equations, rather than solving the equations as they are, better results are obtained when a Poisson equation for the pressure is used in place of either one of Equations 53 and 55. This equation for the pressure is obtained by differentiating Equations 53 and 55 with respect to \( r \) and \( z \), respectively; multiplying Equation 53 by \( 1/r \) and adding the resulting three equations. The equation can be written as:
When a steady state solution of Equations 53-57 is being sought it is better if one solves an equation for the tangential vorticity and an equation for the stream function rather than Equations 53, 55 and 56. The reason for choosing the tangential vorticity and stream function equations is that they enable one to use all the boundary conditions that need to be specified to get a steady state solution while at the same time encompassing all of Equations 53, 55 and 56. The tangential vorticity equation is obtained by differentiating Equations 53 and 55 with respect to \( z \) and \( r \), respectively, and subtracting one of the resulting equations from the other. The tangential vorticity equation can be written as:

\[
\begin{align*}
\frac{1}{r^2} p_{rr} + \frac{1}{r} p_r + p_{zz} &= -\left( \frac{u^2}{r^2} + \frac{v^2}{r} - \frac{2v}{r} v_r + 2u_z w_r + w_z^2 \right) \\
+ K_r \left\{ u_{rr} + \frac{1}{r} u_r + u_{zz} + u_{zr} \right\} + 2K_z \left\{ w_{rr} + \frac{1}{r} w_r + w_{zz} \right\} \\
+ 2K_{rz} (u_z + w_z) + K_{rr} u_r + 2K_{zz} w_z
\end{align*}
\]

(63)
where, the tangential vorticity, $\Omega$ is defined by

$$\Omega = U_z - W_r$$  \hspace{1cm} (65)

The stream function, $\psi$ is defined by

$$U = -\frac{\psi_z}{r} \quad \text{and} \quad W = \frac{\psi_r}{r}$$  \hspace{1cm} (66)

$\psi$ satisfies the continuity equation and inserting the relations of Equation 66 into Equation 65, an equation for $\psi$ is obtained as:

$$\nabla^2 \psi = -\Omega$$  \hspace{1cm} (67)

With this, the development of the equations, that govern a turbulent vortex flow is completed. Solutions to appropriate combinations of the equations of this chapter with boundary conditions appropriate for a vortex flow describe the turbulent flow of a vortex. The combination of equations that are chosen depends on the kind of solution that is being sought.
III. ANALYTICAL AND SEMI-ANALYTICAL SOLUTION OF THE
GOVERNING EQUATIONS OF A VORTEX FLOW

A. Introduction

As it was mentioned in Chapter I, a practical and useful solution of the equations of Chapter II has to be carried out by a numerical method. But even for numerical methods, in order to obtain a unique solution, boundary conditions have to be specified. During the solution of the governing equations for a vortex flow near a surface normal to the axis of the vortex, the outer boundaries of the flow field region that is examined are usually part of the vortex flow field. So, the boundary conditions that are specified at these boundaries must be solutions to the governing equations. For this reason, prior to solving the governing equations in the flow field region that is being examined, the governing equations must be examined and if necessary solved at the outer boundaries.

In this chapter, the governing equations will be solved along a radial line. The solutions that are obtained will be used as the boundary conditions at the axial direction's outer boundary. In section B some simple and restricted analytical solutions are obtained and discussed and in section C a method of obtaining general solutions is developed.
B. Some Simple Solutions to the Governing Equations of a Vortex Flow

For an axisymmetric vortex flow, easily solvable equations are obtained from Equations 53-57 when \( q \) and all the first and second derivatives of the flow variables in the axial direction except \( W_z \) and the derivatives of \( P \) are assumed to be zero. These equations are:

\[
U_t + U U_r - \frac{V^2}{r} = - \frac{P}{r} + 2\mu \{ \frac{1}{r} U_{rr} + \frac{1}{r^2} U_r - \frac{U}{r^2} \} \tag{68}
\]

\[
V_t + U V_r + \frac{U V}{r} = \mu \{ \frac{1}{r} V_{rr} + \frac{1}{r^2} V_r - \frac{V}{r^2} \} \tag{69}
\]

\[
W_t + U W_r + W W_z = - \frac{P}{r} + \mu \{ \frac{1}{r} W_{rr} + \frac{1}{r^2} W_r \} \tag{70}
\]

\[
U_r + \frac{U}{r} + W_z = 0 \tag{71}
\]

Equation 71 can be rewritten as:

\[
\frac{1}{r} (U_r)_r = - W_z (r) \tag{72}
\]

Solving this equation for \( U \) gives:

\[
U(r) = \frac{A_1}{r} - \frac{1}{r} \int_0^r W_z (r) dr \tag{73}
\]

where \( A_1 \) is an integration constant.
For a steady state flow Equation 69 can be rewritten as:

\[
\left( V_r + \frac{V}{r} \right)_r - \frac{U}{\mu} \left( V_r + \frac{V}{r} \right) = 0
\]  
(74)

Integrating this equation gives:

\[
\frac{1}{r} (V_r)_r = A_3 \ e^{b(r)}
\]  
(75)

where, \( A_3 \) is a constant and \( b(r) = \int_0^r \frac{U}{\mu} \ dr \)  
(76)

Integrating Equation 76 \( V(r) \) is obtained as:

\[
V(r) = \frac{A_4}{r} + \frac{A_3}{r} \int_0^r e^{b(r)} r \cdot dr
\]  
(77)

From Equation 68, for a steady state flow the pressure is obtained as:

\[
P(r) = \int_0^r \left( 2\mu \left( U_{rr} + \frac{1}{r} U_r - \frac{U}{r^2} \right) - UU_r + \frac{V^2}{r} \right) dr
\]  
(78)

With a steady state flow assumption Equation 70 becomes:

\[
W_{rr} + \left( \frac{1}{r} - \frac{U}{\mu} \right) W_r - \frac{1}{\mu} W_z \cdot W = \frac{1}{\mu} p_z(r)
\]  
(79)

This is a linear, second order ordinary differential equation for \( W(r) \). Its solution depends on the functions
$W_z(r)$ and $P_z(r,z)$. But there is no distinct general form in which the solution of Equation 79 can be written. So, the solution of Equations 68-71 starts by assuming a function for $W_z(r)$. Then $U(r)$ is determined from Equation 73 and $V(r)$ is determined from Equation 77. If a determination of $W(r)$ is needed, first, a function is chosen for $P_z(r,z)$ and then Equation 79 is solved for $W(r)$ either analytically or by a numerical method.

As an example a simple solution can be obtained by assuming

$$W_z(r) = \text{constant} = a \quad (80)$$

Then, Equation 73 gives:

$$U(r) = -ar \quad (81)$$

Equation 76 gives:

$$b(r) = -\frac{a}{2\mu} r^2 \quad (82)$$

and Equation 77 gives:

$$V(r) = \frac{A_6}{r} (1 - e^{-b_1 r^2}) \quad (83)$$

where $A_6$ is a constant and $b_1 = a/2\mu$.

With this, Equation 79 becomes:

$$W_{rr} + \left(\frac{1}{r} - 2b_1 r\right)W_r - 2b_1 W = \frac{1}{\mu} P_z(r) \quad (84).$$
and it can have many solutions, depending on the function that is chosen for \( P_z(r,z) \).

From Equation 83, it is seen that the rotational velocity, \( V(r) \), in this simple solution, is increasing with increasing \( r \) for small \( r \) and decreasing with increasing \( r \) for large \( r \). Therefore the solution is a solution for a general vortex. The functions for the flow variables can be written together as:

\[
U(r) = -ar
\]

\[
V(r) = \frac{A_0}{r} (1 - e^{-br^2})
\]

\( W(r) \) is arbitrary

\[
P(r) = \int_0^r \left\{ 2\mu (U_{rr} + \frac{1}{r} U_r - \frac{U}{r^2}) - UU_r + \frac{V^2}{r} \right\} dr
\]

\[
q(r) = 0
\]

By choosing \( W_z(r) \) differently, many other solutions can be obtained. As it can be seen from Equation 77, a solution for a general vortex can be obtained only when \( W_z(r) \) is chosen such that \( U(r) \) will be linear near \( r = 0 \), and a magnitude wise monotonously increasing function for large \( r \).

Equations 68-71 have been examined and solved by other investigators. For example, a detailed examination and
several solutions are given by Bellamy-Knights (1970 and 1971). But all the solutions along a radial line to the governing equations of Chapter II that have been obtained up to the present time are solutions to Equations 68-71. There are no solutions for more general equations that have been obtained by other investigators.

The solutions of Equations 68-71 are of restricted applicability. In many vortex flows, for most of the vortex flow field, the axial derivatives of the flow variables are nonzero. So, a physically more realistic solution would be a solution to a set of equations which include the axial derivatives of the flow variables. In the following section a method for obtaining such a solution is developed.

C. Generalized Solutions Along a Radial Line to the Governing Equations of a Vortex Flow

The simple solutions of section B are obtained by making assumptions on the axial derivatives of the flow variables and solving the governing equations with these assumptions included. Then, from the many possible solutions the ones that appertain to a vortex flow are chosen. The assumptions on the axial derivatives of the flow variables can not be made arbitrarily. The variations of the axial derivatives of the flow variables are governed by equations that are obtained by differentiating the governing equations with respect to z.
The assumptions that were made to obtain Equations 68-71 satisfy the equations for the axial derivatives of the flow variables. But, other solutions for the axial derivatives of the flow variables are not easily discernable. Moreover, the equations that govern the axial derivatives of the flow variables are nonlinear, infinite in number and coupled with each other and the equations for the flow variables. So, when the axial derivatives of the flow variables are nonzero, the approach that is used to obtain the simple solutions of section B is impracticable.

In this thesis, a practical and unrestricted method of solving the governing equations of Chapter II along a radial line is suggested. The basic idea behind the method is that the solution of the governing equations along a radial line can be accomplished by starting with a specification of the flow variables and some of their axial derivatives in a general form. Then, what the rest of the axial derivatives of the flow variables should be so that the specified solutions would satisfy the governing equations can be determined from the equations for the flow variables and their axial derivatives. In this way, the problem of solving an infinite number of nonlinear and coupled ordinary differential equations is changed to the problem of solving an infinite number of algebraic equations in a series. Of course, for any practical problem it is not necessary to determine all of the
axial derivatives of the flow variables and so it is not necessary to solve an infinite number of algebraic equations. It is only a small number of the axial derivatives of the flow variables that are determined and then the solutions for the axial derivatives are used to determine whether the whole solution to the governing equations is physically plausible or not. The whole solution process is outlined by the following algorithm.

**Algorithm 1**

1. Choose the axial derivatives that are determined in the solution process.
2. Decide on which of these axial derivatives should be specified and which of them are evaluated. This should be done in such a way so that the algebraic equations that have to be solved can be solved in a sequence.
3. Specify the functions for the flow variables and the axial derivatives that have to be specified.
4. Determine the axial derivatives that have to be determined.
5. Evaluate the solution to the governing equations that is obtained.
6. If the solution that is obtained is satisfactory terminate the solution process. If the solution that is obtained is not satisfactory adjust the specified functions
and go through steps 4 and 5 repeatedly until a satisfactory solution is obtained.

This algorithm is of general applicability. It can be applied to any kind of vortex flow, laminar or turbulent, axisymmetric or nonaxisymmetric and with or without temperature and compressibility effects. As an example of its application, in the following, a way of solving the governing equations of a steady-state, incompressible, laminar and axisymmetric vortex flow along a radial line is developed.

To begin with the first, second and third axial derivatives of the flow variables will be determined during the solution process. The equations that are used to determine the axial derivatives of the flow variables are Equations 53-56 and equations that are obtained by differentiating Equations 53-56 with respect to z the necessary number of times. The equations can be written in a convenient form as:

\[
\mu U_{zzz}(r) = U_z U_r + UU_z r + W_z U_z - WU_{zz} - 2V V_z / r \\
+ \frac{p}{r} - \mu \{U_{zrr} + \frac{1}{r} U_r z - \frac{1}{r^2} U_z\} 
\]

\[
\mu V_{zz}(r) = UV_r + WV_z + \frac{UV}{r} - \mu \{V_{rr} + \frac{1}{r} V_r - \frac{V}{r^2}\} 
\]
\[ V_{zzz}(r) = U_z V_r + U V_{zr} + W_z V_z + W V_{zz} + \frac{V}{r} U_z \]

\[ + \frac{U}{r} V_z - \mu \{ V_{zrr} + \frac{1}{r} V_{zr} - \frac{1}{r^2} V_z \} \]  

\[ (88) \]

\[ W_z(r) = - U_r - \frac{U}{r} \]  

\[ (89) \]

\[ W_{zz}(r) = - U_{zr} - \frac{1}{r} U_z \]  

\[ (90) \]

\[ W_{zzz}(r) = - U_{zzr} - \frac{1}{r} U_{zz} \]  

\[ (91) \]

\[ P(r) = \int \left\{ - U U_r - W U_z + \frac{V^2}{r} + \mu \left( U_{rr} + \frac{1}{r} U_r - \frac{U}{r^2} + U_{zz} \right) \right\} dr \]  

\[ (92) \]

\[ P_z(r) = - U W_r - W W_z + \mu (W_{rr} + \frac{1}{r} W_r + W_{zz}) \]  

\[ (93) \]

\[ P_{zz}(r) = - U_z W_r - U W_{zr} - W W_{zz} - W_z^2 \]

\[ + \mu \{ W_{zrr} + \frac{1}{r} W_{zr} - U_{zrr} - \frac{1}{r} U_{zr} \} \]  

\[ (94) \]

\[ P_{zzz}(r) = - 2 U_z W_{zr} - U_{zz} W_r - U W_{zzr} - 3 W_z W_{zz} \]

\[ - W W_{zzz} + \mu \{ W_{zzrr} + \frac{1}{r} W_{zzr} - U_{zzrr} - \frac{1}{r} U_{zzr} \} \]  

\[ (95) \]
As it can be seen from Equations 86-95, in order to solve the equations, the functions $U(r)$, $V(r)$, $W(r)$, $U_z(r)$, $U_{zz}(r)$ and $V_z(r)$ have to be specified.

For a vortex flow, the radial velocity is zero at the axis of the vortex flow. Near the axis of the vortex the radial velocity is a linear function of $r$ and at large distances from the axis the radial velocity decreases with increasing $r$. A function that behaves in this way is the function $A(1 - e^{-br^n})/r^{n-1}$, where, $A$, $b$ and $n$ are constants and $n > 1$. So, the radial velocity, $U(r)$ can be described by:

$$U(r) = \sum_{i=1}^{I_1} A_{1i} (1 - e^{-b_{1i}r^{n_{1i}}})/r^{n_{1i}}$$

where, $I_1$, $A_{1i}$, $b_{1i}$ and $n_{1i}$ are constants and $n_{1i} > 0$.

In the same way as for $U(r)$, functions can be specified for $U_z(r)$, $U_{zz}(r)$, $V(r)$ and $V_z(r)$ as:

$$U_z(r) = \sum_{i=1}^{I_2} A_{2i} (1 - e^{-b_{2i}r^{n_{2i}}})/r^{n_{2i}}$$

$$U_{zz}(r) = \sum_{i=1}^{I_3} A_{3i} (1 - e^{-b_{3i}r^{n_{3i}}})/r^{n_{3i}}$$

$$V(r) = \sum_{i=1}^{I_4} A_{4i} (1 - e^{-b_{4i}r^{n_{4i}}})/r^{n_{4i}}$$
The I's, A's, b's and n's in these equations are constants and all the n's are greater than zero.

At the axis of a vortex, the axial velocity has some arbitrary value and \( W_r \) is zero. At large distances from the axis of a vortex the axial velocity decreases with increasing \( r \). So, the axial velocity \( W(r) \) can be described by:

\[
W(r) = \sum_{i=1}^{I_6} A_{6i} (1 - e^{-a_i(r)} - b_i r^{m_2}) \frac{r^{n_{6i}}}{r^{n_{6i}}}
\]

(101)

where,

\[
 a_i(r) = b_{6i} (r^{n_{6i} + m_1} + r^{n_{6i}})
\]

(102)

Here, \( I_6 \), the \( A_6 \)'s, \( b_6 \)'s, \( b_7 \)'s, \( n_6 \)'s, \( m_1 \) and \( m_2 \) are constants and \( m_1 \geq 2, m_2 \geq 2 \).

So, the solution of the governing equation is started by specifying the constants in Equations 96-102. This gives \( U(r), U_x(r), U_{xx}(r), V(r), V_x(r) \) and \( W(r) \). Then \( W_z(r), W_{zz}(r), V_{zz}(r) \) and \( P(r) \) are determined with Equations 89, 90, 91, 87 and 92, respectively. After this, \( V_{zzz}(r), P_z(r) \) and \( P_{zz}(r) \) are determined with Equations 88, 93 and 94. Then, \( U_{zzz}(r) \) is determined with Equation 86 and finally \( P_{zzz}(r) \) is determined with Equation 95.
To evaluate the solution that is obtained, a knowledge, judgement or observation of the manner in which the variables are changing axially is used. The way this is done is as follows:

For two axial positions \( z \) and \( (z + \Delta z) \), the values of the flow variables at the two axial positions are related by:

\[
X(r, z + \Delta z) = X(r, z) + X_z(r, z) \Delta z + X_{zz}(r, z) \frac{(\Delta z)^2}{2} + X_{zzz}(r, z) \frac{(\Delta z)^3}{6} + \ldots
\]

(103)

where, \( X \) is any of the flow variables.

When \( X(r, z), X_z(r, z), X_{zz}(r, z) \) and \( X_{zzz}(r, z) \) are determined the way the expansion of Equation 103 converges for a reasonable size of \( \Delta z \) is obtained. If the manner of convergence of the expansion of Equation 103 agrees with the known or desired axial variation of the flow variables then the solution is good. If there is no agreement then the solution is deemed to be bad.

When an unsatisfactory solution is obtained, the solution process is repeated with improved specifications for the constants of Equations 96-102. This is done until a satisfactory solution is obtained. Observing Equations 86-95 it can be seen that if \( U(r), U_z(r), U_{zz}(r) \) and \( V_z(r) \) are specified to be of the same of magnitude as \( \mu \), good solutions
can be obtained in as few as one or two solution trials. The relatively most difficult part of the solution process is the determination of the constants in Equations 96-102. Normally, during the determination of the constants it is necessary to solve some nonlinear algebraic equations numerically.

The solution process for axisymmetric vortex flows that has just been described is quite general and it can be applied to any laminar and axisymmetric vortex flow. It can also be extended to turbulent, nonaxisymmetric or unsteady state vortex flows without too much difficulty.

So, with the method for solving the governing equations of a vortex flow along a radial line as described by Algorithm 1 the problem of determining the axial direction's outer boundary condition for a vortex flow near a solid surface is solved.
IV. FINITE DIFFERENCE METHODS FOR THE SOLUTION
OF THE GOVERNING EQUATIONS OF A VORTEX FLOW

A. Introduction

As it was stated in Chapter I, in most finite difference methods for the solution of the governing equations of Chapter II that have been in practice up to the present time, the numerical stability analyses of the methods are linearized analyses. Very often, the linearized stability analyses are found to be inadequate, and to make the finite difference methods workable one has to resort to some amount of numerical experimentation. This can be wasteful and unreliable. So, finite difference methods with complete stability analyses are very desirable.

For two dimensional problems, with one time and one space coordinates, some methods with complete numerical convergence analyses have been developed by Lees (1959) and Douglas and Jones (1963). Along similar lines as those followed by Lees and Douglas and Jones, in this chapter, a family of finite difference methods with a complete convergence analysis is developed for the solution of the governing equations of Chapter II. The methods can be applied to problems in two, three or four coordinates. The convergence proof for the methods is obtained in such a way that the methods are not constrained by boundary conditions. They can be used with
any set of boundary conditions. The family of numerical methods is developed in the following section.

B. A Family of Finite Difference Methods for the Solution of Parabolic Equations

In this section, the development of the finite difference methods will be carried out for the equations of axisymmetric vortex flow, a three-coordinate problem. But the methods are applicable to parabolic equations in any number of coordinates. As a representative equation, the radial momentum equation will be used for the development of the methods. The equation can be written as:

\[ U_t = - UU_r - WU_z + \frac{v^2}{r} - \frac{v}{r} + (\mu + K)U_{rr} - K_r U_r \]

\[ - \frac{1}{r} (\mu + K) \left\{ U_r - \frac{U}{r} \right\} - K_z U_z - K_z W_r - (\mu + K)U_{zz} \]  

(104)

This equation can be written as

\[ U_t = F(t, r, z, U, U_r, U_z, U_{rr}, U_{zz}) \]  

(105)

The expressions in the radial momentum equation which are in terms of variables other than U can be considered to be functions of r, z and t as far as the variable U is concerned. Let the solution of Equation 105 in the region
0 < r ≤ R, 0 < z ≤ Z and 0 ≤ t ≤ T be considered. Let i, j and k be the indices of the radial, axial and temporal grid points, respectively; and let Δr, Δz and Δt be radial, axial and temporal grid interval sizes, respectively. Then, at the grid point \(i, j, k+\frac{1}{2}\), a consistent differencing of Equation 105 can be done as:

\[
\frac{(U_{i,j,k+1} - U_{i,j,k})}{Δt} = F(t_{k+\frac{1}{2}}, r_{i}, z_{j}, U_{i,j,k+\frac{1}{2}}) + O(Δr^2 + Δz^2 + Δt^2)
\]

(106)

where, \(n = 1\) or \(2\),

\[
δ_r U_{i,j,k} = \frac{(U_{i+1,j,k} - U_{i-1,j,k})}{2Δr}
\]

(107)

\[
δ_z U_{i,j,k} = \frac{(U_{i,j+1,k} - U_{i,j-1,k})}{2Δz}
\]

(108)

\[
Δ_r U_{i,j,k} = \frac{(U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k})}{Δr^2}
\]

(109)

\[
Δ_z U_{i,j,k} = \frac{(U_{i,j+1,k} - 2U_{i,j,k} + U_{i,j-1,k})}{Δz^2}
\]

(110)

In Equation 106, it is seen that, on the left side of the equation the difference terms are at the \((k+1)\)th and \(k\)th time points while on the right side of the equation the difference terms are at the \((k+\frac{1}{2})\)th time point. In order that the
U\textsubscript{i,j} quantities at the (k+1)th time point can be solved in terms of the U\textsubscript{i,j} quantities at the kth time point the U\textsubscript{i,j} quantities at the (k+\frac{1}{2})th time point must be expressed in terms of the U\textsubscript{i,j} quantities at the (k+1)th and kth time points. This can be done by interpolating between the (k+1)st and kth time points. The interpolation can be written in a general form as:

\begin{equation}
U_{i,j,k+\frac{1}{2}} = C_1 U_{i,j,k} + C_2 U_{i,j,k+1} + f(x_j, z_j, t_k) \tag{111}
\end{equation}

where, \(C_1\) and \(C_2\) are constants and \(f(x_j, z_j, t_k)\) is an interpolation quantity. These quantities are determined by making use of the Taylor expansions:

\begin{equation}
U_{i,j,k+1} = U_{i,j,k} + U_{i,j,k}^' \Delta t + U_{i,j,k}^{''} \left(\frac{\Delta t^2}{2}\right) + ... \tag{112}
\end{equation}

\begin{equation}
U_{i,j,k+\frac{1}{2}} = U_{i,j,k} + U_{i,j,k}'^' \left(\frac{\Delta t}{2}\right) + U_{i,j,k}^{'''} \left(\frac{\Delta t^2}{8}\right) + ... \tag{113}
\end{equation}

\begin{equation}
U_{i,j,k+\frac{1}{2}} = U_{i,j,k+1} - U_{i,j,k+1}^{'} \left(\frac{\Delta t}{2}\right) + U_{i,j,k+1}^{''} \left(\frac{\Delta t^2}{8}\right) - ... \tag{114}
\end{equation}

\begin{equation}
U_{i,j,k} = U_{i,j,k+1} - U_{i,j,k+1}^{'} \Delta t + U_{i,j,k+1}^{''} \left(\frac{\Delta t^2}{2}\right) - ... \tag{115}
\end{equation}

where, a prime denotes a differentiation with respect to t.

By carrying out the expansion to the desired order and
combining the equations in different ways, different values for $C_1$, $C_2$ and $f(r_i, z_j, t_k)$ are obtained. By inserting Equation 111 into Equation 106 one obtains:

$$(U_{i,j,k+1}) = \mathcal{F}\{t_{k+\frac{1}{2}}, r_i, z_j, (C_1 U_{i,j,k} + C_2 U_{i,j,k+1})\},$$

$$\begin{align*}
(C_1 \delta_r U_{i,j,k} + C_2 \delta_r U_{i,j,k+1}),
(C_1 \delta_z U_{i,j,k} + C_2 \delta_z U_{i,j,k+1}),
(C_1 \Delta_r U_{i,j,k} + C_2 \Delta_r U_{i,j,k+1}),
(C_1 \Delta_z U_{i,j,k} + C_2 \Delta_z U_{i,j,k+1})
\end{align*}$$

+ 0(\Delta r^2 + \Delta z^2 + \Delta t^n) \tag{116}$$

where, $$(U_{i,j,k+1}) = (U_{i,j,k+1} - U_{i,j,k})/\Delta t \tag{117}$$

In Equation 116, it is assumed that the interpolation quantity $f(r_i, z_j, t_k)$ is determined with an inaccuracy $\lesssim 0(\Delta r^2 + \Delta z^2 + \Delta t^n)$.

The difference equation that is used in the computation is

$$(U_{i,j,k+1}) = \mathcal{F}\{t_{k+\frac{1}{2}}, r_i, z_j, (C_1 U_{i,j,k} + C_2 U_{i,j,k+1})\},$$

$$\begin{align*}
(C_1 \delta_r U_{i,j,k} + C_2 \delta_r U_{i,j,k+1}),
(C_1 \delta_z U_{i,j,k} + C_2 \delta_z U_{i,j,k+1}),
(C_1 \Delta_r U_{i,j,k} + C_2 \Delta_r U_{i,j,k+1}),
(C_1 \Delta_z U_{i,j,k} + C_2 \Delta_z U_{i,j,k+1})
\end{align*}$$

+ 0(\Delta r^2 + \Delta z^2 + \Delta t^n) \tag{118}$$
Let $U$ be the solution to Equation 116, let $\bar{U}$ be the solution to Equation 118 and let the truncation error $E$ be defined by:

$$E_{i,j,k} = U_{i,j,k} - \bar{U}_{i,j,k}.$$  \hspace{1cm} (119)

Then, by subtracting Equation 118 from Equation 116 and by using Equation 105 and the Mean Value theorem, an equation for the truncation error is obtained as:

$$(E_{i,j,k+1})_{t} = \frac{\partial F}{\partial U} (C_1 E_{i,j,k} + C_2 E_{i,j,k+1})$$

$$+ \frac{\partial F}{\partial U} (C_1 (E_{i,j,k})_{\hat{r}} + C_2 (E_{i,j,k+1})_{\hat{r}})$$

$$+ \frac{\partial F}{\partial U} (C_1 (E_{i,j,k})_{\hat{z}} + C_2 (E_{i,j,k+1})_{\hat{z}})$$

$$+ \frac{\partial F}{\partial U} (C_1 (E_{i,j,k})_{\hat{r}\hat{r}} + C_2 (E_{i,j,k+1})_{\hat{r}\hat{r}})$$

$$+ \frac{\partial F}{\partial U} (C_1 (E_{i,j,k})_{\hat{z}\hat{z}} + C_2 (E_{i,j,k+1})_{\hat{z}\hat{z}})$$

$$+ 0(\Delta r^2 + \Delta z^2 + \Delta t^2)$$ \hspace{1cm} (120)

where,

$$(E_{i,j,k})_{\hat{r}} = (E_{i,j,k} - E_{i,j,k-1})/\Delta t$$ \hspace{1cm} (121)

$$(E_{i,j,k})_{\hat{t}} = (E_{i,j,k+1} - E_{i,j,k})/\Delta t$$ \hspace{1cm} (122)

$$(E_{i,j,k})_{\hat{z}} = (E_{i,j,k} - E_{i-1,j,k})/\Delta x$$ \hspace{1cm} (123)
\[ (E_{i,j,k})_r = \frac{(E_{i+1,j,k} - E_{i,j,k})}{\Delta r} \] (124)

\[ (E_{i,j,k})_z = \frac{(E_{i,j,k} - E_{i,j-1,k})}{\Delta z} \] (125)

\[ (E_{i,j,k})_z = \frac{(E_{i,j+1,k} - E_{i,j,k})}{\Delta z} \] (126)

\[ (E_{i,j,k})_r = \frac{(E_{i+1,j,k} - E_{i-1,j,k})}{2\Delta r} \] (127)

\[ (E_{i,j,k})_z = \frac{(E_{i,j+1,k} - E_{i,j-1,k})}{2\Delta z} \] (128)

By multiplying Equation 120 by \( E_{i,j,k+1} \) one obtains:

\[
\frac{1}{2}(E_{i,j,k+1})_r^2 + \frac{\Delta t}{2}(E_{i,j,k+1})_t^2 = C_2 \left\{ \frac{\partial F}{\partial U} E_{i,j,k+1}^2 \right\}
\]

\[ + \frac{\partial F}{\partial U_r} (E_{i,j,k+1})_r \cdot E_{i,j,k+1} + \frac{\partial F}{\partial U_z} (E_{i,j,k+1})_z \cdot E_{i,j,k+1} \]

\[ + \frac{\partial F}{\partial U_{rr}} (E_{i,j,k+1})_{rr} \cdot E_{i,j,k+1} + \frac{\partial F}{\partial U_{zz}} (E_{i,j,k+1})_{zz} \cdot E_{i,j,k+1} \]

\[ + C_1 \frac{\partial F}{\partial U} B_{i,j,k} B_{i,j,k+1} + \frac{\partial F}{\partial U_r} (E_{i,j,k})_r \cdot E_{i,j,k+1} \]

\[ + \frac{\partial F}{\partial U_z} (E_{i,j,k})_z \cdot E_{i,j,k+1} + \frac{\partial F}{\partial U_{rr}} (E_{i,j,k})_{rr} \cdot E_{i,j,k+1} \]

\[ + \frac{\partial F}{\partial U_{zz}} (E_{i,j,k})_{zz} \cdot E_{i,j,k+1} + E_{i,j,k+1} \cdot 0(\Delta r^2 + \Delta z^2 + \Delta t^n) \]

(129)
At the time point $k$, for all $i$ and $j$, let 5 constants $A_k$, $B_k$, $C_k$, $D_k$, and $E_k$ be defined by:

$$A_k = \left| \frac{\partial F}{\partial U} \right|_m, \quad B_k = \left| \frac{\partial F}{\partial r} \right|_m$$

$$C_k = \left| \frac{\partial F}{\partial z} \right|_m, \quad D_k = \left| \frac{\partial F}{\partial rr} \right|_m, \quad E_k = \left| \frac{\partial F}{\partial zz} \right|_m$$

(130)

where the subscript $m$ designates a maximum value for all $i$ and $j$.

By using the Equations 130, from Equation 129 one obtains the relation:

$$\frac{1}{2} (E_{i,j,k+1}^2)_t + \frac{\Delta t}{2} (E_{i,j,k+1})^2_t \leq A_k \{ |C_2| E_{i,j,k+1}^2 \}$$

$$+ |C_1| |E_{i,j,k}||E_{i,j,k+1}| + B_k \{ |C_2| |E_{i,j,k+1}| (E_{i,j,k+1})^2_r \}$$

$$+ |C_1| |E_{i,j,k+1}| (E_{i,j,k})_r + C_k \{ |C_2| |E_{i,j,k+1}| (E_{i,j,k+1})^2_r \}$$

$$+ |C_1| |E_{i,j,k+1}| (E_{i,j,k})_{rr} + E_k \{ |C_2| E_{i,j,k+1} (E_{i,j,k+1})^2_{zz} \}$$

$$+ |C_1| |E_{i,j,k+1}| (E_{i,j,k})_{zz} + \{ E_{i,j,k+1} |0(\Delta r^2 + \Delta z^2 + \Delta t^2)\}$$

(131)

From this relation, by using Schwarz's inequality the following relation is obtained.
\[
\frac{1}{2} (E_{i,j,k+1}^2) + \frac{\Delta t}{2} (E_{i,j,k}^2) \leq A_k \{ |C_2|^2 E_{i,j,k+1}^2 \\
+ \frac{|C_1|}{2} (E_{i,j,k+1}^2 + E_{i,j,k}^2) + \frac{B_k}{2 \Delta x} \{ |C_2|^2 + |C_1|^2 \} E_{i,j,k+1}^2 \\
+ \frac{|C_2|}{2} (E_{i+1,j,k+1}^2 + E_{i-1,j,k+1}^2) + \frac{|C_1|}{2} (E_{i+1,j,k}^2 + E_{i-1,j,k}^2) \}
\]

\[
\frac{C_k}{2 \Delta z} \{ |C_2|^2 + |C_1|^2 \} E_{i,j,k+1}^2 + \frac{|C_2|}{2} (E_{i,j+1,k+1}^2 + E_{i,j-1,k+1}^2) \\
+ \frac{|C_1|}{2} (E_{i+1,j,k+1}^2 + E_{i-1,j,k+1}^2) + \frac{D_k}{\Delta x} \{ (3 |C_2|^2 + 2 |C_1|^2) E_{i,j,k+1}^2 \\
+ E_{i-1,j,k}^2 ) \} + \frac{E_k}{\Delta z} \{ (3 |C_2|^2 + 2 |C_1|^2) E_{i,j,k+1}^2 \\
+ \frac{|C_2|}{2} (E_{i,j+1,k+1}^2 + E_{i,j-1,k+1}^2) + \frac{|C_1|}{2} (E_{i,j+1,k}^2 + E_{i,j,k}^2 + E_{i,j,k+1}^2) \\
+ E_{i,j-1,k}^2 ) \}
\]

\[
+ \frac{1}{2} E_{i,j,k+1}^2 + O(0(\Delta x^2 + \Delta z^2 + \Delta t^4))^2
\]

(132)

Let I, J and M be the number of radial, axial and temporal grid points. Then, by summing the relation (132) from \( i = 1 \) to \( i = I \) and from \( j = 1 \) to \( j = J \), one obtains the relation:
\[ \sum_{i,j} \varepsilon_{i,j,k}^{2} \leq \sum_{i,j} G_{k} \varepsilon_{i,j,k}^{2} + \sum_{i,j} H_{k} \varepsilon_{i,j,k}^{2} + 0 \left( 0 (\Delta r^{2} + \Delta z^{2} + \Delta t^{n}) \right)^{2} \]  

(133)

where,

\[ G_{k} = (|C_{2}| + \frac{|C_{1}|}{2}) A_{k} + \frac{1}{\Delta r} (|C_{2}| + \frac{|C_{1}|}{2}) B_{k} + \frac{1}{\Delta z} (|C_{2}| + \frac{|C_{1}|}{2}) C_{k} \]

\[ + \frac{2}{\Delta r^{2}} (2|C_{2}| + |C_{1}|) D_{k} + \frac{2}{\Delta z^{2}} (2|C_{2}| + |C_{1}|) E_{k} + \frac{1}{2} \]

(134)

\[ H_{k} = \frac{|C_{1}|}{2} \left\{ A_{k} + \frac{B_{k}}{\Delta r} + \frac{C_{k}}{\Delta z} + \frac{4}{\Delta r^{2}} D_{k} + \frac{4}{\Delta z^{2}} E_{k} \right\} \]

(135)

Since \( E_{i,j,k} = 0 \) for all \( i \) and \( j \) when \( k = 1 \), the summing of the relation (133) from \( k = 1 \) to \( k = M - 1 \) gives:

\[ \sum_{i,j} \varepsilon_{i,j,M}^{2} \leq \Delta t_{M-1}^{2} \sum_{i,j} \varepsilon_{i,j,M}^{2} \]

\[ + \sum_{k=1}^{M-1} \Delta t_{k} \sum_{i,j} (G_{k} + H_{k}) \varepsilon_{i,j,k}^{2} + 0 \left( 0 (\Delta r^{2} + \Delta z^{2} + \Delta t^{n}) \right)^{2} \Delta t \]

(136)

where \( \Delta t_{k} = t_{k+1} - t_{k} \)

(137)

From relation (136) one obtains the relation

\[ \sum_{i,j} \varepsilon_{i,j,M}^{2} \leq \sum_{k=1}^{M-1} \frac{(G_{k} + H_{k}) \Delta t_{k}}{(1 - \Delta t_{M-1} G_{M-1})} \cdot \sum_{i,j} \varepsilon_{i,j,k}^{2} \]

\[ + 0 \left( 0 (\Delta r^{2} + \Delta z^{2} + \Delta t^{n}) \right)^{2} \Delta t \]

(138)

For \( M = 2, 3, \) and \( 4 \), this inequality becomes:
\[ \sum \sum E_{i,j,2}^2 \leq 0(0(\Delta r^2 + \Delta z^2 + \Delta t_n)^2)\Delta t \quad (139) \]

\[ \sum \sum E_{i,j,3}^2 \leq \left\{ \frac{(G_2 + H_2) t_2}{(1 - \Delta t_2 G_2)^2 + 1} \right\} 0(0(\Delta r^2 + \Delta z^2 + \Delta t_n)^2)\Delta t \quad (140) \]

\[ \sum \sum E_{i,j,4}^2 \leq \left\{ \frac{(G_2 + H_2)(G_3 + H_3)}{(1 - \Delta t_2 G_2)(1 - \Delta t_3 G_3)} \right\} \Delta t_2 \Delta t_3 + \frac{(G_3 + H_3)}{(1 - \Delta t_3 G_3)} \cdot \Delta t_3 + 1 \right\} 0(0(\Delta r^2 + \Delta z^2 + \Delta t_n)^2)\Delta t \quad (141) \]

From this it can be seen that if \( \Delta t_k \) is chosen for all \( k \) such that

\[ (1 - \Delta t_k G_k) > (G_k + H_k)\Delta t_k \quad (142) \]

then,

\[ \sum \sum E_{i,j,k}^2 \leq 0(0(\Delta r^2 + \Delta z^2 + \Delta t_n)^2)\Delta t \quad (143) \]

Taking the square root of both sides of this inequality gives:

\[ \left\{ \sum \sum E_{i,j,k}^2 \right\}^{\frac{1}{2}} \leq 0(\Delta r^2 + \Delta z^2 + \Delta t_n)\Delta t^{\frac{1}{2}} \quad (144) \]

From this it is easily seen that

\[ |E_{i,j,k}| \leq 0(\Delta r^2 + \Delta z^2 + \Delta t_n)\Delta t^{\frac{1}{2}} \quad (145) \]
for all \( i \) and \( j \). Therefore, if \( \Delta t_k \) is chosen at each time point \( k \) such that the inequality (142) is satisfied, the solution of Equation 118 converges uniformly to the solution of Equation 105 with an inaccuracy of \( O(\Delta r^2 + \Delta z^2 + \Delta t^2) \Delta t_k \).

This completes the convergence proof for the family of finite difference methods in which the interpolation between two time points is described by Equation 111 and the differencing of Equation 105 is described by Equation 116. With different values for \( C_1, C_2 \) and \( f(r_i, z_j, t_k) \) different finite difference methods are obtained. Some of these methods are discussed in the following section.

C. Some Examples of Finite Difference Methods for Parabolic Equations

1) Simple Explicit Method. With \( C_1 = 1, C_2 = 0 \) and \( f(r_i, z_j, t_k) = 0 \) a simple method is obtained. It is a fully explicit method and the truncation error of the method is \( O(\Delta r^2 + \Delta z^2 + \Delta t) \Delta t_k \).

2) Simple Iterative Method. With \( C_1 = 0, C_2 = 1 \) and \( f(r_i, z_j, t_k) = 0 \) another simple method is obtained. It is not as direct as the simple explicit method. For this method, at each time point, some type of iteration is needed. The method can be explicit or implicit in either \( r \) or \( z \), depending on how the iteration is done. The truncation error of the method is \( O(\Delta r^2 + \Delta z^2 + \Delta t) \Delta t_k \).
3) **Explicit Method with Linear Interpolation.** With $C_1 = 1, C_2 = 0$ and $f(r_i, z_j, t_k) = \Delta t u_{i,j,k}'$ another explicit method is obtained. The truncation error of this method is $O(\Delta r^2 + \Delta z^2 + \Delta t^2) \Delta t^{1/2}$.

4) **Iterative Method with Linear Interpolation.** With $C_1 = \frac{1}{2}, C_2 = \frac{1}{2}$ and $f(r_i, z_j, t_k) = 0$ another iterative method is obtained. The truncation error of the method is $O(\Delta r^2 + \Delta z^2 + \Delta t^2) \Delta t^{1/2}$.

5) **Explicit Method with Quadratic Interpolation.** With $C_1 = 1, C_2 = 0$ and $f(r_i, z_j, t_k) = \Delta t u_{i,j,k}' + \Delta t^2 u_{i,j,k}''$ an explicit method is obtained. In this method, the interpolation on the right side of Equation 116 and the interpolation in the centered differencing of the left side of the equation are the same. The truncation error of the method is $O(\Delta r^2 + \Delta z^2 + \Delta t^2) \Delta t^{1/2}$.

6) **Iterative Method with Quadratic Interpolation.** With $C_1 = 3/4, C_2 = 1/4$ and $f(r_i, z_j, t_k) = \Delta t u_{i,j,k}'$ an iterative method is obtained. The truncation error of the method is $O(\Delta r^2 + \Delta z^2 + \Delta t^2) \Delta t^{1/2}$.

Other methods with higher order interpolation on the right side of Equation 116 can be obtained by using Equations 112-115. But, since the interpolation on the left side of Equation 116 is only quadratic there is no benefit from increasing the interpolation order on the right side of the equation to orders higher than quadratic.
From among methods (1)-(6), the choice of the method that is to be used for any particular problem is at the disposition of the user. The direct and explicit methods have only a single step at each time point and so at each time point they are cheaper than the iterative methods. On the other hand, the iterative methods can be made to be implicit in one spatial direction or alternatively implicit in more than one spatial directions. In this way the solution in interior grid points is affected by the boundary conditions faster and so a particular solution that is being sought may be arrived at in fewer time steps with an iterative method rather than with a direct explicit method. The methods with lower order interpolations are cheaper and less complicated than the methods with higher order interpolation. But, for problems where the solution is a time-wise rapidly changing function the methods with higher order interpolations are more accurate. So, depending on the problem that is being solved and cost and accuracy considerations some of the methods are more appropriate to some problems than others.

D. The Numerical Solution of the Governing Equations of a Vortex Flow

The finite difference methods that have been developed in sections B and C of this chapter are methods for parabolic equations. So, they would be used to solve the radial,
tangential and axial momentum equations, the tangential vorticity equation and the equation for the kinetic energy of turbulence that have been developed in Chapter II. The methods can be used to obtain a steady state solution or an unsteady state solution. The Poisson equations for the pressure and the stream function that have been developed in Chapter II are linear elliptic equations. So during the solution of the governing equations the equations for the pressure and the stream function would be solved at each time point by a finite difference method for elliptic equations. Several reliable finite difference methods for linear elliptic equations are discussed in books on numerical methods such as those by Ames (1969) and Roache (1972). Any good method for linear elliptic equations can be used along with the methods for parabolic equations that have been developed in this chapter.

Whether a steady state solution or an unsteady state solution is obtained during the solution of the governing equations in Chapter II depends on the boundary conditions that are used and the combination of equations that are solved. For turbulent axisymmetric flow, to obtain an unsteady state solution, one would solve the radial and tangential momentum equations, the Poisson equation for the pressure, the continuity equation and the equation for the kinetic energy
of turbulence. For turbulent axisymmetric flow, to obtain a steady state solution, one would solve the tangential momentum equation, the tangential vorticity equation, the Poisson equations for the pressure and the stream function, and the equation for the kinetic energy of turbulence. In the following chapter, some unsteady state solutions are obtained and discussed.
V. SOME SOLUTIONS FOR THE UNSTEADY STATE FLOW OF AN
AXISYMMETRIC GENERAL VORTEX NEAR A SOLID SURFACE
NORMAL TO THE AXIS OF THE VORTEX

A. Introduction

In this chapter, as an example of the application of the developments of Chapters II, III and IV, the unsteady state flow of an axisymmetric general vortex near a flat solid surface is examined. The vortex flow that is examined is a flow where the rotational motion at some distance from the surface and the radial inflow at some distance from the axis are maintained. Two sets of solutions, one for a weak inflow and one for a strong inflow, have been obtained. For the case with the strong inflow both the "boundary layer" equations and the whole governing equations have been solved and a comparison between the solutions of the two sets of equations is presented.

In section B, the numerical method that is used to solve the governing equations of the vortex flow and the boundary and initial conditions for which solutions are obtained are described. In section C, the solutions to the flow problem with a strong inflow are presented and discussed. In section D, the solutions to the flow problem with a weak inflow are presented and discussed.
B. The Numerical Method

In order to obtain solutions for the unsteady state flow of an axisymmetric vortex, the set of Equations 53, 54, 56, 57 and 63 and the set of Equations 58-62 have been solved. In both sets of equations, the equations for the radial velocity, tangential velocity and kinetic energy of turbulence have been solved by the Iterative method with linear interpolation of Chapter IV. In the first set of equations, the Poisson equation for the pressure was solved by the PRADI method for elliptic equations. But in the second set of equations, there is no need for the solution of any equation for the pressure. Throughout the flow region that is examined the pressure has the same value as its value at the axial outer boundary. In both sets of equations, the axial velocity is determined by integrating the continuity equation numerically by the simple trapezoidal method. The continuity equation is integrated from the solid surface up to the axial outer boundary.

For the solutions that have been obtained \( v_m^* \) was taken to be the maximum value of the tangential velocity at the axial outer boundary and \( r_m^* \) was taken to be the radial distance from the axis of the vortex to the radial point where \( v_m^* \) exists. With the dimensional quantities chosen this way, \( \mu \) was taken to be \( 2.61 \times 10^{-5} \) and the governing equations of the flow were solved in the region \( 0 \leq r \leq 4 \) and \( 0 \leq z \leq 1 \), with grid interval sizes of \( \Delta r = 0.2 \) and \( \Delta z = 0.05 \).
Within this region, for the equations for U, V and q the following differencing was used.

\[ x_{i,j,k+\frac{1}{2}} = \frac{x_{i,j,k+1}^m + x_{i,j,k}^m}{2} \]  \hspace{1cm} (156)

\[ (x_t)_{i,j,k+\frac{1}{2}} = \frac{x_{i,j,k+1}^{m+1} - x_{i,j,k}^m}{\Delta t} \]  \hspace{1cm} (157)

\[ (x_r)_{i,j,k+\frac{1}{2}} = \frac{x_{i+1,j,k+1}^m - x_{i-1,j,k+1}^m + x_{i+1,j,k+1}^m - x_{i-1,j,k+1}^m}{4\Delta r} \]  \hspace{1cm} (158)

\[ (x_z)_{i,j,k+\frac{1}{2}} = \frac{x_{i,j+1,k+1}^m - x_{i,j-1,k+1}^m + x_{i,j+1,k+1}^m - x_{i,j-1,k+1}^m}{4\Delta z} \]  \hspace{1cm} (159)

\[ (x_{rr})_{i,j,k+\frac{1}{2}} = \frac{x_{i+1,j,k+1}^m - 2x_{i,j,k+1}^m + x_{i-1,j,k+1}^m + x_{i+1,j,k}^m - 2x_{i,j,k}^m + x_{i-1,j,k}^m}{2\Delta r^2} \]  \hspace{1cm} (160)

\[ (x_{zz})_{i,j,k+\frac{1}{2}} = \frac{x_{i,j+1,k+1}^{m+1} - 2x_{i,j,k+1}^{m+1} + x_{i,j-1,k+1}^{m+1} + x_{i,j+1,k}^{m+1} - 2x_{i,j,k}^{m+1} + x_{i,j-1,k}^{m+1}}{2\Delta z^2} \]  \hspace{1cm} (161)

In these difference equations, X is any flow variable, i, j and k are the radial, axial and temporal grid point indices and m is the iteration number, \( \geq 1 \). With the introduction of the difference equations 156-161 into the differential equations for U, V and q, linear coupled algebraic equations are
obtained for \( U, V \) and \( q \) at each time point, \( k + 1 \), radial point, \( i \) and iteration number \((m+1)\). The first iteration was obtained from

\[
X_{i,j,k+1} = X_{i,j,k} + (X_t)_{i,j,k} \cdot \Delta t
\]  

(162)

and \((X_t)_{i,j,k}\) can be obtained from the equations for \( U, V \) and \( q \). For \( m + 1 \geq 1 \) the linear equations that are obtained are of the form:

\[
A_jX_{i,j+1,k+1} + B_jX_{i,j,k+1} + C_jX_{i,j-1,k+1} = D_j
\]  

(163)

Equations of this form are obtained for \( 2 \leq j \leq J - 1 \) and \( 2 \leq i \leq I - 1 \), where \( I \) and \( J \) are the total number of radial and axial grid points. The equations are solved by deriving from them equations of the form:

\[
X_{i,j,k+1} = E_jX_{i,j+1,k+1} + F_j
\]  

(164)

where,

\[
E_j = -A_j/(B_j + C_jE_{j-1})
\]  

(165)

and

\[
F_j = (D_j - C_jF_{j-1})/(B_j + C_jE_{j-1})
\]  

(166)

In order to solve the equations of the form of Equation 164, first, starting with the boundary conditions at \( j = 1 \), \( E_j \) and \( F_j \) are determined for all \( j \quad 2 \leq j \leq J - 1 \) using Equations 165 and 166. Then, starting with the boundary conditions at \( j = J \), \( X_{i,j,k+1} \) is solved for \( J - 1 > j > 1 \) using Equation 164.
At each time point and iteration point, after the determination of \( U, V \) and \( q \), \( W \) is determined by integrating the continuity equation. During the integration, centered differencing was used for \( U_i^\ast \). The difference equation for \( W \) that was used was:

\[
W_{i,j,k} = \sum_{n=2}^{j} \left[ W_{i,n-1,k} - \frac{\Delta z}{2} \left( \frac{1}{r_i} U_{i,n,k} + \frac{1}{2\Delta r}(U_{i+1,n,k} - U_{i-1,n,k}) \right) + \frac{1}{r_i} U_{i,n-1,k} + \frac{1}{2\Delta r}(U_{i+1,n-1,k} - U_{i-1,n-1,k}) \right]
\]  

(167)

After determining \( W \), \( \varepsilon \) was determined using Equation 48 and \( K \) was determined using Equation 43. For all the derivatives of the flow variables and \( Q \) in the equations for \( \varepsilon \) and \( K \), centered differencing was used. With this, for the second set of equations, the determination of the flow variables at a time and iteration point is completed. But for the first set of equations the pressure has to be determined and this was done by solving Equation 63 by the PRADI method. In the application of the PRADI method to Equation 63, for the derivatives of the flow variables other than the pressure, centered differencing was used.

For the numerical solutions that have been obtained, the boundary conditions that were used are the following.
\textbf{At } z = 0: \textbf{ At } z = 0, \textbf{ the no-slip boundary condition was used so that:}

\[ U(r,0), V(r,0), W(r,0), q(r,0) = 0 \quad (168) \]

\[ P_z = \mu (W_{zz})_{z=0} \quad (169) \]

\textbf{At } z = 1: \textbf{ At the axial outer edge of the flow region that was examined, the simple solutions to the governing equations that were obtained in Chapter III, Equations 85 were used so that:}

\[ U(r,1) = -2.512pr \quad (170) \]

\[ V(r,1) = 1.39(1 - e^{-1.256r^2})/r \quad (171) \]

\[ q(r,1) = 0 \quad (172) \]

\[ P(r,1) = \int_0^r \left\{ 2.512\mu rU_x(r,1) + \frac{V^2(r,1)}{r} \right\} dr \quad (173) \]

\textbf{At } r = 0: \textbf{ At the axis of the vortex, the boundary conditions that were used are obtained from the nature of the coordinate system and the governing equations of the flow. The boundary conditions are:}

\[ U(0,z), V(0,z) = 0 \quad (174) \]

\[ U, V \textbf{ are linear near } r = 0 \quad (175) \]

\[ P_x(0,z) = 0 \quad (176) \]
\[ q(0,z) = 0 \text{ or } g(0,z) = cq(4,z)W^2(0,z) \] (177)

where, \( c \) is a constant.

**At \( r = 4 \):** At \( r = 4 \), the radial outer boundary of the flow region that was examined, the boundary conditions that were obtained are obtained from physical considerations and observation. The boundary conditions are:

\[ U(4,z) = Az^{1/7} + Bz^3 + Cz^2 + Dz \] (178)
\[ V(4,z) \sim 1/r \] (179)
\[ q(4,z) = Az^{1/7} + Bz^3 + Cz^2 + Dz \] (180)
\[ P(4,z) \sim 1/r^3 \] (181)

In Equations 178 and 180, \( A, B, C \) and \( D \) are constants and they are chosen such that \( U \) and \( q \) would have a chosen maximum value at a chosen point \( z \) and such that \( U \) and \( q \) would have a chosen value and a chosen axial derivative at \( z = 4 \).

**At \( t = 0 \):** For the solution that have been obtained the initial conditions that were used are:

\[ U(r,z,0) = U(4,z)\{V(r,1)/V(4,1)\} \] (182)
\[ V(r,z,0) = V(r,1)z^{1/7} \] (183)
\[ q(r,z,0) = q(4,z) \] (184)
\[ P(r,z,0) = P(r,1) \] (185)
The convergence of the iteration that is described by Equations 156-161 is a proved fact. For example, a proof of the convergence is given by Lees (1959). In order to determine the number of iterations that are needed, at the first time step, \( k = 2 \), 10 iterations were carried out. The convergence of the iteration was then observed. As a representative example of the convergence of the iteration at all the computational grid points, the convergence of the iteration at the point \( r = 0.8 \) and \( z = 0.5 \) is shown in Table 1 for \( U \), \( V \) and \( W \).

### Table 1. The convergence of the iteration at a time point

<table>
<thead>
<tr>
<th>Iteration number</th>
<th>( U )</th>
<th>( V )</th>
<th>( W )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.3659820</td>
<td>0.8778173</td>
<td>0.9912846</td>
</tr>
<tr>
<td>2</td>
<td>-0.3672625</td>
<td>0.8775676</td>
<td>0.4256893</td>
</tr>
<tr>
<td>3</td>
<td>-0.3646214</td>
<td>0.8782682</td>
<td>0.4255198</td>
</tr>
<tr>
<td>4</td>
<td>-0.3646444</td>
<td>0.8782477</td>
<td>0.4255371</td>
</tr>
<tr>
<td>5</td>
<td>-0.3646449</td>
<td>0.8782479</td>
<td>0.4255351</td>
</tr>
<tr>
<td>6</td>
<td>-0.3646448</td>
<td>0.8782481</td>
<td>0.4255359</td>
</tr>
<tr>
<td>7</td>
<td>-0.3646445</td>
<td>0.8782480</td>
<td>0.4255362</td>
</tr>
<tr>
<td>8</td>
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</tr>
<tr>
<td>9</td>
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<td>0.4255367</td>
</tr>
<tr>
<td>10</td>
<td>-0.3646440</td>
<td>0.8782480</td>
<td>0.4255370</td>
</tr>
</tbody>
</table>
From Table 1, it is seen that three to five iterations are adequate. So, during the application of the numerical method, for the first few time steps, five iterations were used and after a few time steps three iterations were used.

During the solution of the equation for P by the PRADI method two iterations were found to be adequate for a convergence of up to five or six decimal points. The Peaceman Rachford constants that were used were $8 \times \sqrt{5}$ and $0.4\sqrt{5}$.

At the beginning of every time step, $A_k$, $B_k$, $C_k$, $D_k$ and $E_k$ of Equation 130 were computed and the time step size was determined from:

$$\Delta t_k = 0.75/(1.75 G_k + H_k)$$  \hspace{1cm} (186)

This completes the description of the numerical method and boundary and initial condition that were used. The numerical solution process at a time step can be summarized by the following algorithm.

**Algorithm 2**

1) Determine $A_k$, $B_k$, $C_k$, $D_k$, $E_k$, $G_k$, $H_k$ and $\Delta t_k$

2) For $m = 1$, compute the value of the flow variables at $(k + 1)$ using the governing equations and Equation 162.

3) For $m > 1$, compute the values of $U$, $V$ and $q$ at $(k + 1)$ using the governing equations, Equations 156-161 and Equations 164-166. From the governing equations and Equations 156-161 linear coupled equations for $U$, $V$ and $q$ of the form of
Equation 163 are obtained. These equations are then solved using Equations 164-166.

4) Compute the values of \( W \) at \( (k + 1) \) using Equation 167.

5) Compute the values of \( \varepsilon \) and \( K \) at \( (k + 1) \) using Equations 43 and 48. Obviously, \( \varepsilon \) is determined first.

6) Compute the values of \( P \) at \( (k + 1) \) by solving the equation for \( P \) by the PRADI method.

7) Repeat steps 3-6 until a converged solution for the flow variables at the time point \( (k + 1) \) is obtained.

Using this algorithm and the boundary conditions that have been described above, two sets of solutions to the governing of an axisymmetric vortex flow have been obtained. These sets of solutions are presented in the next two sections.

C. Solutions for a Strong Inflow at the Radial Outer Boundary

For the first set of solutions, an inflow per unit area of 0.024 was maintained at \( r = 4 \). The constants \( A, B, C \) and \( D \) of Equations 178 and 180 were chosen such that the radial velocity and the kinetic energy of turbulence would have their maximum values at \( z = 0.2 \). The maximum value of the radial velocity at \( r = 4 \) was chosen to be 0.0445 and the maximum value of the kinetic energy of turbulence at \( r = 4 \) was taken to be equal to \( 10^{-3} \times (U(4,2)^2 + V(4,2)^2 + W(4,2)^2) \). At the
axis of the vortex, for \( q \), the boundary condition \( q_r = 0 \) was used. With these specifications and the boundary and initial conditions specified by Equations 168-184, the whole governing equations of an axisymmetric flow and the "boundary layer" equations were solved by the numerical method of the last section. After a few time steps, when the magnitudes of the values of the flow variables and the magnitudes of the time derivatives of the flow variables became to be of the same order, the observation of the evolution of the vortex flow was started.

The flow starts as a one cell vortex with down flow in the radially outer part of the flow field and upflow in the inner part of the flow field. There is inflow everywhere and the maximum upflow occurs at the axis of the vortex. As time goes on, this upflow at the axis of the vortex increases until it reaches a maximum value and then it starts to decrease. While the upflow at the axis of the vortex decreases, the upflow at radial points that are a short distance away from the axis \((r = 0.3-0.8)\) continues to increase gradually. The nature of the vortex flow at this point of its evolution is shown in detail in Figs. 10-17.

From Figs. 10, 11 and 13, it is seen that the vortex flow is a one cell vortex with upflow for small \( r \) and downflow for large \( r \). The maximum upflow occurs at \( r = 0.4 \). From Fig. 12 it is seen that for small \( r \), when it is observed along the
Fig. 10. The meridional flow pattern for a one cell vortex with a strong inflow at the radial outer boundary.
Fig. 11. The variation of the radial velocity for a one cell vortex with a strong inflow at the radial outer boundary.
Fig. 12. The variation of the tangential velocity for a one cell vortex with a strong inflow at the radial outer boundary
Fig. 13. The variation of the axial velocity for a one cell vortex with a strong inflow at the radial outer boundary.
Fig. 14. The variation of the turbulence energy for a one cell vortex with a strong inflow at the radial outer boundary.
Fig. 15. The variation of the dissipation rate of turbulence energy for a one cell vortex with a strong inflow at the radial outer boundary.
Fig. 16. The variation of the turbulent viscosity for a one cell vortex with a strong inflow at the radial outer boundary.
Fig. 17. The variation of the pressure deficit for a one cell vortex with a strong inflow at the radial outer boundary.
axial direction, the tangential velocity has its maximum value at \( z < 1 \). For some radial points the tangential velocity is also oscillatory. From Fig. 17 it is seen that the pressure does not change much in the axial direction. From Fig. 14, it is seen that \( q \) has not changed much from its initial values. From Fig. 15, it is seen that \( \varepsilon \) has its maximum value in the vicinity of the ground and decreases fast with increasing \( z \). From Fig. 16, it is seen that \( K \) is small where the velocity gradients are large and \( q \) is small and \( K \) is large where the velocity gradients are small and \( q \) is large.

As the integration of the governing equations is continued the upflow near the axis of the vortex decreases continuously until it becomes zero and a downflow near the axis of the vortex begins to appear. As time increases, this downflow near the axis of the vortex increases while an upflow region adjacent to it increases and a clearly two cell vortex flow is obtained. The solution to the whole governing equation at this point of the evolution of the vortex is shown in Figs. 18-25.

Figs. 18, 19 and 21 show that the two cell vortex has a downflow region between \( r = 0 \) and \( r = 0.3 \), an upflow region between \( r = 0.3 \) and \( r = 1.2 \) and a downflow region for \( r > 1.2 \). The maximum downflow occurs at \( r = 0 \) and the maximum upflow occurs at \( r = 0.6 \). Fig. 20 shows that the axially oscillatory variation of the tangential velocity in the radially inner
Fig. 18. The meridional flow pattern for a two cell vortex with a strong inflow at the radial outer boundary.
Fig. 19. The variation of the radial velocity for a two cell vortex with a strong inflow at the radial outer boundary.
Fig. 20. The variation of the tangential velocity for a two cell vortex with a strong inflow at the radial outer boundary
Fig. 21. The variation of the axial velocity for a two cell vortex with a strong inflow at the radial outer boundary.
Fig. 22. The variation of the turbulence energy for a two cell vortex with a strong inflow at the radial outer boundary.
Fig. 23. The variation of the dissipation rate of turbulence energy for a two cell vortex with a strong inflow at the radial outer boundary
Fig. 24. The variation of the turbulent viscosity for a two cell vortex with a strong inflow at the radial outer boundary
Fig. 25. The variation of the pressure deficit for a two cell vortex with a strong inflow at the radial outer boundary
regions of the vortex is getting more pronounced. As it is seen in Figs. 22, 23, 24 and 25, q, ε, K and P for this two cell vortex are of the same nature as the q, ε, K and P for the one cell solution that is shown in Figs. 10-17.

As the integration of the governing equations is continued the downflow near the axis of the vortex increases and the upflow adjacent to it increases. After sometime the downflow reaches a maximum value and starts to decrease. During the same time the downflow region extends out radially. In the upflow region, which is now decreasing in radial extent due to the radial spread of the downflow region, the upflow increases. Also, during the same time the amplitudes of the axial oscillations of the tangential velocity continue to increase and the tangential velocity becomes large and positive at some points and negative at some other points. With further integration, the downflow near the axis of the vortex decreases to zero and near the axis of the vortex an upflow region starts to appear. Adjacent to this upflow region a downflow region persists. During the same time the upflow in the upflow region radially outward and adjacent to this downflow region continues to increase. Also, a narrow downflow region between \( r = 1.1 \) and \( r = 1.3 \) and a narrow upflow region between \( r = 1.3 \) and \( r = 1.5 \) begin to appear. With continued integration, the magnitudes of axial velocities in these different regions increase until finally a clearly
defined five cell vortex flow develops. The solution to the whole governing equations at this point in the evolution of the vortex flow is shown in Figs. 26-33.

As it can be seen in Figs. 26, 27 and 29, the five cell vortex has upflow regions between \( r = 0 \) and \( r = 0.3 \), between \( r = 0.7 \) and \( r = 1.1 \) and between \( r = 1.3 \) and \( r = 1.5 \). Between \( r = 0.3 \) and \( r = 0.7 \), between \( r = 1.1 \) and \( r = 1.3 \) and for \( r > 1.5 \) there are downflow regions. The strongest downflow occurs at \( r = 0.4 \) and the strongest upflow occurs at \( r = 0.8 \). From Fig. 28, it is seen that the axial ascillation of the tangential velocity has increased from what it was when the flow was a two cell vortex. Also, from Figs. 30, 31 and 32, the turbulence quantities \( q, \varepsilon \) and \( k \) are more oscillatory for this five cell vortex than for the two cell vortex. But, the oscillations are not as large as the oscillations of \( q, \varepsilon \) and \( k \) for the five cell solution that was obtained in the exploratory analysis. The pressure, on the other hand, has hardly changed from what it was when the flow was a one cell solution.

The meaning of the negative tangential velocity values that have been obtained is not clear. The negative tangential velocity values could mean that there is a breakdown in the rotational flow pattern, that there is a region within the rotational flow field where there is random mixing rather than an axisymmetric rotation. It could also mean that there are
Fig. 26. The meridional flow pattern for a five cell vortex with a strong inflow at the radial outer boundary
Fig. 27. The variation of the radial velocity for a five cell vortex with a strong inflow at the radial outer boundary.
Fig. 28. The variation of the tangential velocity for a five cell vortex with a strong inflow at the radial outer boundary
Fig. 29. The variation of the axial velocity for a five cell vortex with a strong inflow at the radial outer boundary.
Fig. 30. The variation of the turbulence energy for a five cell vortex with a strong inflow at the radial outer boundary.
Fig. 31. The variation of the dissipation rate of turbulence energy for a five cell vortex with a strong inflow at the radial outer boundary.
Fig. 32. The variation of the turbulent viscosity for a five cell vortex with a strong inflow at the radial outer boundary.
Fig. 33. The variation of the pressure deficit for a five cell vortex with a strong inflow at the radial outer boundary.
some satellite vortices that have developed in the neighborhood of $r = 0.6$. In either case, when this happens, the axisymmetry assumption becomes questionable. So at the point of the integration of the governing equations where substantial negative values for the tangential velocity were obtained the integration of the governing equations was stopped. So for the problem of a vortex flow with a strong inflow at the radial outer boundary the integration of the governing equations was stopped at the point where a five cell vortex was obtained.

The solutions that have been presented with Figs. 10-17, 18-25 and 26-33 are the solutions to the whole governing equations. Up to the point where a two cell vortex was obtained, along with the whole equations, the "boundary layer" equations have also been solved. It has been found that the solutions to the two sets of equations are close. An illustrative comparison of the solutions of the two sets of solutions is given in Tables 2-5. The solutions in Tables 2 and 3 are for a one cell vortex and $r = 0.6$ and 1.6, respectively; and the solutions in Tables 4 and 5 are for a two cell vortex and $r = 0.6$ and 1.6, respectively. As it can be seen from the tables, for most of the flow field points the difference between the solutions of the two sets of equations is less than 5% of the solution to the whole equations. But for a few points the difference gets as large as 15%. Since the
<table>
<thead>
<tr>
<th>z</th>
<th>&quot;Boundary layer&quot; equations</th>
<th>Whole equations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>U</td>
<td>V</td>
</tr>
<tr>
<td>0.05</td>
<td>-0.5539</td>
<td>0.6521</td>
</tr>
<tr>
<td>0.10</td>
<td>-0.5875</td>
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<td>-0.5481</td>
<td>1.0575</td>
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<tr>
<td>0.25</td>
<td>-0.5268</td>
<td>1.0905</td>
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<tr>
<td>0.30</td>
<td>-0.5049</td>
<td>1.1128</td>
</tr>
<tr>
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<td>0.60</td>
<td>-0.2893</td>
<td>1.0985</td>
</tr>
<tr>
<td>0.70</td>
<td>-0.2033</td>
<td>1.0410</td>
</tr>
<tr>
<td>0.80</td>
<td>-0.1033</td>
<td>0.9495</td>
</tr>
<tr>
<td>0.90</td>
<td></td>
<td></td>
</tr>
</tbody>
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Table 3. Comparison of the solutions for the "boundary layer" equations and the whole equations for a one cell vortex at \( r = 1.6 \)

| \( z \) | "Boundary layer" equations | | | Whole equations | | | |
|---|---|---|---|---|---|---|
| | \( U \) | \( V \) | \( q \) | | \( U \) | \( V \) | \( q \) |
| 0.05 | -0.3785 | 0.5298 | 0.2473 \( \times 10^{-4} \) | | -0.3626 | 0.5298 | 0.2528 \( \times 10^{-4} \) |
| 0.10 | -0.4035 | 0.6184 | 0.6207 \( \times 10^{-4} \) | | -0.3976 | 0.6183 | 0.6127 \( \times 10^{-4} \) |
| 0.15 | -0.4071 | 0.6576 | 0.7416 \( \times 10^{-4} \) | | -0.4054 | 0.6576 | 0.7368 \( \times 10^{-4} \) |
| 0.20 | -0.4016 | 0.6856 | 0.7664 \( \times 10^{-4} \) | | -0.4012 | 0.6857 | 0.7646 \( \times 10^{-4} \) |
| 0.25 | -0.3891 | 0.7079 | 0.7625 \( \times 10^{-4} \) | | -0.3890 | 0.7080 | 0.7618 \( \times 10^{-4} \) |
| 0.30 | -0.3707 | 0.7262 | 0.7395 \( \times 10^{-4} \) | | -0.3706 | 0.7263 | 0.7392 \( \times 10^{-4} \) |
| 0.35 | -0.3473 | 0.7418 | 0.7013 \( \times 10^{-4} \) | | -0.3471 | 0.7419 | 0.7010 \( \times 10^{-4} \) |
| 0.40 | -0.3197 | 0.7552 | 0.6511 \( \times 10^{-4} \) | | -0.3195 | 0.7553 | 0.6509 \( \times 10^{-4} \) |
| 0.50 | -0.2552 | 0.7772 | 0.5252 \( \times 10^{-4} \) | | -0.2551 | 0.7773 | 0.5251 \( \times 10^{-4} \) |
| 0.60 | -0.1843 | 0.7945 | 0.3793 \( \times 10^{-4} \) | | -0.1843 | 0.7946 | 0.3792 \( \times 10^{-4} \) |
| 0.70 | -0.1147 | 0.8085 | 0.2370 \( \times 10^{-4} \) | | -0.1146 | 0.8085 | 0.2369 \( \times 10^{-4} \) |
| 0.80 | -0.5504 \( \times 10^{-1} \) | 0.8203 | 0.1151 \( \times 10^{-4} \) | | -0.5508 \( \times 10^{-1} \) | 0.8204 | 0.1150 \( \times 10^{-4} \) |
| 0.90 | -0.1516 \( \times 10^{-1} \) | 0.8314 | 0.3037 \( \times 10^{-5} \) | | -0.1524 \( \times 10^{-1} \) | 0.8314 | 0.3038 \( \times 10^{-5} \) |
Table 4. Comparison of the solutions for the "boundary layer" equations and the whole equations for a two cell vortex at $r = 0.6$

<table>
<thead>
<tr>
<th>$z$</th>
<th>&quot;Boundary layer&quot; equations</th>
<th>Whole equations</th>
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<td>$V$</td>
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<td>1.2132</td>
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<td>-0.4013</td>
<td>1.3096</td>
</tr>
<tr>
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<td>-0.3198</td>
<td>1.3579</td>
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<td>-0.2539</td>
<td>1.3826</td>
</tr>
<tr>
<td>0.35</td>
<td>-0.2052</td>
<td>1.4009</td>
</tr>
<tr>
<td>0.40</td>
<td>-0.1547</td>
<td>1.3884</td>
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<td>-0.1058</td>
<td>1.3285</td>
</tr>
<tr>
<td>0.60</td>
<td>-0.1121</td>
<td>1.2241</td>
</tr>
<tr>
<td>0.70</td>
<td>-0.1001</td>
<td>1.1215</td>
</tr>
<tr>
<td>0.80</td>
<td>-0.7368</td>
<td>$10^{-1}$</td>
</tr>
<tr>
<td>0.90</td>
<td>-0.3791</td>
<td>$10^{-1}$</td>
</tr>
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Table 5. Comparison of the solutions for the "boundary layer" equations and the whole equations for a two cell vortex at r = 1.6

<table>
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<th>z</th>
<th>&quot;Boundary layer&quot; equations</th>
<th></th>
<th>Whole equations</th>
<th></th>
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</thead>
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<td>U</td>
<td>V</td>
<td>q</td>
<td>U</td>
</tr>
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<td>0.6284</td>
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<td>0.15</td>
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<td>0.6674</td>
<td>0.7564 x 10^{-4}</td>
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<td>0.6958</td>
<td>0.7762 x 10^{-4}</td>
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<td>0.7182</td>
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<tr>
<td>0.30</td>
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<td>0.7367</td>
<td>0.7461 x 10^{-4}</td>
<td>-0.3622</td>
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<tr>
<td>0.35</td>
<td>-0.3369</td>
<td>0.7523</td>
<td>0.7068 x 10^{-4}</td>
<td>-0.3370</td>
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<tr>
<td>0.40</td>
<td>-0.3084</td>
<td>0.7657</td>
<td>0.6556 x 10^{-4}</td>
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<tr>
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<td>-0.2440</td>
<td>0.7873</td>
<td>0.5283 x 10^{-4}</td>
<td>-0.2441</td>
</tr>
<tr>
<td>0.60</td>
<td>-0.1745</td>
<td>0.8035</td>
<td>0.3826 x 10^{-4}</td>
<td>-0.1747</td>
</tr>
<tr>
<td>0.70</td>
<td>-0.1069</td>
<td>0.8157</td>
<td>0.2395 x 10^{-4}</td>
<td>-0.1072</td>
</tr>
<tr>
<td>0.80</td>
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<td>0.8256</td>
<td>0.1167 x 10^{-4}</td>
<td>-0.5038 x 10^{-1}</td>
</tr>
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<td>0.8347</td>
<td>0.3108 x 10^{-5}</td>
<td>-0.1410 x 10^{-1}</td>
</tr>
</tbody>
</table>
magnitude of the axial velocity in the radially inner parts of the flow field is not small, normal boundary layer flow concepts cannot be used to explain the closeness of the solutions to the two sets of equations. The explanation must come from some different ideas. The reasons for the closeness of the two sets of solutions are mainly the following two.

i) For much of the flow field, in the governing equations, the convection terms are much larger than the diffusion terms. Since the only difference between the whole equations and the "boundary layer" equations is the absence of radial diffusion terms, in the latter, when the convection terms are much larger than the diffusion terms, the effect of the diffusion terms becomes small and so the solutions of the two sets of equations become close.

ii) An additional reason for the closeness of the two sets of solutions is the nature of the solutions for the flow variables. The nature of the equations that govern a rotating flow and the nature of the radial variation of the flow variables for a vortex flow are such that for an axisymmetric vortex flow the radial diffusion is zero or much smaller than the axial diffusion. This is illustrated well by the diffusion terms in the tangential momentum equation when the diffusivity is constant, $\mu(V_{rr} + V_r/r - V/r^2)$. It is easily seen that when $V \sim 1/r$ or $V \sim r$ the net radial diffusion is zero. So, for parts of the vortex flow field where $V \sim r$ or $V \sim 1/r$
the net radial diffusion would be zero.

For the numerical method that has been used, the difference in the costs of solving the whole equations and the "boundary layer" equations is due to the additional radial diffusion terms in the whole equations and the additional solution of the equation for the pressure during the solution of the whole equations. The difference in the costs depends on the number of iterations at each time point. With five iterations at each time point, on an IBM 365 computer, the difference in the costs is about 20\% the cost of solving "the boundary layer" equations.

The solutions that have been discussed in this section are for one set of boundary conditions. For different sets of boundary conditions different solutions would be obtained. Examining the solutions that have been discussed in this section and the governing equations of the flow it can be seen that the solution to the governing equations that is obtained is greatly affected by the radial inflow per unit area that is maintained at the radial outer boundary. So, in order to get a better understanding of this effect, a second set of solutions has also been obtained. For the second set of solutions, the same boundary conditions as for the solutions that have been discussed in this section were used except for the radial inflow per unit area at the radial outer boundary and the radial boundary conditions for q.
This second set of solutions is presented and discussed in the following section.

D. Solutions for a Weak Inflow at the Radial Outer Boundary

For the second set of solutions, the boundary condition \( q(0,z) = 10^{-2} W(0,z)^2 \left\{ \frac{q(4,z)}{q(4,0.2)} \right\} \) was used and an inflow of 0.0016 per unit area was maintained at \( r = 4 \). Also, the maximum value of \( q \) at \( r = 4 \) was taken to be \( 10^{-2} \{ U(4,0.2) + V^2(4,0.2) + W^2(4,0.2) \} \). The solution was started in the same way the first solution was started. For the earlier periods of the evolution of the vortex flow, the second solution behaves in the same way as the first solution. As in the first solution, the one cell vortex that is started with evolves and changes to a two cell vortex.

The solution to the governing equations at the point of evolution of the vortex flow when the vortex is a one cell vortex that is starting to evolve to a two cell vortex is shown in Figs. 34-41. As it is seen in Figs. 34, 35 and 37, the one cell vortex flow has an upflow region from \( r = 0 \) to \( r = 1.1 \) and a downflow region for \( r > 1.1 \). Also, for this one cell vortex, there is a region of circulating flow near the radial outer boundary. Since the radial inflow at the radial outer boundary is smaller for the second solution than for the first solution, the magnitude of the velocities in
Fig. 34. The meridional flow pattern for a one cell vortex with a weak inflow at the radial outer boundary.
Fig. 35. The variation of the radial velocity for a one cell vortex with a weak inflow at the radial outer boundary.
Fig. 36. The variation of the tangential velocity for a one cell vortex with a weak inflow at the radial outer boundary
Fig. 37. The variation of the axial velocity for a one cell vortex with a weak inflow at the radial outer boundary
Fig. 38. The variation of the turbulence energy for a one cell vortex with a weak inflow at the radial outer boundary
Fig. 39. The variation of the turbulence dissipation rate for a one cell vortex with a weak inflow at the radial outer boundary.
Fig. 40. The variation of the turbulent viscosity for a one cell vortex with a weak inflow at the radial outer boundary.
Fig. 41. The variation of the pressure deficit for a one cell vortex with a weak inflow at the radial outer boundary.
the meridional plane is smaller for this one cell vortex flow than for the one cell vortex flow that is presented in Figs. 10-17. The maximum upflow for this one cell vortex is 0.44 and occurs at \( r = 0.4 \). In Fig. 36, it is seen that the tangential velocity for the one cell vortex has its maximum value at \( z < 1 \) for \( r \leq 1.2 \) but it does not oscillate axially much. It is seen in Figs. 38, 39 and 40 that, because of the boundary conditions that were used for \( q \) at \( r = 0 \) and \( r = 4 \), the value for \( q \), \( \varepsilon \) and \( K \) for this one cell solution of the second set of solutions are larger than those for the one cell solution of the first set of solutions.

The two cell vortex to which the one cell vortex flow evolves is shown in Figs. 42-49. As it is shown in Figs. 42, 43 and 45, the two cell vortex has a downflow region between \( r = 0.0 \) and \( r = 0.3 \), an upflow region between \( r = 0.3 \) and \( r = 1.3 \) and a downflow region for \( r > 1.3 \). Also, there is a region of circulating flow near the radial outer boundary. The maximum downflow is 0.25 and occurs at \( r = 0 \). The maximum upflow is 0.61 and occurs at \( r = 0.6 \). As it is seen in Fig. 44 the tangential velocity is axially oscillatory for small \( r \).

Unlike the first solution, for this second set of solutions, the two cell vortex flow does not evolve to a five cell vortex but to a three cell vortex. The downflow region of the two cell vortex spreads outward radially and an upflow
Fig. 42. The meridional flow pattern for a two cell vortex with a weak inflow at the radial outer boundary
Fig. 43. The variation of the radial velocity for a two cell vortex with a weak inflow at the radial outer boundary.
Fig. 44. The variation of the tangential velocity for a two cell vortex with a weak inflow at the radial outer boundary.
Fig. 45. The variation of the axial velocity for a two cell vortex with a weak inflow at the radial outer boundary.
Fig. 46. The variation of the turbulence energy for a two cell vortex with a weak inflow at the radial outer boundary.
Fig. 47. The variation of the dissipation rate of turbulence energy for a two cell vortex with a weak inflow at the radial outer boundary.
Fig. 48. The variation of the turbulent viscosity for a two cell vortex with a weak inflow at the radial outer boundary.
Fig. 49. The variation of the pressure deficit for a two cell vortex with a weak inflow at the radial outer boundary.
region develops near the axis of the vortex forming a clearly defined three cell vortex. The solution to the governing equations that is obtained when the vortex flow is a three cell vortex is shown in Figs. 50-57. As it is seen from Figs. 50, 51 and 53, for this three cell vortex there is an upflow region from $r = 0$ to $r = 0.3$, a downflow region from $r = 0.3$ to $r = 0.6$, an upflow region from $r = 0.6$ to $r = 1.3$ and a downflow region for $r > 1.3$. There is also a region of circulating flow near the radial outer boundary. As it can be seen in Fig. 52, the axial oscillation of the tangential velocity at $r = 0.6$ has an increased amplitude from what it was when the vortex flow was a two cell vortex.

As the integration of the governing equations is continued the upflow near the axis of the vortex increases until it reaches a maximum value and then starts to decrease towards zero. During the same time the amplitudes of the oscillations of the tangential velocity at $r = 0.6$ increase until a negative tangential velocity is obtained. With further integration, the part of the flow field that was an upflow region when the vortex flow was a three cell vortex becomes a downflow region and at the same time an upflow region develops about $r = 1.6$. Thus by a four cell vortex flow is formed. The solution to the governing equations at this point of the evolution of the vortex flow is shown in Figs. 58-65. From Figs. 58, 59 and 61, it seen that the four cell vortex has a
Fig. 50. The meridional flow pattern for a three cell vortex with a weak inflow at the radial outer boundary.
Fig. 51. The variation of the radial velocity for a three cell vortex with a weak inflow at the radial outer boundary.
Fig. 52. The variation of the tangential velocity for a three cell vortex with a weak inflow at the radial outer boundary
Fig. 53. The variation of the axial velocity for a three cell vortex with a weak inflow at the radial outer boundary
Fig. 54. The variation of the turbulence energy for a three cell vortex with a weak inflow at the radial outer boundary.
Fig. 55. The variation of the dissipation rate of turbulence energy for a three cell vortex with a weak inflow at the radial outer boundary
Fig. 56. The variation of the turbulent viscosity for a three cell vortex with a weak inflow at the radial outer boundary.
Fig. 57. The variation of the pressure deficit for a three cell vortex with a weak inflow at the radial outer boundary
Fig. 58. The meridional flow pattern for a four cell vortex with a weak inflow at the radial outer boundary
Fig. 59. The variation of the radial velocity for a four cell vortex with a weak inflow at the radial outer boundary.
Fig. 60. The variation of the tangential velocity for a four cell vortex with a weak inflow at the radial outer boundary.
Fig. 61. The variation of the axial velocity for a four cell vortex with a weak inflow at the radial outer boundary
Fig. 62. The variation of the turbulence energy for a four cell vortex with a weak inflow at the radial outer boundary.
Fig. 63. The variation of the dissipation rate of turbulence energy for a four cell vortex with a weak inflow at the radial outer boundary.
Fig. 64. The variation of the turbulent viscosity for a four cell vortex with a weak inflow at the radial outer boundary
Fig. 65. The variation of the pressure deficit for a four cell vortex with a weak inflow at the radial outer boundary
downflow region from $r = 0$ to $r = 0.9$, an upflow region from $r = 0.9$ to $r = 1.3$, a downflow region from $r = 1.3$ to $r = 1.5$, an upflow region from $r = 1.5$ to $r = 1.7$ and a downflow region for $r > 1.7$. Also, the region of circulating flow near the radial outer boundary has persisted. As it can be seen from Fig. 60, the negative value for the tangential velocity that is obtained at $r = 0.6$ is quite substantial and so the integration of the governing equations was stopped at this point.

From the solutions that have been presented in this section and in the previous section one can conclude that when the axial flow in a vortex flow field is not constrained the structure of a vortex flow field near a solid surface changes in time continuously. One of the important factors that control the pattern of evolution of the vortex flow is the radial inflow per unit area at the radial outer boundary. For different values for the radial inflow per unit area different patterns of evolution are obtained. But, for any value for the radial inflow per unit area at the radial outer boundary, eventually, the velocities in the inner parts of the flow field become large. Also, eventually, in the inner parts of the flow field, a complexity in the flow structure is created in the form of satellite vortices or the regional breakdown of rotation.
VI. CONCLUSIONS

In Chapter II, a new turbulence model has been developed. In Chapter III, a new method of solving the governing equations of a vortex flow along a radial line has been developed. In Chapter IV, a family of finite difference methods for solving parabolic equations has been developed. For the finite difference methods, a complete convergence analysis is provided. So, with the developments of these three chapters the flow of any general vortex near a solid surface normal to the axis of the vortex can be analyzed thoroughly and dependably.

The turbulence model that has been developed in Chapter II has been developed in a new approach. The model does not make use of the Prandtl's mixing length hypothesis. Instead of following the line of development of Prandtl's mixing length hypothesis, the turbulence diffusivity is determined from a knowledge of the production and dissipation rates of turbulence energy. The relations that are shown in Figs. 7-9 are preliminary results. They are not universally applicable. If those relations are determined in such a way so that the relations are valid for all turbulent flows then the turbulence model would be of universal applicability.

The superiority of such a turbulence model over a mixing length model is obvious. Because all the turbulence
quantities of the present model are expressed in terms of the mean flow variables the present model is of universal applicability while in the mixing length model a function for the mixing length has to be defined for every turbulent flow. The superiority of the present turbulence model over two equation models of turbulence is due to application rather than basic concept. If the nondimensional variables in two equation models can be determined in such a way so that they can have universally correct values, then, conceptually, two equation models would be as good as the present model. At the present time, this cannot be done. Even if it can be done, the present model would still be cheaper because in the present model, one has to solve only one differential equation. Multi-equation models of turbulence are much more expensive to use than the present model. How much more expensive any multi-equation model is depends on the number of differential equations in the multi-equation model. The cost for multi-equation models would be six or more times the cost for the present model. Because in multi-equation models differential equations are provided for each Reynolds stress, in concept, multi-equation models would be more accurate than the present model. But whether any particular multi-equation model is actually more accurate than the present model or not depends on the quality of the experimental data that is used in the model and the way the experimental
data is used. But for most turbulent flows, even if the accuracy of the present model can be improved by using a multi-equation model, most probably, the improved accuracy that may be obtained would not be worth the additional cost that is incurred by the multi-equation model.

The accuracy of the present model can possibly be improved without increasing the cost if generalized expressions for every Reynolds stress term can be obtained from experimental data in the same way that a generalized expression for the turbulence energy dissipation rate has been obtained for the present model. The cost of the present model may be decreased and the accuracy of the present model may also be improved if the differential equation for the kinetic energy of turbulence of the present model would be replaced by a generalized expression for the kinetic energy of turbulence that is obtained from experimental data. These would be very good subjects for further research in turbulence modeling.

The method that has been developed in Chapter III for the solution of the governing equations of a vortex flow is very complete. The method does not involve any assumptions or simplifications. With the method, along a radial line, the governing equations are solved completely. In contrast to the completeness of the method is the simplicity of the method. With the method, complete solutions to the governing equations are obtained quite easily. This is in contrast also
to the solution processes that have been developed by other investigators, for example, Bellamy-Knights (1971). The solution process for obtaining solutions for multi-cell vortex flows that has been developed by Bellamy-Knights (1971) is more time consuming and more expensive than the solution process of Chapter III. Also, his solution process gives restricted solutions while as the method of Chapter III is of unrestricted applicability.

The basic ideas of the method of Chapter III can probably be not improved. But the application of the ideas behind the method can be extended. In the same way that a method of solving the governing equations of a vortex flow along a radial line has been developed, a method of solving the governing equations along an axial line or any other well-defined line can be developed. In addition to obtaining solutions to the governing equations, the method can also be used to estimate values for the flow variables between data points when one has a scanty experimental or observational data set.

The family of finite difference methods that has been developed in Chapter IV is also another complete development that has been presented in this thesis. The family of finite difference methods that has been developed is a large family. For all the finite difference methods that are members of the family a complete convergence analysis has been developed in Chapter IV. The convergence analysis method does not involve
any linearization, simplifications or restrictions due to boundary conditions. This is the main strength of the finite difference methods and what makes the methods dependable and trouble-free. During the application of one of the finite difference methods in the solution of the governing equations of a vortex flow near a solid surface that has been presented in Chapter V no problem of any kind was encountered.

Many finite difference methods for nonlinear parabolic equation that have been in use up to the present time have a linearized numerical stability analysis. This has restricted the application of the methods to problems where the linearized analysis is adequate. So, in addition to the family of finite difference methods that have been discussed in Chapter IV the convergence analysis method that has been developed in Chapter IV can also be used to analyze and extend the application of any finite difference method for nonlinear parabolic equations.

In terms of the tangential velocity at the axial outer boundary, the solutions that have been presented in Chapter V are only for one type of vortex. Before generalized descriptions and explanations for the flow of general vortices near a solid surface can be made solutions to many types of vortices with different sets of boundary conditions must be obtained. Still, from the solutions of Chapter V, in conjunction with the governing equations of a vortex flow one can
make one conclusion on the nature of vortex flows near a solid surface. This is that, when $v_{r}^{m}r_{m}^{*}$, the product of the reference velocity and reference distance, is not very small, from the axis of the vortex up to the radial point where the magnitude of $V_{\infty}(r)$ is of the same order as the magnitude of $(\mu + K)$, the nature of the vortex flow is determined by the convective transport processes of the flow. All the significant and outstanding features of a vortex flow in the radially inner regions of the vortex flow field are due to the convective transport of momentum. This can be seen easily by writing the governing equations in an approximate form as:

$$U_{t} = -UU_{r} - WU_{z} - P_{r} + V^{2}/r$$  \hspace{1cm} (186)

$$V_{t} = -UV_{r} - WV_{z} - \frac{UV}{r}$$  \hspace{1cm} (187)

$$W = -\int_{0}^{2} (U_{r} + U/r)dr$$  \hspace{1cm} (188)

$$P = P_{\infty}(r)$$  \hspace{1cm} (189)

By examining these equations one can follow the processes that create features of vortex flows such as the axial variation of the tangential velocity and the cell structure.

All of the developments in this thesis have been for incompressible flow with negligible temperature effects. The developments can be extended to flows where temperature
effects are important without much difficulty. All that is needed to be done is the inclusion of an equation for the temperature in the set of governing equations for a vortex flow. The developments can also be extended to flows where compressibility effects are important without too much difficulty. This would entail the inclusion of a state equation in the set of governing equations. For both of these extensions experimental turbulence data for flows where temperature and compressibility effects are significant would have to be used. But, the solution processes of Chapters III and IV would remain basically unchanged.
VII. REFERENCES


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IX. APPENDIX

A. Some Remarks on the Application of the Turbulence Model of Chapter II

The relations that are shown in Figs. 7-9 were obtained using the data sets of Laufer (1954), Klebanoff (1955), Townsend (1949) and Townsend (1951). From Laufer's data set, the part for $r/a = 0-0.9$ was used. ($r$ is the radial distance from the center of the pipe and $a$ is the radius of the pipe.) From Klebanoff's data set, the part for $z/\delta = 0.1-1.0$ was used. ($z$ is the distance from the surface and $\delta$ is the boundary layer thickness.) In both of these data sets that have been used, data from the law of the wall region has been excluded. From Townsend's wake data sets, the data for $x/d = 160$ was used. ($x$ is the distance from the axis of a cylinder of diameter $d$.) From Townsend's boundary layer data, the part for the outer half of the boundary layer was used. In all of the data sets, for all of the relations, the velocities are relative to the solid surface associated with each turbulent flow. So, during the application of the turbulence model with the relations of Figs. 7-9 to any wake, boundary layer or pipe flow, the velocities must be relative to the solid surface associated with the particular turbulent flow. The choice of this reference frame is arbitrary, dictated only by the reference frame that was used in the development of the relations.
Since the relations of Figs. 7-9 are functions of a velocity \( \sqrt{Q} \), they are not invariant to some coordinate transformations and particularly to a Galilean transformation. In fluid mechanics, Galilean transformation is applied to the equations of fluid motion frequently, for example, in the analysis of wake flows and boundary layer flows. If the equations of fluid motion are to be uniformly applied to a variety of flow configurations (wake flows, boundary layer flows, vortex flows etc.) without any worry about coordinate systems or reference frames, then the expressions for the Reynolds stresses have to be invariant to a Galilean transformation. "Since the expressions derived in Chapter II depend upon the frame of reference chosen (through the use of variable \( Q \)) they may not be applicable to a vortex flow.\(^1\)"

In addition to not being invariant to a Galilean transformation the present expressions for \( \Lambda/\varepsilon, \varepsilon \) and \( Du \) are not universal. For example, they do not apply to a homogenous flow. The applicability of the expressions to a vortex flow is not known. From a strictly correct point of view, the application of the relations of Figs. 7-9 to the analysis of a vortex flow is valid only if there is experimental evidence

\(^1\)This is a criticism on the applicability of the present turbulence model to a vortex flow that has been made by L. N. Wilson from the Department of Aerospace Engineering at Iowa State University.
that supports the correctness of the application of the relations to a vortex flow. At the present there is no experimental data on the nature of turbulence in a vortex flow near a surface. The present relations for $\Lambda/\varepsilon$, $\varepsilon$ and $D_\mu$ may be correct or may not be correct for a vortex flow. The use of the relations in the analysis of a vortex flow is therefore the same as guessing or estimating relations for $\Lambda/\varepsilon$, $\varepsilon$ and $D_\mu$ for a vortex flow. In the use of the relations of Figs. 7-9 the guess or estimation is being guided by data from a boundary layer flow, a pipe flow and a wake flow. Because the relations are applicable to three different types of flow it is estimated that they are also applicable to a vortex flow. But there is nothing in the development of the relations that guarantees the correct applicability of the relations to vortex flow near a solid surface.

The accuracy or inaccuracy of the solutions for $q$, $\varepsilon$ and $K$ that have been obtained in Chapter V is not known. From a qualitative point of view, the solutions that have been obtained are what they would be expected to be. But, as far as the quantitative accuracy of the solutions is concerned, without a comparison with experimental results, there is nothing that can be said about it.

If there is any inaccuracy in the solutions for $K$, it may affect the quantitative accuracy of the solutions for $U$, $V$, $W$ and $P$ to some extent, especially for large $r$. But, because of
the small magnitude of $K$ compared with the magnitudes of the velocities, the qualitative nature of the solutions for $U$, $V$, $W$ and $P$ would not be affected by errors in the solution for $K$.

Finally, in this thesis, as far as the turbulence model is concerned, what is being emphasized is not the presently obtained relations for $\Lambda/\varepsilon$, $\varepsilon$ and $Du$. The present relations are preliminary results. The emphasis is on the form of the turbulence model itself, a differential equation for $q$ and generalized empirical expressions for $\Lambda/\varepsilon$, $\varepsilon$, $Du$ and any other turbulence quantity of interest. As far as $\Lambda/\varepsilon$, $\varepsilon$ and $Du$ are concerned, probably, many different relations can be obtained for them, and further research to obtain relations for $\Lambda/\varepsilon$, $\varepsilon$ and $Du$ that are universally applicable and invariant to a Galilean transformation is encouraged.

**B. Some Remarks on the Application of the Finite Difference Methods of Chapter IV**

The family of finite difference methods that have been developed in Chapter IV are of a very wide applicability. They can be used to solve any set of equations of the form:

$$\dot{\vec{y}}_t = F(\vec{x}, t, \vec{y}, \dot{\vec{y}}_x, \ddot{\vec{y}}_x, \ddot{\vec{y}}_x)$$ \hspace{1cm} (A4)

where, $\vec{y}$ is the vector variable, $\vec{x}$ is the space vector and a subscript of $\vec{x}$ designates the differentiation of the components of the vector variable in terms of the components of the space variable.
Such equations are encountered very frequently in the mathematical description of physical processes. The equations could contain only first order derivatives, only second order derivatives or both first and second order derivatives. They can be linear or nonlinear. The application of the finite difference methods of Chapter IV to solve such equations is encouraged.

Also, during the development of numerical methods for such equations for any particular purpose the application of the convergence analysis method that has been developed in Chapter IV is encouraged. Furthermore, if any existing method for such equations does not have a complete convergence analysis, the application of the convergence analysis method of Chapter IV to the numerical method and the expansion of its applicability is suggested.