Non-linear resonances in spin-orbit coupling problems with three degrees of freedom

Bradley Alan Eucker

Iowa State University

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NON-LINEAR RESONANCES IN SPIN-ORBIT COUPLING PROBLEMS WITH THREE DEGREES OF FREEDOM

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Non-linear resonances in spin-orbit coupling problems with three degrees of freedom

by

Bradley Alan Eucker

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of the Requirements for the Degree of DOCTOR OF PHILOSOPHY

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I. INTRODUCTION

Prior to 1962, the prevailing theory concerning the rotation of Mercury about its axis held that the same side continually faced the sun. More recent radar doppler measurements, however, indicate a sidereal period of 58.65 ± .006 days, almost exactly 2/3 of the determined orbital period of 87.969 days. Peale and Gold [18] first published a possible explanation solely in terms of tidal torques. They showed that with an orbital eccentricity of .2, the average tidal torque could vanish for a rotation period of about 59 days, resulting in a stable spin state. It was Colombo [1] who first pointed out that the rotational period was almost precisely 2/3 of the orbital period. He suggested that such a situation might actually be stabilized if Mercury possessed a slight permanent equatorial asymmetry. Approximate analyses first published by Colombo and Shapiro [2] and soon after by others (Liu and O'Keefe [13]; Goldreich and Peale [6]; Laslett and Sessler [12]; Jefferys [10]) confirmed that such a spin-orbit resonance could, under certain conditions, be stabilized.

The analysis of Colombo and Shapiro [2] assumed that the spin axis remained perpendicular to the orbit plane, and that the eccentricity of the orbit was fixed. The only apparent problem with this theory is that the probability of Mercury being despun through frictional forces and becoming trapped in the 2/3 spin-orbit ratio is too small to be believable. Counselman and Shapiro [3] developed a theory which encompasses possible variations in orbital eccentricity as well as core-mantle coupling. Without assuming large unexplained variations in the direction of the spin axis, the largest probability was determined to be only .02. Thus, it seemed that a theory was needed which did not require that the spin-axis remain perpendicular to the plane of the orbit.

Hamill and Blitzer [9] developed a unified theory of orbital and rotational resonances encompassing the situation of the Mercury-sun system as a special case. Their study, while restricting neither the eccentricity nor the spin axis, was concerned primarily with resonances and not with long term evolution of the orbit. Consequently, any tidal torques that may have been present were assumed to be negligible, with the result
that their study is of little use to those intent on studying the probability of Mercury attaining the 3:2 spin-orbit resonance.

The model studied by Counselman and Shapiro [3] and several of his predecessors fixed the spin axis to be perpendicular to the plane of the orbit. This assumption leads ultimately to an equation of the form

$$\dot{\theta} = \varepsilon f(t, \theta, \dot{\theta})$$  \hspace{1cm} (I.1)$$

where $\theta$ is an angular variable and $\varepsilon f$ is a small periodic torque which decomposes into a term due to the tidal drag and a term due to the permanent asymmetry of the planet. Murdoch [17] presented a detailed study of this model in which he showed that the results obtained by previous authors are largely independent of the qualitative features of the model.

In this paper, a system of differential equations is developed that is based upon an orbital model which does not require that the spin axis remain perpendicular to the plane of the orbit. Chapter II contains a summary of the notations and terminology that will be used throughout this dissertation. The rest of the main body may be divided into two parts. Chapters III through V contain the development and theoretical study of a system of differential equations which is more or less separate from the problem of resolving Mercury's spin-orbit state. The final two chapters present physical problems that generate systems of equations of the form studied in Chapters III, IV, and V.

In keeping with the spirit of Murdock [17], we observe that the technical results of Chapter V are dependent upon the average torque and not explicitly upon the fine tuned details of the model.
II. NOTATIONS AND TERMINOLOGY

In the development that follows, it will be necessary to deal with several coordinate systems simultaneously. In an effort to minimize potential confusion, we will adhere to the following conventions.

The term "geometric vector" will refer to the physical quantity characterized by a magnitude and direction in three-dimensional space independent of all coordinate frames. The space of geometric vectors is equipped with an inner product and we will denote the associated norm by \(|\cdot|\). Geometric vectors will be denoted by lower case letters with an arrow over the top (i.e. \(\vec{v}, \vec{w}, \) etc.).

Script capital letters such as \(A\) and \(B\) will be used to denote coordinate frames which are orthonormal with respect to the inner product. A geometric vector \(\vec{v}\) expressed as the column vector of its components in a frame \(A\) will be denoted by \([\vec{v}, A]\). All coordinate frames are assumed to be right-handed.

Linear transformations and matrices will both be denoted by non-script capital letters but the difference should be clear from the context. We will follow the usual custom of reserving the letter \(I\) to denote the inertia tensor of a rigid body. Consequently, the letter \(E\) will be used to denote the identity matrix, and the letters \(I_d\) will stand for the identity transformation. When the inertia tensor \(I\) of a rigid body is expressed as a matrix in a specified coordinate frame \(A\), the same convention will be followed as that for vectors. Thus, \([I, A]\) will denote the matrix representation of the inertia tensor expressed in the \(A\)-frame.

For linear transformations, the notation is extended to include both the initial and final coordinate frames by writing: [linear transformation; initial frame, final frame]. Thus, if \(T\) is a linear transformation of geometric vectors, its matrix representation will be denoted by \([T; A, B]\), and hence, \([T; A, B] [\vec{v}, A] = [T \vec{v}, B]\). Only two types of linear transformations will appear in this paper. The first type moves geometric vectors around with respect to a coordinate frame but does not change frames; i.e. \([T; A, A]\). Such a transformation is completely determined by its matrix representation in any specified coordinate frame and we will
abbreviate our notation by writing $[T;A,B] = [T;A]$. The second type of transformation that will appear is merely the identity transformation from one coordinate frame to another. Its matrix representation will be denoted by $[\text{Id};A,B]$, and satisfies $[\text{Id};A,B] [\hat{v},A] = [\hat{v},B]$ for all geometric vectors $\hat{v}$.

The norm used for linear transformations will be the standard operator norm, and will be denoted by $|\cdot|$. Thus, if $T$ is a linear transformation of geometric vectors, then $|T| \equiv \sup_{||x||=1} ||Tx||$. Clearly, any length preserving transformation $R$, such as a rotation, satisfies $|R| = 1$. Thus, for the purpose of calculation, one may exhibit a transformation $T$ in any specific orthonormal coordinate frame $A$ and determine $|T| = \sup_{||[x,A]||=1} ||[Tx,A]||$. As a further consequence, one may interpret any matrix $A$ as the representation of a linear operator in some coordinate frame $A$ and thereby calculate $|A|$. With this in mind, we will freely abuse the terminology and refer to the linear operator norm and the associated matrix norm as being equivalent. The absolute value function for real numbers will also be denoted by $|\cdot|$, but since real numbers will usually be denoted by lower case letters, this should not generate any confusion.

For any fixed coordinate frame $A$, there exists an isomorphism between column vectors and skew symmetric matrices; i.e.

$$[\tilde{w},A] = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \xrightarrow{\text{vect}([\hat{w},A])} \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \Omega.$$  \hspace{1cm} (II.1)

As a notational convenience, we will write $[\tilde{w},A] = \text{vect}(\Omega)$ and $\Omega = \text{skew}([\hat{w},A])$. When this correspondence is employed, the vector in question will always be denoted with a lower case greek letter and the corresponding skew-symmetric matrix will be denoted with the same greek letter in upper case, as with $\tilde{w}$ and $\Omega$ above. The coordinate frame $A$, with respect to which the correspondence is defined, will be clear from the context.
If \([\hat{\mathbf{V}}, A]\) is any column vector, and if \(\hat{\Omega} = \text{skew}([\hat{\omega}, A])\), then it follows that \([\hat{\omega} \times \hat{\mathbf{V}}, A] = \hat{\Omega} [\hat{\mathbf{V}}, A]\). Furthermore, if \(B = [\text{Id}; A, A^\top]\), then the skew-symmetric matrix corresponding to \([\hat{\omega}, A^\top] = B[\hat{\omega}, A]\) is \(B\hat{\Omega}B^{-1}\); this shows the effect of a change in coordinate frame on the correspondence.

Finally, we observe that the right-hand derivative of a real-valued function \(f(t)\) will be denoted by \(D_r f(t)\). Hence,

\[
D_r f(t) = \lim_{h \to 0^+} \frac{f(t+h) - f(t)}{h}.
\]
III. THE EQUATION OF MOTION

In this chapter, a system of differential equations is derived which is applicable to a physical situation such as that formed by the Mercury-sun system. To this end, consider two masses \( M \) and \( m \) with \( m \ll M \). Assume for now that \( m \) is rigid and rotates about an axis \( \hat{\omega} \) through its center of mass. Note: the instantaneous axis of rotation will always be denoted by a lower case greek letter.

We are not concerned here with perturbations in the orbit of \( m \) about \( M \) and will assume that its orbit is known and obeys Kepler's laws. Thus, \( M \) and \( m \) move in a plane which will be referred to as "the plane of the orbit", and their center of mass may be regarded as being at rest.

Let \( I \) be an inertial coordinate frame centered at the center of mass of the \( M,m \) system such that the 3-axis is normal to the plane of the orbit. Define \( A \) to be the coordinate frame parallel to \( I \) whose origin is located at the center of mass of \( m \). If \( \mathbf{r}_m \) is the center of mass of \( m \) as measured from the origin of the \( I \) frame, then the angular momentum \( \mathbf{L}_m \) of \( m \) about \( \mathbf{r}_m \) satisfies

\[
\frac{d\mathbf{L}_m}{dt} = \mathbf{N}_q^e + \mathbf{N}_q^i
\]  

(III.1)

where \( \mathbf{N}_q^e \) is the total external torque exerted on \( m \) by \( M \), and \( \mathbf{N}_q^i \) is the net internal torque of \( m \); (see for example, Symon [19, pp. 163-164]).

From an intuitive standpoint, one would expect \( \mathbf{N}_q^i \) to be zero for any rigid body. The following lemma reveals that, under a relatively strong version of Newton's third law, this is indeed the case.

**Lemma III.1:** Let \( R \) denote the region in space occupied by a rigid body with center of mass \([\mathbf{r}_m, I]\), and let \([\mathbf{F}(\mathbf{x}, \mathbf{y}), I]\) be the "force density" at \([\mathbf{x}, I]\) due to a point mass at \([\mathbf{y}, I]\). If \([\mathbf{F}(\mathbf{x}, \mathbf{y}), I] = -[\mathbf{F}(\mathbf{y}, \mathbf{x}), I]\), and if the force is directed along the line from \([\mathbf{y}, I]\) to \([\mathbf{x}, I]\) then \([\mathbf{N}_q^i, I] = 0\).
Proof: 2[N^,q,I]
= \int \int R_{x,y} \left\{ [(\hat{x}_q - \hat{x}),I] \times \left[ (\hat{x}_q - \hat{x}),I \right] + [(\hat{y}_q - \hat{y}),I] \times \left[ (\hat{y}_q - \hat{y}),I \right] \right\} \, dx \, dy
= \int \int R_{x,y} \left\{ [(\hat{x}_q - \hat{y}),I] \times \left[ (\hat{x}_q - \hat{y}),I \right] \right\} \, dx \, dy
= 0
since \left[ (\hat{x}_q - \hat{y}),I \right] is parallel to \left[ (\hat{x}_q - \hat{y}),I \right].

Thus, under a standard assumption of classical mechanics, it follows that

\[ \left[ \frac{d{\tilde{N}}^q}{dt}, I \right] = \left[ \tilde{N}^e, I \right]. \]

Since the A frame is parallel to the I frame, one obtains

\[ \left[ \frac{dA}{dt}, A \right] = \left[ {\tilde{N}}^c, A \right] \quad (III.2) \]

where \( \tilde{L} \) is the angular momentum of m and \( \tilde{N}^c \) is the torque exerted on m by M. Therefore, although A is not an inertial frame, it may be treated as inertial insofar as the spin of m is concerned.

Most of what is to follow is viewed within the A frame. A notational convenience, defined at this time, is to denote the representation of a geometric vector expressed as a column vector in the A frame with an underscore. Thus, \([\hat{v}_i,A] = \hat{v}\) for all geometric vectors \(\hat{v}\).

Introduce now a new coordinate frame \(B\) whose axes are the principal axes of m such that the 3-axis is the principal axis corresponding to the largest moment of inertia. Let \(\hat{f}^i\) (i=1,2,3) be unit vectors in the directions of the principal axes of m. Then \([\hat{f}^1,B] = (1,0,0)^T\), \([\hat{f}^2,B] = (0,1,0)^T\), and \([\hat{f}^3,B] = (0,0,1)^T\); (the superscript T denotes transpose). With \(A(t)\) defined by \(A(t) = [I_d;B,A]\), (the dependence on time \(t\) entering through the rotation of \(B\) with respect to \(A\)), and since \(A(t)[\hat{f}^i,B] = [\hat{f}^i,A]\) for \(i=1,2,3\), it is clear that the columns of \(A(t)\) are precisely the A-frame representations of \(\hat{f}^i\) (i=1,2,3).
Let \( \hat{\omega}(t) \) be the instantaneous angular velocity of \( m \), and suppose that \( \omega(t) \equiv [\hat{\omega}(t), \Lambda] = (\omega_1(t), \omega_2(t), \omega_3(t))^T \). Then, as described in Chapter II, define \( \Omega(t) = \text{skew}(\omega(t)) \). According to Goldstein [7, p. 133],

\[
\frac{d}{dt} [\hat{\nu}, \Lambda] = \Lambda \left( \frac{d}{dt} [\hat{\nu}, B] \right) + [\hat{\omega} \times \hat{\nu}, \Lambda]
\]  

(III.3)

holds for all geometric vectors \( \hat{\nu} \). Since \( \frac{d}{dt} [\hat{r}, B] = 0 \), it therefore follows that \( \frac{d}{dt} \hat{r} = \frac{d}{dt} [\hat{r}, A] = \hat{\Omega}[\hat{r}, A] = \hat{\Omega}_i \) \((i=1,2,3)\), and hence, the matrix \( \Lambda(t) \) satisfies

\[
\frac{d\Lambda}{dt} = \hat{\Omega}\Lambda.
\]  

(III.4)

Equation (III.4) forms part of a coupled system of differential equations for \( \Lambda(t) \) and \( \hat{\omega}(t) \) as functions of time. To complete the system of equations, consider once again equation (III.2)

\[
\frac{d\Lambda}{dt} = \hat{N}.
\]

(III.5)

The angular momentum \( \hat{L} \) of \( m \) about its center of mass is determined by

\[
\hat{L} = I\hat{\omega}
\]

(III.6)

where \( I \) is the inertia tensor for \( m \). It is assumed that \( m \) is nearly spherical so that the matrix representation of the inertia tensor written in the \( B \) frame may be written in the form

\[
[I, B] = mE + e[m[D, B]]
\]

(III.6)

where \( e \) is a small positive real number. The term \( e[m[D, B]] \) represents the distortion of \( m \) from a sphere.

When the inertia tensor is expressed in the \( A \)-frame, one obtains

\[
[I, A] = m[A(E + \varepsilon[D, B])^{-1}]
\]

\[= mE + e[mA[D, B]^{-1}],
\]

and hence,

\[
\hat{N} = \frac{d}{dt} \left((mE + e[mA[D, B]^{-1}])\omega\right)
\]

\[= e[m\left(\frac{d}{dt} (A[D, B]^{-1})\right) + mE + eA[D, B]^{-1}A]\frac{d\omega}{dt}
\]  

(III.7)
This is to be regarded as a differential equation for \( A \) and \( \omega \) which, when combined with equation (III.4), forms a complete system of differential equations. To use this equation, \( N \) must be replaced by a specific formula for the torque exerted on \( m \) by \( M \). For the moment, it is merely assumed that \( N \) is small, and the notation is accordingly changed from \( N \) to \( \varepsilon N \) to obtain the equation

\[
\varepsilon N = \varepsilon m \left( \frac{d}{dt} (A[D,B]A^{-1}) \right) \omega + m(E + \varepsilon A[D,B]A^{-1}) \frac{d\omega}{dt} \quad (\text{III.8})
\]

For sufficiently small \( \varepsilon \), \( E + \varepsilon A[D,B]A^{-1} \) may be inverted; solving for \( \frac{d\omega}{dt} \) in equation (III.8) yields

\[
\frac{d\omega}{dt} = \varepsilon \sum_{n=0}^{\infty} \left[ (-1)^n \varepsilon^n A[D,B]^{n} A^{-1} \left( \frac{1}{m} N - \Omega A[D,B]A^{-1} \omega \right) \\
- A\frac{d}{dt} [D,B] A^{-1} \omega + A[D,B]A^{-1}\omega \right] \quad (\text{III.9})
\]

For physical systems in which \( m \) is rigid, the torque, \( \varepsilon N \), is a function of time \( t \), a function of \( A \), and a function of \([D,B]\). Furthermore, \([D,B]\) will be constant. With this in mind, and in view of equations (III.4) and (III.9), the following coupled system of differential equations is presented for study:

\[
\frac{dA}{dt} = \Omega A \quad (\text{III.10})
\]

\[
\frac{d\omega}{dt} = \varepsilon f_1(t,\omega,A) + \varepsilon^2 f_2(t,\omega,A,\varepsilon)
\]

with initial conditions \( A(0) = A^* \), \( \omega(0) = \omega^* \).

It will be assumed that \( f_1 \) and \( f_2 \) are \( 2\pi \)-periodic in time \( t \), and \( \mathcal{C}^3 \) in all arguments.
Although system (III.10) was developed under the assumption that \( m \) was rigid, system (III.10) is equally valid for systems in which \( m \) is not rigid. In these cases, \([D,B]\) must specify the effects of a non-rigid \( m \). Unfortunately, the true expression for \([D,B]\) and, hence, \( N \) will depend upon \( \omega \), \( A \), the position of \( m \), and perhaps other factors in some unknown manner. The best that may be accomplished is to assume a reasonable and hopefully representative model for the interactions between the masses. Effectively, this is equivalent to choosing a convenient form for \([D,B]\).

In the application to the problem of determining Mercury's spin-orbit ratio, (Chapter VII), several models for \([D,B]\) are briefly discussed. All of these models explicitly describe the functional dependence of \([D,B]\) on \( \omega \), \( A \), and the position of the sun \( s \).
IV. EXISTENCE OF THE AVERAGING TRANSFORMATION

A. Preparation for Averaging

In Chapter III a coupled system of differential equations was developed which is repeated here in the form

\[
\frac{dA}{dt} = \Omega A
\]

\[
\frac{dw}{dt} = \epsilon f_1(t, w, A) + \epsilon^2 f_2(t, w, A, \epsilon)
\]

(A.1)

\[
A(0) = A_\ast; \quad w(0) = w_\ast.
\]

Recall that \( \Omega = \text{skew}(\omega) \), and that \( f_1 \) and \( f_2 \) are both assumed to be 2\( \pi \)-periodic in \( t \) and \( C^3 \) in all arguments.

Of primary concern in this study are quasiperiodic steady state solutions of system (IV.1) for small \( \epsilon \). Obtaining solutions of the general coupled system is a formidable task and will not be attempted here. Instead, a study will be made of the corresponding "averaged" system of equations, ((IV.17), and (IV.18)), and results applicable to solutions of system (IV.1) will be deduced from the information gained.

The goal of this chapter is to prove the existence of a transformation from system (IV.1) to either system (IV.17) or system (IV.18) depending upon the rationality or irrationality of \(||\omega_\ast|||\). To achieve this goal, it will be necessary to define local coordinates on \( \text{SO}(3) \), the Lie group of matrices \( M \) satisfying \( MM^T = E \) and \( \det(M) = 1 \).

We begin by considering the first equation of system (IV.1):

\[
\frac{dA}{dt} = \Omega A
\]

(IV.2)

where \( \Omega = \text{skew}(\omega) \), and \( A(0) = A_\ast \). The following theorem due to Wronski, (see, for example, Hale [8, p. 82]), will be used in the proof of lemma IV.2.

**Theorem IV.1:** If \( X(t) \) is any matrix solution of the equation

\[
\frac{dx}{dt} = S(t) X,
\]

then \( \det(X(t)) = \det(X(0)) \exp \left\{ \int_0^t \text{trace}(S(t)) \, dt \right\} \).
Lemma IV.2: If \( A(t) \) is any matrix solution of system (IV.1) such that \( A \) is an element of \( \text{SO}(3) \), then \( A(t) \) remains in \( \text{SO}(3) \) for all time.

Proof: If \( A(t) \) satisfies equation (IV.2) for all time, then, since 
\[
\Omega^T = -\Omega,
\]
\[
\frac{d}{dt} (A^T A) = \left[ \frac{d}{dt} (A^T) \right] A + A^T \left( \frac{d}{dt} A \right)
\]
\[
= A^T \Omega A + A^T \Omega A
\]
\[
= 0
\]
and hence, \( A^T A \) is constant. Since \( A \) is an element of \( \text{SO}(3) \), \( A^T A = I \) for all time. Furthermore, since trace(\( \Omega(t) \)) = 0 for all time, by theorem IV.1 it follows that \( \det(A(t)) = 1 \) for all time.

If \( A \) is an arbitrary element of \( \text{SO}(3) \) and if \( \omega \) is an arbitrary column vector with \( \Omega = \text{skew}(\omega) \), then \( A(t) = e^{\Omega t} A \), \( \omega(t) = \omega \) is the zero-th order approximation, (i.e. the solution when \( \varepsilon = 0 \)), to the solution of system (IV.1). According to lemma IV.2, \( A(t) \) defines a path remaining in \( \text{SO}(3) \) for all time. Physically, this solution corresponds to the situation in which \( m \) rotates about the stationary axis \( \omega \) with constant angular velocity \( ||\omega|| \).

By the very nature of system (IV.1), one expects that solutions of the full system for arbitrary small \( \varepsilon \) and initial conditions \( \omega, A \) will "remain close to" the zero-th order solution, at least for a certain length of time. We expand about the zero-th order solution by making two changes of variables. The first change of variables localizes attention near \( \omega \) via the substitution \( w(t) = \omega + \delta(t) \) with the corresponding \( \Omega(t) = \Omega + \Delta(t) \). Substituting into system (IV.1) and expanding \( f(t, w, A) \) around \( w = \omega \) yields
\[
\frac{dA}{dt} = (\Omega + \Delta) A
\]
\[
\frac{d\delta}{dt} = \varepsilon f_1(t, \omega, A) + O(\varepsilon ||\delta||) + O(\varepsilon^2)
\]
(IV.3)
\[
\delta(0) = 0; A(0) = A.
\]
Here \( O(h) \) represents a term which is \( 2\pi \)-periodic in \( t \) and, when divided by \( h \), remains bounded as \( h \to 0 \) uniformly for \( \delta \) and \( A \) in compact sets.
It has already been observed that the zero-th order approximation corresponds to a rotation about a fixed axis. As such, it is completely described by specification of the axis and the angle of rotation. One would expect then, that solutions that are "near" the zero-th order solution may be characterized by a "fast" coordinate, corresponding to the angle of rotation, and certain other "slow" coordinates which specify the "distance", in some sense, from the zero-th order approximation. These notions will be made more precise in the next section, but for now serve as motivation for the second change of variables. Its purpose is to localize near \( A \) in such a way as to facilitate the definitions of the next section.

Define a new coordinate frame \( \tilde{A} \) to be a coordinate frame that has the same origin as \( A \), is stationary with respect to \( A \), and whose 3-axis is parallel to \( \omega_\infty \). Let \( \tilde{A} = [\text{Id}; \tilde{A}, A] \) and set \( B(t) = A^{-1}(t)A^{-1}_\infty A \) to obtain the system

\[
\frac{d\mathbf{\delta}}{dt} = (\tilde{\Omega}_\infty + \tilde{\Omega}) \mathbf{B}
\]

\[
\frac{\mathbf{\delta}}{d\tau} = \sum \mathbf{g}_1(t, \omega_\infty, A_\infty, A, B) + O(\varepsilon ||\mathbf{\delta}||) + O(\varepsilon^2)
\]  

(IV.4)

\( \mathbf{\delta}(0) = \mathbf{0}; \ B(0) = E \)

uniformly for \( \mathbf{\delta} \) and \( E \) in compact sets. Here \( \tilde{\Omega}_\infty = \tilde{A}^{-1} \Omega_\infty \tilde{A}, \tilde{\Omega} = \tilde{A}^{-1} \Delta \tilde{A} \), and \( \mathbf{g}_1(t, \omega_\infty, A_\infty, A, B) = \mathbf{g}_1(t, \omega_\infty, A, B) \).

B. Local Coordinates on \( \text{SO}(3) \)

Consider two arbitrary coordinate frames \( C \) and \( D \) which have a common origin. For fixed \( C \), there is a 1-1 correspondence between coordinate frames \( D \) and the matrices in \( \text{SO}(3) \); namely \( D \leftrightarrow [\text{Id}; D, C] \). While it is not possible to define coordinates which are valid on all of \( \text{SO}(3) \), it is possible to define local coordinates which are valid "near" the path in \( \text{SO}(3) \) corresponding to a rotation about the 3-axis; i.e. "near"

\[
\begin{pmatrix}
\cos \upsilon & -\sin \upsilon & 0 \\
\sin \upsilon & \cos \upsilon & 0 \\
0 & 0 & 1
\end{pmatrix} \quad 0 \leq \upsilon \leq 2\pi
\]
The standard Eulerian angles are a set of local coordinates, but are inadequate for our purpose because of the ambiguity involved in defining the Keplerian elements $\Omega$ and $\omega$ when the inclination $i$ is zero; see Figure 2. The path in $\text{SO}(3)$ listed above corresponds precisely to the case in which $i = 0$.

A set of smooth local coordinates will now be defined which move the ambiguity away from the path of interest. The positive direction for all angles is determined by the right-hand rule.

Given two coordinate frames $C$ and $D$ with a common origin such that the angle between the respective 3-axes of the two frames is less than $\pi/2$, the matrix $[\text{Id};D,C]$ is determined in three steps:

1. Let $\ell_1$ be the intersection of the $D$-frame 1,3-plane and the $C$-frame 1,2-plane; (the intersection is a distinct line since the respective 3-axes are not perpendicular). Define $\theta$, (see Figure 3), to be the angle between $\ell_1$ and the $C$-frame 1-axis, and denote the coordinate frame obtained by rotating the $C$-frame through the angle $\theta$ about its 3-axis by $\mathring{C}$. Then $\ell_1$ is the 1-axis of the $\mathring{C}$-frame.

2. Since $\ell_1$ lies in the $D$-frame 1,3-plane it is perpendicular to the $D$-frame 2-axis. Define $\phi$, (see Figure 4), to be the angle between the $\mathring{C}$-frame 2-axis and the $D$-frame 2-axis as measured about $\ell_1$. Note: it is possible for these axes to coincide, in which case $\phi \equiv 0$. Define $\mathring{D}$ to be the coordinate frame obtained from $\mathring{C}$ by rotating through the angle $\phi$ about $\ell_1$; (= the $\mathring{C}$-frame 1-axis). Observe that the 2-axis of the $\mathring{D}$-frame now coincides with the 2-axis of the $D$-frame.

3. Define $\psi$ to be the angle between $\ell_1$ and the $D$-frame 1-axis as measured about the $\mathring{D}$-frame 2-axis. Then the $D$-frame is obtained from the $\mathring{D}$-frame by a rotation through the angle $\psi$ about the $\mathring{D}$-frame 2-axis.

In Appendix A, the matrices $[\text{Id};D,C]$ are calculated for the orientations in which $D$ is obtained from $C$ by a rotation about any of the $C$-frame coordinate axes. Interpreting the results of the appendix for the more arbitrary orientation outlined above gives
\[ [\text{Id}; \vec{D}, \vec{C}] = \begin{pmatrix} \cos \psi & 0 & \sin \psi \\ 0 & 1 & 0 \\ -\sin \psi & 0 & \cos \psi \end{pmatrix}, \quad [\text{Id}; \vec{D}, \vec{C}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}, \]

and

\[ [\text{Id}; \vec{C}, \vec{C}] = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Thus, \[ [\text{Id}; \vec{D}, \vec{C}] = [\text{Id}; \vec{C}, \vec{C}] [\text{Id}; \vec{D}, \vec{C}] [\text{Id}; \vec{C}, \vec{D}] = T(\theta)S(\phi)R(\psi) \]

\[
\begin{pmatrix}
cos \theta & cos \psi & -sin \theta & sin \psi \\
-sin \theta & cos \phi & sin \theta & sin \phi \\
-sin \theta & cos \phi & sin \theta & sin \phi \\
0 & 0 & 1
\end{pmatrix}.
\]

(IV.5)

where

\[ T(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}, \]

and

\[ R(\psi) = \begin{pmatrix} \cos \psi & 0 & \sin \psi \\ 0 & 1 & 0 \\ -\sin \psi & 0 & \cos \psi \end{pmatrix}. \]

Observe that if a linear transformation \( R \) is defined by \([\vec{v}, \vec{C}] = [R\vec{v}, \vec{D}]\) for all geometric vectors \( \vec{v} \), then

\[ [R\vec{v}, \vec{D}] = [\text{Id}; \vec{C}, \vec{D}] [R; \vec{C}] [\vec{v}, \vec{C}] \]

\[ = [\text{Id}; \vec{C}, \vec{D}] [R; \vec{C}] [R\vec{v}, \vec{D}], \]

and hence,

\[ [R; \vec{C}] = [\text{Id}; \vec{C}, \vec{D}]^{-1} \]

\[ = [\text{Id}; \vec{D}, \vec{C}]. \]

Thus, \([\text{Id}; \vec{D}, \vec{C}]\) may also be interpreted as the matrix representation of the linear transformation \([R; \vec{C}]\) that rotates geometric vectors \( \vec{v} \) in
such a way that the representation of the final position of \( \hat{v} \) expressed in the D-frame is identical to the representation of the initial position of \( \hat{v} \) expressed in the C-frame (i.e. \([\hat{v}, \hat{C}] = [R\hat{v}, D]\)). This second interpretation will be of immediate use in the next section.

C. System (IV.4) Expressed in Local Coordinates for SO(3)

Consider once more system (IV.4) which we list here again as

\[
\frac{dB}{dt} = (\hat{O}_* + \hat{A}) B
\]

\[
\frac{d\hat{O}}{dt} = eB_1(t, \omega_x, \hat{A}_x, A, B) + O(\varepsilon |\hat{O}|) + O(\varepsilon^2)
\]

\[
\hat{O}(0) = 0; \quad B(0) = E.
\]

Recall that \( \hat{A} \) is a coordinate frame whose 3-axis is parallel to \( \omega_x \), so \( \hat{A} = [Id; \hat{A}, A], \hat{\Omega}_x + \hat{\Lambda} = \hat{A}^{-1}(\Omega_x + \Delta)\hat{A} \), and that \( O(h) \) represents a term which is \( 2\pi \)-periodic in \( t \), and, when divided by \( h \), remains bounded as \( h \to 0 \) uniformly for \( \hat{A} \) and \( B \) in compact sets.

Suppose that \( B(t), \hat{O}(t) \) is the solution of system (IV.4). Interpret \( \hat{A} \) as the initial position of a coordinate frame \( \hat{B} \) which is fixed in m. Then \( B(t) \) may be interpreted as the matrix representation of the linear transformation that rotates geometric vectors from their position with respect to \( \hat{A} \) into the same relative position with respect to \( \hat{B} \). Since \( B(0) = E \), there exists an interval \( [0, t_0] \) such that the conditions of the previous section are satisfied by \( C = \hat{A}, \) and \( D = \hat{B} \) for \( 0 \leq t \leq t_0 \).

Therefore, \( B(t) \) implicitly defines functions \( \theta(t), \phi(t) \) and \( \psi(t) \) for \( t \) on some interval \( [0, t_0] \). The length of the interval is the length of time that the angle between \( \omega_x \) and the 3-axis of the \( \hat{B} \)-frame is less than \( \pi/2 \). Thus, from the previous section,

\[
B(t) = \begin{pmatrix}
\cos\theta \cos\phi & -\sin\theta \sin\phi & \sin\psi \\
\sin\theta \cos\psi + \cos\theta \sin\phi & \cos\theta \cos\psi - \sin\theta \sin\phi & \cos\phi \\
-\cos\phi \sin\psi & -\sin\phi \cos\phi & \cos\phi \cos\psi
\end{pmatrix}
\]

where \( \theta = \theta(t), \phi = \phi(t), \) and \( \psi = \psi(t) \).
If \( \Gamma(t) = \begin{pmatrix} 0 & -\gamma_3(t) & \gamma_2(t) \\ \gamma_3(t) & 0 & -\gamma_1(t) \\ -\gamma_2(t) & \gamma_1(t) & 0 \end{pmatrix} \) is any skew-symmetric matrix, direct computation reveals that

\[
\Gamma(t) B(t) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}
\]

where, (omitting unnecessary terms),

\[
a_{12} = \gamma_2 \sin\phi - \gamma_3 \cos\theta \cos\phi,
\]
\[
a_{22} = -\gamma_3 \sin\theta \cos\phi - \gamma_1 \sin\phi,
\]
\[
a_{31} = [\sin\theta \cos\psi + \cos\theta \sin\phi \sin\psi] \gamma_1 - [\cos\theta \cos\psi - \sin\theta \sin\phi \sin\psi] \gamma_2,
\]
\[
a_{32} = \gamma_1 \cos\theta \cos\phi + \gamma_2 \sin\theta \cos\phi,
\]
and
\[
a_{33} = [\sin\theta \sin\psi - \cos\theta \sin\phi \cos\psi] \gamma_1 - [\cos\theta \sin\psi + \sin\theta \sin\phi \cos\psi] \gamma_2.
\]

Since

\[
\frac{dB}{dt} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}
\]

where, (again omitting unnecessary terms),

\[
b_{12} = -\left(\frac{d\theta}{dt}\right) \cos\theta \cos\phi + \left(\frac{d\phi}{dt}\right) \sin\theta \sin\phi,
\]
\[
b_{22} = -\left(\frac{d\theta}{dt}\right) \sin\theta \cos\phi - \left(\frac{d\phi}{dt}\right) \cos\theta \sin\phi,
\]
\[
b_{31} = \left(\frac{d\phi}{dt}\right) \sin\phi \sin\psi - \left(\frac{d\psi}{dt}\right) \cos\phi \cos\psi,
\]
\[
b_{32} = \left(\frac{d\phi}{dt}\right) \cos\phi,
\]
and
\[
b_{33} = -\left(\frac{d\phi}{dt}\right) \sin\phi \cos\psi - \left(\frac{d\psi}{dt}\right) \cos\phi \sin\psi,
\]
equating coefficients in the equation

\[ \frac{dB}{dt} = \Gamma B \]

yields the following equations:

\[
(\frac{d\phi}{dt}) \cos \phi = \gamma_1 \cos \theta \cos \phi + \gamma_2 \sin \theta \cos \phi,
\]

\[
-(\frac{d\phi}{dt}) \sin \theta \cos \phi - (\frac{d\psi}{dt}) \cos \theta \sin \phi = - \gamma_3 \sin \theta \cos \phi - \gamma_1 \sin \phi,
\]

and

\[
(\frac{d\phi}{dt}) \sin \phi \sin \psi - (\frac{d\psi}{dt}) \cos \phi \cos \psi = [\sin \theta \cos \psi + \cos \theta \sin \phi \sin \psi] \gamma_1 - [\cos \theta \cos \psi - \sin \theta \sin \phi \sin \psi] \gamma_2.
\]

Solving these equations for \( (\frac{d\phi}{dt}), (\frac{d\psi}{dt}), \) and \( (\frac{d\theta}{dt}) \) gives

\[
\frac{d\theta}{dt} = [\gamma_1 \sin \theta - \gamma_2 \cos \theta] \tan \phi + \gamma_3, \quad \text{(IV.7)}
\]

\[
\frac{d\phi}{dt} = \gamma_1 \cos \theta + \gamma_2 \sin \theta,
\]

and

\[
\frac{d\psi}{dt} = [-\gamma_1 \sin \theta + \gamma_2 \cos \theta] \sec \phi.
\]

Before replacing \( B(t) \) in system (IV.4) with the product \( T(\theta) S(\phi) R(\psi) \), where

\[
T(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
S(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix},
\]
and

\[ R(\psi) = \begin{pmatrix} \cos \psi & 0 & \sin \psi \\ 0 & 1 & 0 \\ -\sin \psi & 0 & \cos \psi \end{pmatrix}, \]

recall from Chapter II that, if \( \Omega = \text{skew}(\omega) \) and \( \tilde{\mathbf{A}} = [\mathbf{I}; \tilde{\mathbf{A}}, \mathbf{A}] \), then \( \tilde{\Omega} = \tilde{\mathbf{A}}^{-1} \omega \tilde{\mathbf{A}} = \text{skew}(\mathbf{A}^{-1} \omega) \). Let \( \tilde{\omega}_1 \) and \( \tilde{\delta}_1 \) denote the i-th components of \( (\hat{\mathbf{A}})^{-1} \mathbf{w}_x \) and \( (\hat{\mathbf{A}})^{-1} \mathbf{o} \) respectively. Then substituting \( B(t) = T(\theta(t)) \)

\[ S(\phi(t)) R(\psi(t)) \]

in system (IV.4) yields the system

\[ \begin{align*}
\frac{d\theta}{dt} &= [(\tilde{\omega}_1 + \tilde{\delta}_1) \sin \theta - (\tilde{\omega}_2 + \tilde{\delta}_2) \cos \theta] \tan \phi + \tilde{\omega}_3 + \tilde{\delta}_3 \\
\frac{d\phi}{dt} &= (\tilde{\omega}_1 + \tilde{\delta}_1) \cos \theta + (\tilde{\omega}_2 + \tilde{\delta}_2) \sin \theta \\
\frac{d\psi}{dt} &= [-(\tilde{\omega}_1 + \tilde{\delta}_1) \sin \theta + (\tilde{\omega}_2 + \tilde{\delta}_2) \cos \theta] \sec \phi \\
\frac{d\delta}{dt} &= \varepsilon h_1(t, \omega_x, \mathbf{\tilde{A}}, \mathbf{A}_x, \theta, \phi, \psi) + O(\varepsilon |\delta|) + O(\varepsilon^2)
\end{align*} \]

(IV.8)

\[ \theta(0) = 0, \phi(0) = 0, \psi(0) = 0, \delta(0) = 0, \]

where the identification

\[ h_1(t, \omega_x, \mathbf{\tilde{A}}, \mathbf{A}_x, \theta, \phi, \psi) = \varepsilon_1(t, \omega_x, \mathbf{\tilde{A}}, \mathbf{A}_x, T(\theta)S(\phi)R(\psi)) \]

has been made.

It will be emphasized at this time that system (IV.8) is equivalent to system (IV.4) only where \( \theta, \phi, \) and \( \psi \) are defined. Thus, solutions of system (IV.8) may not exist, in general, for all time. Care will be taken, however, to ensure that solutions do exist on certain intervals of interest.

Recall now that the \( \tilde{\mathbf{A}} \)-frame was chosen so that the 3-axis lies along \( \omega_x \). Thus, since \( \tilde{\omega} = (\hat{\mathbf{A}})^{-1} \omega_x = [\tilde{\omega}_x, \mathbf{\tilde{A}}] \), it follows that \( \tilde{\omega}_1 = 0, \tilde{\omega}_2 = 0, \) and \( \tilde{\omega}_3 = ||\omega_x|| \). Observe also that the solution of system (IV.8) with \( \theta(0) = 0, \phi(0) = 0, \psi(0) = 0, \) and \( \delta(0) = 0 \) is precisely the same as the solution of system (IV.4) with \( \mathbf{A}(0) = \mathbf{A}_x \) and \( \omega(0) = \omega_x \). To facilitate the study of solutions near this solution, initial values for \( \theta, \phi, \) and \( \psi \) will be allowed to be nonzero, and the system
\[
\frac{d\theta}{dt} = |\omega_\ast| + \dot{\gamma}_2 + [\dot{\gamma}_1 \sin\theta - \dot{\gamma}_2 \cos\theta] \tan\phi
\]
\[
\frac{d\phi}{dt} = \dot{\gamma}_1 \cos\theta + \dot{\gamma}_2 \sin\theta
\]
\[
\frac{d\psi}{dt} = [-\dot{\gamma}_1 \sin\theta + \dot{\gamma}_2 \cos\theta] \sec\phi
\]

\[
(IV.9)
\]
\[
\frac{d\delta}{dt} = \varepsilon k(t,\theta,\phi,\psi) + O(\varepsilon|\delta|) + O(\varepsilon^2)
\]

\[
\theta(0) = \theta_0, \phi(0) = \phi_0, \psi(0) = \psi_0, \delta(0) = \delta_0
\]
will be studied. Here

\[
k(t,\theta,\phi,\psi) = h_3(t,\omega_\ast,\bar{A},A_\ast,\theta,\phi,\psi).
\]

D. The Averaging Transformation

By assumption, \( k(t,\theta,\phi,\psi) \) is \( 2\pi \)-periodic and of class \( C^3 \) in each of the first two arguments. Therefore, there exists a Fourier series of the form

\[
k(t,\theta,\phi,\psi) = \sum_{m,n} a_{mn}(\phi,\psi) e^{imt+in\theta}. \tag{IV.10}
\]

Implicit in the definitions of \( k, B, \theta, \phi, \) and \( \psi \) is specification of \( \omega_\ast \). We define a special average of \( k \), denoted by \([k]\), to be

\[
[k](\nu,\phi,\psi) = \frac{1}{2\pi q} \int_0^{2\pi q} k(\tau,\nu + \frac{P}{q} \tau,\phi,\psi) d\tau
\]

if \( |\omega_\ast| = \frac{P}{q} \) is rational in lowest terms, and

\[
[k](\phi,\psi) = a_{\infty}(\phi,\psi)
\]

if \( |\omega_\ast| \) is irrational. Note that \([k]\) is a function of three variables in the rational case and only two in the irrational case. This is due to the presence of the resonance variable \( \nu \) in the rational case.

Since

\[
\frac{1}{2\pi q} \int_0^{2\pi q} k(\tau,\nu + \frac{P}{q} \tau,\phi,\psi) d\tau = \frac{1}{2\pi} \int_0^{2\pi} k(q\tau,\nu + q\tau,\phi,\psi) d\tau,
\]

it follows that

\[
[k](\nu,\phi,\psi) = \sum_{k=-\infty}^{\infty} a_{kp-kq}(\phi,\psi) e^{-ik\nu}. \tag{IV.11}
\]
Theorem IV.3: If \( |\omega_\ast| = \frac{P}{q} \) is rational in lowest terms, then there exists a near identity transformation \( \delta \) of the form \( \delta = \gamma + \varepsilon \nu(t, \theta, \phi, \psi) \), where \( \nu(t, \theta, \phi, \psi) \) is 2\( \pi \)-periodic in \( t \) and \( \theta \), which takes system (IV.9) into the system

\[
\begin{align*}
\frac{d\theta}{dt} &= |\omega_\ast| + (\dot{\gamma}_3 + \varepsilon \dot{\nu}_3) + [(\dot{\gamma}_1 + \varepsilon \dot{\nu}_1) \sin \theta - (\dot{\gamma}_2 + \varepsilon \dot{\nu}_2) \cos \theta] \tan \phi \\
\frac{d\phi}{dt} &= (\dot{\gamma}_1 + \varepsilon \dot{\nu}_1) \cos \theta + (\dot{\gamma}_2 + \varepsilon \dot{\nu}_2) \sin \theta \\
\frac{d\psi}{dt} &= -[(\dot{\gamma}_1 + \varepsilon \dot{\nu}_1) \sin \theta + (\dot{\gamma}_2 + \varepsilon \dot{\nu}_2) \cos \theta] \sec \phi \\
\frac{d\gamma}{dt} &= \varepsilon \left[ k(t, \theta, \phi, \psi) - \frac{\partial \nu}{\partial t} - |\omega_\ast| \frac{\partial \nu}{\partial \theta} \right] + O(\varepsilon |\gamma|) + O(\varepsilon^2)
\end{align*}
\]

If \( |\omega_\ast| \) is irrational, then for each integer \( N > 0 \) there exists a near identity transformation of the same form as above which takes system (IV.9) into the system having the same first three equations and the same initial conditions as system (IV.12) but which has

\[
\begin{align*}
\frac{d\gamma}{dt} &= \varepsilon [a_{\infty}(\phi, \psi) + h_N(t, \theta, \phi, \psi)] + O(\varepsilon |\gamma|) + O(\varepsilon^2)
\end{align*}
\]

where \( h_N(t, \theta, \phi, \psi) = \sum_{m,n}^{(N)} a_{mn}(\phi, \psi) e^{i(\theta t + \theta \phi)} \) and where \( \sum \) means the summation is extended over the set \( \{(m, n) | |m| > N \text{ or } |n| > N\} \). As usual, \( \dot{\gamma}_1 \) and \( \dot{\nu}_1 \) denote the components of \( \lambda^{-1} \gamma \) and \( \lambda^{-1} \nu \) respectively.

**Proof:** When the substitution \( \delta = \gamma + \varepsilon \nu \) is made in system (IV.9) the system

\[
\begin{align*}
\frac{d\theta}{dt} &= |\omega_\ast| + (\dot{\gamma}_3 + \varepsilon \dot{\nu}_3) + [(\dot{\gamma}_1 + \varepsilon \dot{\nu}_1) \sin \theta - (\dot{\gamma}_2 + \varepsilon \dot{\nu}_2) \cos \theta] \tan \phi \\
\frac{d\phi}{dt} &= (\dot{\gamma}_1 + \varepsilon \dot{\nu}_1) \cos \theta + (\dot{\gamma}_2 + \varepsilon \dot{\nu}_2) \sin \theta \\
\frac{d\psi}{dt} &= -[(\dot{\gamma}_1 + \varepsilon \dot{\nu}_1) \sin \theta + (\dot{\gamma}_2 + \varepsilon \dot{\nu}_2) \cos \theta] \sec \phi \\
\frac{d\gamma}{dt} &= \varepsilon \left\{ k(t, \theta, \phi, \psi) \frac{\partial \nu}{\partial t} - \frac{\partial \nu}{\partial \theta} \right\} + O(\varepsilon |\gamma|) + O(\varepsilon^2)
\end{align*}
\]
\( \theta(0) = \theta_0, \phi(0) = \phi_0, \psi(0) = \psi_0, \gamma(0) = -c_0(0, \theta_0, \phi_0, \psi_0) \)
is obtained.

Consider the partial differential equation for \( \dot{a}(t, \theta) \):
\[
\frac{\partial \dot{a}}{\partial t} + |w_a| \frac{\partial \dot{a}}{\partial \theta} = \ddot{b}(t, \theta) = \sum_{mn} \hat{b}_{mn} e^{i(mt + n\theta)}
\]
Equation (IV.14) admits the formal Fourier series solution \( a(t, \theta) \)
equal to \( \sum \hat{a}_{mn} e^{i(mt + n\theta)} \) with \( \hat{a}_{mn} \equiv \hat{b}_{mn} \). Two cases exist:

i. If \( |w_a| = p/q \) is rational in lowest terms, some of the coefficients \( m + |w_a| n \) will vanish, and the corresponding \( \hat{a}_{mn} \) cannot be defined unless the \( \hat{b}_{mn} \) also vanish. The \( \hat{a}_{mn} \) in question may then be taken arbitrarily. The necessary and sufficient condition that \( \hat{b}_{mn} \) vanish for \( m + |w_a| n = 0 \) is precisely the condition that \( \{ \hat{b} \} = 0 \), and the indeterminacy in \( \hat{a}(t, \theta) \) is just \( \{ \hat{a} \} \). The inequality \( |m + p/n| > 1/q \) for \( m \) and \( n \) such that \( m + p/n \neq 0 \) together with the convergence of \( \ddot{b}(t, \theta) \) proves that \( \hat{a}(t, \theta) \) with \( \{ \hat{a} \} = 0 \) also converges. It is clear that if \( \ddot{b}(t, \theta) \) is of class \( C^3 \), then so is \( \hat{a}(t, \theta) \).

ii. If \( |w_a| \) is irrational, the only vanishing denominator occurs for \( m = n = 0 \), so that \( \hat{b}_{00} = 0 \) is necessary for a solution and \( \hat{a}_{00} \) is arbitrary. The resulting Fourier series may not converge, however, unless \( |w_a| \) is "badly irrational." This problem does not exist if the Fourier series for \( \ddot{b}(t, \theta) \) is finite; in which case, \( \ddot{b}(t, \theta) \) and \( \hat{a}(t, \theta) \) are analytic and hence, certainly \( C^3 \).

Returning to system (IV.13), observe that, when \( |w_a| = p/q \) is rational in lowest terms, case (i) applies and it is possible to solve the equation
\[
\frac{\partial v}{\partial t} + |w_a| \frac{\partial v}{\partial \theta} = k(t, \theta, \phi, \psi) - \{ [k] \}(\theta - \frac{p}{q} t, \phi, \psi)
\]
with \( \{ [v] \}(\theta - \frac{p}{q} t, \phi, \psi) = 0 \) attaining system (IV.12). Similarly, when \( |w_a| \) is irrational, case (ii) applies and it is possible to solve the equation
yielding system (IV.12) in the irrational case. \[ \Box \]

A final observation to be made is that, when \(|\omega_*| = \frac{p}{q}\) is rational,

\[
[[k]](\nu, \phi, \psi) = \frac{1}{2\pi q} \int_0^{2\pi q} k(\tau, \nu + \frac{p}{q} \tau, \phi, \psi) \, d\tau
\]

\[
= \frac{1}{2\pi q} \int_0^{2\pi q} f_1(\tau, \omega_*, \hat{\omega}^* A^T \nu, \phi, \psi) \, d\tau
\]

\[
= \frac{1}{2\pi q} \int_0^{2\pi q} f_1(\tau, \omega_*, \hat{\omega}^* \nu, \phi, \psi) \, d\tau
\]

\[
= \frac{1}{2\pi q} \int_0^{2\pi q} \hat{f}_1(\tau, \omega_*, \hat{\omega}^* \nu, \phi, \psi) \, d\tau
\]

where \(C = \hat{\omega}^* R(\psi) \hat{\omega}^{-1}\). When \(|\omega_*|\) is irrational,

\[
[[k]](\phi, \psi) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} k(t, \theta, \phi, \psi) \, dt \, d\theta
\]

\[
= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \hat{f}_1(\tau, \omega_*, \hat{\omega}^* \nu, \phi, \psi) \, dt \, d\theta.
\]

Notice that, if \(\phi = \psi = 0\), it follows that \(C = E\).

For the purpose of implementation, the average is most conveniently calculated directly in terms of the original function \(f_1(t, \omega, A)\). In order to reduce the apparent complexity of the equations in the remainder of this dissertation, we define a new average of \(f_1(t, \omega, A)\) to be

\[
\langle f_1 \rangle (\nu, \omega_*, A_*, C) \equiv \frac{1}{2\pi q} \int_0^{2\pi q} f_1(\tau, \omega_*, \hat{\omega}^* \nu, \phi, \psi) \, d\tau
\]

when \(|\omega_*| = \frac{p}{q}\) is rational in lowest terms, and
\( \left\langle f_{-1} \right\rangle (\omega, A, C) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_{-1}(t, \omega, e^{i\theta}CA) \, d\theta \, dt \)

when \( ||\omega|| \) is irrational. The connection between the two averages is summarized by the equation

\[
[k](v, \phi, \psi) = \left\langle f_{-1} \right\rangle \left( v, \omega, A, \gamma_s(\phi)R(\psi)\gamma^{-1} \right)
\]

when \( ||\omega|| \) is rational, and by the equation

\[
[k](\phi, \psi) = \left\langle f_{-1} \right\rangle \left( \omega, A, \gamma_s(\phi)R(\psi)\gamma^{-1} \right)
\]

when \( ||\omega|| \) is irrational.

When working directly in terms of the original function \( f_{-1}(t, \omega, A) \), it is preferable not to be forced to deal directly with the variables \( \Theta(t) \), \( \Phi(t) \), and \( \Psi(t) \). However, in order to avoid disguising the fact that the system of equations is still nonautonomous in the rational case, some usage of these variables is required. Since the variables \( \Theta(t) \), \( \Phi(t) \), and \( \Psi(t) \) are defined implicitly by the matrix \( B(t) \), when they are used in the development to follow, we will write \( \Theta(B) \), \( \Phi(B) \), and \( \Psi(B) \).

**Theorem IV.4:** If \( ||\omega|| = \frac{P}{q} \) is rational in lowest terms, and if \( |A - A_0| < 1 \), then there exists a coordinate transformation of the form

\[
A(t) = \gamma B(t)\gamma^{-1}A
\]

\( \omega(t) = \omega + \gamma(t) + \epsilon \omega(t, B) \)

which takes the system

\[
\frac{dA}{dt} = \Omega A
\]

\[
\frac{d\omega}{dt} = f_{-1}(t, \omega, A) + O(\epsilon^2)
\]

\( A(0) = A_0, \omega(0) = \omega \)
into the system
\[
\frac{d\mathbf{b}}{dt} = \left( \lambda_\ast + \gamma + cU \right) \mathbf{b}
\]
\[
\frac{d\gamma}{dt} = \varepsilon \left( \int_{-1}^{1} (0(B) - \frac{2}{\varepsilon} \omega_\ast \mathbf{A}_\ast \mathbf{C}(B)) + 0(\varepsilon ||\gamma||) + O(\varepsilon^2) \right)
\]
(IV.17)
\[
B(0) = A^{-1} \mathbf{A}_\ast^{-1} \gamma, \quad \gamma(0) = -\varepsilon \mathbf{u}(0, B(0)).
\]

Here \(C(B) = \lambda_\ast S(\phi(B)) R(\gamma(B)) \lambda_\ast^{-1}\) and, as usual, \(\lambda_\ast = \text{skew}(\lambda_\ast^{-1} \omega_\ast)\), \(\gamma = \text{skew}(\lambda_\ast^{-1} \gamma)\), \(U = \text{skew}(\lambda_\ast^{-1} u)\), and \(O(h)\) represents a term which is 2\(\pi\)-periodic in \(t\) and, when divided by \(h\), remains bounded as \(h \to 0\), uniformly for \(\gamma\) and \(B\) on compact sets.

If \(||\omega_\ast||\) is irrational, then for each integer \(N > 0\), there exists a function \(w(N, t, B)\) satisfying \(||w(N, t, B)|| \to 0\) as \(N \to \infty\) uniformly for \(t\) and \(B\) on compact sets and a coordinate transformation of the form (IV.15) which takes system (IV.16) into the system
\[
\frac{d\mathbf{b}}{dt} = \left( \lambda_\ast + \gamma + \varepsilon U \right) \mathbf{b}
\]
\[
\frac{d\gamma}{dt} = \varepsilon \left[ \int_{-1}^{1} (\omega_\ast \mathbf{A}_\ast \mathbf{C}(B)) + \mathbf{w}(N, t, B) \right] + O(\varepsilon ||\gamma||) + O(\varepsilon^2)
\]
(IV.18)
\[
B(0) = A^{-1} \mathbf{A}_\ast^{-1} \gamma, \quad \gamma(0) = -\varepsilon \mathbf{u}(0, B(0)).
\]

**Proof:** The initial conditions \(\theta_0, \phi_0, \text{ and } \psi_0\) of system (IV.9) are defined by \(B(0) = A^{-1} \mathbf{A}_\ast^{-1} \gamma\). Define \(u\) and \(w\) by \(u(t, B) = \nu(t, \theta, \phi, \psi)\) and \(w(N, t, B) = h_N(t, \theta, \phi, \psi)\) where \(\nu\) and \(h_N\) are the functions defined in Theorem IV.3. Then Theorem IV.4 follows from Theorem IV.3 and the definitions above. □

The theorems of Chapter V which are the most useful in the application of Chapter VII only require the calculation of
\[ \left\langle f_{\Omega} \right\rangle (0, \omega, A, E) = \frac{1}{2\pi q} \int_0^{2\pi q} f_{\Omega}(\tau, \omega, e^{\Omega^* \tau A}) d\tau \]

when \(|\omega| = \frac{p}{q}\) is rational, and

\[ \left\langle f_{\Omega} \right\rangle (\omega, A, E) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_{\Omega}(t, \omega, e^{\Omega^* \Theta A}) d\Theta dt \]

when \(|\omega|\) is irrational. In both cases, the average is obtained by making the substitution \(A = e^{\Omega^* t A}\) and integrating over one period.

As an abbreviation of our notation which will be useful in Chapter VII, when the appropriate average above is to be performed on an expression \(f(t, \omega, A)\), we will write \(\left\langle f(t, \omega, A) \right\rangle\) or possibly just \(\left\langle f \right\rangle\).
V. INFERENCES ABOUT SOLUTIONS TO SYSTEM (IV.1)

A. Lemmas and Preliminaries

Recall from Chapter II that the right hand derivative of a real-valued function \( f(t) \) is denoted by \( D_r f(t) \).

**Lemma V.1:** If \( C(t) \) is a continuously differentiable matrix, then \( D_r |C(t)| \) exists and \( D_r |C(t)| \leq \left| \frac{dC}{dt} \right| \).

**Proof:** Let \( C(t) \) be a continuously differentiable matrix and let \( U(t) \) be any matrix. Let \( 0 < \theta < 1 \) and \( h > 0 \). Then

\[
|C(t) + \theta hU(t)| = |C(t) - \theta C(t) + \theta C(t) + \theta hU(t)| \\
\leq (1-\theta)|C(t)| + \theta |C(t) + hU(t)|
\]

and hence,

\[
\frac{|C(t) + \theta hU(t)| - |C(t)|}{\theta h} \leq \frac{|C(t) + hU(t)| - |C(t)|}{h}.
\]

Thus, the difference quotient

\[
\frac{|C(t) + hU(t)| - |C(t)|}{h}
\]

is a non-decreasing function of \( h \). Since

\[
\frac{|C(t) + hU(t)| - |C(t)|}{h} \leq \frac{|C(t)| + h|U(t)| - |C(t)|}{h} = |U(t)|,
\]

it follows that \( \lim_{h \to 0^+} \frac{|C(t) + hU(t)| - |C(t)|}{h} \)

exists for any \( U(t) \). In particular,

\[
\lim_{h \to 0^+} \frac{|C(t) + h\left( \frac{dC}{dt} \right)| - |C(t)|}{h} \text{ exists. Furthermore,}
\]

\[
\lim_{h \to 0^+} \frac{|C(t + h)| - |C(t)|}{h} \leq \frac{|C(t)| + h\left( \frac{dC}{dt} \right)| - |C(t)|}{h}
\]
\[ \lim_{h \to 0^+} \frac{|C(t + h)| - |C(t)|}{h} = \lim_{h \to 0^+} \frac{|C(t + h) - C(t)|}{h} \leq \lim_{h \to 0^+} \left| \frac{C(t + h) - C(t)}{h} - \frac{dC}{dt} \right| = 0. \]

Therefore \( D_r |C(t)| \) exists, and

\[
D_r |C(t)| = \lim_{h \to 0^+} \frac{|C(t + h)| - |C(t)|}{h} = \lim_{h \to 0^+} \frac{|C(t) + h \left( \frac{dC}{dt} \right)| - |C(t)|}{h} \leq \lim_{h \to 0^+} \left| \frac{C(t)}{h} + \frac{dC}{dt} \right| - |C(t)| = \frac{|dC|}{dt} \]

The next lemma, required in the proof of Lemma V.3, is fairly standard and will be stated without proof. For a proof, see Hale [8, pp. 31-32].

**Lemma V.2:** Let \( w(t,u) \) be continuous on an open connected set \( D \subset \mathbb{R}^2 \), and suppose the initial value problem for the scalar equation \( \frac{du}{dt} = w(t,u) \) has a unique solution. If \( u(t) \) is a solution of \( \frac{du}{dt} = w(t,u) \) on the interval \( a \leq t \leq b \), and if \( v(t) \) is a solution of \( D_r v(t) \leq w(t,v) \) on the interval \( a \leq t < b \) satisfying \( v(a) \leq u(a) \), then \( v(t) \leq u(t) \) for \( a \leq t \leq b \).

The next lemma is useful for obtaining an estimate of just how near solutions of system (IV.1) remain to solutions of the unperturbed system. This estimate, while not necessarily very sharp, is valid for all time.

**Lemma V.3:** Let \( B(t), \delta(t) \) be the solution of system (IV.4) with initial conditions \( B(0) = \delta(0) = 0^* \), and let
\[ \hat{\Omega}_* = \hat{A}^{-1} \hat{\Omega}_* \hat{A} = \text{skew}(\hat{A}^{-1} \omega_*) . \]

Then
\[
\left| B(t) - e^{\hat{\Omega}_* t} B_* \right| \leq \exp \left\{ \int_0^t |\Delta(\tau)| d\tau \right\} - 1
\]

for all \( t > 0 \).

**Proof:** Let \( B(t), \delta(t) \) be the solution of system (IV.4) with initial conditions \( B(0) = B_* \) and \( \delta(0) = 0 \). Define \( P(t) \) implicitly by the equation
\[
B(t) = e^{\hat{\Omega}_* t} [B_* + P(t)] .
\]

Differentiating equation (V.1) with respect to time yields
\[
\frac{dB}{dt} = \hat{\Omega}_* B + e^{\hat{\Omega}_* t} \left( \frac{dP}{dt} \right) .
\]

Substituting equation (V.2) into the first equation of system (IV.4) and solving algebraically for \( \frac{dP}{dt} \) gives
\[
\frac{dP}{dt} = \left( e^{\hat{\Omega}_* t} \alpha e^{\hat{\Omega}_* t} \right) [B_* + P(t)] .
\]

Since \(|R| = 1\) for any orthogonal matrix \( R \), it follows that
\[
\left| \frac{dP}{dt} \right| = |\hat{\Omega}| |B_* + P| \\
\leq |\Delta| (1 + |P|) \\
= |\Delta| + |\Delta| |P| ,
\]

and therefore, by Lemma V.1,
\[
D \left| P(t) \right| \leq |\Delta(t)| + |\Delta(t)| |P(t)| .
\]

Observe that \( v(t) = \exp \left\{ \int_0^t |\Delta(\tau)| d\tau \right\} - 1 \)

is the unique solution of the scalar equation
\[ \frac{dv}{dt} = |\Delta(t)| + |\Delta(t)| v(t) \]

with initial condition \( v(0) = 0 \). Since \( B(0) = B_\lambda \), it follows that \( |P(0)| = 0 \), and so, by Lemma V.3,

\[ |P(t)| \leq \exp \left\{ \int_0^t |\Delta(\tau)| \, d\tau \right\} - 1 \]

for all \( t \geq 0 \). Thus,

\[ |B(t) - e^{\hat{A}_\lambda^t B_\lambda}| = |e^{\hat{A}_\lambda^t} P(t)| = |P(t)| \leq \exp \left\{ \int_0^t |\Delta(\tau)| \, d\tau \right\} - 1 \]

for \( t \geq 0 \). ■

The next rather simple lemma is of potential use in several areas of mathematics and physics. It will be used frequently throughout the remainder of this dissertation.

**Lemma V.4:** Let \( \mathbf{y}^T = (y_1, y_2, y_3) \) be an arbitrary unit vector, and set \( Y = \text{skew}(\mathbf{y}) \). Then the matrix representation \( e^{Y\theta} \) of the rotation about the \( y \) axis through the angle \( \theta \) may be determined by

\[ e^{Y\theta} = e^{y_1\theta} + (\sin \theta) Y + (1 - \cos \theta) Y^2 \]  

_(V.3)_

**Proof:** Recall that

\[
Y = \begin{pmatrix}
0 & -y_3 & y_2 \\
y_3 & 0 & -y_1 \\
-y_2 & y_1 & 0 \\
\end{pmatrix}, \quad \text{and}
\]
hence,

\[
\mathbf{y}^2 = \begin{pmatrix}
-y_2^2 - y_3^2 & y_1 y_2 & y_1 y_3 \\
y_1 y_2 & -y_1^2 - y_3^2 & y_2 y_3 \\
y_1 y_3 & y_2 y_3 & -y_1^2 - y_2^2
\end{pmatrix},
\]

and \( y^3 = -Y \). Thus, by direct calculation,

\[
e^\mathbf{Y}^0 = E + \sum_{k=1}^{\infty} \frac{1}{k!} (\mathbf{Y}^0)^k
\]

\[
= E + \sum_{k=1}^{\infty} \left[ \frac{1}{(2k-1)!} (-1)^{k-1} y_1^{2k-2} \right] - (\mathbf{Y}^0)^2
\]

\[
= E + \sum_{k=1}^{\infty} \left[ \frac{1}{(2k-1)!} (-1)^k y_1^{2k} \right] - (\mathbf{Y}^0)^2
\]

\[
= E + \left[ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \theta y_1^{2k+1} \right] - \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} y_1^{2k}
\]

\[
= E + (\sin \theta) Y + (1 - \cos \theta) y^2
\]

The final lemma for this section utilizes several of the previous lemmas to develop bounds on the solution \( \theta(t), \phi(t), \psi(t) \) of the first part of system (IV.9). These bounds will be important to the development in the next section.

**Lemma V.5A:** Let \( \theta(t), \phi(t), \psi(t) \), and \( \delta(t) \) be the solution of system (IV.9) with initial conditions \( \theta(0) = \theta_0, \phi(0) = \phi_0, \psi(0) = \psi_0 \), and \( \delta(0) = 0 \),
Furthermore, suppose that
\[
\exp \left\{ \int_0^t |\Delta(t)| \, dt \right\} - 1 < \alpha \leq \frac{\pi}{24}
\]
for \( t \in [0, t_0] \). Then, if \( |\sin \phi_o| < \alpha \), the bounds \( |\phi(t) - \phi_o| < 6\alpha \), \( |\psi(t) - \psi_o| < 3\alpha \), and \( |\theta(t) - \theta_o - ||\omega_k||t| < 3\alpha \) hold for \( 0 \leq t \leq t_0 \).

**Proof:** By Lemma V.3, \( |B(t) - e^{\frac{\partial}{\partial t} B_k}| < \alpha \) for \( 0 \leq t \leq t_0 \). Using Lemma V.4 and equation (IV.6), direct calculation reveals that
\[
B(t) - e^{\frac{\partial}{\partial t} B_k} = \begin{pmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]
where, (listing only the pertinent terms),
\[
\begin{align*}
a_{12} &= -\sin \theta \cos \phi + \sin(\theta_o + ||\omega_k||t) \cos \phi_o \\
a_{22} &= \cos \theta \cos \phi - \cos(\theta_o + ||\omega_k||t) \cos \phi_o \\
a_{31} &= -\cos \phi \sin \psi + \cos \phi_o \sin \psi_o \\
a_{32} &= \sin \phi - \sin \phi_o
\end{align*}
\]
and
\[
a_{33} = \cos \phi \cos \psi - \cos \phi_o \cos \psi_o.
\]
Since, for any matrix \( A = (a_{ij}) \), \( |A| < \alpha \) implies that \( |a_{ij}| < \alpha \) for all \( i \) and \( j \), it follows that
\[
\begin{align*}
-\alpha < a_{32} < \alpha & \\
-\sqrt{2} \alpha < a_{12} \cos \theta + a_{22} \sin \theta < \sqrt{2} \alpha \quad \text{(V.4)} \\
-\sqrt{2} \alpha < a_{31} \cos \psi + a_{33} \sin \psi < \sqrt{2} \alpha.
\end{align*}
\]
Therefore,
\[ -\alpha < \sin\theta - \sin\theta_0 < \alpha, \]
\[ -\sqrt{2}\alpha < \sin(\theta - \theta_0 - |\omega_\star||t|) \cos\phi_0 < \sqrt{2}\alpha, \]
and
\[ -\sqrt{2}\alpha < \cos\phi_0 \sin(\psi - \psi_0) < \sqrt{2}\alpha \]
for \(0 \leq t \leq t_0\).

Since \(|\sin\phi_0| < \alpha\), it follows that \(|\sin\phi| < 2\alpha\) and hence,
\[ |\sin(\theta - \phi_0)| \leq |\sin\phi| + |\sin\phi_0| < 3\alpha \]
for \(0 \leq t \leq t_0\).

Also, since
\[ |\cos\phi_0| > \sqrt{1 - \alpha^2} \]
\[ > \frac{2\sqrt{2}}{3}, \]
it follows that \(|\sin(\theta - \theta_0 - |\omega_\star||t|)| < \frac{3}{2}\alpha\) and \(|\sin(\psi - \psi_0)| < \frac{3}{2}\alpha\).

Observe that \(f(x) = \sin x - \frac{1}{2}x\) is a monotone increasing function
for \(0 \leq x \leq \frac{\pi}{3}\). Thus, \(\sin x \geq \frac{1}{2}x\) for \(0 \leq x \leq \frac{\pi}{3}\). Since \(\alpha \leq \frac{\pi}{24} < \frac{1}{2\sqrt{3}}\),
we have \(3\alpha < \frac{\sqrt{3}}{2}\), and hence
\[ |\phi - \phi_0| < 6\alpha, \]
\[ |\psi - \psi_0| < 3\alpha, \]
and
\[ |\theta - \theta_0 - |\omega_\star||t| < 3\alpha \]
for \(0 \leq t \leq t_0\).

Lemma V.5B: Let \(B(t), \phi(t)\) be the solution of system (IV.4) with
initial conditions \(B(0) = B_\star, \phi(0) = 0\). If \(|B(t) - e^{\mathbf{A}\star t}| \leq \alpha < \frac{1}{2}\)
for \(0 \leq t \leq t_0\), then \(|\sin(\theta - |\omega_\star||t|) < 2\alpha, |\sin\phi| < \alpha,\) and \(|\sin\psi| < 2\alpha\)
for \(0 \leq t \leq t_0\).
Proof: Using Lemma V.4 and equation (IV.6), direct calculation reveals that

\[ B(t) - e^{\Omega t} = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix} \]

where, (listing only the pertinent terms),

\[
\begin{align*}
  a_{12} &= -\sin \theta \cos \phi + \sin(|\omega| t), \\
  a_{22} &= \cos \theta \cos \phi - \cos(|\omega| t), \\
  a_{31} &= -\cos \phi \sin \psi,
\end{align*}
\]

and

\[ a_{32} = \sin \phi. \]

As in Lemma V.5A, it follows that

\[-\sqrt{2} \alpha < \sin(\theta - |\omega| t) < \sqrt{2} \alpha, \]

\[-\frac{\alpha}{\sqrt{1-\alpha^2}} < \sin \psi < \frac{\alpha}{\sqrt{1-\alpha^2}}\]

and

\[-\alpha < \sin \phi < \alpha.\]

If \( \alpha < \frac{1}{2} \), then

\[ |\sin(\theta - |\omega| t)| < 2\alpha, \]

\[ |\sin \psi| < 2\alpha, \]

and

\[ |\sin \phi| < \alpha \]

for \( 0 \leq t \leq t_0. \)
B. Inferences When $|\omega_*|$ is Rational

In this section, a condition on the averaged system of equations is studied which precludes the existence of steady state quasiperiodic solutions of the original system. The case in which $|\omega_*|$ is irrational will be treated separately from the case in which $|\omega_*| = \frac{p}{q}$ is rational in lowest terms.

We begin with the theorem which has the strongest results but which also requires the strongest hypotheses. In successive theorems, the hypotheses are weakened resulting in correspondingly weaker results.

Theorem V.6: Suppose that $|\omega_*| = \frac{p}{q}$ is rational in lowest terms so that form (IV.17) of the averaged equations applies. If $\mathcal{B}(t), \gamma(t)$ is a solution of system (IV.17) such that $|\mathcal{B} - e^{\Omega t^*}| < \tau < \frac{1}{2}$ for all $t > 0$, and if there exists a unit vector $\hat{a}$ such that

$$a \cdot \langle f \rangle (\mathcal{Q}(B) - \frac{p}{q} t, \omega_* A, C(B)) > 3m > 0$$

for all $t > 0$, then there exist positive constants $\epsilon, \eta$, and $T$ such that the solution of system (IV.1) for $0 < \epsilon < \epsilon_0$ satisfies

$|\omega(t) - \omega_*| > \eta$

for some $t$ on the interval $[0, T/\epsilon]$.

Proof: Let $\mathcal{B}(t), \gamma(t)$ be the solution of system (IV.17) with initial conditions $\mathcal{B}(0) = \mathcal{A}^{-1} A_0 A_*^{-1} \mathcal{A}, \gamma(0) = -c_u(0, B(0))$. Suppose that

$$a \cdot \langle f \rangle (\mathcal{Q}(B) - \frac{p}{q} t, \omega_* A, C(B)) > 3m > 0$$

and

$$|\mathcal{B}(t) - e^{\Omega t^*}| < \tau < \frac{1}{2}$$

for all $t > 0$. Then, since

$$\frac{d\gamma}{dt} = \epsilon \langle f \rangle (\mathcal{Q}(B) - \frac{p}{q} t, \omega_* A, C(B)) + O(\epsilon |\gamma|) + O(\epsilon^2)$$
where the terms $0(h)$ are $2\pi$-periodic in $t$ and, when divided by $h$, remain bounded as $h \to 0$ uniformly for $\gamma$ and $B$ on compact sets, there exist positive constants $\varepsilon_0$ and $\eta$ such that, for $\varepsilon \leq \varepsilon_0$, 

$$||\gamma|| < 2\eta,$$

and for $t \geq 0$,

$$\varepsilon < \frac{\eta}{2\max|u(t,B(t))|},$$

$$||0(\varepsilon ||\gamma||)|| < \varepsilon m,$$

and

$$||0(\varepsilon^2)|| < \varepsilon m.$$

The Proof is completed by contradiction. Let $T = \frac{3n}{m}$, and suppose that $||\gamma(t)|| < 2\eta$ for $0 \leq t \leq T/\varepsilon$. Then

$$a \cdot \frac{dy}{dt} < \varepsilon m$$

for $0 \leq t \leq T/\varepsilon$ implies that

$$a \cdot [\gamma(t) - \gamma(0)] > \varepsilon mt$$

for $0 \leq t \leq T/\varepsilon$. Thus,

$$a \cdot [\gamma(T/\varepsilon) - \gamma(0)] > \varepsilon m(T/\varepsilon)$$

$$= mT$$

$$= 3\eta,$$

and hence,

$$||\gamma(T/\varepsilon)|| > 2\eta$$

contradicting the assumption that $||\gamma(t)|| \leq 2\eta$ for $0 \leq t \leq T/\varepsilon$. Therefore, $||\gamma(t)|| > 2\eta$ for some $t$ on the interval $[0,T/\varepsilon]$.

Since $\omega - \omega_* = \gamma + \varepsilon u$ and $\varepsilon ||u|| < \frac{n}{2}$, it follows that

$$||\omega - \omega_*|| > \frac{3\eta}{2} > \eta$$

for some $t$ on the interval $[0,T/\varepsilon]$. ■
Corollary to Theorem V.6: If the hypotheses of Theorem V.6 are satisfied by

\[ a = \hat{\omega}_k, \]

then there exist positive constants \( \varepsilon_0, \eta, \) and \( T \) such that the solution of system (IV.1) for \( 0 < \varepsilon < \varepsilon_0 \) satisfies

\[ ||\omega(t)|| - ||\omega_k|| > \eta \]

for some \( t \) on the interval \([0, T/\varepsilon]\).

Proof: Let \( m, \varepsilon_0, \eta, \) and \( T \) be as in the Proof of Theorem V.6. Then, by the Proof of Theorem V.6,

\[ \hat{\omega}_k \cdot [\omega(t) - \omega_k] = \hat{\omega}_k \cdot [\gamma(t) + \varepsilon \omega(t, B(t))] > 3\eta - \eta > \eta \]

for some \( t \) on the interval \([0, T/\varepsilon]\). Thus, the component of \( \omega(t) - \omega_k \) in the direction of \( \omega_k \) is greater than \( \eta \), and hence,

\[ ||\omega(t)|| - ||\omega_k|| > \eta. \]

Theorem V.6 and its corollary are, in general, of limited direct value since \( B(t) \) is usually unknown. In Theorem V.7, the knowledge required a priori of \( B(t) \) is weakened resulting in a less specific conclusion.

Theorem V.7: Suppose that \( ||\omega_k|| = \frac{p}{q} \) is rational in lowest terms so that form (IV.17) of the averaged equations applies. If there exists a constant unit vector \( \tilde{a} \) such that

\[ \tilde{a} \cdot \left< f_1 \right> (0, \omega_k, A_k, E) > 0, \]

then there exist positive constants \( \varepsilon_0, \tau, \eta, \) and \( T \) such that the solution of system (IV.1) for \( 0 < \varepsilon < \varepsilon_0 \) and initial conditions \( A(0) = A_0, \omega(0) = \omega_k \) satisfying

\[ |A_0 - A_k| < \tau \]
also satisfies at least one of the following conditions for some \( t \) on the interval \([0, T/e]\):

1. \(|A - e^{\Omega_* t A_*}| > \tau\)
2. \(|\omega(t) - \omega_*| \geq \frac{3}{2} \eta\).

**Proof:** Since \( (v, \omega_* A_* C) \) is continuous in \( v \) and \( C \), there exist positive constants \( m \) and \( \tau \) such that

\[
\mathbf{a}^{*} \left\langle \frac{d}{dt} \right\rangle (v, \omega_* A_* C) \geq 3m
\]

for all \( v \) and \( C \) satisfying \(|v| \leq 4\tau < \frac{1}{2}\) and \(|C - E| \leq 5\tau\). Let \( \eta, \varepsilon_0, \) and \( T \) be as in Theorem V.6 and let \( 0 < \varepsilon \leq \varepsilon_0 \). If

\[
|A(t) - e^{\Omega_* t A_*}| > \tau
\]

for some \( t \) on the interval \([0, T/e]\), we are done. Therefore, suppose that

\[
|A(t) - e^{\Omega_* t A_*}| = |B(t) - e^{\Omega_* t}|
\]

for \( 0 \leq t \leq T/e \). Then, by Lemma V.5B,

\[
|\sin(\Theta(B) - \frac{p}{q} t)| = |\sin v| < 2\tau
\]

for \( 0 \leq t \leq T/e \). Since \( \sin x \geq \frac{1}{2} x \) for \( 0 \leq x \leq \frac{\pi}{2} \), \(|v| < 4\tau\) for

\( 0 \leq t \leq T/e \). Furthermore,
\[ |C(B) - E| = |e^{\hat{\mathcal{A}}t} - e^{\hat{\mathcal{A}}t}e^{\hat{\mathcal{V}}V} - e^{\hat{\mathcal{A}}t}e^{\hat{\mathcal{V}}V}| \]
\[ < |B - e^{\hat{\mathcal{V}}V}| + |E - e^{\hat{\mathcal{V}}V}| \]
\[ < \tau + [2(1 - \cos V)]^{1/2} \]
\[ < \tau + 2|\sin V| \]
\[ < 5\tau \]

for \(0 < t < \frac{T}{\varepsilon}\), and hence,

\[ a^* \langle \mathcal{F}_{\|} \rangle \mathcal{G}(B) - \frac{p}{q} t, \omega, A, C(B) \rangle > 3m \]

for \(0 < t < \frac{T}{\varepsilon}\). Thus, by the proof of Theorem V.6,

\[ ||\omega(t) - \omega|| > \frac{3}{2} \eta \]
\[ > \eta \]

for some \(t\) on the interval \([0, T/\varepsilon]\). \[\square\]

The proof of the following corollary, as well as the proof of the corollary to Theorem V.8, is nearly identical to the proof of the corollary to Theorem V.6 and will not be repeated.

**Corollary to Theorem V.7:** Suppose that the hypotheses of Theorem V.7 are satisfied by \(a = \hat{\omega}_\chi\). Then there exist positive constants \(\varepsilon_0, \tau, \eta,\) and \(T\) such that the solution of system (IV.1) for \(0 < \varepsilon \leq \varepsilon_0\) and initial conditions \(A(0) = A_0, \omega(0) = \omega_\chi\) satisfying \(|A_0 - A_\chi| < \tau\) also satisfies at least one of the following conditions for some \(t\) on the interval \([0, T/\varepsilon]\):

i. \(|A(t) - e^{\Omega t}A_\chi| > \tau\)

ii. \(|\omega(t)| - |\omega_\chi| > \eta\).
In Theorem V.7 and its corollary, the knowledge required a priori of the matrix $A(t)$ was reduced to the initial value. The cost of relaxing this requirement was a loss of the ability to force $\omega$ to leave a fixed neighborhood of $\omega_*$ independent of $\varepsilon$. Instead, either $\omega$ left a fixed neighborhood of $\omega_*$ or $A(t)$ left a fixed neighborhood of $e^{\Omega_0 t} A_*$. In all cases, the neighborhoods were of a fixed size for $0 < \varepsilon < \varepsilon_0$.

The final theorem of this section shows that, under certain conditions, knowledge of the initial value of $A(t)$ is sufficient to force $\omega(t)$ to leave a shrinking neighborhood of $\omega_*$, i.e., a neighborhood of $\omega_*$ whose radius is a function of $\varepsilon$.

**Theorem V.8:** Suppose that $||\omega_*|| = \frac{p}{q}$ is rational in lowest terms so that form (IV.17) of the averaged equations applies. If there exists a constant unit vector $a$ such that

$$a \cdot \left< \eta_1 \right> (0,\omega_*,A_*,E) > 0,$$

then there exist positive constants $\tau$, $\varepsilon_0$, $\eta$, and $T$ such that the solution of system (IV.1) for $0 < \varepsilon < \varepsilon_0$ and initial conditions $A(0) = A_0$, $\omega(0) = \omega_*$ satisfying $|A_0 - A_*| < \tau$ also satisfies $||\omega(t) - \omega_*|| > \eta \sqrt{\varepsilon}$ for some $t$ on the interval $[0,T/\sqrt{\varepsilon}]$.

**Proof:** Suppose that $a \cdot \left< \eta_1 \right> (0,\omega_*,A_*,E) > 0$ for some constant unit vector $a$. Since $\left< \eta_1 \right> (v,\omega_*,A_*,C)$ is continuous in $v$ and $C$, there exist positive constants $m$ and $T$ such that

$$a \cdot \left< \eta_1 \right> (v,\omega_*,A_*,C) \geq 3m$$

for $|v| \leq 8\tau < 1$ and $|C - E| \leq 10\tau$.

Let $M_1$, $M_2$, and $M_3$ be positive constants such that, as long as

$$|B(t) - e^{\Omega_0 t} A_0 A_* A_*^{-1} A_*| < \tau,$$

the following bounds hold:

$$||0(\varepsilon ||\gamma||)|| < \varepsilon ||\gamma|| M_1,$$

$$||0(\varepsilon^2||)|| < \varepsilon^2 M_2,$$
and

\[ |\mathbf{u}(t, B(t))| \leq M_3 \]

where \( \mathbf{u}(t, B) \) is the function of equation (IV.15).

Observe that \( \|v\|_1 = |\text{skew}(v)| \) defines a new norm for column vectors expressed in the \( A \)-frame. Therefore, there exists a constant \( C_1 > 0 \) such that \( \|v\|_1 \leq C_1 \|v\| \) for all column vectors \( v \).

Set

\[ \eta = \min \left\{ \sqrt{\frac{m}{2C_1}}, \sqrt{\frac{mT}{2eC_1}} \right\} \]

\[ \varepsilon_0 = \min \left\{ \left( \frac{\eta}{2M_3} \right)^2, \left( \frac{m}{2\eta M_1} \right)^2, \frac{m}{M_2} \right\} \]

and

\[ T = \frac{2\eta}{m} \].

We proceed by contradiction. Assume that \( ||\delta|| = ||\mathbf{u} - \mathbf{u}_*|| < \eta \sqrt{\varepsilon} \) for \( 0 \leq t \leq T/\sqrt{\varepsilon} \). Then

\[ ||\mathbf{y}|| = ||\delta - \varepsilon \mathbf{u}|| \leq ||\delta|| + \varepsilon ||\mathbf{u}|| < \eta \sqrt{\varepsilon} + \varepsilon \left( \sqrt{eM_3} \right) < 2\eta \sqrt{\varepsilon} < \frac{m}{M_1}, \]

and

\[ \int_0^t |A(s)|ds \leq \int_0^{T/\sqrt{\varepsilon}} |C_1||\delta(s)||ds \]

\[ < C_1 \eta T \]

\[ = \frac{2C_1}{m} \eta ^2 \]
for $0 \leq t \leq T/\sqrt{\varepsilon}$. Since $\eta^2 \leq \frac{m}{2C_1}$, 

$$\int_0^t |\Delta(s)| \, ds < 1$$

and hence, the bound

$$\exp \left\{ \int_0^t |\Delta(s)| \, ds \right\} - 1 \leq \frac{2eC_1}{m} \eta^2$$

holds for $0 \leq t \leq T/\sqrt{\varepsilon}$.

By Lemma V.3, the solution of system (IV.4) with initial conditions $B(0) = A^{-1}A_oA_o^{-1}A^\nu$, $\delta(0) = 0$ satisfies

$$|B(t) - e^{\gamma A^\nu t}A_oA_o^{-1}A^\nu| < \frac{2eC_1}{m} \eta^2$$

for $0 \leq t \leq T/\sqrt{\varepsilon}$. Using $\eta^2 \leq \frac{mt}{2eC_1}$, it follows that

$$|B(t) - e^{\gamma A^\nu t}A_oA_o^{-1}A^\nu| < \tau$$

for $0 \leq t \leq T/\sqrt{\varepsilon}$, and hence, the bounds $||0(\varepsilon)||Y|| \leq \varepsilon||Y||M_1$, $||0(\varepsilon^2)|| \leq \varepsilon^2 M_2$, and $||u(t,B(t))|| \leq M_3$ hold for $0 \leq t \leq T/\sqrt{\varepsilon}$.

Furthermore,

$$|B(t) - e^{\gamma A^\nu t}| \leq |B(t) - e^{\gamma A^\nu t}A_oA_o^{-1}A^\nu| + |A^\nu A_o^{-1}A^\nu - E|$$

$$\leq \tau + |A_o^{-1}A_o - E|$$

$$\leq 2\tau$$

for $0 \leq t \leq T/\sqrt{\varepsilon}$, and so, by Lemma V.5B, $|\sin(\Theta(B) - \frac{p}{q} t)| = |\sin\nu| < 4\tau$. Using $\sin X \geq \frac{1}{2} X$ for $0 \leq X \leq 1$, it follows that $|\nu| < 8\tau$ for $0 \leq t \leq T/\sqrt{\varepsilon}$. Thus,

$$|C-E| = |e^{\gamma A^\nu t}A_oA_o^{-1}A^\nu - e^{\gamma A^\nu t}e^{\hat{\delta} \nu} A_oA_o^{-1}A^\nu|$$

$$\leq |B - e^{\gamma A^\nu t}| + |E - e^{\hat{\delta} \nu}|$$

$$\leq 2\tau + [2(1 - \cos\nu)]^{1/2}$$

$$\leq 2\tau + 2|\sin\nu|$$

$$< 10\tau$$
for $0 \leq t \leq T/\sqrt{\varepsilon}$, so that
\[
\mathbf{a} \cdot \left( \frac{d\mathbf{y}}{dt} \right) \geq \mathbf{a} \cdot \left( \mathbf{f}_1 - \frac{p}{q} t, \omega_*, \mathbf{A}_*, \mathbf{C}(\mathbf{B}) \right) - \varepsilon \|\mathbf{w}\|_1 - \varepsilon^2 M_2
\]
for $0 \leq t \leq T/\sqrt{\varepsilon}$. Therefore,
\[
\|\mathbf{y}(T/\sqrt{\varepsilon}) - \mathbf{y}(0)\| \geq \int_0^{T/\sqrt{\varepsilon}} \mathbf{a} \cdot \left( \frac{d\mathbf{y}}{dt} \right) dt > \varepsilon \sqrt{\varepsilon} = 2n^3
\]
and hence,
\[
\|\mathbf{w}(T/\sqrt{\varepsilon}) - \omega_*\| \geq \|\mathbf{y}(T/\sqrt{\varepsilon}) - \mathbf{y}(0)\| - \varepsilon \|\mathbf{u}(T/\sqrt{\varepsilon}, \mathbf{B}(T/\sqrt{\varepsilon}) - \mathbf{u}(0, \mathbf{B}(0))\| > 2n^3 - \varepsilon \sqrt{\varepsilon} (2n^3)
\]
contradicting $\|\hat{\mathbf{a}}(t)\| \leq \eta \sqrt{\varepsilon}$ for $0 \leq t \leq T/\sqrt{\varepsilon}$. Thus,
\[
\|\mathbf{w}(t) - \omega_*\| > \eta \sqrt{\varepsilon}
\]
for some $t$ on the interval $0 \leq t \leq T/\sqrt{\varepsilon}$.

**Corollary to Theorem V.8:** If the hypotheses of Theorem V.8 are satisfied by $\mathbf{a} = \hat{\omega}_*$, then there exist positive constants $\tau, \varepsilon_0, \eta,$ and $T$ such that the solution of system (IV.1) for $0 < \varepsilon < \varepsilon_0$ and initial conditions $\mathbf{A}(0) = \mathbf{A}_0, \omega(0) = \omega_*$ satisfying $|\mathbf{A}_0 - \mathbf{A}_*| < \tau$ satisfies
\[
\|\mathbf{w}(t)\| - \|\omega_*\| > \eta \sqrt{\varepsilon}
\]
for some $t$ on the interval $[0, T/\sqrt{\varepsilon}]$.

**C. Inferences When $\|\omega_*\|$ is Irrational**

The case in which $\|\omega_*\|$ is irrational yields results that are essentially identical to those obtained in the rational case. The only differences occur in the presence and handling of the term $\mathbf{w}(N, t, \mathbf{B})$ which involves the higher harmonics of the Fourier expansion of $f_1(t, \omega_*, \mathbf{A})$. Since the proofs for the theorems in the irrational case all differ from the corresponding proofs in the rational case in precisely the same way, only the last theorem will be proved in detail. The rest are listed without proof.
Theorem V.9: Suppose that $|\omega_x|$ is irrational so that form (IV.18) of the averaged equations applies. If there exists a constant unit vector $a$ such that $a \cdot \left< \varepsilon_{\varepsilon_1} \right> (\omega_x, A_x, C(B)) > 3m > 0$ for all $t \geq 0$, and if the solution $B(t), \delta(t)$ of system (IV.18) satisfies

$$|B(t) - e^{\Omega \varepsilon_1 t}| \leq \tau < 1/2$$

for all $t \geq 0$, then there exist positive constants $\varepsilon_0$, $\eta$, and $T$ such that the solution of system (IV.1) for $0 < \varepsilon \leq \varepsilon_0$ satisfies $|\omega(t) - \omega_x| > \eta$ for some $t$ on the interval $[0,T/\varepsilon]$.

Corollary to Theorem V.9: If the hypotheses of Theorem V.9 are satisfied by $a = \hat{\omega}_x$, then there exist positive constants $\varepsilon_0$, $\eta$, and $T$ such that the solution of system (IV.1) for $0 < \varepsilon \leq \varepsilon_0$ satisfies $|\omega(t) - \omega_x| > \eta$ for some $t$ on the interval $[0,T/\varepsilon]$.

Theorem V.10: Suppose that $|\omega_x|$ is irrational so that form (IV.18) of the averaged equations applies. If there exists a constant unit vector $a$ such that $a \cdot \left< \varepsilon_{\varepsilon_1} \right> (\omega_x, A_x, E) > 0$, then there exist positive constants $\varepsilon_0$, $\tau$, $\eta$, and $T$ such that the solution of system (IV.1) for $0 < \varepsilon \leq \varepsilon_0$ and initial conditions $A(0) = A_0$, $\omega(0) = \omega_x$ satisfying $|A_x - A_0| < \tau$ also satisfies at least one of the following conditions for some $t$ on the interval $[0,T/\varepsilon]$:

i. $|A - e^{\Omega \varepsilon_1 t} A_x| > \tau$

ii. $|\omega(t) - \omega_x| > \frac{3}{2} \eta$.

Corollary to Theorem V.10: If the hypotheses of Theorem V.10 are satisfied by $a = \hat{\omega}_x$, then there exist positive constants $\varepsilon_0$, $\tau$, $\eta$, and $T$ such that the solution of system (IV.1) for $0 < \varepsilon \leq \varepsilon_0$ and initial conditions $A(0) = A_0$, $\omega(0) = \omega_x$ satisfying $|A_x - A_0| < \tau$ also satisfies at least one of the following conditions for some $t$ on the interval $[0,T/\varepsilon]$:

i. $|A - e^{\Omega \varepsilon_1 t} A_x| > \tau$

ii. $|||\omega(t) - \omega_x||| > \eta$. 
Theorem V.11: Suppose that \(|\omega|\) is irrational so that form (IV.18) of the averaged equations applies. If there exists a constant unit vector \(\mathbf{a}\) such that \(a \cdot \langle \mathbf{f}_1 \rangle (\omega_\star, A_\star, E) > 0\), then there exist positive constants \(\varepsilon_0\), \(\tau\), \(\eta\), and \(T\) such that the solution of system (IV.1) for \(0 < \varepsilon < \varepsilon_0\) with initial conditions \(A(0) = A_0\), \(\mathbf{w}(0) = \omega_\star\) satisfies \(|A_\star - A_\star| < \tau\) also satisfies
\[
|\mathbf{w}(t) - \omega_\star| > \eta \sqrt{\varepsilon} \text{ for some } t \text{ on the interval } [0, T/\sqrt{\varepsilon}].
\]

Proof: Assume that \(a \cdot \langle \mathbf{f}_1 \rangle (\omega_\star, A_\star, E) > 0\) for some constant unit vector \(\mathbf{a}\). Since \(\langle \mathbf{f}_1 \rangle (\omega_\star, A_\star, E)\) is continuous in \(\mathbf{C}\), there exists \(0 < \tau_0 < \frac{\pi}{24}\) such that \(a \cdot \langle \mathbf{f}_1 \rangle (\omega_\star, A_\star, E) > 4m > 0\) for \(|C - E| < \tau_0\). Recall that
\[
\mathbf{w}(N, t, B) = h_N(t, \theta, \phi, \psi),
\]
where
\[
h_N(t, \phi, \theta, \psi) = \sum_{m,n} a_{mn}(\phi, \psi) e^{i(mt+n\theta)}
\]
and
\[
k(t, \theta, \phi, \psi) = \sum_{m,n} a_{mn}(\phi, \psi) e^{i(mt+n\theta)}
\]
is an absolutely convergent Fourier series. Thus, there exists a \(N_1\) such that
\[
|\sum_{m,n} a_{mn}(\phi, \psi) e^{i(mt+n\theta)}| \leq m
\]
for all \(N \geq N_1\) provided that \(|B - e^{\frac{\tau_0}{2}}t| < \tau_1\). Let \(N \geq N_1\), and let \(u(t, B)\) be part of the coordinate transformation (IV.15) which takes system (IV.16) into system (IV.18) with \(N\) as above. Then there exist positive real numbers \(h_1\), \(h_2\), \(\eta_1\), \(\tau_2\), \(M_1\), \(M_2\), and \(M_3\) such that, for \(\varepsilon \|\gamma\| < h_1\), \(\varepsilon^2 < h_2\), \(|\gamma\| < \eta_1\), and \(|B - e^{\frac{\tau_0}{2}}t| < \tau_2\), it follows that the function \(u(t, B)\) satisfies \(|u| \leq M_3\) and the terms
\[
O(\varepsilon \|\gamma\|) \text{ and } O(\varepsilon^2)
\]
satisfy \(|O(\varepsilon \|\gamma\|)| \leq \varepsilon \|\gamma\| M_1\) and \(|O(\varepsilon^2)| \leq \varepsilon^2 M_2\) for all \(t \geq 0\).
As in the proof of Theorem V.8, let $C_1$ be such that $|\text{skew}(\nu)| \leq C_1 |\nu|$ for all column vectors $\nu$. Then define

$$
\tau = \frac{1}{2} \min\{\tau_1, \tau_2\},
$$

$$
\eta = \min \left\{ \eta_1, \sqrt{\frac{m}{2C_1}}, \sqrt{\frac{\ln m}{480C_1}}, \sqrt{\frac{m}{2eC_1}}, \frac{m}{2M_1} \right\},
$$

$$
\varepsilon_0 = \min \left\{ \frac{m}{M_2}, \sqrt{\frac{h_2}{1 + \eta}}, \frac{\eta}{2M_3}, \left( \frac{\eta}{4M_3} \right)^2, \frac{m}{2M_1M_3} \right\}
$$

and

$$
T = \frac{2\eta}{m}.
$$

The proof is completed by a contradiction argument. Assume that $|\delta| = |\omega - \omega_\ast| < \eta \sqrt{\varepsilon}$ for $0 \leq t \leq T/\sqrt{\varepsilon}$. Then

$$
\int_0^t |\Delta(s)| \, ds < \int_0^{T/\sqrt{\varepsilon}} C_1 |\delta(s)| \, ds < C_1 \eta T = \frac{2C_1}{m} \eta^2.
$$

Since

$$
\eta^2 < \frac{m}{2C_1} \int_0^t |\Delta(s)| \, ds < 1
$$

for $0 < t < T/\sqrt{\varepsilon}$ and, hence,

$$
\exp \left\{ \int_0^t |\Delta(s)| \, ds \right\} - 1 < \frac{2eC_1}{m} \eta^2
$$

holds for $0 \leq t \leq T/\sqrt{\varepsilon}$. By Lemma V.3, the solution of system (IV.4) for $0 < \varepsilon \leq \varepsilon_0$ with initial conditions $B(0) = \Lambda^{-1} A_0 \Lambda^{-1} A$, $\delta(0) = 0$ such that
\[ |B(0) - E| = |A_0 - A_k| \leq \tau \text{ satisfies} \]

\[ |B(t) - e^{\bar{\Omega} t} B(0)| < \frac{2eC_1}{m} \eta^2 \]

for \( 0 \leq t \leq T/\sqrt{\varepsilon} \). Thus,

\[ |B - e^{\bar{\Omega} t}| \leq |B - e^{\bar{\Omega} t} B(0)| + |e^{\bar{\Omega} t} B(0) - e^{\bar{\Omega} t}| \]

\[ < \frac{2eC_1}{m} \eta^2 + \tau \]

\[ \leq 2\tau \]

\[ \leq \tau_2 \]

for \( 0 \leq t \leq T/\sqrt{\varepsilon} \), and it follows that the bounds \( ||u|| \leq M_3 \), \( ||0(e||\gamma||)|| \leq \varepsilon ||\gamma||M_1 \), and \( ||0(e^2)|| \leq \varepsilon^2 M_2 \) hold for \( 0 \leq t \leq T/\sqrt{\varepsilon} \).

Therefore,

\[ a^* \frac{dY}{dt} \geq \varepsilon a^* \left< \frac{e}{t} \right> (\omega^*, A_k, C) - \varepsilon m - \varepsilon ||\gamma||M_1 - \varepsilon^2 M_2 \]

\[ > \varepsilon (4m - m - m - m) \]

\[ = \varepsilon m \]

for \( 0 \leq t \leq T/\sqrt{\varepsilon} \). Thus,

\[ ||Y(T/\sqrt{\varepsilon}) - Y(0)|| \geq \int_0^{T/\sqrt{\varepsilon}} \left( a^* \frac{dY}{dt} \right) dt \]

\[ > mTV\varepsilon \]

\[ = 2n\sqrt{\varepsilon} \]

and, hence,

\[ ||\omega(T/\sqrt{\varepsilon}) - \omega^*|| \geq ||Y(T/\sqrt{\varepsilon}) - Y(0)|| - \varepsilon ||u(T/\sqrt{\varepsilon}, B) - u(0, B_k)|| \]

\[ > 2n\sqrt{\varepsilon} - \frac{1}{2} n\sqrt{\varepsilon} \]

\[ = \frac{3}{2} n\sqrt{\varepsilon} \]
Corollary to Theorem V.11: If the hypotheses of Theorem V.11 are satisfied by \( a = \bar{a} \), then there exist positive constants \( \varepsilon_0, \tau, \eta, \) and \( T \) such that the solution of system (IV.1) for \( 0 < \varepsilon \leq \varepsilon_0 \) with initial conditions \( A(0) = A_0, \omega(0) = \omega_0 \) satisfying \( |A_0 - A_*| < \tau \) also satisfies

\[
|\|\omega(t)\| - |\|\omega_*\|\| > \eta \sqrt{\varepsilon}
\]

for some \( t \) on the interval \([0, T/\sqrt{\varepsilon}]\).
VI. HYPOTHETICAL PROBLEMS

A. Introduction

If any practical application is to be made of the theorems of Chapter V to a system of equations of the form

\[
\frac{dA}{dt} = \Omega A
\]

(VI.1)

\[
\frac{d\omega}{dt} = \varepsilon f_1(t, \omega, A) + \varepsilon^2 f_2(t, \omega, A, \varepsilon)
\]

having initial conditions \(A(0) = A^0, \omega(0) = \omega^0\), one must first calculate \(f_1(t, \omega, A)\) as a function of \(\omega^0\) and \(A^0\).

The zero-th order solution of system (VI.1), (i.e. the solution of system (VI.1) when \(\varepsilon = 0\)), is \(\omega(t) = \omega^0, A(t) = e^{\Omega t}\). The theorems of chapter V imply that whenever \(f_1(t, \omega, A) \neq 0\), any solution of system (VI.1), (for small enough \(\varepsilon\)), must leave a fixed neighborhood of the zero-th order solution in a finite length of time. Furthermore, such a solution must satisfy the requirement that \(\omega(t)\) leave an \(\varepsilon\)-dependent neighborhood of \(\omega^0\).

The problem of locating solutions of system (VI.1) with constant, (or nearly constant), \(\omega\) therefore becomes one of locating initial conditions \(\omega^0, A^0\) such that \(f_1(t, \omega, A) = 0\). It is only for these initial conditions that solutions may exist which remain "near" their respective zero-th order solutions.

It is possible to show with the use of asymptotic estimates that if \(\omega^0, A^0\) are such that \(f_1(t, \omega, A) = 0\), then \(\omega\) stays near \(\omega^0\) for a long time. This is insufficient, however, to actually obtain quasiperiodic solutions to system (VI.1).

The remaining two sections of this chapter deal with examples which illustrate the opposite extreme situations that may be encountered in analyzing \(f_1(t, \omega, A)\). These examples are somewhat abstract, and only serve to demonstrate the type of general implications that may be derived from the location of \(\omega^0\) such that \(f_1(t, \omega, A) = 0\).
B. $f_1(t, \omega, A) = \omega \times f(t, \omega, A)$

Suppose that $f_1(t, \omega, A)$ has the form $\omega \times f(t, \omega, A)$ so that $f_1$ is always directed perpendicular to $\omega$. According to the proofs of theorems V.8 and V.11, if $\langle f_1(t, \omega, A) \rangle \neq 0$ then the solution $\omega(t), A(t)$ to system (VI.1) must satisfy the requirement that $\omega(t)$ leave a neighborhood of $\omega_\star$ by a change in direction; Note: the length of $\omega(t)$ remains constant.

If $\omega_\star \neq 0$, then $\langle f_1(t, \omega, A) \rangle$ can be zero in only two ways:

1. $\langle f(t, \omega, A) \rangle \parallel \omega_\star$

2. $\langle f(t, \omega, A) \rangle = 0$

Thus, either $f(t, \omega, A)$ itself averages to zero or the average of $f(t, \omega, A)$ is parallel to $\omega_\star$.

C. $f_1(t, \omega, A) = f(t, \omega, A)\hat{\omega}$

Suppose that $f_1(t, \omega, A)$ is directed parallel to $\omega$ so that $f(t, \omega, A)$ may be written in the form $f_1(t, \omega, A) = f(t, \omega, A)\hat{\omega}$. Then, if $\langle f(t, \omega, A) \rangle \neq 0$, one may conclude from the corollaries to theorems V.8 and V.11 that the solution $\omega(t), A(t)$ to system (VI.1) must satisfy the requirement that $\omega(t)$ leave a neighborhood of $\omega_\star$ through a change in magnitude. The direction of $\omega(t)$ remains unchanged.

This conclusion can be particularly useful for situations, such as that discussed in the next chapter, in which the rate of spin is of primary importance. Thus, nearly constant spin rates for nearly fixed axes of rotation may only exist near $\omega_\star$ such that $\langle f(t, \omega, A) \rangle = 0$. Elsewhere, the magnitude of $\omega(t)$ must change with time.
VII. APPLICATION TO THE MERCURY PROBLEM

A. Setup

It will be assumed throughout this chapter that the system composed of Mercury and the sun satisfies Kepler's laws, and that there exists an inertial coordinate frame \( I \) such that the Mercury-sun system lies in the \( 1,2 \)-plane of the \( I \)-frame. Let \( A \) be the coordinate frame that is parallel to \( I \) and which has its origin at the center of mass of Mercury.

According to Kepler's laws, the orbit of Mercury about the sun is that of an ellipse with the sun at one focus. When viewed in the \( A \)-frame, however, the sun appears to be moving in the elliptical orbit with Mercury at one focus. Let \( \alpha(t) \) be the true anomaly, measured from perihelion, under the interpretation that Mercury is located at a focus of the ellipse; (see Figure 8). Then, if \( a, b, \) and \( e \) are the semi-major axis, semi-minor axis, and eccentricity of the ellipse, it follows from Kepler's laws that

\[
\dot{s}^t = [\dot{s}, A]^t = (s_1, s_2, 0) \quad \text{and} \quad \frac{d\alpha}{dt} = (1 - e^2)^{-3/2}(1 + e \cos \alpha)^2 = \frac{ab}{|s|}^2
\]

where

\[
s_1 = ||s|| \cos \alpha, \quad s_2 = ||s|| \sin \alpha, \quad \text{and} \quad ||s|| = a(1 - e^2)(1 + e \cos \alpha)^{-1}
\]

Let \( B \) be a coordinate frame fixed in Mercury that has its origin at the center of mass of Mercury. If \( \mathcal{H} \) and \( m \) are the masses of the sun and Mercury respectively, and if \( A = [I; B, A] \), then the system of equations

\[
\frac{dA}{dt} = [\text{skew}(\omega)] A
\]

(VII.1)

\[
\frac{d\omega}{dt} = \varepsilon \sum_{n=0}^{\infty} \left[ (-1)^n \epsilon^n (A[D, B]A^{-1})^n \left( \frac{1}{m} N(t) - \Omega A[D, B]A^{-1} \omega \right) - A \frac{d}{dt} [D, B]A^{-1} \omega + A[D, B]A^{-1} \Omega \omega \right]
\]

with initial conditions \( \omega(0) = \omega_0 \), and \( A(0) = A_0 \) developed in Chapter III.
for the case of a rigid $m$ will be shown to remain valid for a nonrigid $m$. Here the inertia tensor $[I,B] = m(E + \varepsilon[D,B])$.

Comparison of system (VII.1) with system (IV.1) leads to the identification

$$f_{1}(t,\omega,\Lambda) = \frac{1}{m} \Lambda - \Omega [D,B]A^{-1} \Lambda - A \frac{d}{dt} [D,B]A^{-1} \omega + A[D,B]A^{-1} \Omega \omega.$$  

Our goal is to apply the theorems developed in Chapter V to the average of $f_{1}$ in an effort to obtain qualitative results concerning solutions of system (VII.1) with non-steady $\omega$. Of obvious necessity in determining the average of $f_{1}$ is specification of the precise functional forms for $N(t)$ and $D$.

According to Colombo and Shapiro [2], only two of the torques acting on Mercury appear to be significant enough to warrant consideration: the torques exerted by the sun on a permanent equatorial asymmetry of Mercury's moment of inertia ellipsoid and on the tidal bulge raised by the sun. Unfortunately, the actual mechanisms involved in the internal dynamics of a not totally rigid body are largely unknown even for the earth, and completely unknown for Mercury. Consequently, a model is introduced for describing the gross order manifestations of the torques without regard for the internal dynamics of the problem.

The only validity claimed for this model is that it does not seem totally unreasonable as a first order approximation in the eyes of the authors, and it results in manageable equations. It should be mentioned, however, that the actual functional form is unimportant. It is the average of $f_{1}$ that is utilized in the theorems of Chapter V, not the true functional form.

B. The Planetary Model

For the purpose of specifying the functional forms of $N(t)$ and $D$, the contributions to the moment of inertia ellipsoid due to the permanent asymmetry and tidal bulge are conceived as being derived from two separate bodies which may be treated independently. Let $m_{1}$ and $m_{2}$ be the masses of the two bodies so that $m = m_{1} + m_{2}$. 
One body, of mass $m_1$, is conceived of as being totally rigid and includes the contribution due to the permanent asymmetry of Mercury. The amount of deviation from a sphere is assumed to be small and, hence, will be denoted by $\varepsilon P$. Since the rigid portion is fixed with respect to the body frame $B$, the inertia tensor for the rigid portion expressed in the $A$-frame becomes $m_1(E + \varepsilon A[P,B]A^{-1})$.

The second body, of mass $m_2$, accounts for the tidal distortion of Mercury and is conceived of as being a pseudoelastic sphere. Under the gravitational influence of the sun, this body takes the form of an ellipsoid of revolution. If Mercury were a mechanically lossless planet, the major axis of the ellipsoid would always lie along the Mercury-sun axis, and, by symmetry, the sun's attraction would produce no net torque. The presence of internal dissipation is modeled by causing the major axis of the ellipsoid to lag behind the Mercury-sun axis by an angle $\kappa$. A model for $\kappa$ will be developed in section E.

The internal dynamics of the tidal model are assumed to be such that, instantaneously, the tidal bulge rotates with the body. The sun's gravitational field then redeforms the bulge into the shape of an ellipsoid of revolution whose major axis is as specified by the model for $\kappa$. These assumptions produce two major consequences.

First, the inertia of the tidal bulge enters the equations as resistance to turning even though, strictly speaking, the bulge does not turn with rate $||\omega||$.

Secondly, the torque generated by the tidal distortion may be calculated from a potential even though the system is not conservative. The instantaneous torque is determined by the instantaneous configuration of the system and then inserted into the independently derived equations of motion.

Similar to the handling of the permanent asymmetry, the deviation of the ellipsoid from a sphere is assumed to be small. We therefore denote the tidal bulge term by $\varepsilon T$. This term will be most easily expressed directly in the $A$-frame and will be written as $m_2(E + \varepsilon[T,A])$. 
A major discrepancy between the assumptions of this chapter and
the derivation of system (IV.1) lies in the time dependence of \([T,\beta]\).
In particular, \([T,A]\) is assumed to be a function of \(\frac{du}{dt}\) and hence, \(\varepsilon\).
Thus, strictly speaking, not all of \(\frac{d}{dt} [T,A]\) belongs within \(f_1(t,\omega,\Lambda)\).
Due to the complexity involved in separating \(\frac{d}{dt} [T,A]\) into the appropriate pieces, however, we will write the equations as if all of \(\frac{d}{dt} [T,A]\)
does belong in \(f_1(t,\omega,\Lambda)\). In Lemma VII.2 we show that the average of
the portion of \(\frac{d}{dt} [T,A]\omega\) which does belong in \(f_1(t,\omega,\Lambda)\) is zero.

C. The Apparent Solar Angular Velocity \(\hat{\Omega}\)

A necessary element in our model for calculating the position of the
major axis of the tidal ellipsoid is the apparent solar angular velocity
\(\hat{\Omega}\); (i.e., the solar angular velocity as viewed from the surface of
Mercury). In this section, we determine \(\hat{\Omega}\) as a function of time.

Let \(A = [\text{Id}; B,A]\) with \(A\) and \(B\) as defined in section VII.A. Then
\[ \frac{d}{dt} A = \text{skew}(\omega)A, \]
where \(\omega\) is the instantaneous angular velocity of \(B\) with
respect to \(A\) as expressed in the \(A\)-frame. Define the coordinate frame
\(C\) to have the same origin and \(3\)-axis as \(A\) and to be such that the \(1\)-axis of
the \(C\)-frame always passes through the sun. Then, if \(B = [\text{Id}; C,B]\), it
follows that \(\frac{d}{dt} B = \text{skew}([\hat{\Omega},B])B\), where \([\hat{\Omega},B]\) is the instantaneous
angular velocity as viewed from the surface of Mercury.

Since \([\hat{s},C] = (1,0,0)^t\) and \([\hat{s},A] = (\cos \alpha, \sin \alpha, 0)^t\), where \(\hat{s} = \frac{1}{||\hat{s}||} \hat{s}\), and since \(C\) is obtained from \(A\) by a rotation about the \(3\)-axis of
the \(A\)-frame, it follows that
\[
R = [\text{Id}; C,A] = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Thus,
\[ B = [\text{Id}; C, B] = [\text{Id}; A, B][\text{Id}; B, C] = A^{-1}R, \]
and hence
\[ \frac{d}{dt} B = A^{-1}\left(\frac{d}{dt} R\right) - A^{-1} \text{skew}(\omega) R \]
\[ = A^{-1}\left[\left(\frac{d}{dt} R\right) R^{-1} - \text{skew}(\omega) \right] A(A^{-1} R). \]
Therefore,
\[ \text{vect} \left[\left(\frac{d}{dt} R\right) R^{-1}\right] = \omega \]
is the instantaneous solar angular velocity expressed in the \( A \)-frame.

Since \( \left(\frac{d}{dt} R\right) R^{-1} = \frac{d\alpha}{dt} \text{skew}(k) \), it follows that
\[ \Psi = \frac{d\alpha}{dt} k - \omega. \quad (\text{VII.2}) \]

D. The Torque \( N(t) \)

In section VII.B, the inertia tensor for the rigid portion of Mercury was expressed as \( m_1(E + \varepsilon A[P, B]A^{-1}) \) where \([P, B]\) is constant. Define the body frame \( B \) to be the coordinate frame defined by the principal axes of the rigid portion such that the \( B \)-frame 3-axis corresponds to the principal axis having the largest moment of inertia. Then \([P, B]\) will be diagonal and one may write
\[ m_1(E + \varepsilon[P, B]) = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad (\text{VII.3}) \]
where \( I_1 \leq I_2 \leq I_3 \).

The inertia tensor for the tidally distorted portion of Mercury was expressed in section VII.B in the form \( m_2(E + \varepsilon[T, A]) \). Let the boundary of the ellipsoid of revolution satisfy \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1 \) with respect to the major axes of the ellipsoid. The assumption that the ellipsoid is a small perturbation of a sphere is implemented by writing \( a = \rho + \varepsilon \rho_1 \) and
b = \rho - \varepsilon \rho_2 \text{ where } \rho \text{ is the mean radius of Mercury. It will be further assumed that the volume and density of the ellipsoidally distorted sphere remain constant. Thus } \rho^3 = ab^2. 

In Appendix B, an approximation to the potential energy of Mercury as a function of the position of the sun \( \mathbf{s} \) is calculated using a derivation similar to that employed in obtaining MacCullagh's formula. The result, under the assumptions of this chapter, is listed here in the form

\[
V(\mathbf{s}) = -\frac{MG}{|\mathbf{s}|} \left[ m + \frac{1}{2|\mathbf{s}|^2} \left( C_1 - 3 \mathbf{a}^T \mathbf{A}^{-1} \mathbf{a} + \frac{3m_2}{5} \left( a^2 - b^2 \right) \cos^2 \kappa \right) \right]
\]

where

\[
C_1 = 2 \left[ I_1 + I_2 + I_3 \right] + \frac{4m_2}{5} \left( a^2 + 2b^2 \right) - \frac{3m_2}{5} \left( a^2 + b^2 \right)
\]

and

\[
I_\rho = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.
\]

For the case in which the system is conservative, the generalized force \( Q_1 \) corresponding to a rotation coordinate \( q_1 \) measured about an axis \( \mathbf{n} \) is the component of the total applied torque about that axis (see Goldstein [7, pp. 50-51]). Obviously, the situation presented above is not conservative. However, as argued in section B, under our model, the instantaneous torque may be determined from the instantaneous configuration of the system. This is accomplished by assuming that the instantaneous torque is equivalent to the torque induced in a conservative system having the same instantaneous configuration.

Let \( \hat{\mathbf{b}}^t = (b_1, b_2, b_3) \) be an arbitrary unit vector, and let \( \mathbf{B} = \text{skew}(\hat{\mathbf{b}}) \). Then \([R(\theta); \mathbf{A}] = \exp(\mathbf{B})\) is the rotation about the \( \hat{\mathbf{b}} \)-axis through the angle \( \theta \). Observe that a rotation of Mercury through an angle \( \theta \) about the \( \hat{\mathbf{b}} \)-axis is equivalent to a rotation of the position of the sun \( \mathbf{s} \) through an angle \(-\theta\) about the same axis.
Since the kinetic energy of a rotating body is independent of the angular variable used to measure position about the axis of rotation, \( \Theta \) enters into the Lagrangian only through \( V(s) \). Therefore, the generalized force \( \Theta \), corresponding to the rotation coordinate \( \Theta \) may be calculated by

\[
\Theta = -\left\{ \frac{\partial}{\partial \Theta} \left[ V(e^{-B \Theta/j}) \right] \right\}_{\Theta=0}.
\] (VII.5)

Let \( \hat{a} \) be a unit vector along the major axis of the ellipsoid. Then, using \( \|e^{-B \Theta/j}\| = \|s\| \), it follows that

\[
e_N \cdot \hat{a} = + \frac{3GM}{2\|s\|^3} \left[ \frac{m_2}{5} \left( a^2 - b^2 \right) \left( e^{-B \Theta/j} \cdot \hat{a} \right)^2 - m_1 \|s\|^2 \right]
\]

\[
- \epsilon m_1 (e^{-B \Theta/j} \cdot \hat{a}) \left( e^{-B \Theta/j} \cdot \hat{a}, \hat{a} \right) - \epsilon m_1 (e^{-B \Theta/j} \cdot \hat{a}) \left( e^{-B \Theta/j} \cdot \hat{a}, \hat{a} \right)
\]

\[
- \epsilon m_1 \left( e^{-B \Theta/j} \cdot \hat{a}, \hat{a} \right) \left( e^{-B \Theta/j} \cdot \hat{a}, \hat{a} \right) - \epsilon m_1 \left( e^{-B \Theta/j} \cdot \hat{a}, \hat{a} \right) \left( e^{-B \Theta/j} \cdot \hat{a}, \hat{a} \right)
\]

Thus, since \( \hat{a} \) was arbitrary,

\[
e_N = \frac{3GM}{\|s\|^3} \left[ \frac{m_2}{5} \left( b^2 - a^2 \right) \cos(k) \left( \hat{a}, \hat{a} \right) - \epsilon m_1 \left( e^{-B \Theta/j} \cdot \hat{a}, \hat{a} \right) \left( e^{-B \Theta/j} \cdot \hat{a}, \hat{a} \right) \right].
\] (VII.6)

Recall that, under the assumption that the ellipsoid of revolution is a small perturbation of a sphere, we may write \( a = \rho + \epsilon \rho_1 \), \( b = \rho - \epsilon \rho_2 \), and \( \rho^2 = ab^2 \). Thus, \( b^2 - a^2 = -6\epsilon \rho_2 + O(\epsilon^2) \) and hence, to first order in \( \epsilon \), we obtain

\[
e_N = -\epsilon \frac{3GM}{\|s\|^3} \left[ \frac{6m_2 \rho_2}{5} \cos(k) \left( \hat{a}, \hat{a} \right) + \epsilon m_1 \left( A[P,B]A^{-1} \hat{a}, \hat{a} \right) \left( A[P,B]A^{-1} \hat{a}, \hat{a} \right) \right].
\] (VII.7)
Furthermore, \( \hat{s} = \exp \left\{ \text{skew}(\hat{\mathbf{v}}) \right\} \hat{s} \) where \( \mathbf{v} = \frac{d\alpha}{dt} \) is the apparent solar angular velocity derived in section C. Thus

\[
\hat{s} \times \hat{s} = \hat{s} \times \left\{ e^{\text{skew}(\hat{\mathbf{v}}) \hat{s}} \right\} = \sin(\kappa) [\hat{\mathbf{v}} - (\hat{s} \cdot \hat{\mathbf{v}}) \hat{s}] + (\hat{s} \cdot \hat{\mathbf{v}})(1 - \cos(\kappa))(\hat{s} \times \hat{\mathbf{v}})
\]

and hence

\[
N = -\frac{3GM}{||s||^3} \left\{ \frac{3m_s\rho_2}{5} \sin(2\kappa) [\hat{\mathbf{v}} - (\hat{s} \cdot \hat{\mathbf{v}}) \hat{s}] + \frac{6m_s\rho_2}{5} \cos(\kappa)(1 - \cos(\kappa))(\hat{s} \cdot \hat{\mathbf{v}})(\hat{s} \times \hat{\mathbf{v}}) + m_1(A[\mathbf{P},\mathbf{B}]A^{-1}\hat{s}) \times \hat{s} \right\} \tag{VII.8}
\]

E. The Lag Angle Model

Recall that the model for the tidal bulge is an ellipsoid of revolution whose major axis is located by a rotation about the apparent solar angular velocity axis \( \mathbf{v} \). The lag angle \( \kappa \) was defined as the angle of rotation about that axis.

The true functional dependence of \( \kappa \) on quantities such as the angular velocity and the amplitude of the tidal strain is unknown. For the one degree of freedom problem in which the spin axis is held fixed perpendicular to the plane of the orbit, Counselman and Shapiro [3] considered three models which they claimed were representative:

1) \( \kappa \) constant,
2) \( \kappa \) proportional to \( \frac{d\alpha}{dt} - \omega \)

and

3) \( \kappa \) proportional to the tidal strain.

For our model, the shape of the ellipsoid, and hence the tidal strain, is assumed to be constant. Thus, model (3) is equivalent to model (1). One perhaps philosophic problem with model (1), however, is the instantaneous jump in the position of the tidal bulge when \( \frac{d\alpha}{dt} - \omega \) changes sign. Therefore, model (2) will be the only model considered in this dissertation.
In the one degree of freedom case, \( \omega \) is a one dimensional variable and is equivalent to the angular velocity of the planet. The generalization employed in moving to the three degree of freedom case is to take \( \kappa \) proportional to \( \| \frac{d\alpha}{dt} k - \omega \| \). Notice that this reduces to the one degree of freedom case when \( \omega \) is perpendicular to the plane of the orbit.

Thus, for the remainder of this dissertation, \( \kappa = c \| \nu \| \) for some positive real constant \( c \).

**F. The Average of \( f_1(t,\omega,A) \)**

In this section, the hypotheses and models developed in sections A through E will be combined to obtain a form for \( f_1(t,\omega,A) \) which may be averaged without resorting to numerical techniques.

In section A, the equation

\[
\frac{d}{dt} f_1(t,\omega,A) = -m_1 \lambda (\Omega A^*[P,B] A^{-1} - A^*[P,B] A^{-1} \Omega) \omega
\]

was obtained and it was observed that not all of \( \frac{d}{dt} f_1(t,\omega,A) \) \( \omega \) belongs in \( f_1(t,\omega,A) \). The next two lemmas show that the average of \( \frac{d}{dt} f_1(t,\omega,A) \) is just the average of \( \lambda N \).

**Lemma VII.1:** \( \langle (\Omega A^*[P,B] A^{-1} - A^*[P,B] A^{-1} \Omega) \omega \rangle = 0. \)

**Proof:** Case I: \( \| \omega_x \| = p/q \) is rational.

\[
\langle (\Omega A^*[P,B] A^{-1} - A^*[P,B] A^{-1} \Omega) \omega \rangle
\]

\[
= \frac{1}{2\pi q} \int_0^{2\pi q} (\Omega e^{\omega_x t} A^*[P,B] A^{-1} e^{-\Omega_x t} - e^{\omega_x t} A^*[P,B] A^{-1} e^{-\Omega_x t}) \omega_x dt
\]

where \( A_x \) and \( \omega_x \) are constants. Since \( [P,B] \) is constant, and since \( e^{\Omega_x t} \) is 2\( \pi q \)-periodic in \( t \),

\[
\langle (\Omega A^*[P,B] A^{-1} - A^*[P,B] A^{-1} \Omega) \omega \rangle
\]

\[
= \frac{1}{2\pi q} \int_0^{2\pi q} \frac{d}{dt} \left[ (e^{\omega_x t} A^*[P,B] A^{-1} e^{-\Omega_x t}) \omega_x \right] dt
\]

\[
= 0.
\]
Case II: $||\omega_*||$ is irrational.

$$\left<(\Omega A[P,B]A^{-1} - A[P,B]A^{-1}\Omega)\omega\right> \equiv a_{oo}(0,0)$$

$$= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left[ \left( \Omega e^{\hat{\Omega}\theta} A_*[P,B]A_*^{-1} e^{-\hat{\Omega}\theta} - e^{\hat{\Omega}\theta} A_*[P,B]A_*^{-1} e^{-\hat{\Omega}\theta} \right) \omega_* \right] d\theta dt$$

$$= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left[ \int_0^{2\pi} \frac{d}{d\theta} \left( e^{\hat{\Omega}\theta} A_*[P,B]A_*^{-1} e^{-\hat{\Omega}\theta} \right) d\theta \right] \omega_* dt$$

$$= 0$$

since $e^{\hat{\Omega}\theta}$ is $2\pi$-periodic in $\theta$.\[\square\]

Lemma VII.2: Let $\omega = g(t,\omega,\alpha) + \epsilon h(t,\omega,\alpha,\epsilon)$.

Then $\left<g(t,\omega,\alpha)\right> = 0$.

Proof: Direct computation reveals that

$$E + \epsilon[T,A] = \frac{a^2 + b^2}{2} E + \left(\frac{b^2 - a^2}{2}\right) e \text{ skew}(\nu) \frac{\partial}{\partial \alpha} \frac{t}{2} e^{-c \text{ skew}(\nu)}.$$

Thus,

$$\epsilon \frac{d}{dt} [T,A] = \frac{1}{5} (b^2 - a^2) \frac{d}{dt} \left[ e^{c \text{ skew}(\nu)} \frac{\partial}{\partial \alpha} \frac{t}{2} e^{-c \text{ skew}(\nu)} \right]$$

where $\nu = \frac{\partial}{\partial \alpha} k - \omega = (1 - e^2)^{-1/2} (1 + e \cos \alpha)^{-1} k - \omega$.

Since $\left||\omega_*\right|| = o(\epsilon)$,

$$g(t,\omega,\alpha) = \frac{b^2 - a^2}{5\epsilon} \frac{\partial}{\partial \alpha} \left[ e^{c \text{ skew}(\nu)} \frac{\partial}{\partial \alpha} \frac{t}{2} e^{-c \text{ skew}(\nu)} \right] \frac{d\alpha}{dt} \omega.$$

Thus, for $\omega = \omega_*$ and since $\alpha$ is $2\pi$-periodic,

$$\int_0^{2\pi} \left\{ \frac{\partial}{\partial \alpha} \left[ e^{c \text{ skew}(\nu)} \frac{\partial}{\partial \alpha} \frac{t}{2} e^{-c \text{ skew}(\nu)} \right] \right\} \frac{d\alpha}{dt} dt = 0.$$

It follows, therefore, that $\left<g(t,\omega,\alpha)\right> = 0$.\[\square\]

Thus, according to Lemmas VII.1 and VII.2,

$$\left<\hat{f}_1(t,\omega,\alpha)\right> = \frac{1}{m} \left<\hat{N}(t)\right>$$

(VII.10)
In Appendix C, the average of $N(t)$ is calculated for arbitrary $\omega_n$. Due to the complexity of $\langle N(t) \rangle$ in the general case, our goal of applying the theorems of Chapter V by first locating the zeroes of $\langle N(t) \rangle$ analytically does not appear to be feasible in the most general case. One might hope to ascertain some knowledge through numerical techniques, but this will be reserved for potential future study. We will instead content ourselves with a study of two special cases:

1. $\omega_n = ||w_n|| k$
2. The lag angle $\kappa$ is sufficiently small that $\frac{\sin \kappa}{\kappa} \equiv 1$, $\cos \kappa \equiv 1$, and $\frac{1 - \cos \kappa}{\kappa^2} \equiv 1/2$.

**Case (1):** $\omega_n = ||w_n|| k$

Then, from Appendix C,

$$\langle f_1(t, \omega, A) \rangle = \frac{9GMm_w^2 \rho_2}{5ma^3} \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)!}{(2n+1)!} \left[ \sum_{x=0}^{n} \sum_{y=0}^{n} \frac{x!y!(n-x-y)!}{n!} \left( -2\omega_3 \right)^{x+y} \right]^{2(n-x-y)}$$

$$\left( -2\omega_3 \right)^{x+y} \left( 1 - e^2 \right)^{x+y} \frac{\omega_n}{G_x, y(e)} \left[ \begin{array}{c} p_{23} \\ p_{13} \\ 0 \end{array} \right]$$

$$\begin{cases} \left( \text{if } ||w_n|| = \text{an integer} \right) \\ \left( \frac{3GMm_w}{4ma^3} \right) \right.$$
\[
\begin{align*}
&\begin{cases}
\text{if } 2 | \omega_k | = n \text{ is an integer}, \\
\text{if } 2 | \omega_k | = n + 1 \text{ is an integer},
\end{cases} \\
&\left( \frac{3GM_1}{2ma} \right) \left\{ \begin{array}{l}
\left[ G_{2,0,-n-2}(e) + G_{2,0,n-2}(e) \right] \hat{p}_{12}^k \\
\left[ G_{2,0,n-2}(e) - G_{2,0,-n-2}(e) \right] \hat{p}_{12}^k
\end{array} \right\}
\end{align*}
\]

where \( \hat{G}_{x,y}(e) = (1-e^2)^{1/2} G_{2(x+2y+2),x+2y+2,0}(e) - \omega_3 G_{2(x+2y+1),x+2y+1,0}(e) \).

Since \( G_{2,1,n}(e) = G_{2,1,-n}(e) \), we may write

\[
\begin{align*}
&\left( \frac{6m_c p_p}{5} \right) \sum_{n=0}^{\infty} (-1)^n \frac{(2c)^{2n+1}}{(2n+1)!} \left[ \sum_{x=0}^{n} \sum_{y=0}^{n} \frac{n! | \omega_k |^{2(n-x-y)}}{x! y! (n-x-y)!} \right] \\
&\left( -2\omega_3 \right)^{x+2y} \left( 1-e^2 \right)^{2(n-x-y)} \hat{G}_{x,y}(e) \right\}^k
\end{align*}
\]

\[
= -\frac{3GM}{2ma} \left\{ \begin{cases}
\text{if } 2 | \omega_k | = n \text{ is an integer}, \\
\text{if } 2 | \omega_k | = n + 1 \text{ is an integer},
\end{cases} \\
\left[ G_{2,1, \ell}(e) - G_{2,0, \ell-2}(e) \right] \hat{p}_{23} \\
\left[ G_{2,1, \ell}(e) - G_{2,0, \ell-2}(e) \right] \hat{p}_{13} \\
0
\right\}
\]

VII.10

Case (2): Small lag angle.

Suppose that the lag angle \( \kappa = c ||v|| \) is sufficiently small that the approximations \( \sin \kappa \approx 1 \), \( \cos \kappa \approx 1 \), and \( \frac{1-\cos \kappa}{\kappa^2} \approx \frac{1}{2} \) can be made.

Then, from Appendix C,

\[
\begin{align*}
&\left\langle \sin(2c ||v||) \left( \hat{\mathbf{s}} - (\hat{\mathbf{s}} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}} \right) \right\rangle \\
&\quad + \cos(c ||v||) \left( 1-\cos(c ||v||) \right) \left( \hat{\mathbf{s}} \cdot \hat{\mathbf{v}} \right) (\hat{\mathbf{s}} \times \hat{\mathbf{v}}) \\
&= \frac{c}{2b} \left[ (2 + 3e^2)(1-e^2)^{-3/2} - \omega_3 \right] \left( \frac{c}{2b} \omega_k \right)
\end{align*}
\]
The assumption of a small lag angle obviously has no effect on the rigid portion of the planet, and thus does not reduce the complexity of the averaged rigid term. If $2||\omega_3||$ is not an integer, however, the simple expression

$$\left< f_1(t,\omega, A) \right> = \left( \begin{array}{c} \frac{3\omega_3\hat{\rho}_2}{5} + c \left[ \frac{(2 + 3e^2)(1-e^2)^{-3/2} - \omega_3}{2} \right] \frac{k - \omega_3}{\omega_1} \\ - c \left[ \frac{(1 + \frac{9e^2}{4})(1-e^2)^{-3/2} - \omega_3}{2} \right] \omega_1 \end{array} \right)$$

(VII.12)

is obtained.

G. Discussion of Results for the Mercury Problem

Locating the zeroes of $\left< f_1(t,\omega, A) \right>$ analytically does not appear to be feasible for the most general case. Consequently, the theorems of
chapter V cannot be applied in that instance. However, two special cases where some information can be obtained concerning the location of zeroes are worth considering.

For the case in which \( \omega_\kappa = \left| \omega_\kappa \right| \kappa \), the axis of rotation is perpendicular to the plane of the orbit. This is the situation studied by Counselman and Shapiro [3] and most of their predecessors. Our main concern in studying this case is to verify that to first order in \( \varepsilon \), our model reduces to the model studied by Murdock [17] in the one degree of freedom case.

Consider first the second term of equation (VII.10). Table 2 in Kaula [11, p. 38] shows that \( G_{2,1,\ell}(\varepsilon) \neq G_{2,0,\ell-2}(\varepsilon) \) for \( \ell = 0, 1, 2 \) and \( G_{2,1,\ell}(\varepsilon) \neq G_{2,0,-\ell-2}(\varepsilon) \) for \( \ell = -2, -1, 0 \). Furthermore, examination of the expressions for \( G_{2,1,\ell}(\varepsilon) \), \( G_{2,0,\ell-2}(\varepsilon) \), and \( G_{2,0,-\ell-2}(\varepsilon) \) makes it appear unlikely that equality will exist for any integer \( \ell \). In any event, if \( G_{2,1,\ell}(\varepsilon) \neq G_{2,0,\ell-2}(\varepsilon) \) and the product of inertia \( P_{23} \) of the original orientation of the rigid portion with respect to the \( A \)-frame 2,3-axes is non-zero, or if \( G_{2,1,\ell}(\varepsilon) \neq G_{2,0,-\ell-2}(\varepsilon) \) and the product of inertia \( P_{13} \) of the original orientation of the rigid portion with respect to the \( A \)-frame 1,3-axes is non-zero, then \( \left< x_1(t,\omega,A) \right> \neq 0 \) for \( \left| \omega_\kappa \right| = \kappa \). Hence, by Theorem V.8, for sufficiently small \( \varepsilon \) there exists a neighborhood \( N_\varepsilon \) of \( \omega_\kappa = \kappa \kappa \) such that \( \omega \) leaves \( N_\varepsilon \) in a finite length of time.

Observe that, in the situation described above, \( \left< x_1(t,\omega,A) \right> \) will have a component that is perpendicular to \( \omega_\kappa \). Then, by the proof of Theorem V.8, \( \omega \) not only leaves a neighborhood of \( \omega_\kappa \), it also changes direction.

This is not surprising once it is pointed out that if the inertia tensor for the rigid portion is not identical to that for a sphere, then \( \tilde{P}_{23} = 0 \) and \( \tilde{P}_{13} = 0 \) if and only if \( \omega_\kappa \) is the largest principal axis of the rigid portion. Since it is already known from classical mechanics that rotation about an axis which is not a principal axis of the rigid body is unstable, it will be assumed that \( \tilde{P}_{13} = 0 \) and \( \tilde{P}_{23} = 0 \). When \( \tilde{P}_{13} \) and \( \tilde{P}_{23} \) both vanish, equation (VII.10) has the form

\[
\left< x_1(t,\omega,A) \right> = f(t,\omega,A)\kappa
\]

which agrees to first order in \( \varepsilon \) with the model studied by Murdock [17].
The first term of equation (VII.10) is obtained from the average of the expression

$$\frac{\sin(2c|\omega|)}{|\omega| |s|} \left[ \frac{d\alpha}{dt} k - \omega_0 \right].$$

For $\omega_0 = 0$ and a sufficiently small lag angle, this expression is positive for all time and hence, its average is positive. For Mercury, $\frac{d\alpha}{dt} = (1-e^2)^{-3/2} (1+e \cos \alpha)^2 \leq 1.6$. Thus, for $||\omega_0|| > 2$, the expression above will be positive for all time and hence, so will its average. Therefore, for some $\omega_0 = |\omega_0| k$ satisfying $0 < ||\omega_0|| < 2$, the first term of equation (VII.10) will be zero. If such an $\omega_0$ satisfies $||\omega_0||$ being irrational, then $\langle \xi_1(t,\omega,A) \rangle = 0$ and solutions for $\omega$ stay near $\omega_0$ for a long time. Furthermore, quasi-periodic solutions to system (IV.1) may exist.

If $||\omega_0|| = \frac{n}{2}$ for some integer $n$, then an equation is obtained which may be solved for $\rho_{12}$. If an orientation exists in which the product of inertia, $(\mathbf{I}_{12})$, of the rigid portion with respect to the $A$-frame 1,2-axes equals this value, then quasi-periodic solutions may exist satisfying $\omega = \frac{n}{2} k$. This agrees with earlier findings, ([2], [6], [10], [12] and [13]), that the situation in which $||\omega_0|| = \frac{3}{2} k$ could be stabilized if Mercury possesses a small permanent equatorial asymmetry.

For special case (2), in which a small lag angle is assumed, the much simpler approximation (VII.11) is obtained for the average tidal torque. If $\omega_0 = |\omega_0| k$, the first term in equation (VII.10) may be replaced by $\frac{c}{2b^3} (2 + 3e^2)^{-3/2} (1-e^2)^{-3/2} - 2||\omega_0|| k$ and the same analysis as above applies. If it is not assumed that $\omega_0$ is perpendicular to the plane of the orbit, the completely general equations are formidable. If $2||\omega_0||$ is not an integer, however, equation (VII.12) is obtained. This equation is zero if and only if

$$\omega_3 = (1 + \frac{3}{2} e^2)(1-e^2)^{-3/2},$$
\[
\frac{3}{5} \text{cm}_2 \rho \rho_2 \left[ (-\omega_1) - \frac{3e^2}{4} (1-e^{-2})^{-3/2} \omega_2 \right] \\
+ \frac{m_1}{4} \left[ \text{trace}([P,B]) - 3 \left( \Omega_x \Omega_y \right) \right] \omega_2 \omega_3 = 0,
\]

and
\[
\frac{3}{5} \text{cm}_2 \rho \rho_2 \left[ (-\omega_2) - \frac{3e^2}{4} (1-e^{-2})^{-3/2} \omega_1 \right] \\
- \frac{m_1}{4} \left[ \text{trace}([P,B]) - 3 \left( \Omega_x \Omega_y \right) \right] \omega_1 \omega_3 = 0.
\]

Multiplying the second equation by \(\omega_1\), the third equation by \(\omega_2\) and adding yields the equation
\[
\omega_1^2 + \omega_2^2 + c \left( \frac{3e^2}{2} \right) (1-e^{-2})^{-3/2} \omega_1 \omega_2 = 0.
\]

Clearly, \(\omega_1 = 0\), \(\omega_2 = 0\), and \(\omega_3 = \left( 1 + \frac{3e^2}{2} \right) (1-e^{-2})^{-3/2}\) is a solution of the equation \(\langle \hat{f}_1 (t, \omega, A) \rangle = 0\) when equation (VII.12) is taken as the expression for \(\langle \hat{f}_1 (t, \omega, A) \rangle\). If \(c < \frac{4}{3e^2} (1-e^{-2})^{+3/2}\), \(\Omega^2 29.6\) for Mercury), then this is the only real solution. Thus, for a sufficiently small lag angle, the only possible solution of the form \(\omega = \Omega_k\), where \(|\Omega_k|\) is not a half-integer, is \(\omega = \left( 1 + \frac{3e^2}{2} \right) (1-e^{-2})^{-3/2} k\).

The consideration of \(\Omega_k\) for which \(2|\Omega_k|\) is an integer does not appear to be amenable to analytic study. Presumably, one might proceed with a numerical approach to solving the equations for \(\Omega_k\), but such speculations will be reserved for future study.
VIII. CONCLUSIONS

A. Summary

Examination of a physical system consisting of a large mass $M$ and a smaller mass $m$ in orbit about it leads to a system of matrix and vector differential equations of the form

$$\frac{dA}{dt} = \Omega A$$

(VIII.1)

$$\frac{d\omega}{dt} = \varepsilon f_1(t, \omega, A) + \varepsilon^2 f_2(t, \omega, A, \varepsilon) .$$

Analysis of system (VIII.1) leads to a system having a slightly different form than that investigated in earlier papers by Murdock ([14], [15], [16], and [17]). A development parallel to that in Murdock [14], however, yields comparable qualitative results. The main thrust of these results is negative in the sense that no claims are made advocating the existence of solutions of interest. Rather, a condition is obtained which, when satisfied by initial conditions $\omega_0^r$ and $A_0^r$, prohibits the existence of solutions to system (VIII.1) that have the property that $\omega$ remain arbitrarily close to $\omega_0^r$.

The theorems obtained from the theoretical discussion are applied to the physical problem of locating potential angular velocity vectors which might be achieved by Mercury as a steady-state. The planetary model used in the application consists of several simplifying assumptions.

(1) The dynamics of the non-rigid planet are such that the inertia tensor may be treated as the sum of two independent terms. One term is treated as totally rigid, and the other term is treated as arising from an ellipsoid of revolution representing the tidally distorted portion of Mercury.

(2) For the purpose of determining the potential energy of the system, the sun may be treated as a point mass.

(3) The instantaneous torque is completely determined by the instantaneous orientations of the rigid and tidally distorted portions of Mercury within the gravitational field of the sun.
(4) The amplitude of the tidal distortion is assumed to be constant.

(5) The lag angle is proportional to the length of the apparent solar angular velocity vector.

After calculation of a special average of $f(t,\omega,A)$, the theorems from the theoretical discussion allow the following conclusions.

1. Solving the completely general equations $f(t,\omega,A) = 0$ for initial conditions $\omega,A$ may not be possible except through numerical techniques.

2. For the special case in which the spin axis is held fixed perpendicular to the plane of the orbit, two major observations are readily available which agree with the current understanding of the problem.
   a) Unless the largest principal axis of Mercury is perpendicular to the plane of the orbit, integer spin rates cause the spin axis to leave its perpendicular orientation with respect to the orbit plane.
   b) Stable spin rates of $\frac{n}{2}$ for odd integers $n$ may exist if Mercury possesses a small permanent equatorial asymmetry.

3. For the special case in which the lag angle is small enough so that $\frac{\sin \kappa}{\kappa} \approx 1$, the general equations are probably still too complicated to analyze analytically. Several observations are possible, however, when the general case is restricted.
   a) If it is again assumed that $\omega$ is perpendicular to the orbit plane, the same analysis as in paragraph (2) applies.
   b) If $\omega$ is not restricted to be perpendicular to the orbit plane, and if $||\omega|| \neq \frac{n}{2}$ for some odd integer $n$, then for a sufficiently small lag angle the only solution of $f_N(t,\omega,A) = 0$ is $\omega = (1 + \frac{3}{2} \epsilon^2)(1-\epsilon^2)^{-3/2}$. (For Mercury $\epsilon = .2056$, thus this solution is $\omega \approx 1.13 \kappa$.

B. Discussion and Suggestions for Future Study

More work is required to determine if the three dimensional approach developed here is useful for resolving the conflicts involved in determining Mercury's spin-orbit status. The theorems proved in this dissertation are useful for determining whether or not it is possible for
Mercury to evolve into a spin-orbit state in which its angular velocity vector is arbitrarily near a specific $\omega_\star$, but they shed no light on the precise evolution of Mercury's spin axis.

The relatively simple model used to describe the internal dynamics of Mercury enabled us to calculate a special average of the total torque in closed form. This is crucial to the process of searching analytically for the $\omega_\star$ such that the torque averages to zero. Even for the simple model considered, however, the general problem appears too difficult to analyze completely in an analytic fashion. One possible avenue for future study is to further analyze the averaged torque, either analytically or with numerical techniques, in an effort to more accurately locate all possible $\omega_\star$ for which the torque averages to zero.

Another perhaps more reasonable approach is to create a totally new model which more accurately reflects the true internal dynamics of a not totally rigid body. In particular, our model assumed that the rigid and tidally distorted portions could be treated independently. Also, the amplitude of the tidal distortion was assumed to be constant. This latter assumption is believed to be responsible for the value $\omega_\star = 1.13 \ k$ in the special case of a small lag angle. It was expected that the ground state would be closer to the resonance value of $\omega_\star = (3/2) \ k$. This problem could possibly be remedied in our model by making $\rho_2$ depend upon $|\rho_2|$ in some manner, and this will be pursued in future studies. Whether such a realistic and presumably more complicated model will meet with equal success in obtaining a closed form expression for the average torque remains to be seen. However, the more accurate, (and thus more defendable), the planetary model is, the more reliable one's results should be.

The question of stability of solutions near an $\omega_\star$ for which the torque averages to zero has not been addressed directly by our theorems. One possible approach to this problem is to consider a second order averaging scheme such as that found in Murdock [17]. This would further reduce the number of $\omega_\star$ which Mercury might attain as a steady state axis of rotation.

Finally, it would be interesting to see if the one-dimensional
methods developed by Counselman and Shapiro [3] for calculating the probability of Mercury being trapped in a specific spin state can be extended to our three degree of freedom model. Such an analysis could help to resolve several of the conflicts surrounding the question of Mercury's spin orbit status.
# IX. TABLE OF SYMBOLS

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<td>Semi-major axis of the orbit</td>
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<td>(a_{mn})</td>
<td>Fourier coefficients of (f_1(t,\omega,\Delta))</td>
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<td>(a)</td>
<td>Major axis of the tidal ellipsoid</td>
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<td>(\hat{a})</td>
<td>Unit vector in the direction of (a)</td>
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<td>(A,A(t))</td>
<td>Identity transformation from the body frame (B) to the space frame (A)</td>
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<td>Identity transformation from an intermediate frame to the space frame</td>
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<td>Coordinate frame parallel to an inertial frame</td>
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<td>(\hat{A})</td>
<td>Coordinate frame fixed with respect to (A)</td>
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<tr>
<td>(\alpha,\alpha(t))</td>
<td>The true anomaly</td>
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<td>(b)</td>
<td>Semi-minor axis of the orbit</td>
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<tr>
<td>(B)</td>
<td>(B(t) = \frac{\gamma^{-1}}{A} \Delta A_k^{-1} A)</td>
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<td>Initial value for (B(t)) in a generalized system</td>
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<td>The right-hand derivative of a function (f(t))</td>
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<td>Eccentricity of the orbit</td>
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<td>E</td>
<td>The identity matrix</td>
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<td>$\varepsilon$</td>
<td>A small positive real number</td>
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<td>$f_1$</td>
<td>The first term of the second equation of system (IV.1)</td>
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<td>$f_2$</td>
<td>The second term of the second equation of system (IV.1)</td>
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<td>One of the angles used to parameterize $B(t)$</td>
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<tr>
<td>m</td>
<td>The small mass in orbit about M</td>
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<tr>
<td>m°</td>
<td>The portion of m which may be treated as a pseudoelastic sphere</td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>The larger mass</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>The torque exerted on m by M</td>
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<tr>
<td>N°</td>
<td>The external torque on m</td>
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<tr>
<td>V</td>
<td>The apparent solar angular velocity</td>
<td></td>
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<tr>
<td>0</td>
<td>A unit vector in the direction of y</td>
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<tr>
<td>0^</td>
<td>The angular velocity of m expressed in the A-frame</td>
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<td>0ω(t)</td>
<td>The angular velocity of m</td>
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<td>L</td>
<td>Angular momentum about the center of mass of m</td>
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<td>K</td>
<td>The lag angle</td>
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<tr>
<td>L</td>
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An inertial coordinate frame whose 3-axis is perpendicular to the plane of the orbit.
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<td>A geometric vector representing the angular velocity</td>
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<td>(\hat{\Omega}_0 = \hat{\Lambda}^{-1}\Omega_0\hat{\Lambda})</td>
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<td>Distortion of the inertia tensor due to the permanent asymmetry</td>
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<td>(\hat{P} = \hat{A}_0 [P,B]A_0^{-1})</td>
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<td>The components of (\hat{P})</td>
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<td>One of the expressions used in defining (G_{ijk}(e))</td>
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XII. ACKNOWLEDGMENTS

I would like to express my gratitude to Dr. James A. Murdock for his guidance, assistance, and above all his patience and belief that this dissertation would be completed.

I would also like to give special thanks to my wife, Shelly, whose constant encouragement enabled me to persist.
Consider four coordinate frames $A$ and $B_1^i$, ($i = 1, 2, 3$), which have a common origin. We will determine here $[\text{Id}; B_1^i, A]$ for the three situations in which

1) $B_1^1$ is obtained from $A$ by rotating $A$ through an angle $\nu$ about the $A$ frame 1-axis,

2) $B_1^2$ is obtained from $A$ by rotating $A$ through an angle $\nu$ about the $A$ frame 2-axis,

3) $B_1^3$ is obtained from $A$ by rotating $A$ through an angle $\nu$ about the $A$ frame 3-axis.

Let $\hat{u}$, $\hat{v}$, and $\hat{w}$ be unit vectors parallel to the coordinate axes of the $A$-frame, and let $\hat{i}$, $\hat{j}$, and $\hat{k}$ be unit vectors parallel to the corresponding coordinate axes of the $B_1^i$-frame. Suppose that $B_1^i$ is obtained from $A$ by a rotation through an angle $\nu$ about the $A$-axis. Then $[\hat{i}, A] = (\cos \nu)[\hat{u}, A] + (\sin \nu)[\hat{v}, A]$, $[\hat{j}, A] = (-\sin \nu)[\hat{u}, A] + (\cos \nu)[\hat{v}, A]$, and $[\hat{k}, A] = [\hat{w}, A]$; see Figure 1. Furthermore, if $\hat{r}$ is an arbitrary vector with $[\hat{r}, B_1^i] = a[\hat{i}, B_1^i] + b[\hat{j}, B_1^i] + c[\hat{k}, B_1^i]$, then $[\hat{r}, A] = (a \cos \nu - b \sin \nu) [\hat{u}, A] + (a \sin \nu + b \cos \nu) [\hat{v}, A] + c[\hat{w}, A]$, and, hence,

$$[\text{Id}; B_1^3, A] = \begin{pmatrix} \cos \nu & -\sin \nu & 0 \\ \sin \nu & \cos \nu & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

In a similar fashion, we obtain

$$[\text{Id}; B_1^1, A] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \nu & -\sin \nu \\ 0 & \sin \nu & \cos \nu \end{pmatrix}, $$

and

$$[\text{Id}; B_1^2, A] = \begin{pmatrix} \cos \nu & 0 & \sin \nu \\ 0 & 1 & 0 \\ -\sin \nu & 0 & \cos \nu \end{pmatrix}. $$
XIV. APPENDIX B: APPROXIMATING THE POTENTIAL USING MACCULLAGH'S FORMULA

Figure 7 depicts the situation in which a body B of mass m, (Mercury), is located a distance R from a large attracting point mass S of mass M, (the sun). Let R denote the region in space occupied by the body B. If R is large compared with the dimensions of B, then the potential energy of the body B with respect to the attracting point mass located at S may be approximated by a derivation similar to that used in obtaining MacCullagh's formula (see, for example, Danby [4, pp. 97-98]).

$$V(S) = -MG \int_{R} \frac{1}{r} \, dm$$

$$= -MG \int_{R} \left[ \frac{1}{r^2} + \frac{1}{r^2} - 2\frac{r}{r} \cos \theta \right]^{-1/2} \, dm$$

$$= -MG \int_{R} \frac{1}{r^2} \left[ 1 - 2(\frac{R}{r}) \cos \theta + (\frac{R}{r})^2 \right]^{-1/2} \, dm$$

$$= -\frac{MG}{R} \int_{R} \left[ 1 + \frac{1}{R} (r \cos \theta) + \frac{1}{2R^2} (2r^2 - 3r^2 \sin^2 \theta) \right] \, dm$$

$$= -\frac{MG}{R} \left[ \frac{1}{R} \int_{R} (r \cos \theta) \, dm + \frac{1}{2R^2} \int_{R} (2r^2 - 3r^2 \sin^2 \theta) \, dm \right]$$

$$= -\frac{MG}{R} \left[ m + \frac{1}{2R^2} \int_{R} (2r^2 - 3r^2 \sin^2 \theta) \, dm \right]$$

if $\theta$ is the center of mass of the body B.

The final integral is evaluated by introducing the specific assumptions and conventions developed in chapter VII. Let $A$ and $B$ denote the space and body frames of chapter VII, and let $\mathbf{s} = [\hat{s}, A]$ and $\mathbf{a} = [\hat{a}, A]$ be the position of the sun and major axis of the ellipsoid respectively. Let $R_1$ correspond to the region in space occupied by the rigid portion of mass $m_1$ and $R_2$ correspond to the region occupied by the pseudoelastic portion of mass $m_2$. Then

$$\int_{R_1} r^2 \, dm = I_1 + I_2 + I_3,$$
\[ \int_{R_1} r^2 \sin^2 \theta \, dm = \text{the moment of inertia of the rigid portion about the } \hat{s}-\text{axis} \]
\[ = \hat{s}^T A \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} A^{-1} \hat{s}, \]

and

\[ \int_{R_2} r^2 \, dm = \frac{2}{5} m_2 \left( a^2 + 2b^2 \right). \]

To evaluate the last term of the final integral over the region \( R_2 \) the following lemma will be needed:

**Lemma B.1:** Let \( m' \) be the mass of an ellipsoid of revolution with constant density whose boundary satisfies \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1 \) with respect to the principal axes of the ellipsoid. If \( \mathbf{u} = \left( u_1, u_2, u_3 \right)^T \) is any unit vector that passes through the origin of the ellipsoid, then the moment of inertia of the ellipsoid about the \( \mathbf{u} \)-axis is \( I_u = \frac{m'}{5} \left( (a^2 + b^2) + (b^2 - a^2)u_1^2 \right) \). Here, \( \left( u_1, u_2, u_3 \right)^T \) is the representation of \( \mathbf{u} \) with respect to the principal axes of the ellipsoid.

**Proof:** Direct calculation \( \blacksquare \)

Thus,

\[ \int_{R_2} r^2 \sin^2 \theta \, dm = \text{the moment of inertia of the ellipsoid about the } \hat{s}-\text{axis} \]
\[ = \frac{m_2}{5} \left[ (a^2 + b^2) + (b^2 - a^2) (\hat{s} \cdot \hat{s})^2 \right] \]
\[ = \frac{m_2}{5} \left[ (a^2 + b^2) + (b^2 - a^2) \cos^2 \kappa \right]. \]
Collecting all of the individual terms yields:

\[ V(g) = -\frac{MG}{R} \left[ m + \frac{1}{2R^2} \left( 2 \left[ I_1 + I_2 + I_3 \right] + \frac{4m_2}{5} (a^2 + 2b^2) - 3 \hat{E} \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} A^{-\frac{1}{3}} \right) \right. 

\left. - \frac{3m_2}{5} \left[ (a^2 + b^2) + (b^2 - a^2 \cos^2 \kappa) \right] \right]. \]
A. Preliminaries

Averaging of terms involving both $\alpha$ and $t$ requires that the dependence on $\alpha$ be replaced by an explicit dependence on $t$. This is most commonly achieved via the substitution

$$\frac{1}{r+1} \left[ \cos \left( (\ell - 2p)\alpha \right) \right] = \frac{1}{a} \sum_q G_{\ell pq} (e) \left[ \cos \left( (\ell - 2p + q) t \right) \right]$$

where $r = |q|$ and $a$ is the semi-major axis of the elliptical orbit. The term $G_{\ell pq} (e)$ is, in general, quite complicated. One solution for the general case in which $\ell - 2p + q \neq 0$, due to Tisserand [20, p. 256] and as quoted by Kaula [11, p. 37] is

$$G_{\ell pq} (e) = (-1)^{|q|} \beta^{|q|} \sum_{k=0}^{\infty} p_{\ell pqk} Q_{\ell pqk} \beta^{2k}$$

where

$$\beta = \frac{e}{1 + \sqrt{1 - e^2}}$$;

$$p_{\ell pqk} = \sum_{r=0}^{h} \left( \frac{2p' - 2\ell}{h - r} \right) \frac{(-1)^r}{r!} \left( \frac{(\ell - 2p' + q')e}{2\beta} \right)^r$$

$$Q_{\ell pqk} = \sum_{r=0}^{h} \left( \frac{-2p'}{h - r} \right) \frac{1}{r!} \left( \frac{(\ell - 2p' + q')e}{2\beta} \right)^r ,$$

and

$$h = k + q', \ q' > 0; \ h = k, \ q' < 0;$$

$$p' = p, \ q' = q \text{ for } p \leq \ell/2; \ p' = \ell - p, \ q' = -q \text{ for } p > \ell/2.$$
For the special case in which \( \lambda - 2p + q = 0 \), Kaula [11, p. 36] obtains the simpler form

\[
G_{\lambda p}(2p-\lambda) (e) = \frac{1}{(1-e^2)^{\lambda-1/2}} \sum_{d=0}^{p'-1} \frac{(-1)^{d}}{(2d+\lambda-2p')!} (e^2)^{2d+\lambda-2p'},
\]

in which \( p' = p \) for \( p \leq \lambda/2 \) and \( p' = \lambda - p \) for \( p \geq \lambda/2 \).

A final task to perform before averaging is to complete some algebraic manipulations. Let \( \dot{P} = A_{*}[P,B]A_{*}^{-1} \) where \( A_{*} = [\text{Id}; S, A] \) at time \( t = 0 \), and let

\[
\dot{P} = \begin{pmatrix}
\nu_{11} & \nu_{12} & \nu_{13} \\
\nu_{12} & \nu_{22} & \nu_{23} \\
\nu_{13} & \nu_{23} & \nu_{33}
\end{pmatrix}
\]

and \( \Omega_{*} = \text{skew}(\omega_{*}) = \begin{pmatrix}
0 & -\omega_{3} & \omega_{2} \\
\omega_{3} & 0 & -\omega_{1} \\
-\omega_{2} & \omega_{1} & 0
\end{pmatrix} \)

Then,

\[
\Omega_{*} \dot{P} = \begin{pmatrix}
\nu_{13} - \omega_{3} \nu_{12} & \nu_{23} - \omega_{3} \nu_{22} & \nu_{33} - \omega_{3} \nu_{32} \\
-\omega_{1} \nu_{13} + \omega_{3} \nu_{11} & -\omega_{1} \nu_{23} + \omega_{3} \nu_{12} & -\omega_{1} \nu_{33} + \omega_{3} \nu_{13} \\
\nu_{12} - \omega_{2} \nu_{11} & \nu_{22} - \omega_{2} \nu_{21} & \nu_{32} - \omega_{2} \nu_{31}
\end{pmatrix},
\]

\[
\Omega_{*} \dot{P} \Omega_{*}^{-1} = \begin{pmatrix}
\left(\nu_{13} - \omega_{3} \nu_{12}\right) - \omega_{3} \nu_{13} & \left(\nu_{23} - \omega_{3} \nu_{22}\right) - \omega_{3} \nu_{23} & \left(\nu_{33} - \omega_{3} \nu_{32}\right) - \omega_{3} \nu_{33} \\
\left(\nu_{13} - \omega_{3} \nu_{11}\right) - \omega_{3} \nu_{13} & \left(\nu_{23} - \omega_{3} \nu_{21}\right) - \omega_{3} \nu_{23} & \left(\nu_{33} - \omega_{3} \nu_{31}\right) - \omega_{3} \nu_{33} \\
\left(\nu_{12} - \omega_{2} \nu_{11}\right) - \omega_{2} \nu_{12} & \left(\nu_{22} - \omega_{2} \nu_{21}\right) - \omega_{2} \nu_{22} & \left(\nu_{32} - \omega_{2} \nu_{31}\right) - \omega_{2} \nu_{32}
\end{pmatrix},
\]

\[
\begin{pmatrix}
\nu_{13} - \omega_{3} \nu_{12} & \nu_{23} - \omega_{3} \nu_{22} & \nu_{33} - \omega_{3} \nu_{32} \\
\nu_{13} - \omega_{3} \nu_{11} & \nu_{23} - \omega_{3} \nu_{21} & \nu_{33} - \omega_{3} \nu_{31} \\
\nu_{12} - \omega_{2} \nu_{11} & \nu_{22} - \omega_{2} \nu_{21} & \nu_{32} - \omega_{2} \nu_{31}
\end{pmatrix},
\]

\[
\begin{pmatrix}
\omega_{2} \nu_{13} - \omega_{3} \nu_{12} & \omega_{2} \nu_{23} - \omega_{3} \nu_{22} & \omega_{2} \nu_{33} - \omega_{3} \nu_{32} \\
\omega_{3} \nu_{13} - \omega_{1} \nu_{11} & \omega_{3} \nu_{23} - \omega_{1} \nu_{21} & \omega_{3} \nu_{33} - \omega_{1} \nu_{31} \\
\omega_{1} \nu_{12} - \omega_{2} \nu_{11} & \omega_{1} \nu_{22} - \omega_{2} \nu_{21} & \omega_{1} \nu_{32} - \omega_{2} \nu_{31}
\end{pmatrix},
\]

\[
\begin{pmatrix}
\omega_{2} \nu_{13} - \omega_{3} \nu_{12} & \omega_{2} \nu_{23} - \omega_{3} \nu_{22} & \omega_{2} \nu_{33} - \omega_{3} \nu_{32} \\
\omega_{3} \nu_{13} - \omega_{1} \nu_{11} & \omega_{3} \nu_{23} - \omega_{1} \nu_{21} & \omega_{3} \nu_{33} - \omega_{1} \nu_{31} \\
\omega_{1} \nu_{12} - \omega_{2} \nu_{11} & \omega_{1} \nu_{22} - \omega_{2} \nu_{21} & \omega_{1} \nu_{32} - \omega_{2} \nu_{31}
\end{pmatrix}.\]
$$\Omega_n^2 \bar{P} = \begin{pmatrix}
\left(\omega_1^\wedge P_{12} - \omega_2^\wedge P_{11}\right) \omega_2 - \left(\omega_3^\wedge P_{11} - \omega_1^\wedge P_{13}\right) \omega_3 & \left(\omega_1^\wedge P_{22} - \omega_2^\wedge P_{12}\right) \omega_2 - \left(\omega_3^\wedge P_{12} - \omega_1^\wedge P_{23}\right) \omega_3 \\
\left(\omega_2^\wedge P_{13} - \omega_3^\wedge P_{12}\right) \omega_3 - \left(\omega_1^\wedge P_{12} - \omega_2^\wedge P_{11}\right) \omega_1 & \left(\omega_2^\wedge P_{23} - \omega_3^\wedge P_{22}\right) \omega_3 - \left(\omega_1^\wedge P_{22} - \omega_2^\wedge P_{12}\right) \omega_1 \\
\left(\omega_3^\wedge P_{11} - \omega_1^\wedge P_{13}\right) \omega_1 - \left(\omega_2^\wedge P_{13} - \omega_3^\wedge P_{12}\right) \omega_2 & \left(\omega_3^\wedge P_{12} - \omega_1^\wedge P_{23}\right) \omega_1 - \left(\omega_2^\wedge P_{23} - \omega_3^\wedge P_{22}\right) \omega_2
\end{pmatrix}$$

$$\Omega_n^2 \bar{P} \Omega_n = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}$$

where

$$a_{11} = \left[\left(\omega_1^\wedge P_{22} - \omega_2^\wedge P_{12}\right) \omega_3 - \left(\omega_1^\wedge P_{23} - \omega_2^\wedge P_{13}\right) \omega_2 \right] \omega_2$$

$$- \left[\left(\omega_3^\wedge P_{11} - \omega_1^\wedge P_{13}\right) \omega_3 - \left(\omega_3^\wedge P_{12} - \omega_1^\wedge P_{23}\right) \omega_2 \right] \omega_3 ,$$

$$a_{12} = \left[\left(\omega_1^\wedge P_{23} - \omega_2^\wedge P_{13}\right) \omega_1 - \left(\omega_1^\wedge P_{12} - \omega_2^\wedge P_{11}\right) \omega_3 \right] \omega_2$$

$$- \left[\left(\omega_3^\wedge P_{13} - \omega_1^\wedge P_{33}\right) \omega_1 - \left(\omega_3^\wedge P_{11} - \omega_1^\wedge P_{13}\right) \omega_3 \right] \omega_3 ,$$

$$a_{13} = \left[\left(\omega_1^\wedge P_{12} - \omega_2^\wedge P_{11}\right) \omega_2 - \left(\omega_1^\wedge P_{22} - \omega_2^\wedge P_{12}\right) \omega_1 \right] \omega_2$$

$$- \left[\left(\omega_3^\wedge P_{11} - \omega_1^\wedge P_{13}\right) \omega_2 - \left(\omega_3^\wedge P_{12} - \omega_1^\wedge P_{23}\right) \omega_1 \right] \omega_3 ,$$

$$a_{21} = \left[\left(\omega_2^\wedge P_{23} - \omega_3^\wedge P_{22}\right) \omega_3 - \left(\omega_2^\wedge P_{33} - \omega_3^\wedge P_{23}\right) \omega_2 \right] \omega_3$$

$$- \left[\left(\omega_1^\wedge P_{22} - \omega_2^\wedge P_{12}\right) \omega_3 - \left(\omega_1^\wedge P_{23} - \omega_2^\wedge P_{13}\right) \omega_2 \right] \omega_1 ,$$

$$a_{22} = \left[\left(\omega_2^\wedge P_{23} - \omega_3^\wedge P_{22}\right) \omega_3 - \left(\omega_2^\wedge P_{33} - \omega_3^\wedge P_{23}\right) \omega_2 \right] \omega_3$$

$$- \left[\left(\omega_1^\wedge P_{22} - \omega_2^\wedge P_{12}\right) \omega_3 - \left(\omega_1^\wedge P_{23} - \omega_2^\wedge P_{13}\right) \omega_2 \right] \omega_1 ,$$

$$a_{23} = \left[\left(\omega_2^\wedge P_{23} - \omega_3^\wedge P_{22}\right) \omega_3 - \left(\omega_2^\wedge P_{33} - \omega_3^\wedge P_{23}\right) \omega_2 \right] \omega_3$$

$$- \left[\left(\omega_1^\wedge P_{22} - \omega_2^\wedge P_{12}\right) \omega_3 - \left(\omega_1^\wedge P_{23} - \omega_2^\wedge P_{13}\right) \omega_2 \right] \omega_1 ,$$

$$a_{31} = \left[\left(\omega_3^\wedge P_{11} - \omega_1^\wedge P_{13}\right) \omega_3 - \left(\omega_3^\wedge P_{12} - \omega_1^\wedge P_{23}\right) \omega_2 \right] \omega_3$$

$$- \left[\left(\omega_2^\wedge P_{22} - \omega_3^\wedge P_{12}\right) \omega_3 - \left(\omega_2^\wedge P_{33} - \omega_3^\wedge P_{23}\right) \omega_2 \right] \omega_1 ,$$

$$a_{32} = \left[\left(\omega_3^\wedge P_{11} - \omega_1^\wedge P_{13}\right) \omega_3 - \left(\omega_3^\wedge P_{12} - \omega_1^\wedge P_{23}\right) \omega_2 \right] \omega_3$$

$$- \left[\left(\omega_2^\wedge P_{22} - \omega_3^\wedge P_{12}\right) \omega_3 - \left(\omega_2^\wedge P_{33} - \omega_3^\wedge P_{23}\right) \omega_2 \right] \omega_1 ,$$

$$a_{33} = \left[\left(\omega_3^\wedge P_{11} - \omega_1^\wedge P_{13}\right) \omega_3 - \left(\omega_3^\wedge P_{12} - \omega_1^\wedge P_{23}\right) \omega_2 \right] \omega_3$$

$$- \left[\left(\omega_2^\wedge P_{22} - \omega_3^\wedge P_{12}\right) \omega_3 - \left(\omega_2^\wedge P_{33} - \omega_3^\wedge P_{23}\right) \omega_2 \right] \omega_1 ,$$
\[ a_{22} = \left( \omega_2 \omega_3 P_{33} - \omega_3 \omega_2 P_{23} \right) \omega_1 - \left( \omega_2 \omega_1 P_{13} - \omega_3 \omega_2 P_{12} \right) \omega_1, \]

\[ a_{23} = \left( \omega_2 \omega_3 P_{13} - \omega_3 \omega_2 P_{12} \right) \omega_2 - \left( \omega_2 \omega_1 P_{23} - \omega_3 \omega_2 P_{22} \right) \omega_1, \]

\[ a_{31} = \left( \omega_3 \omega_1 P_{12} - \omega_1 \omega_3 P_{23} \right) \omega_3 - \left( \omega_3 \omega_1 P_{13} - \omega_1 \omega_3 P_{33} \right) \omega_2, \]

\[ a_{32} = \left( \omega_3 \omega_1 P_{13} - \omega_1 \omega_3 P_{22} \right) \omega_2 - \left( \omega_3 \omega_2 P_{12} - \omega_1 \omega_3 P_{23} \right) \omega_1, \]

and

\[ a_{33} = \left( \omega_3 \omega_2 P_{13} - \omega_2 \omega_3 P_{12} \right) \omega_2 - \left( \omega_3 \omega_2 P_{13} - \omega_2 \omega_3 P_{22} \right) \omega_1, \]

and

\[ \Omega_\alpha^2 \rightleftarrows \Omega_\alpha^2 = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}, \]

where

\[ b_{11} = \left( \omega_2^2 + \omega_3^2 \right) P_{11} - 2 \left( \omega_2^2 + \omega_3^2 \right) \omega_1 \omega_2 P_{12} - 2 \left( \omega_2^2 + \omega_3^2 \right) \omega_1 \omega_3 P_{13} + \omega_1 \omega_2 \omega_3 P_{22}, \]

\[ + 2 \omega_1 \omega_2 \omega_3 P_{22} + \omega_1 \omega_2 \omega_3 P_{22}, \]

\[ b_{12} = - \left( \omega_2^2 + \omega_3^2 \right) \omega_1 \omega_2 P_{11} + \left( \omega_1^2 + 2 \omega_1 \omega_2 \omega_3 \right) P_{12} - \left( \omega_2^2 + \omega_3^2 \right) \omega_1 \omega_2 P_{22}, \]

\[ - \left( \omega_1^2 + \omega_2^2 \right) \omega_1 \omega_2 P_{22} - \left( \omega_1^2 - \omega_2^2 + \omega_3^2 \right) \omega_1 \omega_3 P_{23} + \omega_1 \omega_2 \omega_3 P_{33}, \]
\[ \begin{align*}
\mathbf{b}_{13} &= -\left( \omega_2^2 + \omega_3^2 \right) \omega_1 \omega_3 p_{11} + \left( \omega_1^2 - \omega_2^2 - \omega_3^2 \right) \omega_2 \omega_3 p_{12} + \left( |\omega_x| \| \omega_2^2 + 2 \omega_1^2 \omega_3^2 \right) p_{13} \\
&\quad + \omega_1^2 \omega_2 \omega_3 p_{22} - \left( \omega_1^2 + \omega_2^2 - \omega_3^2 \right) \omega_1 \omega_2 \omega_3 p_{23} - \left( \omega_1^2 + \omega_2^2 \right) \omega_1 \omega_3 p_{33},
\mathbf{b}_{22} &= \omega_1^2 \omega_2 \omega_3 p_{11} - 2 \left( \omega_1^2 + \omega_2^2 \right) \omega_1 \omega_2 \omega_3 p_{12} + 2 \omega_2 \omega_3 \omega_1 p_{13} + \left( \omega_1^2 + \omega_3^2 \right) p_{22} \\
&\quad - 2 \left( \omega_1^2 + \omega_3^2 \right) \omega_2 \omega_3 p_{23} + \omega_2 \omega_3 p_{33},
\mathbf{b}_{23} &= \omega_1^2 \omega_2 \omega_3 p_{11} - \left( \omega_1^2 - \omega_2^2 + \omega_3^2 \right) \omega_1 \omega_3 p_{12} - \left( \omega_1^2 + \omega_2^2 - \omega_3^2 \right) \omega_2 \omega_3 p_{13} \\
&\quad - \left( \omega_1^2 + \omega_3^2 \right) \omega_2 \omega_3 p_{22} + \left( |\omega_x| \| \omega_1^2 + \omega_2^2 \omega_3^2 \right) p_{23} - \left( \omega_1^2 + \omega_2^2 \right) \omega_2 \omega_3 p_{33},
\end{align*} \]

and
\[ \begin{align*}
\mathbf{b}_{33} &= \omega_1^2 \omega_2 \omega_3 p_{11} + 2 \omega_1 \omega_2 \omega_3 p_{12} - 2 \left( \omega_1^2 + \omega_2^2 \right) \omega_1 \omega_3 p_{13} + \omega_2 \omega_3 p_{22} \\
&\quad - 2 \left( \omega_1^2 + \omega_2^2 \right) \omega_2 \omega_3 p_{23} + \left( \omega_1^2 + \omega_3^2 \right) p_{33}.
\end{align*} \]

B. The Permanent Asymmetry Term

Given \( \omega_x, A_x, \) and \([P,B]\), let \( \mathbf{\hat{y}} = A_x [P,B] A_x^{-1} \). Two cases exist.

Case I: \( |\omega_x| = p/q \) is rational in lowest terms. Then

\[ \begin{align*}
\left\langle \frac{1}{|s|^3} \left( (A[P,B] A^{-1}) S \right) x \right\rangle \\
&= \frac{1}{2\pi q} \int_0^{2\pi q} \frac{1}{|s|^3} \left[ e^{\Omega_x t} \mathcal{F} e^{-\Omega_x t} \right] x \hat{s} dt \\
&= \frac{1}{2\pi q} \int_0^{2\pi q} \frac{1}{|s|^3} \left[ \begin{array}{c}
E + \frac{\sin(|\omega_x| |t|)}{|\omega_x|} \Omega_x + \frac{[1-\cos(|\omega_x| |t|)]}{|\omega_x|^2} \Omega_x^2 \\
E - \frac{\sin(|\omega_x| |t|)}{|\omega_x|} \Omega_x + \frac{[1-\cos(|\omega_x| |t|)]}{|\omega_x|^2} \Omega_x^2 \end{array} \right] x \hat{s} dt.
\end{align*} \]
\[
\begin{align*}
&= \frac{1}{2\pi q} \int_0^{2\pi q} \frac{1}{||s||^3} \left[ (\Omega^2 \Omega^2 - \Omega^2 \Omega^2) \hat{s} \right] \times \hat{s} \left( \frac{\sin(||\omega_k|| t)}{||\omega_k||} \right) dt \\
&\quad + \frac{1}{2\pi q} \int_0^{2\pi q} \frac{1}{||s||^3} \left( \gamma \hat{s} \right) \times \hat{s} dt \\
&\quad - \frac{1}{2\pi q} \int_0^{2\pi q} \frac{1}{||s||^3} \left[ \Omega^2 \Omega^2 \hat{s} \right] \times \hat{s} \left( \frac{\sin^2(||\omega_k|| t)}{||\omega_k||^2} \right) dt \\
&\quad + \frac{1}{2\pi q} \int_0^{2\pi q} \frac{1}{||s||^3} \left[ \left( \Omega^2 \Omega^2 - \Omega^2 \Omega^2 \right) \hat{s} \right] \times \hat{s} \left( \frac{1-\cos(||\omega_k|| t)}{||\omega_k||^2} \right) dt \\
&\quad + \frac{1}{2\pi q} \int_0^{2\pi q} \frac{1}{||s||^3} \left[ \left( \Omega^2 \Omega^2 - \Omega^2 \Omega^2 \right) \hat{s} \right] \times \hat{s} \left( \frac{\sin(||\omega_k|| t) \left[ 1-\cos(||\omega_k|| t) \right]}{||\omega_k||^3} \right) dt \\
&\quad + \frac{1}{2\pi q} \int_0^{2\pi q} \frac{1}{||s||^3} \left[ \left( \Omega^2 \Omega^2 - \Omega^2 \Omega^2 \right) \hat{s} \right] \times \hat{s} \left( \frac{\left[ 1-\cos(||\omega_k|| t) \right]^2}{||\omega_k||^4} \right) dt
\end{align*}
\]

Observe that if \( A_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \) is symmetric, then

\[
(A \hat{s}) \times \hat{s} = \begin{pmatrix} -a_{13} \sin \alpha \cos \alpha - a_{23} \sin^2 \alpha \\ a_{13} \cos^2 \alpha + a_{23} \sin \alpha \cos \alpha \\ (a_{11} - a_{22}) \sin \alpha \cos \alpha + a_{12} \left( \sin^2 \alpha - \cos^2 \alpha \right) \end{pmatrix} \] (c.2)
Using the calculations that are summarized in Appendix D and omitting the algebra, one obtains

\[
\left\langle \frac{1}{|\Omega|^{3/2}} \left[ \left( A[P,B] \ A^{-1} \right) \hat{\Omega} \right] \times \hat{\Omega} \right\rangle
\]

\[
= \left( \frac{1-e^2}{4a^3} \right)^{-3/2} \left[ \text{trace}(P) - 3 \left( \hat{\omega}_k \hat{\nu} \hat{\omega}_k \right) \right] \begin{pmatrix}
\omega_2 & \omega_3 \\
-\omega_1 & \omega_3 \\
0 & 0
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
a_1 \\
\frac{a_2}{a_3} \\
\frac{b_1}{b_2}
\end{pmatrix}
(\text{if } \frac{P}{q} = k \text{ is an integer})
\]

\[
+ \begin{pmatrix}
\frac{a_1}{a_3} \\
\frac{b_1}{b_3}
\end{pmatrix}
(\text{if } \frac{2P}{q} = n \text{ is an integer})
\]

where

\[
a_1 = \frac{1}{4a^3} \left[ G_{2,0,\ell-2}(e) + G_{2,0,\ell-2}(e) - G_{2,1,-\ell}(e) - G_{2,1,\ell}(e) \right] p_{23}
\]

\[
+ \frac{1}{4k^2a^3} \left[ G_{2,1,\ell}(e) + G_{2,1,-\ell}(e) \right]
\]

\[
+ \frac{1}{4k^2a^3} \left[ G_{2,0,\ell-2}(e) - G_{2,0,-\ell-2}(e) \right]
\]

\[
\left[ 2\omega_2\omega_3 \left( \hat{\omega}_k \hat{\nu} \hat{\omega}_k \right) - \omega_1\omega_3 \hat{\nu}_{12} \\
-\omega_1\omega_2 \hat{\nu}_{13} - \omega_2\omega_3 \hat{\nu}_{23} + \omega_1^2 \hat{\nu}_{23} - \omega_2\omega_3 \hat{\nu}_{33}
\right]
\]

\[
+ \frac{1}{4k^2a^3} \left[ G_{2,0,\ell-2}(e) - G_{2,0,-\ell-2}(e) \right]
\]

\[
\left[ \frac{\omega_1^2 \hat{\nu}_{11} - \left( \omega_1^2 - \omega_2^2 - \omega_3^2 \right) \omega_1 \hat{\nu}_{12} - \left( \omega_1^2 - \omega_2^2 \right) \omega_2 \hat{\nu}_{22} \\
- \left( \omega_1^2 + \omega_2^2 - \omega_3^2 \right) \omega_3 \hat{\nu}_{23} - \omega_2\omega_3 \hat{\nu}_{33}
\right]
\]
\[ a_2 = \frac{1}{4a^3} \left[ G_{2,1,-\ell}(e) + G_{2,1,\ell}(e) - G_{2,0,\ell-2}(e) - G_{2,0,-\ell-2}(e) \right] \hat{p}_{13} \]

\[ + \frac{1}{4\ell^2 a^3} \left[ G_{2,1,-\ell}(e) + G_{2,1,\ell}(e) ight. \]

\[ + G_{2,0,\ell-2}(e) + G_{2,0,-\ell-2}(e) \]

\[ \left[ -2\omega_1\omega_3 \left( \frac{\chi_{\omega_2}}{\omega_2} \hat{p}_{12} \right) + \omega_1\omega_3 \hat{p}_{11} + \omega_2\omega_3 \hat{p}_{12} \right] \]

\[ - \omega_2^2 \hat{p}_{13} + \omega_1\omega_2 \hat{p}_{23} + \omega_1\omega_3 \hat{p}_{33} \]

\[ + \frac{1}{4\ell^3 a^3} \left[ G_{2,0,\ell-2}(e) - G_{2,0,-\ell-2}(e) \right] \]

\[ \left[ \left( \omega_2^2 - \omega_3^2 \right) \omega_1 \hat{p}_{11} + \left( \omega_1^2 - \omega_2^2 + \omega_3^2 \right) \omega_2 \hat{p}_{12} \right] \]

\[ - \left( \omega_1^2 + \omega_2^2 - \omega_3^2 \right) \omega_3 \hat{p}_{13} - \omega_1\omega_2 \hat{p}_{22} - \omega_1\omega_3 \hat{p}_{33} \]

\[ a_3 = \frac{1}{4\ell^2 a^3} \left[ G_{2,0,\ell-2}(e) + G_{2,0,-\ell-2}(e) \right] \]

\[ \left[ -2\ell^2 \hat{p}_{12} + 4\omega_1\omega_2 \left( \frac{\chi_{\omega_2}}{\omega_2} \hat{p}_{12} \right) - 2\omega_1\omega_2 \hat{p}_{11} \right] \]

\[ + 2\omega_3^2 \hat{p}_{12} - 2\omega_2\omega_3 \hat{p}_{13} - 2\omega_1\omega_2 \hat{p}_{22} - 2\omega_1\omega_3 \hat{p}_{23} \]

\[ \left[ \omega_1\omega_2\omega_3 \left( \hat{p}_{11} + \hat{p}_{22} - 2\hat{p}_{33} \right) + \left( \omega_1^2 + \omega_2^2 \right) \omega_3 \hat{p}_{12} \right] \]

\[ - \left( \omega_1^2 - \omega_3^2 \right) \omega_2 \hat{p}_{13} - \left( \omega_2^2 - \omega_3^2 \right) \omega_1 \hat{p}_{23} \]
\[
\begin{align*}
\mathbf{b}_1 &= \frac{1}{2n^a_3} \left[ G_{2,0,-n-2}(e) + G_{2,0,n-2}(e) \right] \\
&\quad - G_{2,1,-n}(e) - G_{2,1,n}(e) \\
&= \frac{1}{2n^a_3} \left[ \begin{pmatrix} \omega_2 \omega_3 \left( \frac{\omega_2}{\omega_3} \mathbf{e} + \frac{\omega_3}{\omega_2} \mathbf{e} \right) + \omega_2 \omega_3 \left( \mathbf{p}_{11} - \mathbf{p}_{22} - \mathbf{p}_{33} \right) \\ -2\omega_1 \omega_3 \mathbf{p}_{12} - 2\omega_1 \omega_2 \mathbf{p}_{13} + 2\omega_1 \mathbf{p}_{23} \end{pmatrix} \right] \\
&\quad + \frac{1}{3n^a_3} \left[ G_{2,0,-n-2}(e) - G_{2,0,n-2}(e) \right] \\
&= \frac{1}{2n^a_3} \left[ \begin{pmatrix} (\omega_2^2 + \omega_3^2) \omega_2 \mathbf{p}_{11} - 2 (\omega_2^2 + \omega_3^2) \omega_1 \mathbf{p}_{12} + (\omega_1^2 - \omega_3^2) \omega_2 \mathbf{p}_{22} \\ 2 (\omega_1^2 + \omega_2^2) \omega_2 \mathbf{p}_{23} - (\omega_1^2 + \omega_2^2) \omega_2 \mathbf{p}_{33} \end{pmatrix} \right] \\
\mathbf{b}_2 &= \frac{1}{2n^a_3} \left[ G_{2,0,-n-2}(e) + G_{2,0,n-2}(e) \right] \\
&\quad + G_{2,1,-n}(e) + G_{2,1,n}(e) \\
&= \frac{1}{2n^a_3} \left[ \begin{pmatrix} \omega_1 \omega_3 \left( \frac{\omega_1}{\omega_3} \mathbf{e} + \frac{\omega_3}{\omega_1} \mathbf{e} \right) - \omega_1 \omega_3 \left( \mathbf{p}_{11} - \mathbf{p}_{22} + \mathbf{p}_{33} \right) \\ -2\omega_1 \omega_3 \mathbf{p}_{12} + 2\omega_2 \mathbf{p}_{13} - 2\omega_1 \mathbf{p}_{23} \end{pmatrix} \right] \\
&\quad + \frac{1}{3n^a_3} \left[ G_{2,0,-n-2}(e) - G_{2,0,n-2}(e) \right] \\
&= \frac{1}{2n^a_3} \left[ \begin{pmatrix} (\omega_2^2 - \omega_3^2) \omega_1 \mathbf{p}_{11} - 2 (\omega_2^2 + \omega_3^2) \omega_1 \mathbf{p}_{12} + 2 (\omega_1^2 + \omega_2^2) \omega_3 \mathbf{p}_{13} \\ + (\omega_1^2 + \omega_3^2) \omega_3 \mathbf{p}_{22} - (\omega_1^2 + \omega_2^2) \omega_2 \mathbf{p}_{33} \end{pmatrix} \right]
\end{align*}
\]
and
\[ b_3 = - \frac{1}{n^3} \left[ G_{2,0,-n-2}(e) + G_{2,0,n-2}(e) \right] \]
\[ + \frac{2}{n^3} \left[ G_{2,0,-n-2}(e) - G_{2,0,n-2}(e) \right] \]
\[ = \omega_1 \omega_2 \left( \hat{p}_{11} + \hat{p}_{22} - \hat{p}_{33} \right) - \left( \frac{1}{n^3} \omega_1^2 + \omega_2^2 \right) \omega_3 \hat{p}_{12} \]
\[ + \left( \frac{1}{n^3} \omega_1^2 - \omega_2^2 - 2 \omega_3^2 \right) \omega_2 \hat{p}_{13} + \left( \frac{1}{n^3} \omega_1^2 - \omega_2^2 + 2 \omega_3^2 \right) \omega_1 \hat{p}_{23} \]

Case II: \( ||w_\star|| \) is irrational.

Then,
\[ \left\langle \frac{1}{||s||^3} \left[ (A[P,S]A^{-1})_\star \right] \times \hat{s} \right\rangle \]
\[ = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{||s||^3} \left[ \left( \hat{e} 
abla \times \hat{e} - \hat{\omega} \hat{e} \right) \hat{s} \right] \times \hat{s} \, d\theta dt \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{||s||^3} \left[ \left( \frac{1}{2\pi} \int_0^{2\pi} \hat{e} \hat{\omega} \left( \int_0^{2\pi} \hat{e} \times \hat{e} - \hat{\omega} \hat{e} \right) d\theta \right) \hat{s} \right] \times \hat{s} \, dt \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{||s||^3} \left[ \left( \frac{1}{2\pi} \int_0^{2\pi} \left[ E + (\sin\theta) \hat{\omega} + [1-\cos\theta] \hat{\omega}^2 \right] \hat{s} \right) \times \hat{s} \right] \, dt \]
\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|s|} \left[ \left( P' + [\hat{\Omega}_* P + P\hat{\Omega}_*] - \frac{1}{2} \hat{\Omega}_* P \hat{\Omega}_* + \frac{3}{2} \hat{\Omega}_* \hat{\Omega}_* \right) \hat{s} \right] \times \hat{s} \, dt \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|s|} (A_1 \hat{s}) \times \hat{s} \, dt \]

where

\[ A_1 = P' + [\hat{\Omega}_* P + P\hat{\Omega}_*] - \frac{1}{2} \hat{\Omega}_* P \hat{\Omega}_* + \frac{3}{2} \hat{\Omega}_* \hat{\Omega}_* \]

and

\[ \hat{\Omega}_* = \text{skew} \left( \hat{\Omega}_* \right). \]

Observe that \( A_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \) is symmetric and

\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|s|} \, dt = \frac{1}{a^3} (1-e^2)^{-3/2}, \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin 2\alpha}{|s|} \, dt = 0, \]

and

\[ \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos 2\alpha}{|s|} \, dt = \left( \frac{1}{a^3} \right) G_{2,0,-2}(e) = 0. \] Thus by equation (C.2),

\[ \left\langle \frac{1}{|s|} \left[ (A [P, B] A^{-1}) \hat{s} \right] \times \hat{s} \right\rangle = \frac{1}{2a^3} (1-e^2)^{-3/2} \begin{pmatrix} -a_{23} \\ a_{13} \\ 0 \end{pmatrix}. \]
Omitting the algebra,

\[
\left\langle \frac{1}{|s|^3} \left[ (A [P,B] A^{-1}) \hat{s} \right] \times \hat{s} \right\rangle
\]

\[
= \frac{(1-e^2)^{-3/2}}{4a^3} \left[ \text{trace}(P) - 3 \left( \hat{\omega}_3 \hat{\omega}_3 \right) \omega_3 \begin{pmatrix} \omega_2 \\ \omega_1 \\ 0 \end{pmatrix} \right].
\]

Notice that this result is the same as that for case I when \(2||\omega_x||\) is not an integer.

C. The Tidal Term

Let \(\omega_x\) and \(A_x\) be non-zero constants such that \(A_x^{-1} = A_x^t\). Then, under the model for the tidal torque developed in Chapter VII, the tidal torque is \(2\pi\)-periodic in \(t\) and is independent of \(\Theta = |\omega_x|t\) for all \(\omega_x\). Thus,

\[
\left\langle \sin(2c|v|) \frac{\sin(2c|v|)}{|v| |s|} \left( \hat{v} - (\hat{s} \cdot \hat{v}) \hat{s} \right) \right\rangle
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\sin(2c|v|)}{|v| |s|} \left[ \frac{d\alpha}{dt} \right] \left( k - \omega_x \right) + (\hat{s} \cdot \omega_x) \hat{s} \right] dt
\]

\[
= \frac{1}{2\pi} \left( \int_{0}^{2\pi} \frac{\sin(2c|v|)}{|v| |s|} d\alpha \right) \left( k - \frac{1}{2\pi} \left( \int_{0}^{2\pi} \frac{\sin(2c|v|)}{ab |v| |s|} d\alpha \right) \omega_x \right)
\]

\[
+ \frac{1}{2\pi} \left( \int_{0}^{2\pi} \frac{\sin(2c|v|)}{ab |v| |s|} \left( \omega_1 \cos\alpha + \omega_2 \sin\alpha \right) \hat{s} \right) d\alpha \right).
Since $\alpha$ enters $|\mathbf{v}|$ and $|\mathbf{s}|$ only in the form $\cos \alpha$, and since
\[
\int_0^{2\pi} f(\cos \alpha) \sin \alpha \, d\alpha = 0
\]
for any continuous function $f(x)$, it follows that
\[
\left\langle \frac{\sin(2\pi |\mathbf{v}|)}{|\mathbf{s}|^3} \left( 0 - (\mathbf{s} \cdot \mathbf{v}) \mathbf{s} \right) \rightangle
\]
\[
= \frac{1}{2\pi} \left( \int_0^{2\pi} \frac{\sin(2\pi |\mathbf{v}|)}{|\mathbf{v}| |\mathbf{s}|^3} \, d\alpha \right) - \frac{1}{2\pi} \left( \int_0^{2\pi} \frac{\sin(2\pi |\mathbf{v}|)}{|\mathbf{v}| |\mathbf{s}|} \left( \frac{\omega_1(1-\cos^2 \alpha)}{\omega_2(1-\sin^2 \alpha)} \right) \, d\alpha \right)
\]
\[
= \frac{1}{2\pi} \left[ \int_0^{2\pi} \left( \frac{1}{\mathbf{v} \cdot \mathbf{b}} \right)^3 \sum_{n=0}^{\infty} (-1)^n \frac{(2c)^{2n+1}}{(2n+1)!} \left( \frac{|\mathbf{v}|^2}{2} \right)^n (1+\epsilon \cos \alpha)^3 \, d\alpha \right] k
\]
\[
- \frac{1}{2\pi} \left[ \int_0^{2\pi} \frac{1}{\mathbf{b} \cdot \mathbf{v}} \sum_{n=0}^{\infty} (-1)^n \frac{(2c)^{2n+1}}{(2n+1)!} \left( \frac{|\mathbf{v}|^2}{2} \right)^n (1+\epsilon \cos \alpha) \left( \frac{\omega_1 \sin^2 \alpha}{\omega_2 \cos^2 \alpha} \right) \, d\alpha \right].
\]
Writing $\|v\|^2 = \|\omega\|^2 - 2\omega_3 (1-e^2)^{-3/2} (1+\cos \alpha)^2 + (1-e^2)^{-3} (1+\cos \alpha)^4$, one obtains

$$\left(\|v\|^2\right)^n = \sum_{x=0}^{n} \sum_{y=0}^{x} \frac{n!}{(n-x-y)! x! y!} \left(\|\omega\|^2\right)^{n-x-y}$$

$$\left[-2\omega_3 (1-e^2)^{-3/2} (1+\cos \alpha)^2\right]^x \left[(1-e^2)^{-3} (1+\cos \alpha)^4\right]^y$$

$$= \sum_{x=0}^{n} \sum_{y=0}^{x} \frac{n!}{x! y! (n-x-y)!} \left[\left(\|\omega\|^2\right)^{2(n-x-y)} \left(-2\omega_3\right)^x \left(1-e^2\right)^{-6y-3x}\right]$$

$$(1+\cos \alpha)^{2x+4y}.$$ 

Since $\frac{1}{2\pi} \int_0^{2\pi} (1+\cos \alpha)^{2n+1} d\alpha$

$$= \sum_{x=0}^{2n+1} \binom{2n+1}{x} e^x \left(\frac{1}{2\pi} \int_0^{2\pi} \cos^x \alpha d\alpha\right)$$
\[
\begin{align*}
&= \sum_{x=0}^{n} \left( \frac{2n+1}{2x} \right) e^{2x} \left[ \frac{1}{2\pi} \int_{0}^{2\pi} \cos^{2x} \alpha \, d\alpha \right] \\
&= \sum_{x=0}^{n} \left( \frac{2n+1}{2x} \right) e^{2x} \frac{2x!(2x)!}{2^{2x} (x!)^2} \\
&= \sum_{x=0}^{n} \left( \frac{2n+1}{2x} \right) \binom{2x}{x} \left( \frac{e}{2} \right)^{2x} \\
&= (1-e^2)^{2n+3/2} G_2(n+1), n+1, 0 (e), \\
&= \left\langle \frac{\sin(2\pi |\mathbf{v}|)}{|\mathbf{g}|^3} \right| (\mathbf{0} - (\mathbf{\hat{a}} \cdot \mathbf{0})\mathbf{\hat{a}}) \left\rangle \\
&= \left( \frac{b}{4a} \right) \sum_{n=0}^{\infty} (-1)^n \frac{(2c)^{2n+1}}{(2n+1)!} \left[ \sum_{x=0}^{n} \sum_{y=0}^{x} \frac{n!}{x!y!(n-x-y)!} \right] \frac{x+2y}{2} \left( \frac{e}{2} \right)^{2(n-x-y)} G_2(x+2y+2), x+2y+2, 0 (e) \right] k \\
&= -\left( \frac{1}{a^3} \right) \sum_{n=0}^{\infty} (-1)^n \frac{(2c)^{2n+1}}{(2n+1)!} \left[ \sum_{x=0}^{n} \sum_{y=0}^{x} \frac{n!}{x!y!(n-x-y)!} \right] \frac{x+2y}{2} \left( \frac{e}{2} \right)^{2(n-x-y)} G_2(x+2y+1), x+2y+1, 0 (e) \right] \left( \begin{array}{c}
\omega_1 \\
0 \\
\omega_3
\end{array} \right)
\end{align*}
\]
\[- \left( \frac{1}{b^3} \right) \sum_{n=0}^{\infty} (-1)^n \frac{(2c)^{2n+1}}{(2n+1)!} \left[ \sum_{x=0}^{n} \sum_{y=0}^{x} \frac{n!}{x!y!(n-x-y)!} \cdot |\omega_x|^2(n-x-y) \right] (2n+1) \]

Similarly,

\[
\left\langle \frac{\cos(c|\nu||1-\cos(c|\nu||))}{|s|^3} (s \cdot 0)(s \times 0) \right\rangle
\]

\[= \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(c|\nu|)[1-\cos(c|\nu|)]}{|\nu|^2 |s|^3} \left(-\omega_1 \cos\alpha - \omega_2 \sin\alpha\right)
\]

\[
\left\langle \left\langle \frac{d}{dt} - \omega_3 \right| \frac{\sin\alpha}{\left(-\omega_3 \cos\alpha - \omega_1 \sin\alpha\right)} \right\rangle dt
\]

\[= \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(c|\nu|)^2 \cos^2 \alpha}{|\nu|^2 |s|^3} \left( \omega_2 \sin^2 \alpha \right) \frac{d\alpha}{0}
\]

\[+ \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(c|\nu|)[1-\cos(c|\nu|)]}{|\nu|^2 ab |s|^3} \left( \frac{\omega_2 \omega_3 \sin^2 \alpha}{\omega_1 \omega_2 \left(\sin^2 \alpha - \cos^2 \alpha\right)} \right) d\alpha
\]
\[= \left(\frac{a}{b^3}\right) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(c)^{2n}}{(2n)!} \left[ \frac{1}{2\pi} \int_0^{2\pi} (|\mathbf{v}|^2)^{n-1} (1+\cos \alpha)^3 \right. \]

\[
\left. \cos(c|v|) \ d\alpha \right] \begin{pmatrix} -\omega_2 \\ 0 \\ 0 \end{pmatrix}
\]

\[+ \left(\frac{a}{b^3}\right) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(c)^{2n}}{(2n)!} \left[ \frac{1}{2\pi} \int_0^{2\pi} (|\mathbf{v}|^2)^{n-1} (1+\cos \alpha)^3 \right. \]

\[
\left. \left(\cos^2 \alpha\right) \cos(c|v|) \ d\alpha \right] \begin{pmatrix} \omega_2 \\ \omega_1 \\ 0 \end{pmatrix}
\]

\[= \left(\frac{1}{b^3}\right) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(c)^{2n}}{(2n)!} \left[ \frac{1}{2\pi} \int_0^{2\pi} (|\mathbf{v}|^2)^{n-1} (1+\cos \alpha) \right. \]

\[
\left. \cos(c|v|) \ d\alpha \right] \begin{pmatrix} \omega_2 \omega_3 \\ \omega_1 \omega_3 \\ 0 \end{pmatrix}
\]

\[= \left(-\frac{1}{b^3}\right) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(c)^{2n}}{(2n)!} \left[ \frac{1}{2\pi} \int_0^{2\pi} (|\mathbf{v}|^2)^{n-1} (1+\cos \alpha) \right. \]

\[
\left. \left(\cos^2 \alpha\right) \left(\cos(c|v|)\right) \ d\alpha \right] \begin{pmatrix} \omega_2 \omega_3 \\ \omega_1 \omega_3 \\ 2\omega_1 \omega_2 \end{pmatrix}
\]
\[- \left( \frac{a^3}{b^6} \right) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{c^{2n}}{(2n)!} \left[ \sum_{m=0}^{\infty} (-1)^m \frac{c^{2m}}{(2m)!} \left( \frac{1}{2\pi} \int_0^{2\pi} \left( |\mathbf{v}|^2 \right)^{m+n-1} \right. \right. \right. \]
\[ \left. \left. \left. (1+\cos \alpha)^3 \left( -\omega_2 \right) \right] \left( \begin{array}{c} \omega_2 \\ 0 \\ 0 \end{array} \right) \right. \right. \right. \]
\[ + \left( \frac{a^3}{b^6} \right) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{c^{2n}}{(2n)!} \left[ \sum_{m=0}^{\infty} (-1)^m \frac{c^{2m}}{(2m)!} \left( \frac{1}{2\pi} \int_0^{2\pi} \left( |\mathbf{v}|^2 \right)^{m+n-1} \right. \right. \right. \]
\[ \left. \left. \left. (1+\cos \alpha)^3 \cos^2 \alpha \left( \omega_2 \right) \right] \left( \begin{array}{c} \omega_2 \\ \omega_1 \\ 0 \end{array} \right) \right. \right. \right. \]
\[ + \left( \frac{1}{b^3} \right) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{c^{2n}}{(2n)!} \left[ \sum_{m=0}^{\infty} (-1)^m \frac{c^{2m}}{(2m)!} \left( \frac{1}{2\pi} \int_0^{2\pi} \left( |\mathbf{v}|^2 \right)^{m+n-1} \right. \right. \right. \]
\[ \left. \left. \left. (1+\cos \alpha) \left( \begin{array}{c} \omega_2 \omega_3 \\ 0 \\ \omega_2 \omega_2 \end{array} \right) \right. \right. \right. \]
\[ - \left( \frac{1}{b^3} \right) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{c^{2n}}{(2n)!} \left[ \sum_{m=0}^{\infty} (-1)^m \frac{c^{2m}}{(2m)!} \left( \frac{1}{2\pi} \int_0^{2\pi} \left( |\mathbf{v}|^2 \right)^{m+n-1} \right. \right. \right. \]
\[ \left. \left. \left. (1+\cos \alpha) \cos^2 \alpha \left( \begin{array}{c} \omega_2 \omega_3 \\ \omega_1 \omega_3 \\ 2\omega_1 \omega_2 \end{array} \right) \right. \right. \right. \]
\[
- \left( \frac{a^3}{b^6} \right) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(c)^{2n}}{(2n)!} \left[ \sum_{m=0}^{\infty} (-1)^m \frac{(c)^{2m}}{(2m)!} \left( \sum_{x=0}^{m+n-1} \sum_{y=0}^{x} \frac{(m+n-1)!}{x!y!(m+n-x-y-1)!} \right) \right] \left( -2\omega_3 \right)^x \left( 1-e^{2} \right)^{-\frac{3x-6y}{2}} \left[ \frac{1}{2\pi} \int_{0}^{2\pi} (1+\cos\alpha)^{2(x+2y+1)+1} \cos^2 \alpha \ d\alpha \right] \right) \left( \omega_2 \right) \left( \omega_1 \right)
\]

\[
+ \left( \frac{a^3}{b^6} \right) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(c)^{2n}}{(2n)!} \left[ \sum_{m=0}^{\infty} (-1)^m \frac{(c)^{2m}}{(2m)!} \left( \sum_{x=0}^{m+n-1} \sum_{y=0}^{x} \frac{(m+n-1)!}{x!y!(m+n-x-y-1)!} \right) \right] \left( -2\omega_3 \right)^x \left( 1-e^{2} \right)^{-\frac{3x-6y}{2}} \left[ \frac{1}{2\pi} \int_{0}^{2\pi} (1+\cos\alpha)^{2(x+2y+1)+1} \cos^2 \alpha \ d\alpha \right] \right) \left( \omega_2 \right) \left( \omega_1 \right)
\]

\[
+ \left( \frac{1}{3} \right) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(c)^{2n}}{(2n)!} \left[ \sum_{m=0}^{\infty} (-1)^m \frac{(c)^{2m}}{(2m)!} \left( \sum_{x=0}^{m+n-1} \sum_{y=0}^{x} \frac{(m+n-1)!}{x!y!(m+n-x-y-1)!} \right) \right] \left( -2\omega_3 \right)^x \left( 1-e^{2} \right)^{-\frac{3x-6y}{2}} \left[ \frac{1}{2\pi} \int_{0}^{2\pi} (1+\cos\alpha)^{2(x+2y+1)+1} \cos^2 \alpha \ d\alpha \right] \right) \left( \omega_2 \right) \left( \omega_1 \right)
\]

\[
- \left( \frac{1}{3} \right) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(c)^{2n}}{(2n)!} \left[ \sum_{m=0}^{\infty} (-1)^m \frac{(c)^{2m}}{(2m)!} \left( \sum_{x=0}^{m+n-1} \sum_{y=0}^{x} \frac{(m+n-1)!}{x!y!(m+n-x-y-1)!} \right) \right] \left( -2\omega_3 \right)^x \left( 1-e^{2} \right)^{-\frac{3x-6y}{2}} \left[ \frac{1}{2\pi} \int_{0}^{2\pi} (1+\cos\alpha)^{2(x+2y+1)+1} \cos^2 \alpha \ d\alpha \right] \right) \left( \omega_2 \right) \left( \omega_1 \right)
\]
D. Special Case II: Small Lag Angle \( \kappa \)

If the substitutions \( \sin(c||v||) = c||v|| \), \( \cos(c||v||) = 1 \), and \( 1 - \cos(c||v||) = (1/2)c^2||v||^2 \) are made in equations (VII.8), then the average of the tidal terms are more simply calculated to be

\[
\left\langle \frac{\sin(2c||v||)}{3} \right\rangle \left[ 0 - (b \cdot \hat{s}) \right]
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{2c}{||s||^3} \left[ v - (b \cdot v)b \right] dt
\]

\[
= \frac{c}{\pi} \int_0^{2\pi} \left[ \frac{1}{||s||} \left( \frac{d\alpha}{dt} \right)^2 - \frac{1}{||s||^3} \omega_1 \frac{\cos \alpha + \omega_2 \sin \alpha}{||s||^3} \right] dt
\]

\[
= \frac{c}{\pi} \left\{ \int_0^{2\pi} \frac{ab}{||s||^3} dt - \int_0^{2\pi} \left( \frac{1}{||s||^3} \right) \omega_s dt + \int_0^{2\pi} \frac{1}{||s||^3} \left( \omega_1 \frac{\cos^2 \alpha + \omega_2 \sin \alpha \cos \alpha}{||s||^3} \right) dt \right\}
\]

\[
= \frac{c}{\pi} \left\{ \int_0^{2\pi} \frac{1}{a^3} \left[ \sum_q G_{4,2,q}(e) \cos qt \right] dt \right\}
\]

\[
= \frac{c}{\pi} \left\{ - \left( \int_0^{2\pi} \frac{1}{a^3} \left[ \sum_q G_{2,1,q}(e) \cos qt \right] dt \right) \left( \omega_1 - \frac{1}{2} \frac{\omega_1}{\omega_2} \right) \right\}
\]

\[
+ \frac{1}{2} \left( \int_0^{2\pi} \frac{1}{a^3} \left[ \sum_q G_{2,0,q}(e) \cos[(q+2)t] \right] dt \right) \left( -\omega_2 \right)
\]
\[
\begin{align*}
\frac{c}{\pi} & \left[ \frac{b}{a^4} (2\pi) G_{4,2,0}(\epsilon) \right] k \\
& - \left[ \frac{1}{a^3} (\pi) G_{2,1,0}(\epsilon) \right] \begin{pmatrix} \omega_1 \\ \omega_2 \\ 2\omega_3 \end{pmatrix} \\
& + \left[ \frac{1}{a^3} (\pi) G_{2,0,-2}(\epsilon) \right] \begin{pmatrix} \omega_1 \\ -\omega_2 \\ 0 \end{pmatrix} \\
& \left[ \frac{2b}{a} \left( 1 + \frac{3e^2}{2} \right) \right] (1-e^2)^{-7/2} k \\
& \left\{ \begin{array}{c} (2+3e^2)(1-e^2)^{-3/2} \end{array} \right\} - (1-e^2)^{-3/2} \begin{pmatrix} \omega_1 \\ \omega_2 \\ 2\omega_3 \end{pmatrix} + 0 \\
& = \frac{c}{a^3} \left\{ \begin{array}{c} (2+3e^2)(1-e^2)^{-3} \end{array} \right\} - \omega_3 (1-e^2)^{-3/2} \begin{pmatrix} \omega_1 \\ \omega_2 \\ -\omega_3 \end{pmatrix} k - (1-e^2)^{-3/2} \omega_k \\
& = \frac{c}{b^3} \left\{ \begin{array}{c} (2+3e^2)(1-e^2)^{-3/2} \end{array} \right\} - \omega_3 \begin{pmatrix} \omega_1 \\ \omega_2 \\ -\omega_3 \end{pmatrix} k - \omega_k \\
\end{align*}
\]

and

\[
\begin{align*}
& \left< \cos(c ||\hat{v}||)[1-\cos(c ||\hat{v}||)] \right> \\
& \left( \hat{g} \cdot \hat{v} \right) (\hat{g} \times \hat{v}) \\
& \approx \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{1}{||g||^3} \left( \frac{c^2}{2} \right) (\hat{g} \cdot \hat{v}) (\hat{g} \times \hat{v}) \right] dt \\
& = \frac{c^2}{4\pi} \int_0^{2\pi} \left[ \frac{1}{||g||^3} (\omega_1 \cos \alpha + \omega_2 \sin \alpha) \right] \begin{pmatrix} \sin \alpha \\ \frac{d\alpha}{dt} - \omega_3 \\ -\cos \alpha \\ \frac{d\alpha}{dt} - \omega_3 \\ -\omega_2 \cos \alpha + \omega_1 \sin \alpha \end{pmatrix} dt \\
\end{align*}
\]
\[ \begin{aligned}
&= \frac{c^2}{4\pi} \int_0^{2\pi} \frac{ab}{||s||^5} \begin{pmatrix}
\omega_1 \sin \alpha \cos \alpha + \omega_2 \sin^2 \alpha \\
-\omega_1 \cos^2 \alpha - \omega_2 \sin \alpha \cos \alpha \\
0
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
-\omega_1 \omega_3 \sin \alpha \cos \alpha - \omega_2 \omega_3 \sin^2 \alpha \\
\omega_1 \omega_3 \cos^2 \alpha + \omega_2 \omega_3 \sin \alpha \cos \alpha \\
\omega_1 \omega_3 \sin^2 \alpha - \cos^2 \alpha + (\omega_1^2 - \omega_2^2) \sin \alpha \cos \alpha
\end{pmatrix} \, dt \\
+ \frac{c^2}{4\pi} \int_0^{2\pi} \frac{1}{||s||^3} \begin{pmatrix}
\omega_1 \\
-\omega_1 \omega_3 \sin \alpha \cos \alpha - \omega_2 \omega_3 \sin^2 \alpha \\
\omega_1 \omega_3 \cos^2 \alpha + \omega_2 \omega_3 \sin \alpha \cos \alpha \\
\omega_1 \omega_2 \sin^2 \alpha - \cos^2 \alpha + (\omega_1^2 - \omega_2^2) \sin \alpha \cos \alpha
\end{pmatrix} \, dt
\end{aligned} \]

\[ \begin{aligned}
&= \frac{c^2}{4\pi} \left\{ \begin{array}{l}
ab \int_0^{2\pi} \left[ \sum_q G_{4,2,q}(e) \cos(qt) \right] \begin{pmatrix}
\omega_2 \\
-\omega_1 \\
0
\end{pmatrix} \, dt \\
- ab \int_0^{2\pi} \left[ \sum_q G_{4,1,q}(e) \cos(2+q)t \right] \begin{pmatrix}
\omega_2 \\
\omega_1 \\
0
\end{pmatrix} \, dt \\
+ \int_0^{2\pi} \left[ \sum_q G_{2,1,q}(e) \cos(qt) \right] \begin{pmatrix}
-\omega_2 \omega_3 \\
\omega_1 \omega_3 \\
0
\end{pmatrix} \, dt \\
+ \int_0^{2\pi} \left[ \sum_q G_{2,0,q}(e) \cos((2+q)t) \right] \begin{pmatrix}
\omega_2 \omega_3 \\
\omega_1 \omega_3 \\
-\omega_1 \omega_3
\end{pmatrix} \, dt
\end{array} \right\}
\end{aligned} \]
\[
\begin{align*}
&= \frac{c^2}{4\pi} \left\{ 
\frac{1}{a^5} \left[ a b (2\pi G_{2,0,0}^{e}) \right] \begin{pmatrix}
\omega_2 \\
-\omega_1 \\
0
\end{pmatrix}
\right.
- \frac{1}{a^5} \left[ a b (2\pi G_{4,1,-2}^{e}) \right] \begin{pmatrix}
\omega_2 \\
\omega_1 \\
0
\end{pmatrix}
+ \frac{1}{a^3} \left[ (2\pi G_{2,1,0}^{e}) \right] \begin{pmatrix}
-\omega_2 \omega_3 \\
\omega_1 \omega_3 \\
0
\end{pmatrix}
+ \frac{1}{a^3} \left[ (2\pi G_{2,0,-2}^{e}) \right] \begin{pmatrix}
\omega_2 \omega_3 \\
\omega_1 \omega_3 \\
-\omega_1 \omega_2
\end{pmatrix}
\right. \\
&= \frac{c^2}{2\pi^3} \left\{ 
\frac{b}{a} \left( 1 + \frac{3e^2}{2} \right) (1-e^2)^{-7/2} \begin{pmatrix}
\omega_2 \\
-\omega_1 \\
0
\end{pmatrix}
\right.
- \frac{1}{a} \left( \frac{3e^2}{4} \right) (1-e^2)^{-7/2} \begin{pmatrix}
\omega_2 \\
\omega_1 \\
0
\end{pmatrix}
+ (1-e^2)^{-3/2} \begin{pmatrix}
-\omega_2 \omega_3 \\
\omega_1 \omega_3 \\
0
\end{pmatrix}
+ 0
\right. 
\end{align*}
\]
\[
\frac{c^2}{2a^3} \left\{ \left[ \left( 1 + \frac{3e^2}{4} \right) \left( 1 - e^2 \right)^{-3} - \omega_3 (1 - e^2)^{-3/2} \right] \right\} (\omega_{24}) \\
- \left[ \left( 1 + \frac{9e^2}{4} \right) \left( 1 - e^2 \right)^{-3} - \omega_3 (1 - e^2)^{-3/2} \right] (\omega_{14}) \\
\frac{c^2}{2b^3} \left\{ \left[ \left( 1 + \frac{3e^2}{4} \right) \left( 1 - e^2 \right)^{-3/2} - \omega_3 \right] \right\} (\omega_{24}) \\
- \left[ \left( 1 + \frac{9e^2}{4} \right) \left( 1 - e^2 \right)^{-3/2} - \omega_3 \right] (\omega_{14}) 
\]
XVI. APPENDIX D: SUMMARY OF CALCULATIONS FOR AVERAGING

OF THE TERM \( \frac{1}{|s|} (A[P,B]A^{-1}s) \times \hat{s} \)

Let \( \hat{s} = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \), \( |s| = \frac{b^2}{a} (1 + \epsilon \cos \alpha)^{-1} \), and

\[
\frac{d\alpha}{dt} = \left( \frac{a}{b} \right)^3 (1-\cos \alpha)^2 = \frac{ab}{|s|^2} \, . \text{ Then}
\]

\[
\int_0^{2\pi q} \frac{s_1^2}{|s|^5} dt = \int_0^{2\pi} \frac{s_2^2}{|s|^5} dt = \frac{\pi q}{a^3} (1-\epsilon^2)^{-3/2}
\]

\[
\int_0^{2\pi q} \frac{s_1 s_2}{|s|^5} dt = 0
\]

\[
\int_0^{2\pi q} \frac{s_1^2}{|s|^5} \sin \left( \frac{p}{q} \right) dt = \int_0^{2\pi q} \frac{s_2^2}{|s|^5} \sin \left( \frac{p}{q} \right) dt = 0 \quad \forall \, p, q.
\]

\[
\int_0^{2\pi q} \frac{s_1 s_2}{|s|^5} \sin \left( \frac{p}{q} \right) dt = \begin{cases} \frac{\pi q}{2a^3} \left[ G_{2,0,m-2}(e) - G_{2,0,-m-2}(e) \right] & \text{if } \frac{p}{q} = m \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}
\]

\[
\int_0^{2\pi q} \frac{s_1^2}{|s|^5} \cos \left( \frac{p}{q} \right) dt = \begin{cases} \frac{\pi q}{2a^3} \left[ G_{2,1,-m}(e) + G_{2,1,m}(e) \right] & \text{if } \frac{p}{q} = m \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}
\]
\[
\int_{0}^{2\pi q} \frac{s_2}{|s|^5} \cos \left( \frac{p}{q} \right) dt = 0 \quad \forall \ p, q
\]
\[
\int_{0}^{2\pi q} s_2^2 \frac{1}{|s|^5} \cos \left( \frac{p}{q} \right) dt = \begin{cases} 
\frac{\pi q}{2a^3} \left[ G_{2,1,-m}(e) + G_{2,1,m}(e) \right] & \text{if } p = m \text{ is an integer} \\
0 & \text{otherwise}
\end{cases}
\]
\[
\int_{0}^{2\pi q} \frac{s_1}{|s|^5} \sin^2 \left( \frac{p}{q} \right) dt = \begin{cases} 
\frac{\pi q}{2a^3} (1-e^2)^{-3/2} - \frac{\pi q}{4a^3} \left[ G_{2,0,-m-2}(e) + G_{2,0,m-2}(e) \right] & \text{if } \frac{2p}{q} = m \text{ is an integer} \\
\frac{\pi q}{2a^3} (1-e^2)^{-3/2} & \text{otherwise}
\end{cases}
\]
\[
\int_{0}^{2\pi q} s_1 s_2 \frac{1}{|s|^5} \sin^2 \left( \frac{p}{q} \right) dt = 0 \quad \forall \ p, q
\]
\[
\int_0^{2\pi q} \frac{s_2^2}{||s||^5} \sin^2 \left( \frac{p_t}{q} \right) \, dt = \begin{cases} 
\frac{nq}{2a} (1-e^2)^{-3/2} \\
- \frac{nq}{4a^3} \left[ G_{2,1,-m}(e) \right] + G_{2,1,m}(e) \\
\frac{nq}{2a^3} (1-e^2)^{-3/2}
\end{cases} \quad \text{if } \frac{2p}{q} = m \text{ is an integer}
\]

\[
\int_0^{2\pi q} \frac{s_1^2}{||s||^5} \cos^2 \left( \frac{p_t}{q} \right) \, dt = \begin{cases} 
\frac{nq}{3} (1-e^2)^{-3/2} + \frac{nq}{4a} \left[ G_{2,1,-m}(e) \right] + G_{2,0,-m-2}(e) + G_{2,0,m-2}(e) \\
\frac{nq}{2a^3} (1-e^2)^{-3/2}
\end{cases} \quad \text{if } \frac{2p}{q} = m \text{ is an integer}
\]

\[
\int_0^{2\pi q} \frac{s_1 s_2}{||s||^5} \cos^2 \left( \frac{p_t}{q} \right) \, dt = 0 \quad \forall \ p, q
\]
\[
\int_0^{2\pi q} \frac{s_2^2}{\|s\|^5} \cos^2 \left( \frac{Pt}{q} \right) \, dt = \begin{cases} \frac{\pi q}{2a^3} (1-e^2)^{-3/2} + \frac{\pi q}{4a^3} \left[ G_{2,1,-m}(e) \right] \\ -G_{2,0,-m-2}(e) - G_{2,0,0-2}(e) + G_{2,1,m}(e) \end{cases} \quad \text{if } \frac{2p}{q} = m \text{ is an integer}
\]

\[
\int_0^{2\pi q} \frac{s_2^2}{\|s\|^5} \sin \left( \frac{2pt}{q} \right) \, dt = 0 \quad \forall \ p, q
\]

\[
\int_0^{2\pi q} \frac{s_1 s_2}{\|s\|^5} \sin \left( \frac{2pt}{q} \right) \, dt = \begin{cases} \frac{\pi q}{2a^3} \left[ G_{2,0,m-2}(e) - G_{2,0,-m-2}(e) \right] \quad \text{if } \frac{2p}{q} = m \text{ is an integer} \\ 0 \quad \text{otherwise} \end{cases}
\]