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Elastodynamic ray theory and asymptotic methods for direct and inverse scattering problems

Jer-Shi Chen

Iowa State University

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ELASTODYNAMIC RAY THEORY AND ASYMPTOTIC METHODS FOR DIRECT AND INVERSE SCATTERING PROBLEMS

Iowa State University

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Jer-Shi Chen

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1. INTRODUCTION

Ultrasonics is the study of how high frequency mechanical waves propagate in and interact with materials. One common application of ultrasonics is in the nondestructive evaluation (NDE) of materials for defects. In such an application, a wave is injected through the surface of a material to be inspected and allowed to interact with whatever flaws may be present. These interactions produce additional (scattered) waves that travel back to the surface and can be measured. A prime objective of ultrasonic NDE is to use the measurements of these scattered waves to deduce the nature of the defects which produced them. Making a deduction of this sort is often called the solution of an "inverse" problem [1].

The present work seeks to find such an inversion scheme that can be used in conjunction with ultrasonic nondestructive evaluation tests. In view of this connection, our main concern will be on the scattering of elastodynamic waves by arbitrarily shaped flaws (obstacles). Throughout this dissertation, the scatterer (flaw) will be assumed to be an isolated obstacle embedded in an infinite, homogeneous, isotropic medium. Also, the scatterer will be assumed to have a smooth boundary and be acoustically "opaque". Representative scatterers of this type include acoustically hard obstacles in acoustic scattering problems and traction free voids in elastodynamic problems. The (smooth) boundary of the scatterer will be denoted by $B$, and the regions interior and exterior to $B$ will be denoted by $V_{\text{int}}$ and $V_{\text{ext}}$ respectively.
Typically the incident wave is generated either by sources located at a finite distance from the scatterer or by sources at infinity. In this dissertation, however, all incident fields will be assumed to be emitted by sources of the latter type. A familiar example of this type of source is a plane incident wave.

For convenience, most of the detailed derivations in this thesis are done for time-harmonic problems. However, the major end results are also obtained for transient problems through the use of Fourier synthesis. This type of transformation is important because most ultrasonic NDE tests are in fact carried out with transient pulses [2].

In general, there are two classes of scattering problems, namely, the direct and inverse scattering problems. These two classes of problems are briefly described in what follows.

**THE DIRECT SCATTERING PROBLEM:** An obstacle, after being struck by an incident wave, acts as a secondary source and radiates waves outward from itself. These newly generated waves, known as the scattered waves, are superimposed on the otherwise undisturbed incident wave. The resulting wave field, which consists of the incident field and the scattered field, is referred to as the total field. Mathematically, all obstacles can be characterized by prescribing some boundary condition(s) on their boundaries. For instance, an acoustically hard obstacle can be characterized by prescribing the boundary condition $\partial u/\partial n = 0$ on its boundary $B$, where $u$ may be pressure or velocity potential and $\partial / \partial n$ is the normal derivative. For a void in an elastic solid, the boundary condition is that the traction vector vanishes everywhere on the
Simply stated, the direct problem is as follows. A time-harmonic incident wave \( u^\text{in}(x) \) is propagating toward an obstacle bounded by a smooth surface \( B \). Determine the scattered wave \( u^\text{sc}(x) \) such that

\[
\begin{align*}
(1.1a) & \quad L_{EQ} [u^\text{sc}(x)] = 0 & \text{in } \Omega \setminus B \\
(1.1b) & \quad L_{BC} [u^\text{in}(x) + u^\text{sc}(x)] = 0 & \text{on } B \\
(1.1c) & \quad [u^\text{sc}(x)] = 0 & \text{as } |x| \to \infty
\end{align*}
\]

where \( L_{EQ}, L_{BC}, \) and \( L_{RC} \) are linear differential operators dependent on the problem being considered. Eq. (1.1a) corresponds to the governing partial differential equation (possibly a vector equation), and Eq. (1.1b) serves to specify the boundary condition(s). Finally, Eq. (1.1c) is the radiation condition. Physically, this condition requires that the scattered waves are indeed outgoing waves at infinity and is sufficient to guarantee a unique solution to our scattering problem. In scalar wave problems, for example, the three differential operators are explicitly given by

\[
\begin{align*}
(1.2a) & \quad L_{EQ} [u] = (\nabla^2 + k^2)u \\
(1.2b) & \quad L_{BC} [u] = \alpha u + \beta \partial u / \partial n, \quad u = u^\text{in} + u^\text{sc} = \text{total field} \\
(1.2c) & \quad L_{RC} [u^\text{sc}] = \partial u^\text{sc} / \partial r - i ku^\text{sc}, \quad r = |x|
\end{align*}
\]
where $k$ is the wave number; and, $\alpha = 0$ and $\beta = 0$ correspond to the Neumann and Dirichlet boundary conditions, respectively. If neither $\alpha$ nor $\beta$ vanishes, (1.2b) becomes an impedance boundary condition. On the other hand, in elastodynamic scattering problems, $L_{EQ}$ corresponds to the Navier (vector) equation of motion, $L_{BC}$ depends on the character of the scatterer, and $L_{RC}$ are the elastodynamic radiation conditions [3,4].

**THE INVERSE SCATTERING PROBLEM:** Generally speaking, there are two types of inverse problems. For the first type, one is given the incident field together with the scattered field. The objective is to determine the geometrical and, possibly, the material properties of the unknown scatterer. For obvious reason, this type of problem is classified as the "flaw characterization" problem [5]. On the other hand, in the second type of problem, one is given the complete (geometrical and material) information concerning the scatterer in conjunction with the scattered field (typically the far-field pattern). The aim, then, is to uncover the unknown source(s) that is (are) responsible for causing the known scattered field. This type of problem is called the "inverse source" problem [1,6]. The inverse source problem is fundamentally different from the flaw characterization problem, and, only under certain rather stringent conditions will it possess a unique solution [1,7].

In the present work, we will be concerned with the flaw characterization problem, as our ultimate goal is to search for an inversion scheme that can be used in NDE applications. In fact, we will specialize the flaw characterization problem even further to the simpler...
"flaw sizing" problem. In other words, we will confine ourselves to obtaining the geometrical properties of the scatterer's surface. Stated more precisely, for scalar wave problems, we shall postulate that the obstacle is acoustically hard so that Neumann boundary condition applies. And, for elastodynamic wave problems, we shall assume that the obstacle is a traction free void. In either case the material properties of the scatterer, by assumption, are known a priori.

One way of attacking the difficult inverse (flaw sizing) problem is to learn how information on the geometry of the scatterer is contained in the scattered waves by solving the direct scattering problem. Thus much of this dissertation is concerned with the solution of the direct scattering problem. In fact, until Chapter 8 no inverse problem will be addressed at all.

As mentioned earlier, emphasis will be placed on the time-harmonic problems. For this class of problems, all wave frequencies will be assumed to be "sufficiently high" so that the asymptotic methods are applicable. By high frequency (short wavelength) we mean that all wavelengths arising in the problems are small compared with the characteristic length of the scatterer. The asymptotic methods to be used are stationary phase [8,9,10] and the ray methods [11-12].

The first step in applying the method of stationary phase is to formulate the problem in an integral form. Accordingly, after reviewing the relevant literature in Chapter 2, we begin our study in Chapter 3 with the integral formulation for our scattering problems. Then, in Chapter 4, we give details for the asymptotic expansion of these
integrals. In this regard, several methods have been devised for the one dimensional (definite) integrals; such as the well known Laplace method, method of stationary phase, and method of steepest descent \cite{8,9,13,14}. Based upon these methods many important results have been accumulated \cite{8,15,16}. For two dimensional integrals defined on a "planar" region, such as a finite domain in the xy-plane, a variety of results are also well known \cite{17-23}. Even for three dimensional integrals, the asymptotic theory is well developed to some extent \cite{17-23}. However the application of asymptotics to integrals defined on "curved" surfaces, which are of 2-D in nature (in the sense of surface coordinates), has not been as adequately treated in the literature. Unfortunately, it is this type of surface integral that often arises in scattering theory \cite{24,25}. To fill in this gap, Chapter 4 is devoted to this subject. Results derived therein are then applied to both scalar wave and elastodynamic wave problems.

Chapter 5 begins with our introduction to the ray theory. The scalar wave case is considered first. Despite its inherent simplicity, this case conveys the essential ideas of ray theory. More importantly, it aids in the discussion of elastodynamic ray theory presented in the second half of Chapter 5. In this chapter we also investigate an important problem in ray theory, viz., the calculation of curvature tensors of the reflected and refracted wavefronts at the point of incidence.

Chapter 6 is on the scalar wave scattering by an acoustically hard obstacle. In this chapter we shall show that the classical Kirchhoff
(or physical optics) approximation [26] can be interpreted as the first term in the ray series. The second term in the ray series, which is a correction to the Kirchhoff approximation, is also obtained.

Chapter 7 is concerned with the elastodynamic scattering by a traction free void and forms the heart of this dissertation. Both longitudinal and transverse plane incident waves (abbreviated as L-wave and T-wave in our work) are considered. This chapter begins with the introduction of ray pencil coordinate system which is a fundamental building block of this lengthy chapter. Next, by using the ray methods, we consider the behavior of waves at the scattering surface. An immediate consequence of this consideration leads us to the familiar generalized Snell's law [27]. Then we proceed to find the zeroth order elastodynamic solution which, just like the scalar wave case, turns out to be coincident with the result of the Kirchhoff approximation. Having obtained the zeroth order solution, we continue to calculate the tangential derivatives of the zeroth order solution at the scattering surface, which are necessary for the calculation of the first order solution. This seemingly trivial task is in fact a crucial and difficult step in the analysis. It is probably because of this difficulty that first order solutions have never been previously obtained for the elastodynamic scattering problems. The success of our method is based upon an idea originally used by Lee for electromagnetic wave scattering problems [28]. The importance of this result lies in the fact that it further uncovers the local geometry of the scattering surface near the specular point. More specifically, the first order
solution can be processed to extract out two principal curvatures of the scattering surface at the specular point. An inversion scheme based upon the usage of this curvature information is discussed in Chapter 8. Chapter 9 summarizes our main results and contains a discussion and some conclusions.

During the course of our research some detailed concepts and results derived from differential geometry have been used. For the sake of completeness, we have included in Appendix A a brief summary of differential geometry. Also, in order to avoid detracting too much from our main concerns, some derivations in the main body of this thesis have been deliberately removed; these omitted steps can be found in Appendix B. Finally, the equivalent flaw sizing algorithms described in Chapter 8 depend heavily on the geometrical properties of ellipsoids, these properties are discussed in Appendix C.
2. LITERATURE REVIEW

Elastodynamics, the study of wave propagation in elastic solids, has its roots in the even older field of geometrical optics. Geometrical optics, in turn, was erected on the foundation of three simple laws: the laws of propagation, reflection, and refraction. Although these fundamental laws were all deduced from experimental observations, they were remarkably successful in describing a wide variety of optical phenomena. A great number of optical instruments, which are still in use today, were in fact designed based upon these three laws. Therefore, some scientists in the period of the seventeenth to nineteenth century sought to explain why light obeyed these laws [29].

In the seventeenth century, two physical theories of light were created. Christian Huygens formulated the geometrical wave theory and Newton formulated a mechanical corpuscular theory of light. On the whole, Newton's theory was crude for the variety of phenomena he tried to embrace and he made many ad hoc assumptions [30,31]. Nevertheless, Newton developed his corpuscular theory so thoroughly that its completeness and Newton's own great reputation caused scientists, except for Euler and a few others, to accept it for almost one hundred years [31]. Meanwhile, Huygens' work was ignored. An interesting story about the strong influence of Newton's authority was told by J. B. Keller [32] in an annual lecture organized by the American Mathematical Society: In 1818 Poisson devised a reductio ad absurdum argument to show that the wave theory was untenable. He considered an opaque circular disk, such
as a coin, illuminated from one side by a distant source of light. According to the wave theory, waves would emanate from each point on the rim of the circular disk and they would all arrive at the axis of the circular disk in phase, so the axis would be very light (due to constructive interference). Therefore, a cross section of the shadow would contain a bright spot at its center. Poisson thought that this was ridiculous, and concluded that the wave theory must be wrong. When Arago performed this experiment [33] in 1818 and found the bright spot, Poisson was immediately converted and began working enthusiastically on the wave theory.

In the search for the correct equations to describe the propagation of light, the elastodynamic (Navier) equations [34] were discovered; and light was interpreted as a disturbance propagating in a hypothetical elastic medium, called aether [33]. Although the elastic aether theory of light was later replaced by the now well known Maxwell’s electromagnetic wave theory, it managed to attract a number of the greatest mathematicians and scientists of that time to contribute their enthusiasm. Modern elastodynamics rests on the foundations laid by these pioneer’s works. For examples, the essential ideas of stresses and strains were introduced by Cauchy, and the (scalar and vector) displacement potentials were first introduced and used by Poisson.

After the emergence of Maxwell’s electromagnetic wave theory and modern physics, most of the research in elastodynamics was, for some fifty years, connected primarily to a single field—geophysics. The publication of Kolsky’s monograph "Stress Waves in Solids" in 1953 [35]
marked the revival of elastodynamics research in non-geophysical problems. This revival was accelerated in the sixties by intense governmental interest (and funding) in missile silo protection studies [36]. Since then, the research activity on elastodynamic waves has continued to grow rapidly within the communities of applied mathematics, geophysics [37], and applied mechanics [36,38-40]. In the 1970s this growth was spurred even more by the U.S. Air Force/DARPA (Defense Advanced Research Projects Agency) program in quantitative NDE and its strong interest in improving the state of the art in the use of ultrasonics for insuring safety and reliability of materials and structures [41]. Because of this program, today NDE ranks equal to other scientific disciplines, such as underwater acoustics and seismology, as a major user/developer of research in elastodynamics.

At the start of 1970s, the elastodynamic wave scattering theory relevant to NDE was rather limited. Besides the scattering of plane waves by a sphere [42-49] and a few simple shaped scatterers, such as a (infinitely long) circular cylinder [49-55] and a Griffith crack [56-59], no other relevant exact solutions were available. Apart from the integral transform technique [60,61], these exact solutions were primarily based upon the classical technique of wave function expansions (separation of variables) [62] which Rayleigh had used previously in his study on acoustic scattering problems [63]. However, as one may conceive, this technique suffers from some inherent limitations and will work only for a limited number of special cases. Morse and Feshbach [62] pointed out that this limitation is even stringent for
elastodynamic problems, since they involve vector wave equations. Hence, for a general elastodynamic scattering problem, some sort of approximations are unavoidable.

One of the earliest approximate methods used in NDE was the Born approximation [64,65], pioneered by Gubernatis et al. [64,65], Rose [66-72], Richardson [69,73], and others [74-80]. The Born approximation used in elastodynamics is not too much different from that used previously in quantum mechanics [62], acoustics and electromagnetics (where it is called the Rayleigh-Gans approximation [81]). The starting point is the volumetric integral formulation [82]. This formulation gives the scattered field in terms of a volume integral extended over the region occupied by the scatterer; the integrand of the volume integral being composed of the unknown displacement and strain fields inside the scatterer. The Born approximation consists of replacing these unknown fields by their corresponding incident fields, thus reducing the difficult problem to simply a "computation" problem. As noted by Gubernatis et al. [65] and Hudson and Heritage [80], the result of Born approximation can be interpreted as the leading term of the Neumann-Born series [83]. This approximation has been assessed by comparisons of its predictions with exact and experimental results [84-86]. After extensive analysis, it was found that this approximation becomes progressively bad as changes in the material properties become large or the wave length become small (compared with the characteristic length of the scatterer). Physically, this is to be expected, since the displacement and strain fields inside the scatterer will not be much
different from their corresponding incident fields only if the scatterer has similar material properties as the host medium, and only if the scatterer's characteristic dimension is small relative to the incident wave length. Thus, the Born approximation is classified as a weak scattering, long wavelength (or low frequency) approximation. Another approximation, which is solely a long wavelength approximation, is called the quasi-static approximation [87]. Essentially, this approximation consists of replacing the displacement field inside the scatterer by the amplitude of the incident displacement field, and the strain field inside the scatterer by the strain field that would present in the static problem when the strain at infinity equals the amplitude of the incident strain field. Some other approximations which are related to the Born approximation include the distorted Born approximation [88] and those described in [86].

In the high frequency (short wavelength) regime, the prevalent approximate methods are the Kirchhoff (or physical optics) approximation and ray methods. We will discuss the former approximation first. Like the Born approximation, the Kirchhoff approximation begins with the integral formulation. However, instead of the volumetric formulation, the Kirchhoff approximation employs a surface formulation [24] where the integral contains only unknown surface displacement and strain fields. Motivated by the experience with geometrical optics, the Kirchhoff approximation assumes that the scattering surface can be divided sharply into a lit side and a dark side. On the dark side the displacement field is assumed to be identically zero. Further, at every point of the
lit side the scattered field is approximated by \( R(x)u^{\text{in}}(x) \), where \( R(x) \) is the reflection coefficient [12] of a corresponding canonical problem at \( x \) and \( u^{\text{in}}(x) \) is the incident field evaluated at the same point \( x \). In principle, this approximation can be used equally well in problems of elastic wave scattering by both volumetric flaws and cracks. But, to date, it has been mainly used in studying crack scattering problems [12,89-91].

In addition to elastodynamics, the Kirchhoff approximation also finds applications in other fields, such as acoustics [26] and, in particular, electromagnetics [26,92-94] where it has also been used to formulate solutions to inverse problems. In fact, Bojarski’s celebrated POFFIS (physical optics far field inverse scattering) identity, derived (1967) from this approximation, was for electromagnetic waves [94]. Since then, a variety of similar POFFIS identities have been obtained [95]. Bojarski’s idea laid the basis for a series of works on inverse scattering problems later done by Bleistein [6,96-98]. However, it should be pointed that Bleistein’s theory has all been based on the scalar wave model. For vector (e.g., electromagnetic and elastodynamic) wave problems, Bleistein’s theory does not seem to be readily applicable. In this regard, the ray methods seem to be more promising.

The ray theory has its origins in the classical WKB method [99] originally used in solving ordinary differential equations. In the nineteenth century many authors tried to solve certain linear ordinary differential equations containing a large parameter \( k \) by writing the solution in the form
They substituted this series into some particular equation and equated to zero the coefficient of each power of \( k \). In this way they obtained a recursive system of ordinary differential equations for the successive determination of the phase \( S(x) \) and the amplitude \( A_n(x) \). However, they were unable to justify whether the resulting series is convergent or not. In 1885 Poincare proved that in general the resulting series diverges. Instead, he suggested that the series is asymptotic to the solution in the sense that for each \( N \)

\[
(2.2) \quad u(x,k) - \exp[ikS(x)] \sum_{n=0}^{N} (ik)^{-n}A_n(x) = o(k^{-N}) \quad \text{as} \quad k \to \infty
\]

The use of such series is often called the WKB method after Wentzel, Kramers, and Brillouin who used them in quantum mechanics [100,101]. In the 1950s some applied mathematicians in the Courant Institute of Mathematical Sciences of New York University, such as Keller, Kline, Lewis, and Seckler extended the WKB method to solve a class of linear partial differential equations by using the same series (2.2) except replacing the 1-D variable \( x \) in (2.2) with \( x \). The equations they studied included the Helmholtz [102,103] and Maxwell [104] equations. Using this extended WKB method, they succeeded in solving a great number
of scattering problems [102,105]. In some cases where the exact solutions were available, they found that the asymptotic expansions of the known exact solutions coincided with their asymptotic solutions [102,105]. During the course of their research, Keller observed that only the waves associated with the rays of geometrical optics appeared in their asymptotic solutions, but no diffracted waves arose. However, diffracted waves were present in the asymptotic expansions of the exact and approximate solutions of various problems. For example, edge diffracted waves occurred in Sommerfeld's solutions of wedge and half-plane scattering problems [25,106]. They also appeared in Rubinovicz's approximate solution of diffraction by an aperture [107]. Vertex diffracted rays occurred in the diffraction by a semi-infinite circular cone, analyzed by Felsen [108], by Hansen and Schiff [109], and by others [26]. Surface diffracted waves were found by Watson in his study of diffraction by a sphere [110], by Franz in the solution of diffraction by a circular cylinder [26], and by Friedlander in the analysis of the diffraction of pulses by cylinders with smooth convex cross sections [103]. All these results and others made Keller believe [32] that their asymptotic theory should be further extended so as to include diffracted waves. Some of this further extension was reported in [111-116].

From the fact that the solution is asymptotic to the exact solution at high frequencies, Keller surmised that the leading term in the extended WKB series solution might correspond to the geometrical optics solution and that subsequent terms account for diffraction effects which
geometrical optics fails to describe. In 1958, Keller presented his "Geometrical Theory of Diffraction" [117] in two different but equivalent forms. The first formulation enumerated different species of diffracted rays and provided explicitly the characterization for each of these rays. The second formulation was based upon the extension of Fermat’s principle [118] and was shown to be completely equivalent to the first formulation from considerations of the calculus of variations. Later in 1962 [119] Keller reviewed the ray theory of diffraction with particular attention paid to specific applications and experimental confirmation of the theory. Three years later (1965), Keller and Hansen [120] also provided an exhaustive survey of the theory of short-wave diffraction by thin screens and other objects with edges. The theory as applied to the diffraction by smooth objects is contained in [105,121,122].

Keller and his colleague’s interest was not solely confined to the acoustic and electromagnetic wave problems. In 1959 Karal and Keller published the now classic paper "Elastic Wave Propagation in Homogeneous and Inhomogeneous Media" [123], in which a general method was developed for the solution of linearized equations of elastodynamics. Time harmonic waves, transient pulses, and rapidly changing wave forms were all treated in this paper. However, this important paper only discusses the "propagation" problem and excludes the cases where the medium may contain boundaries, interfaces, or obstacles. In these cases, in addition to the incident ray, there will be reflected, refracted, and possibly diffracted and surface rays. And, each family of the new rays
will be accompanied with a (new) wave. One major difficulty in treating these new waves is that of finding the initial value for the amplitude function \( A_n(x) \) of the ray series at each point of incidence. In Chapter 7 of this thesis, one way for circumventing this difficulty will be discussed. Some other works relating to Keller's theory that discuss elastodynamic problems can be found in [124-126]. In general, the advantage of Keller's method is that it does not rely on separation of variables (or other similar procedures), and is therefore especially useful for problems involving scatterers with complicated shapes. This is particularly true for the scalar wave case.

Starting in the 1970s Achenbach and his coworkers became involved in this field, and further extended Keller's theory to solve many important elastodynamic scattering problems. However, these extensions all involved crack scattering problems [12,127-139]; scattering of elastodynamic waves by volumetric type flaws was not included. Although some other authors have also used the ray theory to investigate the elastodynamic scattering by volumetric type flaws [140,141], these studies were all confined to the two dimensional problems. For general three dimensional elastodynamic wave scattering problems, explicit ray theory solutions are apparently nonexistent. The success of this thesis in obtaining such solutions is therefore an important first step which, as will be shown, sheds new light on both direct and inverse scattering problems involving volumetric scatterers.
3. INTEGRAL FORMULATION FOR SCATTERING PROBLEMS

The main objective of this chapter is to discuss the surface integral formulation for steady-state wave scattering problems. The volumetric formulation [24,82], though often seen in the literature, is not pertinent to our study. Hence, these volumetric-type integral formulas will not be discussed at all. Most of the results given in this chapter are not completely new, and have been included here primarily for the sake of completeness and convenience. Insofar as the steady-state waves are concerned, only the Helmholtz-type integral formulas are required. Therefore, the Kirchhoff-type formulas [24], which are mainly used in the transient problems, will not be addressed in this chapter. This, however, does not mean that the transient problems will be excluded from our study. As a matter of fact, one of the foremost aims of this thesis is to obtain the (far-field) scattered waves caused by a variety of impulsive incident waves. Obviously, all these waves are transients. In order to avoid complicating matters, however, the details regarding how the steady-state solutions can be used to obtain transient solutions will be postponed until Chapter 6. Integral formulas for both scalar and elastodynamic waves will be presented. The former, due to its simplicity, will be taken up first in Section 3.1. Then, in Section 3.2 we will consider the elastodynamic waves. As one shall see, the overall development in Section 3.2 essentially parallels that developed in Section 3.1. Before our discussion on integral formulas, we agree to adopt the following convention for steady-state waves: Whenever a time-harmonic wave is
referred to, the time dependence will be assumed to be $e^{-i\omega t}$, where $\omega$ and $t$ are circular frequency and time, respectively. With this understanding, most of the time in this thesis, the exponential factor $e^{-i\omega t}$ will be suppressed for brevity.

3.1 Scalar Waves

Consider a time-harmonic incident wave $u^0(x)$ impinging on an obstacle embedded in an otherwise homogeneous infinite medium. Let $B$ be the boundary of the obstacle (scatterer) and let $V_{\text{int}}$ and $V_{\text{ext}}$ be the regions interior and exterior to the scatterer, respectively. In the exterior region $V_{\text{ext}}$, the scattered wave $u^1(x)$ satisfies the homogeneous Helmholtz equation

$$\nabla^2 u^1(x) + k^2 u^1(x) = 0, \quad k = \omega/c$$

where $k$, $\omega$, and $c$ are, respectively, the wave number, frequency, and wave speed associated with the host medium.

The integral formulation begins with the consideration of this equation:

$$\nabla^2 G(x,y) + k^2 G(x,y) + \delta(x - y) = 0$$

where $\delta$ is the Dirac delta function. The solution $G(x,y)$ of Eq. (3.2) is the free space Green's function or fundamental solution. It is known that [142]
\[ G(x, y) = \frac{e^{i k R}}{4 \pi R}, \quad R = |x - y| \]

from which it can be seen that the Green's function is symmetrical, i.e., \( G(x, y) = G(y, x) \), and

\begin{align*}
(3.4a) \quad & G = O(1/R) \\
(3.4b) \quad & R(\partial G/\partial R - i k G) = O(1/R)
\end{align*}

Let \( B_R \) be a "large" sphere of radius \( R \) centered at the coordinate origin \( y \) (\( y = 0 \), see Fig. 3.1), and let \( V' \) be the region interior to \( B_R \) but exterior to \( B \). Now consider the volume integral over \( V' \) (Fig. 3.1) and make use of Green's second identity [143]

\[ \int_{V'} G(x, y) \left[ \nabla^2 u^1(x) + k^2 u^1(x) \right] dV(x) = \int_{V'} u^1(x) \left[ \nabla^2 G(x, y) + k^2 G(x, y) \right] dV(x) + \int_{B + B_R} \left[ G(x, y) \frac{\partial u^1(x)}{\partial n(x)} - u^1(x) \frac{\partial G(x, y)}{\partial n(x)} \right] dB(x) \]

By Eq. (3.1), the left-hand side of this equation vanishes identically. Hence after making use of (3.2) and the sifting property of \( \delta \) function, Eq. (3.5) becomes

\[ \int_{B + B_R} \left[ G(x, y) \frac{\partial u^1(x)}{\partial n(x)} - u^1(x) \frac{\partial G(x, y)}{\partial n(x)} \right] dB(x) = \alpha u^1(y) \]
Fig. 3.1. Scattering geometry and the "large" sphere $B_R$
where

\[
\alpha(y) = \begin{cases} 
1 & \text{if } y \text{ in } V_{\text{ext}} \\
0 & \text{if } y \text{ in } V_{\text{int}} 
\end{cases}
\]

Now recall that, physically, the fundamental solution \(G(y,x)\) represents an outgoing spherical wave at \(y\) due to a point source at \(x\). If one sits at a large distance from the scatterer observing the scattered wave \(u^1(y)\), the scatterer \(B\) would look just like a point source. Hence, we expect the scattered wave \(u^1(y)\) should also possess the same order properties that the fundamental solution \(G(y,x)\) has at infinity, as expressed in Eqs. (3.4a,b). Therefore, as \(R \to \infty\),

\[
(3.7a) \quad u^1 = O(1/R) \\
(3.7b) \quad R(\delta u^1/\delta R - iku^1) = O(1/R)
\]

Eqs. (3.7a) and (3.7b) together are known as the Sommerfeld radiation conditions [6,7,144]. Physically, these two conditions impose on the scattered wave the restriction that it should behave as a spherically diverging wave in the far-field of the scatterer. Mathematically, they serve to provide the sufficient conditions for the uniqueness of our scattering problem [6,7].

One consequence of these conditions is that the integral over \(B_R\) (denoted by \(C\), for short, in the following) in Eq. (3.6a) must go to zero as \(R \to \infty\). To see this, note that
But, by Eqs. (3.4) and (3.7), the integrand in the last integral is of $O(R^{-3})$; accordingly, as $R \to \infty$, $C$ vanishes.

With this result, the integral over $B_R$ in (3.6a) can be discarded. Also, in keeping with the customary convention, we shall reverse the normal direction on the scattering surface $B$ and obtain from (3.6a)

\begin{equation}
\alpha u^1(\gamma) = \int_B \left[ au^1(x) \frac{\partial G(x,\gamma)}{\partial n} - G(x,\gamma) \frac{\partial u^1(\gamma)}{\partial n(\gamma)} \right] dB(\gamma)
\end{equation}

where $\alpha$ is defined as in (3.6b) and $n$ is the outward (to the scatterer) unit normal (see Fig. 3.2).

In the interior region $V_{\text{int}}$, the incident field $u^0(x)$ also satisfies the homogeneous Helmholtz equation (3.1). By following the same procedure employed above, one obtains an equation similar to Eq. (3.10):

\begin{equation}
(\alpha - 1)u^0(\gamma) = \int_B \left[ u^0(x) \frac{\partial G(x,\gamma)}{\partial n(x)} - G(x,\gamma) \frac{\partial u^0(\gamma)}{\partial n(\gamma)} \right] dB(\gamma)
\end{equation}

Here $\alpha$ is the same as that defined in Eq. (3.6b), and again $n$ is the
Fig. 3.2. Outward unit normal $\mathbf{n}$ to the scattering surface $B$
outward (to the scatterer) unit normal.

Adding Eq. (3.11) to Eq. (3.10) and utilizing \( u^0 + u^1 = u \), one obtains

\[
(3.12) \quad u(y) = u^0(y) + \int_B \left[ \frac{u(x) \partial G(x,y)}{\partial n(x)} - G(x,y) \frac{\partial u(x)}{\partial n(x)} \right] dB(x)
\]

One important special case of this equation is when the point \( y \) is in the far-field of the scatterer. In this case, one has (see Fig. 3.3)

\[
(3.13a) \quad |x - y| \sim y - x \cdot \bar{y}
\]

\[
(3.13b) \quad \frac{1}{|x - y|} \sim \frac{1}{y}
\]

\[
(3.13c) \quad G(x,y) \sim \frac{\exp[ik(y - \bar{y} \cdot x)]}{4 \pi y}
\]

\[
(3.13d) \quad \frac{\partial G(x,y)}{\partial n(x)} \sim -ik \bar{y} \cdot n \frac{\exp[ik(y - \bar{y} \cdot x)]}{4 \pi y}
\]

Under these far-field approximations, Eq. (3.12) reduces to

\[
(3.14) \quad u(y) = u^0(y) + f(k) \frac{e^{iky}}{y}
\]

where
Fig. 3.3. A point $\mathbf{y}$ in the far-field of the scatterer and the unit vectors $\mathbf{Y}$ (from coordinate origin $0$ to $\mathbf{y}$), $\mathbf{R}$ (from $\mathbf{y}$ to $\mathbf{x}$)
Eq. (3.14) is important in two aspects. First, it shows that in the far-field the scattered wave, \( u^1 = u - u^0 \), is indeed a spherically spreading wave, as we argued it should be, with amplitude variations determined solely by the \( f(k) \) function defined in Eq. (3.15). Second, the expression of this \( f(k) \) function shows that the behavior of the scattered wave in the far-field is completely determined by the values of \( u \) and its normal derivative \( \partial u / \partial n \) on the boundary \( B \) of the scatterer. In general, on \( B \), either \( u \) or \( \partial u / \partial n \) is unknown; and in some cases (such as inclusion problems) both \( u \) and \( \partial u / \partial n \) may be unknown. Often, these quantities are approximated via assumptions, such as are made in the Born [64,65] and Kirchhoff [6] approximations. With these approximations, the function \( f(k) \) and the far-field scattered wave can be obtained directly. More exact solutions generally rely on numerical schemes, such as the T-matrix methods [145-150], method of optimal truncation [151-153] and boundary element techniques [154-157]. In the boundary element approach, for example, one usually first derives an integral equation for \( u \) and/or \( \partial u / \partial n \) by taking the limit \( y \rightarrow B \) for Eq. (3.10) or Eq. (3.12). If \( B \) is smooth, the limiting process
corresponding to Eqs. (3.10) and (3.12) leads, respectively, to

\begin{equation}
\frac{1}{2} u^1(y) = \int_B \left[ u^1(x) \frac{\partial G(x,y)}{\partial n(x)} - G(x,y) \frac{\partial u^1(x)}{\partial n(x)} \right] dB(x)
\end{equation}

(for $y$ on $B$)

and

\begin{equation}
\frac{1}{2} u(y) = u^0(y) + \left[ \text{same as the above integral} \right] \quad \text{(for $y$ on $B$)}
\end{equation}

In the above two equations, $f$ is Cauchy principal value integral. Under suitable boundary conditions, equations of this type can be solved numerically to yield $u$ and/or $\partial u / \partial n$ on the boundary $B$. Placing these results into the integral terms of $f(k)$ then gives the far-field values immediately. On the other hand, if required, one may also place these results back into Eq. (3.12) for an arbitrary point $y$ exterior to the scatterer to obtain the near-field values.

### 3.2 Elastodynamic Waves

Now we consider a scatterer embedded in an infinite homogeneous isotropic medium with elastic constants $C_{ijkl}$ (Lame's constants $\lambda$ and $\mu$) and density $\rho$. As in Section 3.1 let the boundary of the scatterer be denoted by $B$. Analogous to the scalar wave case, for steady-state waves one can apply the elastodynamic reciprocal theorem (which replaces the Green's second identity for the scalar problem) and the elastodynamic
Sommerfeld radiation conditions [3,4] to obtain [65]

\[(3.19) \quad \alpha(\gamma) u_m(\gamma) = u_0^m(\gamma) + \int_B C_{ijkl} n_j(x) \left[ u_i(x) \frac{\partial G_{km}(x,\gamma)}{\partial x_1} \right. \left. \frac{\partial u_k(x)}{\partial x_1} G_{im}(x,\gamma) \right] dB(x) \]

where, as before, \(\alpha(\gamma) = 1\) for \(\gamma\) in \(V_{\text{ext}}\) and 0 for \(\gamma\) in \(V_{\text{int}}\) and \(u_0^m\) is the incident wave. In Eq. (3.19), \(G_{im}(x,\gamma)\) is the fundamental solution associated with the host medium, which satisfies the following equation:

\[(3.20) \quad C_{ijkl} \frac{\partial G_{km}(x,\gamma)}{\partial x_1 \partial x_j} + \rho \omega^2 G_{im}(x,\gamma) + \delta_{im} \delta(x - \gamma) = 0 \]

Physically, \(G_{im}(x,\gamma)\) is the \(i^{th}\) component of a displacement field produced at \(x\) by the \(m^{th}\) component of a concentrated unit force, with \(exp(-i\omega t)\) time dependence, applied at the point \(\gamma\) in an infinite medium with elastic constants \(C_{ijkl}\) and density \(\rho\). By use of Fourier transforms and contour integration techniques, it can be shown that [65,158]

\[(3.21) \quad G_{ij} = \frac{1}{4\pi \rho \omega^2} \left\{ k^2 \frac{\exp(ik_2R)}{R} \delta_{ij} + \left[ \frac{\exp(ik_1R)}{R} - \frac{\exp(ik_1R)}{R} \right]_j^i \right\} \]

where
\begin{align}
(3.22) \quad R &= |x - y| \\
(3.23) \quad k_\alpha &= \omega c_\alpha, \ (\alpha = 1, 2) \\
(3.24) \quad (c_1)^2 &= (\lambda + 2\mu)/\rho, \quad (c_2)^2 = \mu/\rho
\end{align}

In the above $c_1$ and $c_2$ are, respectively, the longitudinal and transverse wavespeeds.

As in the scalar wave case, one can specialize Eq. (3.19) for a far-field point $y$, obtaining

\begin{align}
(3.25) \quad u_m(y) &= u_m^0 + A_m \frac{\exp(ik_1y)}{y} + B_m \frac{\exp(ik_2y)}{y}
\end{align}

In the above equation,

\begin{align}
(3.26) \quad A_m(y) &= Y_m Y_1 f_i(k_1) \\
(3.27) \quad B_m(y) &= (\delta_{1m} - Y_1 Y_m) f_i(k_2)
\end{align}

with

\begin{align}
(3.28) \quad k_{-\alpha} &= k_\alpha Y \\
(3.29) \quad f_i(k_{-\alpha}) &= -\frac{k_{-\alpha}^2}{4\pi\omega^2} C_{ijkl} \left\{ \int_B u_{k,1} Y_{j,1}^{n_k} \exp(-ik_{-\alpha}x) dB(x) + \int_B u_{k,1} u_{j,1}^{n_k} \exp(-ik_{-\alpha}x) dB(x) \right\}
\end{align}
Now let us observe carefully the correspondence and distinction between the two sets of equations: (3.25, 29) and (3.14, 15). From (3.25), it is clear that the elastodynamic scattered field, in contrast to the scalar wave problem, consists of two spherically spreading waves; one travelling with the longitudinal wavespeed $c_1$ and the other with transverse wavespeed $c_2$. Moreover, the $f$-function in (3.15) is a scalar function, whereas the $f$-function in (3.29) is a vector one.

The numerical schemes mentioned earlier in Section 3.1 can once again be exploited to obtain the boundary values needed in the evaluation of the $f$-vector. Alternatively, approximations can also be made "directly" for the unknown boundary values to facilitate the calculation of the $f$-vector. In the next few chapters, we will be particularly interested in one of the latter type approaches, namely, the Kirchhoff (or physical optics) approximation. Essentially, this approximation, being classified as a high frequency approximation, assumes that the scattering surface behaves locally, at every point, as a flat interface. Solving this canonical problem is much easier and has been studied extensively. We will come back to this approximation after presenting a discussion of stationary phase and ray methods, in which we shall show that the Kirchhoff approximation can be mathematically interpreted as the leading term in a ray series.
4. ASYMPTOTIC EXPANSION OF INTEGRALS: METHOD OF STATIONARY PHASE

4.1 Introduction

The asymptotic expansion of a function is an approximation of a function in the neighborhood of a point. However, unlike Taylor's series, an asymptotic expansion need not converge to the function at any point at all. Nevertheless, asymptotic expansions are quite adequate for approximating functions in a sense to be defined.

A sequence \( \{ \phi_n(k) \} \) \( (n = 0,1,2, \ldots) \) is said to be an asymptotic sequence \[8\], as \( k \to k_o \), if for every \( n = 0,1,2, \ldots \)

\[
\phi_{n+1}(k) = o[\phi_n(k)] , \ k \to k_o
\]

where "o" is the familiar small 0 order notation. If a continuous function \( f(k) \) can be written, as \( k \to k_o \), in a form like

\[
f(k) = \sum_{n=0}^{N} a_n \phi_n(k) + O(\phi_{n+1})
\]

for every \( N \), where \( a_n \) are independent of \( k \), then \( f(k) \) is said to have an asymptotic expansion of the series given on the right hand side of the above equation. Notice that, because of the property \( (4.1) \) of an asymptotic sequence, the error committed in truncating the series \( (4.2) \) after the \( N \)th term is asymptotically negligible compared to the \( N \)th term. Also note that the asymptotic representation of a given function \( f(k) \) is not unique. In fact, \( f(k) \) can be represented by an infinite
number of asymptotic expansions because there exist an infinite number of asymptotic sequences that can be used in the representation. However, once a particular asymptotic sequence is given, the representation of \( f(k) \) is unique [9].

In this chapter, the asymptotic sequence that will be employed is \( \{1, k^{-1}, k^{-2}, \ldots \} \) and \( k_0 \) will be \( \infty \). Furthermore, the function \( f(k) \) will be defined by a definite integral. Thus, we will be interested in expanding integrals in inverse powers of \( k \). Although the resulting series expansions need not be convergent, the error can be made as small as one wishes, provided \( k \) is sufficiently large. In scattering problems, \( k \) usually corresponds to wave number; large \( k \) implies high frequency. It is to be noted that the asymptotic expansions are obtained based upon the assumption of "large" \( k \), and the approximations are expected to yield good results only if \( k \) is large. However, a rather amazing result of asymptotics is that the approximations often work well for \( k \) not really large. After some extensive studies, Bleistein proposed a rule of thumb: \( k = 3 \) is large enough for the purpose of asymptotic approximation [6,8].

The main idea of this technique can best be demonstrated by one dimensional integrals; this is the subject of next section.

4.2 Method of Stationary Phase for 1-D Integrals

To avoid unnecessary mathematical complexities, which may hamper the simplicity of the central idea of asymptotic expansions of integrals, we first consider the generalized Fourier integral:
\begin{equation}
I(k) = \int_{a}^{b} f(t) \exp[-ikh(t)] \, dt, \quad b > a, \quad k \to +\infty
\end{equation}

Although the integrand is complex-valued, the two functions \( f \) and \( h \) are assumed to be real and continuous functions of \( t \) in the interval \([a,b]\). The two functions \( f \) and \( h \) are referred to as amplitude and phase of the integrand, respectively. If \( t_0 \in [a, b] \) is a point such that \( h'(t_0) = 0 \), then it is called a stationary point. Notice that a phase function \( h(t) \) may have more than one stationary point.

If \( h(t) \) never vanishes on the interval \([a,b]\), viz. no stationary points, the asymptotic expansion of the integral \( I(k) \) is trivial; it can be obtained simply by repeated usage of integration by parts [8].

If, however, \( h(t) \) vanishes at any point on \([a,b]\), the integration by parts method fails; and it is in this case that the method of stationary phase applies. The method can be explained heuristically by looking at the following example:

\begin{equation}
J(k) = \int_{0}^{4} t(4-t) \exp \left[ -ik(t-2)^2 \right] \, dt
\end{equation}

which has a stationary point at \( t = 2 \). The real part of the integrand is \( t(4-t)\cos[k(t-2)^2] \), the plot of which for two values of \( k \) is shown in Fig. 4.1. As revealed in Fig. 4.1, to the leading order, the principal contribution to the integral \( J \) comes from the neighborhood of the stationary point; the contribution from regions away from the stationary point are not significant due to the rapid self-cancelling oscillations.
Fig. 4.1. Real part of integrand: \( t(4-t)\cos [k(t-2)^2] \)
As the parameter $k$ gets larger and larger, the contribution from the rapid-oscillation region becomes less and less important; and, as $k$ approaches infinity this contribution becomes asymptotically negligible.

With this observation in mind, the generalized Fourier integral defined in (4.3) can be approximated, to the leading order, by

\[ I(k) \approx \int_{t_0-\varepsilon}^{t_0+\varepsilon} f(t) \exp[-ikh(t)] \, dt \]

where $\varepsilon > 0$ is a "small" quantity. The small quantity $\varepsilon$ need not be assigned any specific value since it only appears in the intermediate steps of analysis; it will not appear in the final result. Expanding the phase function $h(t)$ in Taylor series about $t_o$, one obtains

\[ h(t) = h(t_o) + \frac{1}{2} h''(t_o) (t - t_o)^2 + ... \]

since $h'(t_o) = 0$. Upon substituting (4.6) in (4.5), replacing $f(t)$ with $f(t_o)$, and introducing the change of variable $z = t - t_o$, Eq. (4.5) becomes

\[ I(k) \approx f(t_o) \exp[-ikh(t_o)] \int_{-\varepsilon}^{\varepsilon} \exp \left[ - \frac{1}{2} ikh''(t_o) z^2 \right] \, dz \]

Now, since the dominant contribution arises from the region $-\varepsilon < z < \varepsilon$,
the two integration limits in (4.7) can be replaced by $-\infty$ and $\infty$ at the expense of introducing asymptotically negligible error. The resulting integral then can be evaluated by means of contour integration in the complex $z$-plane. The final result for $I(k)$ is [159]

$$I(k) = \frac{(2\pi)^{1/2} f(t_0) e^{-ik h(t_0)} + (i\pi/4)}{[k|h''(t_0)|]^{1/2}}$$

(4.8)

where $+$ and $-$ signs are for $h(t_0) < 0$ and $> 0$, respectively; the error committed in this approximation is $O(k^{-1})$.

4.3 Method of Stationary Phase: Surface Integrals

The method of stationary phase for 2-D integrals in "planar" regions and 3-D integrals in the Euclidean space have been well studied, too. But sometimes they are not sufficient for the purpose of solving wave scattering problems. Here in this section, the ordinary method of stationary phase will be generalized to treat Fourier type integrals defined on "curved" surfaces. The result derived in this section will provide a rigorous mathematical justification for the classical Kirchhoff (or physical optics) approximation as we shall see in the next few chapters.

Specifically, we shall consider the asymptotic expansion of the following integral [cf. Eq. (3.15) for $f(k)$ and Eq. (3.29) for $f(k)$]:

$$I(k) = \int_S f(x) \exp[-ik h(x)] \, dS(x), \quad h(x) = a \cdot x$$

(4.9)
in which $S$ is a smooth, curved, but otherwise arbitrary surface in the 3-D space, and $a$ is a constant unit vector. In scattering problems, $a$ usually corresponds to the direction of wave propagation and $S$ represents the scattering surface.

The phase function in the above integral is

$$(4.10) \quad h(x) = a \cdot x, \quad x \text{ on } S$$

If the surface $S$ is parametrized by two surface coordinates $(a_1, a_2)$, the phase $h$ can be thought as a function of $a_1$ and $a_2$, and Eq. (4.10) may be written alternatively as

$$(4.11) \quad h(a_1, a_2) = a \cdot x(a_1, a_2)$$

The stationary point is determined by the conditions $h_{,\alpha} = 0 \ (\alpha = 1, 2)$, i.e.,

$$(4.12) \quad a_1 \cdot x_1 = 0 \quad \text{and} \quad a_2 \cdot x_2 = 0$$

Since $x_1$ and $x_2$ are vectors tangent to the $a_1$ and $a_2$ coordinate lines, respectively, Eq. (4.12) simply says that the stationary point occurs at the place where the unit vector $a$ is normal to the surface $S$. For a smooth convex closed surface, there are two such points.

To find the principal contribution from a stationary point having surface coordinates $(b_1, b_2)$, we follow the usual procedure and
accordingly expand the phase function $h$ about the stationary point $(b_1, b_2)$ in a Taylor series,

$$h(a_1, a_2) = a \cdot x^S + \frac{1}{2} (a_\alpha - b_\alpha)(a_\beta - b_\beta) a \cdot x^S_{\alpha\beta} + \ldots$$

Here $\alpha$ and $\beta$ are summed over 1 and 2, and the superscript $s$ signifies functional values at the stationary point. The ellipsis (\ldots) in (4.13) stand for small quantities of at least second order.

Substituting (4.13) in (4.9) and keeping in mind that the major contribution to $I(k)$ comes from the immediate neighborhood of the stationary point, one obtains

$$I(k) \sim f(x^S) \exp \left( -ika \cdot x^S \right) J(x^S) \times$$

$$\int_{b_2 - \epsilon_2}^{b_2 + \epsilon_2} \int_{b_1 - \epsilon_1}^{b_1 + \epsilon_1} \exp \left[ - \frac{1}{2} ik(a_\alpha - b_\alpha)(a_\beta - b_\beta) a \cdot x^S_{\alpha\beta} \right] da_1 da_2$$

where the new notation $J$ is defined as \[J_{11} S_{22} - S_{12} S_{21}\] (see Appendix A) and arises from the differential area $dS = J da_1 da_2$. Since the above integration is to be carried out over a small region containing the stationary point, $f(x)$ and $J(x)$ can be approximated by their respective values at that point. This is why $f$ and $J$ were pulled out of the integral sign in the above equation (4.14).

From differential geometry (Appendix A), the second derivatives of $x$ at the stationary point $x^S$ can be expressed as
\[(4.15) \quad x_{\alpha\beta}^S = \Gamma^\gamma_{\alpha\beta} x_{\gamma}^S + h_{\alpha\beta}^S n(x_{\gamma}^S), \quad \alpha, \beta, \gamma = 1, 2; \text{ sum over } \gamma\]

where the \(\Gamma\)'s are Christoffel's symbols of the second kind \([160]\) in tensor theory, \(h_{\alpha\beta}\) are components of curvature tensor, and \(n\) is the unit normal vector of \(S\) at the stationary point being considered. By stationary phase conditions \((4.12)\) and Eq.\((4.15)\),

\[(4.16) \quad a \cdot x_{\alpha\beta}^S = h_{\alpha\beta} a \cdot n\]

The argument \(x_{\alpha\beta}^S\) for \(h_{\alpha\beta}\) and \(n\) in the above equation has been omitted for brevity. By \((4.16)\), the integrand of the double integral in \((4.14)\) becomes

\[(4.17) \quad -\frac{1}{2} ik(a_\alpha - b_\alpha)(a_\beta - b_\beta) h_{\alpha\beta} a \cdot n\]

Now if the surface coordinates \((a_1, a_2)\) are chosen in such a way that they are arc-length parameters and are coincident with the two principal curvature directions at \(x_{\alpha\beta}^S\), then (see Appendix A)

\[(4.18) \quad h_{12} = h_{21} = 0 \quad \text{at } x_{\alpha\beta}^S\]

Furthermore, \(h_{11}\) and \(h_{22}\) will be equal to the two principal radii, i.e.,

\[(4.19) \quad h_{11} = \kappa_1, \quad h_{22} = \kappa_2\]
It then follows from Eqs. (4.17-19) that the integrand of the integral in Eq. (4.14) can be expressed as

\begin{equation}
(4.20) \quad \frac{1}{2} \, ik \left[ (a_1 - b_1)^2 \kappa_1 + (a_2 - b_2)^2 \kappa_2 \right] a \cdot n
\end{equation}

For convenience, denote the double integral in (4.14) by \( I' \). After some change of dummy variables \( c_1 = a_1 - b_1, \ c_2 = a_2 - b_2 \) and making use of (4.20),

\begin{equation}
(4.21) \quad I' = \int_{-\varepsilon_2}^{\varepsilon_2} \int_{-\varepsilon_1}^{\varepsilon_1} \exp \left\{ -\frac{1}{2} \, ik (\kappa_1 c_1^2 + \kappa_2 c_2^2) \right\} a \cdot n \, dc_1 dc_2
\end{equation}

In the spirit of asymptotics, \( \varepsilon_1 \) and \( \varepsilon_2 \) can be replaced by \( \infty \). Thus

\begin{equation}
(4.22) \quad I' \sim 4 \int_0^\infty \exp \left[ -ik (\kappa_1 a \cdot n/2) c_1^2 \right] dc_1 \int_0^\infty \exp \left[ -ik (\kappa_2 a \cdot n/2) c_2^2 \right] dc_2
\end{equation}

Here we emphasize that \( \kappa_1, \kappa_2, \) and \( n \) are values at the stationary point \( \mathbf{x}_s \) and hence are constant.

Now observe that both integrals in (4.22) are of the same form as the following integral:

\begin{equation}
(4.23) \quad A = \int_0^\infty \exp \left[ -ik \lambda z^2 \right] dz
\end{equation}
For the first integral in (4.22) \( \lambda = \kappa_1 a \cdot \pi/2 \) whereas for the second integral \( \lambda = \kappa_2 a \cdot \pi/2 \). It should be noted that \( \kappa_1 \) and \( \kappa_2 \) are always of the same sign (sign convention for curvatures are defined in Appendix A) for an elliptical point, i.e., a point of positive Gaussian curvature \( \kappa_1 \kappa_2 \). For a smooth convex surface, which we shall assume, every point is an elliptical point.

The integral \( \Lambda \) in (4.23) can be evaluated by means of contour integration in the complex \( z \)-plane. First, we consider the case \( \lambda > 0 \). By Cauchy theorem, the original path of integration (Fig. 4.2) can be rotated clockwise through an angle of \( \pi/4 \), since during the course of contour deformation no singularities have been encountered. On the new path of integration \( C \), \( z = \rho \exp(-i\pi/4) \) and \(-ik\lambda z^2 = -k\lambda \rho^2\). Hence (4.23) becomes

\[
(4.24) \quad \Lambda = e^{-i\pi/4} \int_0^\infty \exp \left[ -k\lambda \rho^2 \right] d\rho
\]

After introducing the change of variable \( s = k\lambda \rho^2 \), the preceding equation becomes

\[
(4.25) \quad \Lambda = (1/2)(k\lambda)^{-1/2} e^{-i\pi/4} \int_0^\infty e^{-s} s^{-1/2} ds
\]

The last integral in the above equation is recognized to be the gamma function \( \Gamma(1/2) \) which is equal to \( \pi^{1/2} \). Using this result in (4.25) one
Fig. 4.2. Path of integration for $\lambda > 0$
then obtains

\[(4.26) \quad \Lambda = \left(\frac{1}{2}\right)\left(\pi/K\right)^{1/2} e^{-i\pi/4} \quad (\lambda > 0)\]

For \(\lambda < 0\), the original path of integration needs to be rotated counterclockwise through an angle of \(\pi/4\). In a similar fashion, one obtains

\[(4.27) \quad \Lambda = \left(\frac{1}{2}\right)\left(-\pi/K\right)^{1/2} e^{i\pi/4} \quad (\lambda < 0)\]

Now, refer to Fig. 4.3, which shows a typical smooth convex surface \(S\), the constant unit vector \(\mathbf{a}\), and the two stationary points \(x_s^{(1)}\), \(x_s^{(2)}\). If \(\mathbf{n}\) is chosen to be the inward normal of \(S\), then, according to our sign convention, both \(\kappa_1\) and \(\kappa_2\) are positive at every point of \(S\). At the "first" stationary point \(x_s^{(1)}\),

\[(4.28) \quad \kappa_1 \mathbf{a} \cdot \mathbf{n}/2 = \kappa_1/2, \quad \kappa_2 \mathbf{a} \cdot \mathbf{n}/2 = \kappa_2/2\]

Thus, at this stationary point, "\(\lambda" > 0" for both of the two integrals in Eq. (4.22). Hence, Eq. (4.26) applies. With these results, Eq. (4.22) then yields

\[(4.29) \quad I' = -(2\pi\mathbf{i}/k)(\kappa_1\kappa_2)^{-1/2}\]

Substituting (4.29) in (4.14), keeping in mind the definition of \(I'\), and
Fig. 4.3. A typical smooth convex surface
noting that $J(x^{S(1)}) = 1$ (due to the way that the surface $S$ is parametrized, see Appendix A), one obtains

$$\left(4.30\right) \quad I(k) \left|_{x^{S(1)}} \right. \sim -(2\pi i/k)(R_1 R_2)^{1/2} f \exp \left[ -ika \cdot x^{S(1)} \right]$$

where $R_1, R_2,$ and $f$ are values at $x^{S(1)}$.

At the second stationary point $x^{S(2)}$, $\lambda < 0$ since $a \cdot n = -1$ there.

In a similar fashion, one can show that

$$\left(4.31\right) \quad I(k) \left|_{x^{S(2)}} \right. \sim (2\pi i/k)(R_1 R_2)^{1/2} f \exp \left[ -ika \cdot x^{S(2)} \right]$$

Eqs. (4.30) and (4.31) give the two principal contributions to $I(k)$; thus, by adding them together,

$$\left(4.32\right) \quad I(k) \sim (2\pi i/k) \left\{ -(R_1 R_2)^{1/2} f \exp \left[ -ika \cdot x^{S(1)} \right] \left|_{x^{S(1)}} \right. \right. \\
+ (R_1 R_2)^{1/2} f \exp \left[ -ika \cdot x^{S(2)} \right] \left|_{x^{S(2)}} \right. \right\}$$

This asymptotic formula plays an important role in wave scattering theory, it finds a variety of applications as will be seen later in this thesis. A simple application of which is given in the next section. When using this equation, some remarks deserve special attention.
First, one must be careful in identifying the two stationary points $\mathbf{x}^{s(1)}$ and $\mathbf{x}^{s(2)}$. If we imagine a plane moving in the direction of $\mathbf{a}$, then $\mathbf{x}^{s(1)}$ is the first point being struck by the plane and $\mathbf{x}^{s(2)}$ is the one being struck later (see Fig. 4.3). Second, in deriving (4.32), we have assumed that $\mathbf{n}$ is the inward normal and hence both $R_1$ and $R_2$ are positive according to our convention (Appendix A). Had we chosen $\mathbf{n}$ as the outward normal, both $R_1$ and $R_2$ would have been negative. But, since $R_1$ and $R_2$ appear in (4.32) in a product form, the signs of $R_1$ and $R_2$ do not matter at all, provided the surface $S$ is convex.

4.4 Kirchhoff Approximation for Scalar Waves

We now consider a simple application of the asymptotic formula (4.32) derived in the preceding section. Suppose a high frequency time-harmonic plane scalar wave $u^0$ with amplitude $A$,

$$u^0(x) = A \exp[i k a^0 \cdot x]$$

is scattered by a Neumann scatterer bounded by a smooth convex surface $B$ ($\partial u / \partial n = 0$ on $B$). Physically, this scatterer corresponds to an acoustically hard obstacle. In (4.33), $k$ is the wave number and $a^0$ is a unit vector defining the incident wave direction. Upon incidence of $u^0$ on the surface $B$, the obstacle gives rise to a scattered wave $u^1$. The far-field scattered wave is of particularly interested to us. To find the far-field scattered wave, we first use the Neumann boundary condition $\partial u / \partial n = 0$ in Eq. (3.15), obtaining
(4.34) \[ f(k) = -\frac{1}{4\pi} \int_B i k \cdot n(x) [u^0(x) + u^1(x)] \exp(-ik \cdot x) \, dB(x) \]

where use has been made of \( u(x) = u^0(x) + u^1(x) \). The Kirchhoff (or physical optics) approximation assumes that the scattering surface \( B \) can be divided, sharply, into two parts; one being the "lit" side \( B_L \) and the other one "dark" side \( B_D \). On the lit side, the unknown scattered field \( u^1 \) in (4.34) is assumed to be equal to the incident field \( u^0 \), while on the dark side \( u^1 \) is assumed to be zero. This approximation was motivated by the experience with geometrical optics, in which the wave frequency can be as high as billion MHz [31]. In the high frequency limit, one can think of the incident wave as illuminating a portion of \( B \) (lit side \( B_L \)) while leaving a deep shadow zone in the remaining part (dark side \( B_D \)) of \( B \). Making use of this approximation and (4.33) in Eq. (4.34), one obtains for the back scattering case (i.e., \( a^0 = -\gamma \), recall Fig. 3.3)

(4.35) \[ f(k) = -\frac{A}{2\pi} \int_{B_L} i k \gamma \cdot n(x) \exp[-i(2k)\gamma \cdot x] \, dB(x) \]

This integral has precisely the same form as the one given in Eq. (4.9); hence it can be evaluated by using the asymptotic formula (4.32), giving

(4.36) \[ f(k) = \frac{A}{2} (R_1 R_2)^{1/2} \exp[2ika^0 \cdot x^s] \]
Here, $\mathbf{x}_s$ is the position vector of the specular point, and $R_1$ and $R_2$ are the principal radii of curvature of $B$ at $\mathbf{x}_s$. Substituting this $f(k)$ function back into Eq. (3.14) then yields immediately the backscattered far-field wave:

$$u^1(y) = \frac{A}{2} \frac{1}{(R_1R_2)^{1/2}} e^{i k y} \exp\left(2i k a_0 \cdot \mathbf{x}_s\right)$$

If the coordinate origin is chosen right at the specular point ($\mathbf{x}_s = 0$) the last exponential factor in (4.37) simply becomes one, and the preceding equation reduces to

$$u^1(y) = \frac{A}{2} \frac{1}{(R_1R_2)^{1/2}} e^{i k y}$$

This result shows that the backscattered wave $u^1(y)$ is completely determined by the local geometry of the scattering surface near the specular point.

In Chapter 6, this same problem will be solved by using the ray method, in which an exactly identical result will be obtained. Thus the result of Kirchhoff approximation—which is suggested by our physical intuition—can be mathematically interpreted as the leading term in an asymptotic series.
4.5 Kirchhoff Approximation for Elastodynamic Waves

In this section we shall describe the Kirchhoff approximation as applied to the problem of an elastodynamic wave scattered by a smooth closed convex (traction free) void. As with the scalar wave problem, we shall confine ourselves to the backscattered far-field case in order to avoid unnecessary complexity, since our primary purpose is to illustrate the usage of the asymptotic formula (4.32).

The approach adopted here will be essentially parallel to that employed in the preceding section. Thus we start with the \( \mathbf{f} \)-vector defined previously in Eq. (3.29). By use of \( \omega = k \alpha c_\alpha \) (\( \alpha = 1, 2; \) no sum on \( \alpha \)) and the traction free boundary condition, Eq. (3.29) yields

\[
(4.39) \quad f_1(k) = -\frac{ik\alpha}{4\pi pc_\alpha} \int_B C_{ijkl} l_j(u^0_1 + u^{sc}_1)n_k \exp(-ik\alpha \cdot x) \, dB(x)
\]

where use has been made of \( u = u^0_1 + u^{sc}_1 \), \( u^{sc}_1 \) being the scattered field. As one shall see later, the scattered field \( u^{sc}_1 \) generally consists of two waves, a P-wave \( u^1 \) and an S-wave \( u^2 \). Under certain circumstances, however, the scattered field may consist of only one wave.

The Kirchhoff approximation for scalar waves remains basically unchanged for elastodynamic waves, with only one minor modification. Besides assuming that the scattering surface \( B \) can be divided into lit \( (B_L) \) and dark \( (B_D) \) zones as in the scalar wave case, it further assumes that the scattering surface behaves locally like a perfect planar interface. To be more precise, let us consider an arbitrary point \( x^B \) on
the scattering surface $B$ as shown in Fig. 4.4. This figure shows that a plane displacement wave $u^0(x)$ is impinging on a smooth convex void bounded by $B$. Insofar as the displacement at the point $x^B$ is concerned, the Kirchhoff approximation allows one to ignore the existence of the real physical surface $B$ and replace it with an imagined infinite plane tangent to $B$ at the point $x^B$ (Fig. 4.4). The solution to this canonical problem is well known [12,161] and is given by

\begin{equation}
(4.40) \quad u^{sc}(x^B) = R(x^B) u^0(x^B)
\end{equation}

In this equation, $R(x^B)$ is the reflection coefficient which, in general, is point dependent; i.e., the reflection coefficient $R$ varies from point to point in most cases. Suppose now $u^0$ is a time-harmonic plane incident wave with amplitude $A$, defined by

\begin{equation}
(4.41) \quad u^0(x) = A D \exp(i k \alpha^0 \cdot x)
\end{equation}

where $D$ and $\alpha^0$ are unit vectors defining the directions of motion and propagation, respectively. Notice that for an incident longitudinal wave $D \cdot \alpha^0 = 1$, whereas for an incident transverse wave $D \cdot \alpha^0 = 0$. By Eqs. (4.40) and (4.41),

\begin{equation}
(4.42) \quad u^{sc}(x^B) = R A D \exp(i k \alpha^0 \cdot x)
\end{equation}

Substituting (4.41) and (4.42) in (4.39) and remembering that $u = 0$ on
Fig. 4.4. An arbitrary point $x^B$ on the scattering surface $B$ and the tangent plane to $B$ at $x^B$. 
the dark side $B_D$, one obtains

$$ f_i(k_\alpha) = -\frac{ik_A}{4\pi oc^2} \int_{B_L} (1 + R)C_{ijkl}Y_{ij}D_{ik}n_k \exp[ik_\alpha(a^0 - Y) \cdot x] \ dB(x) $$

This integral, though it looks different, is actually of the same form as the integral defined in Eq. (4.9). Hence it can be evaluated, again, by using the asymptotic formula (4.32). According to this formula, the dominant contribution to the above $f$-vector is determined entirely by the stationary point (i.e., the specular point) $x^S$. Also, as stated in the beginning of this section, we shall restrict ourselves to the back scattering case. In this special case $a^0 = -Y$ and, regardless of the type of incident wave, the reflection coefficient $R$ at $x^S$ is always equal to one. Consequently, at the stationary point $x^S$ (where $n = Y$), the "amplitude function" of the integrand of (4.43) becomes

$$ (1+R)C_{ijkl}Y_{ij}D_{ik}n_k = 2(\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk})Y_{ij}D_{ik}n_k $$

$$ = 2(\lambda D_i n_Y + \mu Y_D n_i + \mu Y_n D_i) $$

$$ = \begin{cases} 
2(\lambda + 2\mu)D_i & \text{for incident L-wave (} \alpha = 1 \text{)} \\
2\mu D_i & \text{for incident T-wave (} \alpha = 2 \text{)} 
\end{cases} $$

With the aid of this result, the asymptotic formula (4.32) gives for both $\alpha = 1$ and 2,
(4.45) \( f_i (k_{\alpha}) = \frac{A}{2} (R_1 R_2)^{1/2} D_i \exp(2k_{\alpha}^0 \cdot x^S) \) for \( \alpha = 1 \) and 2

where \( R_1 \) and \( R_2 \) are principal radii of curvature of the scattering surface at the specular point \( x^S \). It is of interest to note that the foregoing equation is similar to what we have just found in Section 4.4 [cf. Eq. (4.36)]. Substitution of the above \( f \)-vector in Eqs. (3.26) and (3.27) then gives

\[
A_m^0(Y) = \left\{ \begin{array}{ll}
\frac{A}{2} (R_1 R_2)^{1/2} D_m \exp(2k_{\alpha}^0 \cdot x^S) & \text{for } \alpha = 1 \\
0 & \text{for } \alpha = 2
\end{array} \right.
\]

(4.46)

\[
B_m^0(Y) = \left\{ \begin{array}{ll}
0 & \text{for } \alpha = 1 \\
\frac{A}{2} (R_1 R_2)^{1/2} D_m \exp(2k_{\alpha}^0 \cdot x^S) & \text{for } \alpha = 2
\end{array} \right.
\]

(4.47)

Again, as in Section 4.4, if the coordinate origin is chosen at the specular point, the exponential factors appearing in (4.46) and (4.47) all become unity (since \( x^S = 0 \)). Hence, in this case, it follows immediately from (3.25) that

\[
(4.48) \quad u_{SC}^0(y) = \frac{A}{2} (R_1 R_2)^{1/2} D \times \left\{ \begin{array}{ll}
\frac{\exp(ik_1y)}{y} & \text{for an incident L-wave} \\
\frac{\exp(ik_2y)}{y} & \text{for an incident T-wave}
\end{array} \right.
\]
This result shows that, in the back scattering case, an incident L-wave is reflected as an L-wave, and an incident T-wave is reflected as a T-wave; no mode conversion occurs. Also, as in the scalar wave case, the scattered field is completely determined by the local geometry of the scattering surface at the specular point.
5. RAY METHODS

5.1 Introduction to Ray Methods

Rays were originally defined in optics as the paths along which light travels. In geometrical optics, there were only three kinds of rays: direct, reflected, and refracted rays. In homogeneous materials the three kinds of rays all travel along straight paths, and the propagation of these rays is governed by the laws of reflection and refraction.

Based upon these ancient and basic geometrical laws, many optical instruments had been successfully invented. However, new optical phenomena were discovered which were not accounted for by geometrical optics. Diffraction, for example, is the occurrence of light where there should be none according to geometrical optics. It was not until the late-nineteenth century that the scientific world realized that light was really an electromagnetic phenomenon governed by Maxwell's equations. But then why do the optical instruments, designed solely based upon the old geometrical optics, all work so well? If one observes that the only property distinguishing light from other electromagnetic waves is the short wavelength (i.e., high frequency), one might infer that the geometrical (ray) optics is asymptotically correct in the high frequency limit. As a matter of fact, ray methods were originally suggested by the observation of this asymptotic nature.

A significant advance in ray optics was its extension to diffraction problems. This development, due mainly to Keller [117] and his co-workers, greatly increased the domain of application of ray
methods. In Keller's geometrical (ray) theory of diffraction [117], it was postulated that, along with the usual rays of geometrical optics, there exists a class of diffracted rays which accounts for the phenomenon of diffraction. These rays are produced when the incident rays hit edges, vertices (corners) of scattering surfaces, or when the incident rays impinge tangentially on smoothly curved boundaries. Some of the diffracted rays penetrate into the shadow zones and account for the existence of fields there; other rays modify the field in the illuminated regions. The initial values of the fields on diffracted, reflected, and refracted rays are obtained by multiplying the incident field at the point of incidence by appropriate diffraction, reflection, and refraction coefficients. According to Keller's theory, these coefficients are determined entirely by the local properties of the media and the boundary in the immediate neighborhood of the point of incidence. Further, in regions away from the scattering surfaces, the diffracted rays are assumed to behave just like the ordinary reflected and refracted rays of geometrical optics. Since only the local properties near the point of incidence are important, these coefficients may be determined from the solutions of some simpler boundary value problems having these local properties. Such problems are called canonical problems. A great number of these canonical problems have received extensive study and their solutions are now well known [26]. Alternatively, experimental measurements on canonical configurations can yield these coefficients.

In Keller's theory disturbances are assumed to propagate along
rays, and the interaction of such rays with inhomogeneities (scatterers) obeys some simple geometrical rules. By using these rules, rays can be traced and the signals that are carried by all rays passing through a point of observation can be superimposed to yield the complete wave field.

Mathematically, ray methods are an extension to partial differential equations of the classical WKB method [9,99] for ordinary differential equations. They usually yield a series solution which is asymptotic to the exact one. Physically, they extend the basic concepts of geometrical optics to a wider class of optical wave phenomena and then extend these results to other wave phenomena, such as acoustic, electromagnetic and elastodynamic waves.

Keller’s geometrical theory of diffraction was later extended by Achenbach et al. to treat many important elastodynamic wave problems [12]. However, almost all of Achenbach and his co-workers’ works were concentrated on crack problems. Our study, in contrast, will be devoted to volumetric type of flaw problems. First, owing to its simplicity, the scalar wave case will be discussed in order to convey the the essential ideas of ray theory. Next, the more complicated elastodynamic waves will be considered. Finally, to conclude this chapter, we will investigate one of the most important problems in ray theory, viz., the calculation of the curvatures of reflected and refracted wavefronts.
5.2 Scalar Waves

5.2.1 Ray series and eikonal function

Suppose that a time harmonic wave $U^0$,

\begin{equation}
U^0(x,t) = u^0(x)\exp(-i\omega t)
\end{equation}

is incident on a finite, closed, convex body with boundary $B$. Upon incidence $U^0$ is scattered, giving rise to a new wave $U^1$ outside the scatterer:

\begin{equation}
U^1(x,t) = u^1(x)\exp(-i\omega t)
\end{equation}

In the foregoing two equations, $x$ denotes the spatial coordinate vector, $t$ is time, and $\omega$ is angular frequency.

By linear acoustics, the space-dependent part of the total field $u(x)$,

\begin{equation}
u(x) = u^0(x) + u^1(x)
\end{equation}

and $u^0$, $u^1$ all satisfy, in $V_{\text{ext}}$, the Helmholtz equation

\begin{equation}
\nabla^2 u + (\omega/c)^2 u = 0
\end{equation}

where $c$ being the acoustic wave speed.

We assume that $u$ has an asymptotic expansion of the form
(5.5) \[ u(x) = \exp[i\omega S(x)] \sum_{m=0}^{\infty} (i\omega)^{-m} u_m(x) \]

where \( S(x) \) is the eikonal (phase) function. The surface \( S = \text{constant} \) defines a wavefront, whose orthogonal trajectories are rays.

Inserting (5.5) in (5.4) and using \( (\omega/c)^2 = -(i\omega/c)^2 \), one obtains after some manipulations,

(5.6) \[ \sum_{m=0}^{\infty} (i\omega)^{-m} \left[ (i\omega)^2 N(u_m) + i\omega M(u_m) + L(u_m) \right] = 0 \]

The three operators \( N \), \( M \), and \( L \) appearing in (5.6) are defined by

(5.6a) \( N(u_m) = (\nabla S \cdot \nabla S - c^{-2}) u_m \)

(5.6b) \( M(u_m) = 2 \nabla S \cdot \nabla u_m + (\nabla^2 S) u_m \)

(5.6c) \( L(u_m) = \nabla^2 u_m \)

Since (5.6) holds for any values of \( \omega \), the coefficient of each power of \( \omega \) must vanish. Hence

(5.7) \( N(u_m) + M(u_{m-1}) + L(u_{m-2}) = 0, \quad m = 0,1,2,\ldots \)

In (5.7), it is understood that \( u_{-1} = u_{-2} = 0 \) identically.

Setting \( m = 0 \) in Eq. (5.7) results
(5.8) \[ N(u_0) = (\nabla S \cdot \nabla S - c^{-2})u_0 = 0 \]

Thus

(5.9) \[ \nabla S \cdot \nabla S = 1/c^2 \]

This is just the eikonal equation in geometric optics, which governs the propagation of wavefronts.

Since $\nabla S$ is orthogonal to the surface $S = \text{constant}$, it coincides with the ray direction. If we denote the unit vector along the ray path by $\hat{a}$, and the distance measured from some reference point by $a$. Then (5.9) can be written as

(5.10) \[ \hat{a} \cdot \nabla S = 1/c \]

But the left hand side of this equation is just the directional derivative of $S$ in $\hat{a}$ direction, hence (5.10) is in fact an ordinary differential equation along the ray,

(5.11) \[ dS/da = 1/c \]

Its solution is trivial:

(5.12) \[ S(a) = S(a_0) + (a-a_0)/c \]
where $S(a_o)$ is the phase at some initial point whose distance from the reference point along the ray is $a_o$.

Now as a consequence of the eikonal Eq. (5.9), $N(u_m)$ is identically zero. Accordingly, (5.7) reduces to

\begin{equation}
(5.13) \quad M(u_m) + L(u_{m-1}) = 0, \quad m = 0, 1, 2, ...
\end{equation}

or, by Eqs. (5.6b) and (5.6c),

\begin{equation}
(5.14) \quad 2\nabla S \cdot \nabla u_m + (\nabla^2 S)u_m = -\nabla^2 u_{m-1}
\end{equation}

This important relation is known as the transport equation. It governs the transportation of energy along the ray [102].

5.2.2 Solution of transport equation

By recalling the discussion following Eq. (5.9), we may rewrite the transport equation (5.13) as

\begin{equation}
(5.15) \quad \frac{du_m}{da} + (c\nabla^2 S/2)u_m = (-c/2)\nabla^2 u_{m-1}
\end{equation}

For ease of notations in the rest of this section, we shall replace $u_m$ with $u$ and define

\begin{equation}
(5.16) \quad f = (-c/2)\nabla^2 u_{m-1}
\end{equation}

In terms of these shorthand notations, (5.15) becomes
(5.17) \[ \frac{du}{da} + (c\nabla^2 S/2)u = f \]

This is a linear first order ordinary differential equation, an integrating factor of which is

(5.18) \[ p = \exp \left[ \int_{a_0}^{a} (c\nabla^2 S/2) dy \right] \]

By using this integrating factor, it is easy to find that

(5.19) \[ u(a) = u(a_0) \exp \left[ - \int_{a_0}^{a} (c\nabla^2 S/2) dx \right] + \int_{a_0}^{a} \left\{ f(y) \exp \left[ - \int_{y}^{a} (c\nabla^2 S/2) dx \right] \right\} dy \]

The exponential factor in (5.19) has an interesting geometrical interpretation. To see this, note that

(5.20) \[ c\nabla^2 S/2 = (c/2)\nabla \cdot (\nabla S) = (1/2)\nabla \cdot \n \]

The last equality follows from the fact that \( c = \) constant in a homogeneous medium. Now, consider an infinitesimal volume element \( V \) about the "axial ray" (Fig. 5.1). The lateral surface of \( V \) is composed of "paraxial" rays and the bases are sections of wavefronts normal to the axial ray. If \( B \) denotes the boundary surface of \( V \) and \( n \) denotes
Fig. 5.1. An infinitesimal volume element $V$ "centered" about the axial ray, the lateral surface of $V$ is composed of paraxial rays.
the outward normal of \( B \), then by the invariant definition of divergence,

\[
\nabla \cdot \mathbf{a} = \frac{1}{V} \lim_{V \to 0} \left[ \int_{B} \mathbf{a} \cdot \mathbf{n} \ dB \right] = \frac{1}{A(a) \delta a} \lim_{\delta a \to 0} \left[ \frac{A(a+\delta a) - A(a)}{\delta a} \right] = \frac{1}{A(a)} \frac{dA}{da} = \frac{d}{da} \left[ \ln A(a) \right]
\]

Using this result and (5.20), we obtain

\[
\exp \left[ - \int_{\gamma} \left( c \nabla S/2 \right) dx \right] = \left[ \frac{dA(y)}{dA(a)} \right]^{1/2}
\]

Since we are considering an infinitesimal ray tube, the areas \( A(y) \) and \( A(a) \) are really infinitesimal quantities. This is why we wrote \( dA(y) \) and \( dA(a) \) in the above equation.

Using (5.19) and (5.22) and resuming our unabridged notations, we finally obtain

\[
u_{m}(a) = u_{m}(a_{0}) \left[ \frac{dA(a_{0})}{dA(a)} \right]^{1/2}
\]

\[
\frac{c}{2} \int_{a_{0}}^{a} \left[ \frac{dA(y)}{dA(a)} \right]^{1/2} \nabla^{2} u_{m-1}(y) dy
\]

Since \( dA \) is the area on a wavefront intersected by a ray tube,
\( \frac{dA(a_0)}{dA(a)} \) is a measurement of how a ray tube opens up or closes up.

When \( m = 0 \) (zeroth order solution), the integral in (5.23) is absent since \( u_{-1} \) is identically zero, so \( u_0 \) is given by the first term. This solution shows that \( u_0 \) varies along a ray inversely as the square root of the cross sectional area of a ray tube. Therefore \( (u_0)^2 \) multiplied by this area is constant along each ray, which physically expresses the conservation of energy within such a tube.

There are several ways of expressing the area ratio \( \frac{dA(a_0)}{dA(a)} \) in the literature:

\[
\frac{dA(a_0)}{dA(a)} = \frac{J(a)}{J(a_0)} = \frac{G(a)}{G(a_0)}
\]

where \( J \) is the Jacobian of the mapping via rays and \( G \) is the Gaussian curvature. The first equality in (5.24) follows directly from the definition of the Jacobian, and the second equality can be established with the aid of Fig. 5.2 as follows. If the two principal radii of curvature of a wavefront at point \( a_0 \) are \( r_1 \) and \( r_2 \) (note that the signs of \( r_1 \) and \( r_2 \) depend on how the normal vector to wavefront is defined), then (see Fig. 5.2)

\[
G(a) = \frac{|r_1||r_2|}{G(a_0)} \frac{1}{(|r_1| + a - a_0)(|r_2| + a - a_0)}
\]

Further, from Fig. 5.2, it is clear that
Fig. 5.2. The principal radii $r_1$ and $r_2$ of a wavefront passing the reference point $a_0$ on an axial ray.
Hence, the second equality in (5.24) follows immediately. Usually, it is convenient to choose \( a_0 = 0 \). In this case (5.26) reduces to

\[
\frac{\text{d}A(a_0)}{\text{d}A(a)} = \frac{|r_1||r_2|}{(|r_1| + a - a_0)(|r_2| + a - a_0)}
\]

(5.27)

5.3 Elastodynamic Waves

5.3.1 Ray series and eikonal function

The linearized equations of motion for a homogeneous elastic medium are

\[
\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + \mu) \nabla(\nabla \cdot u) + \mu \nabla^2 u
\]

(5.28)

As in the case of scalar waves, we consider a time harmonic solution of (5.28) and write it in the following form:

\[
u(x,t) = \exp \left\{ i \omega [S(x) - t] \right\} \sum_{m=0}^{\infty} (i \omega)^{-m} u_m(x)
\]

(5.29)

Inserting this equation in (5.28) one obtains (see Appendix B) an equation analogous to (5.6):
But now the operators \( L, M, \) and \( N \) are more complicated, they are defined by

\[
\begin{align*}
(5.31a) \quad N(u_m) &= -\rho u_m + (\lambda + \mu)(u_m \cdot \nabla)\nabla S + \mu(\nabla S)^2 u_m \\
(5.31b) \quad M(u_m) &= (\lambda + \mu)(\nabla u_m) \cdot \nabla S + \nabla (u_m \cdot \nabla S) + \\
&\quad 2\mu(\nabla S \cdot \nabla u_m) + \mu(\nabla^2 S) u_m \\
(5.31c) \quad L(u_m) &= (\lambda + \mu) \nabla (\nabla u_m) + \mu \nabla^2 u_m
\end{align*}
\]

It is clear from the above three expressions that \( L, M, N \) are linear differential operators. This linearity should be always observed.

Now, since Eq. (5.30) holds for all values of \( \omega \), the coefficients of each power of \( \omega \) must vanish. Hence

\[
(5.32) \quad N(u_m) + N(u_{m-1}) + L(u_{m-2}) = 0 \quad (m = 0,1,2,\ldots)
\]

In the above equation, just like the scalar wave case, it is understood that \( u_{-1} = u_{-2} = 0 \) identically.

For \( m = 0 \), (5.22) reduces to \( N(u_0) = 0 \), that is

\[
(5.33) \quad -\rho u_0 + (\lambda + \mu)(u_0 \cdot \nabla S) + \mu(\nabla S \cdot \nabla S) u_0 = 0
\]
By taking scalar and vector products of this equation with $\nabla S$ subsequently, the following two equations are obtained:

\begin{align*}
(5.34a) & \quad \left( c_1^2 \nabla S \cdot \nabla S - 1 \right) (u_0 \cdot \nabla S) = 0 \\
(5.34b) & \quad \left( c_2^2 \nabla S \cdot \nabla S - 1 \right) (u_0 \times \nabla S) = 0
\end{align*}

where

\begin{equation}
(5.35) \quad c_1 = \left[ \frac{\lambda + 2\mu}{\rho} \right]^{1/2}, \quad c_2 = \left[ \frac{\mu}{\rho} \right]^{1/2}
\end{equation}

If we require that neither $u_0$ nor $\nabla S$ be zero, then at most one of the product $u_0 \cdot \nabla S$ and $u_0 \times \nabla S$ can be zero. Also, since $c_1 \neq c_2$, at most one of the bracketed expressions in (5.34) can be zero. As a consequence, the system of equations (5.34a) and (5.34b) has only two solutions. Following Cerveny and Ravindra [11] we shall call these two solutions the P-wave and S-wave solutions.

\begin{align*}
(5.36) & \quad \text{I (P-wave):} \quad \nabla S \cdot \nabla S = \left( \frac{1}{c_1} \right)^2, \quad u_0 \times \nabla S = 0 \\
(5.37) & \quad \text{II (S-wave):} \quad \nabla S \cdot \nabla S = \left( \frac{1}{c_2} \right)^2, \quad u_0 \cdot \nabla S = 0
\end{align*}

Each of the first equations in (5.36) and (5.37) are in the same
form as the eikonal equation (5.9) which was derived for the scalar wave problems. However, there is only one eikonal function in scalar wave problems; now there are two eikonal functions, each governing the propagation of one wavefront. As can be seen from (5.36) and (5.37), the two wavefronts propagate at different speeds \( c_1 \) and \( c_2 \).

For solution I, the displacement \( u_0 \) is parallel to the direction of wave propagation since \( u_0 \times \nabla_{\mathbf{S}} = 0 \). Thus, physically, it corresponds to a pressure (primary) wave. For solution II, on the other hand, the displacement \( u_0 \) is perpendicular to the direction of wave propagation. Hence, it corresponds to a shear (secondary) wave.

Now, consider a certain ray path \( C \). Let this path \( C \) be parametrized by an arc-length parameter \( a \). Also, let the unit vector tangent to \( C \) at an arbitrary point (with coordinate \( a \)) be denoted by \( \mathbf{a}(a) \). Then

\[
(5.38) \quad \mathbf{a} = c \nabla_{\mathbf{S}}
\]

where \( c \) may be \( c_1 \) or \( c_2 \) depending on \( P \)- or \( S \)-wave is considered. The derivative of this unit tangent vector with respect to the arc-length parameter \( a \) is

\[
(5.39) \quad \frac{da}{da} = (\mathbf{a} \cdot \nabla) \mathbf{a} = c (\nabla_{\mathbf{S}} \cdot \mathbf{v}) (c \nabla_{\mathbf{S}})
\]

For a homogeneous isotropic material, which we shall always assume, \( c \) is a constant. It then follows from the preceding equation that
This result means the unit tangent vector $\mathbf{a}$ does not change its direction along a ray. Thus, rays are straight lines in homogeneous isotropic media. Now, since rays are straight lines in such media, it proves useful to introduce three orthonormal vectors $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$. The unit vector $\mathbf{a}_3$ is directed in the ray direction and $\mathbf{a}_2, \mathbf{a}_3$ are any two orthonormal vectors perpendicular to $\mathbf{a}_3$. Note that both $\mathbf{a}_1$ and $\mathbf{a}_2$ are tangent to the wavefront. In the sequel, we shall often choose $\mathbf{a}_1$ and $\mathbf{a}_2$ such that they coincide with the two principal directions of the wavefront.

The coefficients $u_m$ of the ray series in (5.29) can be resolved into three orthogonal components by

$$u_m = u_{m1} \mathbf{a}_1 + u_{m2} \mathbf{a}_2 + u_{m3} \mathbf{a}_3$$

For a P-wave, we shall call $u_{m3}$ the major component and $(u_{m1}, u_{m2})$ the minor components, since for a pure P-wave (longitudinal wave) the component should have only the $\mathbf{a}_3$-component $u_{m3}$. Similarly, for an S-wave, $(u_{m1}, u_{m2})$ will be called the major components and $u_{m3}$ the minor component.

5.3.2 Coefficients of ray series: P-wave

For a P-wave, the phase function $S$ satisfies the eikonal equation (5.36). The determination of the minor components $(u_{m1}, u_{m2})$ are much
easier than the major component \( u_{m3} \). To find \((u_{m1}, u_{m2})\), we take scalar product of (5.31a) with \( a_{\alpha} \) (\( \alpha = 1,2 \)) and obtain

\[
(5.42) \quad a_{\alpha} \cdot N(u_m) = -\rho a_{\alpha} \cdot u_m + \mu(\nabla S)^2 a_{\alpha} \cdot u_m
\]

In the above, we have used \( a_{\alpha} \cdot \nabla S = 0 \). Now, by Eq. (5.41), \( a_{\alpha} \cdot u_m = u_{m\alpha} \). Also, \((\nabla S)^2 = \left(1/c_1^2\right)^2\), thus

\[
(5.43) \quad a_{\alpha} \cdot N(u_m) = \left\{ -\rho + \left[ \frac{\mu}{c_1^2} \right] \right\} u_{m\alpha}
\]

On substitution of (5.32), the above equation then yields the minor components

\[
(5.44) \quad u_{m\alpha} = \left[ \frac{c_1^2}{\lambda + \mu} \right] a_{\alpha} \cdot [M(u_{m-1}) + L(u_{m-2})] \quad (\alpha = 1,2)
\]

If we remember, \( M \) and \( L \) are linear differential operators, hence Eq. (5.44) implies that the minor components \((u_{m1}, u_{m2})\) can be obtained simply by differentiating \( u_{m-1} \) and \( u_{m-2} \) without needing the knowledge of any initial conditions. Later on, we shall see that the determination of major component \( u_{m3} \) requires the knowledge of some initial conditions of \( u_{m3} \). Note that since \( u_{-1} = u_{-2} = 0 \), \( u_{0\alpha} (\alpha = 1,2) \) vanish identically. This means, for the zeroth order solution \((m = 0)\), the P-wave is is purely longitudinal.
Since $a_1$ and $a_2$ are perpendicular to the ray direction $a_3$, it is convenient to introduce a vector quantity

$$u_m^\perp = u_{m1} a_1 + u_{m2} a_2$$

It then follows from (5.44) that

$$u_m^\perp = \left( \frac{c_1^2}{\lambda + \mu} \right) [N(u_{m-1}) + L(u_{m-2})]$$

Now consider the determination of the major component $u_{m3}$. First, we introduce a quantity $u_m'$ defined by

$$u_m' \perp S = u_{m3}' a_3 \quad \text{or} \quad u_m' = c_1 u_{m3}$$

Then,

$$u_m = u_m' \perp S + u_m^\perp$$

It can be seen from Eq. (5.47) that knowing $u_m'$ is as good as knowing $u_{m3}$, the only difference being that $u_{m3}$ tells us the major component directly whereas $u_m'$ indirectly.

From the definition of $N$ it can be shown that

$$N(u_m'). \perp S = 0$$
Combining this result with (5.32), we obtain

\begin{equation}
\mathcal{M}(u_m) \cdot \nabla \mathcal{S} = - \mathcal{L}(u_{m-1}) \cdot \mathcal{S} \quad (m = 0, 1, 2, \ldots)
\end{equation}

On substitution of (5.48) in (5.50), observing the linearities of \( \mathcal{M} \), one obtains

\begin{equation}
\mathcal{M}(u'_m \cdot \nabla \mathcal{S}) \cdot \mathcal{V} = -[\mathcal{L}(u_{m-1}) + \mathcal{M}(u'_{m-1})] \cdot \mathcal{S}
\end{equation}

The left hand side of this equation can be shown to be equal to

\[
\frac{2p}{c_1} \frac{du'_m}{da} + \rho (\nabla^2 \mathcal{S}) u'_m
\]

Placing this result back in (5.51) results in

\begin{equation}
\frac{du'_m}{da} + \frac{c_1}{2} (\nabla^2 \mathcal{S}) u'_m = - \frac{c_1}{2p} [\mathcal{L}(u_{m-1}) + \mathcal{M}(u'_{m-1})] \cdot \nabla \mathcal{S}
\end{equation}

This equation is of the same form as the transport Eq. (5.17) for scalar wave problems, which has already been solved in the previous section. Thus, one can write down immediately the solution for \( u'_m \). Finally, by (5.48), the solution for the major component \( u_{m3} \) can be obtained. The final result is
As noted earlier, the determination of the major component \( u_{m3} \) requires the knowledge of initial condition \( u_{m3}(a_0) \).

### 5.3.3 Coefficients of ray series: S-wave

For an S-wave, the phase function satisfies the eikonal Eq. (5.37). As in the P-wave case, the minor component \( u_{m3} \) will be considered first.

By taking scalar product of (5.31a) with \( a_3 \), one obtains

\[
(5.54) \quad u_{m3} = - \frac{c_2}{\lambda + \mu} a_3 \left\{ \frac{M(u_{m-1})}{u_{m-1}} + \frac{L(u_{m-2})}{u_{m-2}} \right\} \quad (m = 0,1,2,\ldots)
\]

Just like the P-waves, the minor component \( u_{m3} \) is obtained simply by differentiating \( u_{m-1} \) and \( u_{m-2} \). Also observe that \( u_{03} \) vanishes identically, since \( u_{-1} = u_{-2} = 0 \). This means that, for the zeroth order solution \( (m = 0) \), the S-wave is purely transverse.

To obtain the major components \( (u_{m1}, u_{m2}) \), we take the scalar product of (5.31a) with \( a_\alpha \ (\alpha = 1,2) \). It turns out that

\[
(5.55) \quad a_\alpha \cdot N(u_m) = 0
\]
Hence, by Eq. (5.32)

\[ a_{\alpha}M(u_{m}) = - a_{\alpha}L(u_{m-1}) \]  

(5.56)

Now, as before, we write

\[ u_{m} = u_{m}^{1} + u_{m3}a_{3} \]  

(5.57)

where

\[ u_{m}^{1} = u_{m1}a_{1} + u_{m2}a_{2} \]  

(5.58)

Substituting (5.57) in (5.56) and observing the linearities of $M$ and $L$, we obtain

\[ a_{\alpha}M(u_{m}^{1}) = - a_{\alpha}\left\{ L(u_{m-1}) + M(u_{m3}a_{3}) \right\} \]  

(5.59)

Also, after some calculations, it can be shown that

\[ a_{\alpha}M(u_{m}^{1}) = \frac{2\mu}{c_{2}} \frac{du_{m\alpha}}{da} + \mu(\nabla^{2})u_{m\alpha} \]  

(5.60)

Thus, by equating the right hand sides of the preceding two equations,
\[ 5.61 \quad \frac{d u_{m\alpha}}{da} + \frac{c_2}{2} (v^2 S) u_{m\alpha} = - \frac{c_2}{2\mu} a_\alpha \left\{ L(u_{m-1}) + M(u_{m+3} a_3) \right\} \]

Again, this is of the same form as Eq. (5.17); thus the solution of which is readily obtained:

\[ 5.62 \quad u_{m\alpha}(a) = u_{m\alpha}(a_0) \left[ \frac{dA(a_0)}{dA(a)} \right]^{1/2} - \frac{c_2}{2\mu} \left[ \frac{a}{a_0} \left[ \frac{dA(y)}{dA(a)} \right]^{1/2} a_\alpha \left\{ L(u_{m-1}) + M(u_{m+3} a_3) \right\} \right] dy \]

\[ \alpha = 1, 2; \quad m = 0, 1, 2, 3, \ldots \]

5.3.4 Coefficients of ray series for \( m = 0 \) and 1

The coefficients of ray series for both P- and S-waves have been derived in the preceding two sections. The formulas derived there are valid for any order \( (m = 0, 1, 2, \ldots) \). In the subsequent chapters, however, only the zeroth and first order solutions will be considered. For this purpose, some formulas for \( m = 0 \) and 1 are summarized in this section (Table 5.1).

In order to distinguish P- and S-waves, a superscript \( p \) (\( p = 1, 2 \)) will be appended to the coefficients of ray series. \( p = 1 \) and 2 correspond to P- and S-waves, respectively. More specifically, \( u_{ij}^p \) represents the \( a_j \)-component of \( u_i \) for a \( p \)th wave. For example,
$u_{13}^2$ represents the $a_3$-component of $u_1$ for an S-wave.

For the first order solution ($m = 1$), only the minor components are tabulated in Table 5.1, since the major components for $m = 1$ are not used in the future development.

All entries in Table 5.1 are readily obtainable simply by specializing the equations in Sections 5.3.2 and 5.3.3.

Table 5.1. Summary of equations for $m = 0$ and $1$

<table>
<thead>
<tr>
<th>$m$</th>
<th>P-Wave</th>
<th>S-wave</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$u_{03}^1(a) = u_{03}^1(a_0) \left[ \frac{dA(a_0)}{dA(a)} \right]^{1/2}$</td>
<td>$u_{0\alpha}^2(a) = u_{0\alpha}^2(a_0) \left[ \frac{dA(a_0)}{dA(a)} \right]^{1/2}$ $\alpha = 1, 2$</td>
</tr>
<tr>
<td></td>
<td>$u_{01}^1 = u_{02}^1 = 0$</td>
<td>$u_{03}^2 = 0$</td>
</tr>
<tr>
<td>1</td>
<td>$u_{1\alpha}^1 = c_1 a_\alpha \cdot u_{03}^1$, $\alpha = 1, 2$</td>
<td>$u_{13}^2 = -c_2 \nabla \cdot u_0^2$</td>
</tr>
</tbody>
</table>
5.4 Curvatures of Reflected and Refracted Wavefronts

The solutions (5.23), (5.53), and (5.62) of their respective transport equations provide a means for calculating the wave fields along any particular ray. In order to use these results, however, one must know how the ray-tube area associated with the axial ray being considered changes its magnitude. In other words, one has to know how to calculate the area ratio $\frac{dA(a)}{dA(a_0)}$ as one moves out along the axial ray. Since, by Eq. (5.26), this area ratio is related to the principal curvatures of wavefronts, the calculation of this ratio naturally leads one to consider the following problem: An arbitrary smooth three dimensional body $B$ is illuminated by an incident wave, what are the principal curvatures of the reflected and/or refracted wavefronts?

As one shall see later in Chapters 6 and 7, the solution of this problem plays a crucial role in the entire ray theory; without it, all the theories developed so far would be of little use. For "single-wave-speed" problems, such as acoustic and electromagnetic wave scattering problems, the problem posed at the end of the preceding paragraph has been solved by various authors [162-164]. However, for elastodynamic wave problems, this problem remains unsolved. The current section is intended to fill up this gap. To condense the presentation we will focus on analyzing the reflected wave, but in the appropriate place we will point out that all the results of this section are actually applicable to a refracted wave, too.

To start, we begin with Fig. 5.3. According to the law of
reflection (in Chapter 6 we will show that this is a consequence of phase matching) the three unit vectors \( \mathbf{n}_I, \mathbf{n}_R, \) and \( \mathbf{n}_B \) all lie on the plane of incidence \( E_{inc} \). Therefore, at the point \( X_B \) (on \( B \), see Fig. 5.3), it is obvious that

\[
(5.63) \quad \mathbf{n}_B \cdot (\mathbf{n}_I \times \mathbf{n}_R) = 0
\]

Furthermore, if \( c_I \) and \( c_R \) denote the wave speeds of incident and reflected waves, respectively, the law of reflection requires that the \( e_1 \)-components of \( \mathbf{n}_R/c_R \) and \( \mathbf{n}_I/c_I \) be equal. That is

\[
(5.64) \quad e_1 \cdot (\mathbf{n}_R/c_R) = e_1 \cdot (\mathbf{n}_I/c_I)
\]

Now since \( \mathbf{n}_R \) and \( \mathbf{n}_I \) have no "out-of-plane" components, one has (note that \( \mathbf{n}_B = e_3 \), see Fig. 5.3)

\[
(5.65a) \quad \mathbf{n}_R/c_R = [e_1 \cdot (\mathbf{n}_R/c_R)]e_1 + [e_2 \cdot (\mathbf{n}_R/c_R)]n_B
\]

\[
(5.65b) \quad \mathbf{n}_I/c_I = [e_1 \cdot (\mathbf{n}_I/c_I)]e_1 + [e_2 \cdot (\mathbf{n}_I/c_I)]n_B
\]

Subtracting (5.65b) from (5.65a), taking vector cross product of the resulting equation with \( \mathbf{n}_B \), and utilizing Eq. (5.64), one obtains immediately

\[
(5.66) \quad \mathbf{n}_B \times (\mathbf{n}_R/c_R - \mathbf{n}_I/c_I) = 0
\]
Fig. 5.3. Unit vectors $\mathbf{n}_I, \mathbf{n}_R, \mathbf{n}_B$ and the three coordinate systems
$(a_1, a_2, a_3), (b_1, b_2, b_3), (e_1, e_2, e_3)$
It should be noted that Eqs. (5.63) and (5.66) are completely equivalent to the law of reflection and can be regarded as another way for stating the familiar law of reflection. Also, it is important to keep in mind that \( n_I \), \( n_R \), and \( n_B \) are functions of the point of reflection. If one changes the point of reflection \( x_B \) to some other adjacent point \( (x_B + \delta x_B) \in B \), the three unit vectors will undergo small variations \( \delta n_I \), \( \delta n_R \), and \( \delta n_B \), respectively. At point \( x_B + \delta x_B \) the three unit vectors \( n_I \), \( n_R \), and \( n_B \) in Eqs. (5.63) and (5.66) need to be replaced by \( n_I + \delta n_I \), \( n_R + \delta n_R \), and \( n_B + \delta n_B \). Thus at point \( x_B + \delta x_B \) Eqs. (5.63) and (5.66) become, respectively,

\[
\begin{align*}
(5.67a) & \quad \delta n_B \cdot (\delta n_I \times n_R) + n_B \cdot (\delta n_I \times \delta n_R) + n_B \cdot n_I \times \delta n_R = 0 \\
(5.67b) & \quad n_B \times (\delta n_R/c_R - n_I/c_I) + n_B \times (\delta n_I/c_R - \delta n_I/c_I) = 0
\end{align*}
\]

In the preceding two equations, small quantities of second and higher orders have been discarded.

Now since \( n_I \), \( n_R \), and \( n_B \) are unit vectors, it follows that

\[
(5.68) \quad n_I \cdot \delta n_I = n_R \cdot \delta n_R = n_B \cdot \delta n_B = 0
\]

This equation implies that all the three unit vectors, \( \delta n_I \), \( \delta n_R \), and \( \delta n_B \), do not have any "3" -component (see Fig. 5.3). Hence

\[
(5.69a) \quad \delta n_I = (\delta a_1 \cdot \delta n_I) \hat{a}_1 + (\delta a_2 \cdot \delta n_I) \hat{a}_2
\]
(5.69b) \( \delta n_R = (b_1 \cdot \delta n_R) b_1 + (b_2 \cdot \delta n_R) b_2 \)

(5.69c) \( \delta n_B = (e_1 \cdot \delta n_B) e_1 + (e_2 \cdot \delta n_B) e_2 \)

Also, from the geometry of Fig. 5.3, it is clear that

\[
\begin{align*}
(5.70a) \quad & \left\{ a_1 \right\} = \begin{bmatrix} -\cos \Theta_I & -\sin \Theta_I \\ \sin \Theta_I & -\cos \Theta_I \end{bmatrix} \left\{ a_3 \right\} \\
(5.70b) \quad & \left\{ b_1 \right\} = \begin{bmatrix} \cos \Theta_R & -\sin \Theta_R \\ \sin \Theta_R & \cos \Theta_R \end{bmatrix} \left\{ b_3 \right\} \\
(5.70c) \quad & n_I = a_3, \quad n_R = b_3, \quad n_B = e_3, \quad a_2 = b_2 = e_2
\end{align*}
\]

Making use of (5.69) and (5.70) in Eqs. (5.67a,b) one obtains

\[
\begin{align*}
(5.71a) \quad & (b_1 \cdot \delta n_R) + A(a_1 \cdot \delta n_I) - B(e_1 \cdot \delta n_B) = 0 \\
(5.71b) \quad & (b_2 \cdot \delta n_R) - C(a_2 \cdot \delta n_I) - D(e_2 \cdot \delta n_B) = 0
\end{align*}
\]

where

\[
\begin{align*}
(5.71c) \quad & A = \frac{c_R \cos \Theta_I}{c_I \cos \Theta_R}, \quad B = 1 + \lambda, \quad C = \frac{c_R}{c_1}, \quad D = B \cos \Theta_R
\end{align*}
\]

From differential geometry, it can be shown that
(5.72a) \[ a_1 \cdot \delta x_B = h^{I}_{11}(a_1 \cdot \delta x_B) + h^{I}_{12}(a_2 \cdot \delta x_B) \]

(5.72b) \[ a_2 \cdot \delta x_B = h^{I}_{21}(a_1 \cdot \delta x_B) + h^{I}_{22}(a_2 \cdot \delta x_B) \]

(5.72c) \[ b_1 \cdot \delta x_R = h^{R}_{11}(b_1 \cdot \delta x_B) + h^{R}_{12}(b_2 \cdot \delta x_B) \]

(5.72d) \[ b_2 \cdot \delta x_R = h^{R}_{21}(b_1 \cdot \delta x_B) + h^{R}_{22}(b_2 \cdot \delta x_B) \]

(5.72e) \[ e_1 \cdot \delta x_B = h^{B}_{11}(e_1 \cdot \delta x_B) + h^{B}_{12}(e_2 \cdot \delta x_B) \]

(5.72f) \[ e_2 \cdot \delta x_B = h^{B}_{21}(e_1 \cdot \delta x_B) + h^{B}_{22}(e_2 \cdot \delta x_B) \]

where \( h^{I}_{\alpha \beta} \), \( h^{R}_{\alpha \beta} \), and \( h^{B}_{\alpha \beta} \) (\( \alpha, \beta = 1, 2 \)) are, respectively, the curvature tensors of incident wavefront, reflected wavefront, and scattering surface at the point of incidence \( x_B \).

Substituting Eqs. (5.72a-f) in (5.71a,b) yields

\[
\begin{bmatrix}
    h^{R}_{11} & h^{R}_{12} \\
    h^{R}_{21} & h^{R}_{22}
\end{bmatrix}
\begin{bmatrix}
    b_1 \cdot \delta x_B \\
    b_2 \cdot \delta x_B
\end{bmatrix}
= \begin{bmatrix}
    -Ah^{I}_{11} & -Ah^{I}_{12} \\
    Ch^{I}_{21} & Ch^{I}_{22}
\end{bmatrix}
\begin{bmatrix}
    a_1 \cdot \delta x_B \\
    a_2 \cdot \delta x_B
\end{bmatrix}
+ \begin{bmatrix}
    Bh^{B}_{11} & Bh^{B}_{12} \\
    Dh^{B}_{21} & Dh^{B}_{22}
\end{bmatrix}
\begin{bmatrix}
    e_1 \cdot \delta x_B \\
    e_2 \cdot \delta x_B
\end{bmatrix}
\]

Now since \( \delta x_B \) can be any admissible variation (that is, the point \( x_B + \delta x_B \) has to remain on surface B), one may choose \( \delta x_B \) in such a way that it is either parallel to \( e_1 \) or \( e_2 \) (note that \( n_B \cdot \delta x_B = 0 \) since \( \delta x_B \) is
tangent to the surface $B$, see Fig. 5.3). For $\delta x_B$ parallel to $e_1$, i.e.,

$$\delta x_B = \varepsilon e_1$$

(here $\varepsilon$ is a small quantity which needs not be assigned any specific value), Eq. (5.73) yields, with the aid of (5.70),

$$\begin{bmatrix} h_{11}^R & h_{12}^R \\ h_{21}^R & h_{22}^R \end{bmatrix} \begin{bmatrix} I \\ 1 \end{bmatrix} = \begin{bmatrix} -Ah_{11}^I & -Ah_{12}^I \\ Ch_{21}^I & Ch_{22}^I \end{bmatrix} \begin{bmatrix} \cos \Theta_I \\ \cos \Theta_R \end{bmatrix} + \begin{bmatrix} Bh_{11}^B & Bh_{12}^B \\ Dh_{21}^B & Dh_{22}^B \end{bmatrix} \begin{bmatrix} \sec \Theta_R \\ 0 \end{bmatrix}$$

(5.75a)

Similarly, by choosing $\delta x_B = \varepsilon e_2$, one obtains from Eq. (5.73)

$$\begin{bmatrix} h_{11}^R & h_{12}^R \\ h_{21}^R & h_{22}^R \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -Ah_{11}^I & -Ah_{12}^I \\ Ch_{21}^I & Ch_{22}^I \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} Bh_{11}^B & Bh_{12}^B \\ Dh_{21}^B & Dh_{22}^B \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(5.75b)

By utilizing (5.71c), the foregoing two equations (5.75a) and (5.75b) can be combined together to yield

$$\begin{bmatrix} h_{11}^R & h_{12}^R \\ h_{21}^R & h_{22}^R \end{bmatrix} = \frac{c_R}{c_I} \begin{bmatrix} h_{11}^I & \frac{\cos^2 \Theta_I}{\cos^2 \Theta_R} \\ -h_{21}^I & \frac{\cos \Theta_I}{\cos \Theta_R} \end{bmatrix} + \begin{bmatrix} \frac{\cos \Theta_I}{\cos \Theta_R} & -I \\ I & h_{22}^I \end{bmatrix}$$

(5.76)
This is the desired result, which enables one to calculate the curvatures of the reflected wavefront at the point of incidence. It is valid for any type of incident waves and any smooth scattering surface. Furthermore, since the preceding result (5.76) is derived from Eqs. (5.63) and (5.66) and since these two equations still remain valid for refracted waves (law of refraction), Eq. (5.87) can just as well be applied to refracted waves, provided the superscript $R$ is endowed the meaning of "refracted". More specifically, had we drawn the unit vector $b_3$ in Fig. 5.3 in the direction of a refracted ray, we would have still ended up with an equation exactly identical to (5.76).

In scalar (e.g., acoustic) and electromagnetic scattering problems, the reflected wave speed is the same as that of the incident wave, and $\theta_R = \theta_I$. Thus, in these cases, Eq. (5.76) reduces to

\[
\begin{bmatrix}
R_{11} & R_{12} \\
R_{21} & R_{22}
\end{bmatrix} = \begin{bmatrix}
I_{11} & -I_{12} \\
-I_{21} & I_{22}
\end{bmatrix} + 2 \begin{bmatrix}
B_{11} \sec \theta_I & B_{12} \\
B_{21} & B_{22} \cos \theta_I
\end{bmatrix}
\]
which coincides with the known result in literature [162-164]. Now we consider some special cases of the general equation (5.76).

1) Flat reflecting surface: If $B$ is a flat surface then $h_{\alpha\beta}^B = 0$ ($\alpha, \beta = 1, 2$), thus the second matrix on the right hand side of Eq. (5.76) vanishes. This special case finds its typical application in wave propagation problems involving layered media [165].

2) Spherical incident wave (point source): A number of problems in seismology and acoustics, especially the former discipline, contain radiating regions that are so small in comparison with the host media that can be modeled as point sources. In these cases, the first matrix on the right hand side of Eq. (5.76) is diagonal:

\[
\begin{align*}
(5.78) \quad h_{11}^I &= h_{22}^I = 1/R, \\
&= h_{12}^I = h_{21}^I = 0
\end{align*}
\]

where $R$ is the distance between the point of incidence $x_B$ and the point source.

3) Plane incident wave: This is one of the important cases for NDE applications, since in practical NDE measurements the probing disturbance emitted by the transmitter is often modeled as a plane incident wave [166]. In this case, all components of the curvature tensor at every point of the incident wavefront are identically zero. Hence the first matrix on the right hand side of Eq. (5.76) is absent. As a consequence, Eq. (5.76) simplifies to
For a pulse-echo experimental setup, the point of normal incidence (i.e., specular point) is of particular importance. At this point the incident and reflected angles are both equal to zero, hence Eq. (5.79) further reduces to

\[
\begin{bmatrix}
 h_{11}^R & h_{12}^R \\
 h_{21}^R & h_{22}^R
\end{bmatrix}
= \left(1 + \frac{c_R \cos \Theta_R}{c_I \cos \Theta_I}\right) \begin{bmatrix}
 h_{11}^R \sec \Theta_R & h_{12}^R \\
 h_{21}^R & h_{22}^R \cos \Theta_R
\end{bmatrix}
\]

Often, the surface \( B \) is parametrized in such a way that the two surface coordinate lines coincide with the two principal directions of \( B \) at the specular point (according to differential geometry, this is always possible). By choosing this kind of parametrization, the components of the curvature tensor of \( B \) at the specular point become

\[
(5.81) \quad h_{11}^B = \kappa_1, \quad h_{22}^B = \kappa_2, \quad h_{12}^B = h_{21}^B = 0
\]

where \( \kappa_1 \) and \( \kappa_2 \) are principal curvatures of \( B \) at the specular point. Thus, Eq. (5.80) becomes
And, it follows immediately from this equation that

\[
\begin{bmatrix}
h_{11}^R & h_{12}^R \\
h_{21}^R & h_{22}^R
\end{bmatrix} = \begin{bmatrix}
1 + \frac{c_R}{c_I} & 0 \\
0 & 1 + \frac{c_R}{c_I}
\end{bmatrix}
\]

(5.82)

Here \( \kappa_1^R \) and \( \kappa_2^R \) are the principal curvatures of the reflected wavefront at the specular point. Eq. (5.83) shows that the two principal curvature directions of the reflected wavefront at the specular point coincide with those of the reflecting surface at the same point; but the magnitudes of the two principal curvatures associated with the reflected wavefront are equal to \( (1 + c_R/c_I) \) times those associated with the reflecting surface. Since \( (1 + c_R/c_I) \) is always greater than one, the two principal curvatures of the reflected wavefront are always greater than their corresponding principal curvatures of \( B \). A schematic diagram for the reflected wavefront in a small neighborhood of the specular point is shown in Fig. 5.4. Notice that the reflected wavefront looks more "convex" than the reflecting surface.

As one shall see in Chapters 6 and 7, the results of this section, especially Eq. (5.83), are of fundamental importance in the entire ray
theory. The foremost application of them is in conjunction with the determination of the ray-tube area after reflection/refraction, which must be known in advance before one can apply Eqs. (5.23), (5.53), and (5.62) to calculate the coefficients of the ray series.

Fig. 5.4. Shapes of the reflected wavefront and scattering surface in the vicinity of the specular point
6. SCALAR WAVE SCATTERING BY A NEUMANN SCATTERER

The scattering of scalar waves by a Neumann scatterer is considered in the present chapter. In spite of its simplicity, it still demonstrates several important features of ray techniques. This shall aid in the discussion of elastodynamic scattering problems presented in the next chapter.

6.1 Behavior of Waves at Interface

Consider a scalar wave $u^O(x)$ impinging on a Neumann scatterer ($\partial u/\partial n = 0$). As usual, we assume that both the incident and scattered fields have an expression of the form

$$u^p(x) = \exp[i\omega S^p(x)] \sum_{m=0}^{\infty} (i\omega)^{-m} u^p_m(x)$$

where

$$p = \begin{cases} 0 & \text{for incident wave} \\ 1 & \text{for scattered wave} \end{cases}$$

In order that the two phase functions be matched at the boundary $B$, it is necessary that

$$S^0(x) = S^1(x), \quad x \in B$$
Now consider a particular incident ray striking at a point \( O \in B \). In the immediate neighborhood of \( O \), the scattering surface \( B \) can be parametrized by two arc-length parameters \( (a_1, a_2) \). Since (6.2) holds for every points on \( B \), we may differentiate it with respect to these two surface coordinates \( (a_1, a_2) \) obtaining

\[
S_\alpha^0(x) = S_\alpha^1(x) \quad \alpha = 1, 2
\]

where comma (,) means partial differentiation.

By chain rule, this becomes

\[
x_\alpha^1 \nabla S^1 = x_\alpha^0 \nabla S^0
\]

Since \( a_\alpha \) is an arc-length parameter, \( x_\alpha \) is the unit vector \( a_\alpha \) tangent to the \( a_\alpha \)-coordinate line. Hence,

\[
a_\alpha \nabla S^1 = a_\alpha \nabla S^0
\]

It is of interest to note that this is just another form of stating the familiar law of reflection \([27]\). To see this, choose \( a_2 \) in the plane of incidence \( E_{inc} \) and let \( a_1 \) be perpendicular to \( E_{inc} \) (Fig. 6.1). By \( \nabla S^P = a^P/c \), Eq. (6.5) yields

\[
\begin{align*}
(6.6a) \quad a_1 \cdot a_1 &= a_1 \cdot a_0 = 0 \\
(6.6b) \quad a_2 \cdot a_1 &= a_2 \cdot a_0
\end{align*}
\]
Fig. 6.1. Plane of incidence $E_{\text{inc}}$
The last equality in (6.6a) follows from the fact that $a_l$ is perpendicular to $E_{inc}$. Hence, by these two equations, $a_l$ must lie in the plane of incidence $E_{inc}$ and

$$(6.7) \quad \sin \theta_1 = \sin \theta_0 \quad \text{or} \quad \theta_1 = \theta_0$$

Now let us consider the boundary condition, which requires

$$(6.8) \quad \frac{1}{\varepsilon} \sum_{p=0}^{\infty} (\partial u^P/\partial n) = 0 \quad \text{on } B$$

Substituting (6.1) in (6.8) and using (6.2) we obtain

$$(6.9) \quad \sum_{m=0}^{\infty} (i\omega)^{-m} \left[ \frac{1}{\varepsilon} \sum_{p=0}^{\infty} \left[ i\omega (\partial s^P/\partial n)u_m^P + \partial u_m^P/\partial n \right] \right] = 0$$

Since this equation must hold for any value of $\omega$, it is necessary that

$$(6.10) \quad \frac{1}{\varepsilon} \sum_{p=0}^{\infty} \left[ \partial s^P/\partial n \ u_m^P + \partial u_m^P/\partial n \right] = 0$$

or, equivalently,

$$(6.11) \quad n \cdot \nabla^1 u_m^1 + \partial u_m^1/\partial n = - \left[ \begin{array}{c} \text{L.H.S. with the superscript} \\ 1 \text{ replaced by } 0 \end{array} \right]$$
This is a recursive system of equations, which can be solved one by one starting from \( m = 0 \).

6.2 Zeroth and First Order Solutions at Interface

For \( m = 0 \), Eq. (6.11) gives

\[
(6.12) \quad (\mathbf{n} \cdot \mathbf{\nabla})^1 u_0^1 = -(\mathbf{n} \cdot \mathbf{\nabla})^0 u_0^0
\]

But \( \mathbf{n} \cdot \mathbf{\nabla}^1 = -\mathbf{n} \cdot \mathbf{\nabla}^0 \), by the law of reflection. Thus at the point of incidence 0,

\[
(6.13) \quad u_0^1 = u_0^0
\]

Next we set \( m = 1 \) in (6.11) and obtain

\[
(6.14) \quad (\mathbf{n} \cdot \mathbf{\nabla})^1 u_1^1 + \partial u_0^1 / \partial n = -[(\mathbf{n} \cdot \mathbf{\nabla})^0 u_1^0 + \partial u_0^0 / \partial n]
\]

If the incident field is a time-harmonic plane wave of the form

\[
(6.15) \quad u_0^0(\mathbf{r}) = A \exp(i\omega \mathbf{\alpha}^0 \cdot \mathbf{r} / c)
\]

then by comparison with (5.5), one sees immediately that

\[
(6.16) \quad S_0^0(\mathbf{r}) = \mathbf{\alpha}^0 \cdot \mathbf{r} / c; \quad u_0^0(\mathbf{r}) = A, \quad u_k^0(\mathbf{r}) = 0 (k = 1, 2, \ldots)
\]

Thus the right hand side of (6.14) vanishes, and
(6.17) \( u_1^1 = (\partial u_0^1 / \partial n)/(n \cdot \nabla S^0) \)

For the special case of normal incidence,

(6.18) \( n \cdot \nabla S^0 = -1/c \)

Also, since [see (5.27)]

(6.19) \( u_0^1(a) = u_0^1(0) \left[ \frac{r_1 r_2}{(r_1 + a)(r_2 + a)} \right]^{1/2} \)

and since the normal derivative \( \partial / \partial n \) is nothing but \( d/d \alpha \), one obtains

after differentiating (6.19) and using \( r_1 = R_1/2 = 1/(2 \kappa_1) \), \( r_2 = R_2/2 = 1/(2 \kappa_2) \) (\( \kappa_1 \) and \( \kappa_2 \) are absolute values of the curvatures of reflecting surface at point 0),

(6.20) \( \frac{\partial u_0^1(a)}{\partial n} \bigg|_0 = -A(\kappa_1 + \kappa_2) \)

Substitution of (6.18) and (6.20) in (6.17) then yields

(6.21) \( u_1^1(0) = A c(\kappa_1 + \kappa_2) \)

With (6.13), (6.16), and (6.21), the reflected wave at the specular point is, to the first order, given by
\[ u^1 = A \exp(i \omega_0 x/c) \left\{ 1 + \frac{c(K_1 + K_2)}{i \omega} \right\} \]

Strictly speaking, this equation is valid only at the specular point, but since the reflected field is continuous, one may assume that it is valid in the immediate vicinity of the specular point. As will be seen in the next section, this approximation plays an important role in the first order far-field solution.

6.3 Far-Field Results

In many NDE applications a transducer, situated at a far distance from the flaw, is used to emit a plane incident wave. Upon incidence, the incident wave interacts with the flaw and gives rise to a scattered wave. One type of experiment where this scattered wave is measured by the same transducer used to launch the incident wave is known as a pulse-echo setup. A more general setup is called pitch-catch, in which two transducers are used, one being the transmitter and the other one receiver. The technique presented in this section applies equally well to both situations. However, since the latter is not relevant to our future development, it will be omitted. Instead, we shall be concentrating on the pulse-echo case.

First, the far-field result for the zeroth order solution in the frequency domain will be obtained by extending the solution at the specular point along the reflected ray. Then the first order far-field solution will be considered next. The difficulty of using such
extension-along-ray method will be pointed out and a remedy to this problem given.

6.3.1 Frequency-domain

The zeroth order solution can be readily obtained. By choosing \( a_0 = 0 \) and setting \( m = 0 \) in (5.23), one obtains after using (5.27)

\[
(6.23) \quad u_0^1(a) = u_0^1(0) \left[ \frac{R_1 R_2}{(R_1 + a)(R_2 + a)} \right]^{1/2}
\]

But (see Section 5.4) \( R_1^1 = R_1/2, \ R_2^1 = R_2/2 \), where \( R_1 \) and \( R_2 \) are principal radii of the reflecting surface at the specular point 0. Hence, by (6.23), (6.13), and (6.16)

\[
(6.24) \quad u_0^1(a) = A \left[ \frac{R_1 R_2}{(R_1 + 2a)(R_2 + 2a)} \right]^{1/2}
\]

In the far-field \( (a \to \infty) \), Eq. (6.24) becomes (assuming \( R_1 \) and \( R_2 \) are finite)

\[
(6.25a) \quad u_0^1(a) = \frac{A}{2\pi} \left| R_1 R_2 \right|^{1/2}
\]

or,
(6.25b) \[ u_0^1(a) = \frac{A}{2a} \left[ \kappa_1 \kappa_2 \right]^{-1/2} \]

Also, the phase function \( S^1 \) at \( a \) is

\[ (6.26) \quad S^1(a) = S^1(0) + a/c = S^0(0) + a/c = (a^0 \cdot \hat{x}^0 + a)/c \]

Thus, to the zeroth order approximation, Eq. (6.1) gives

\[ (6.27) \quad u^1(a) \sim \exp \left[ i \omega \left( a^0 \cdot \hat{x}^0 + a \right)/c \right] (A/2a) \left[ R_1 R_2 \right]^{-1/2} \]

After some simple calculations, (6.27) can be manipulated into a form exactly the same as (4.37), which was obtained previously by using the Kirchhoff approximation. Thus, mathematically, the Kirchhoff approximation can be regarded as the leading term in a ray series.

The zeroth order solution \((m = 0)\) was easy because the ray integral in (5.23) was absent. For \( m = 1 \), however, that integral does not vanish and one encounters the problem of carrying out that integration. Unfortunately, evaluating the ray integral in a "closed" form turns out to be one of the most difficult problems in ray theory. So far only a limited number of ray integrals have been carried out, the success of which is mainly due to the simple geometrical shapes of the scatterers and the wavefront systems. For a complex geometry, this task is in general impossible. For direct problems, of course, the ray integral can be calculated numerically on the computer; but such
"numerical" results would prove to be of little use in inverse problems.

This difficulty cannot be avoided in the frequency domain. However, if one switches to the time domain, the complete problem can be escaped. In fact, the success of our inversion scheme is based upon this idea.

6.3.2 Time-domain: leading edge approximation

By setting $\alpha = 1$ and using the Neumann boundary condition $\partial u/\partial n = 0$ in Eq. (3.12), we obtain

$$u^1(y) = \int_B u(x) \frac{\partial G(x,y)}{\partial n(x)} dB(x)$$

Now, if $y$ is in the far-field, then the normal derivative of $G$ can be approximated by (3.13d) and the above equation becomes

$$u^1(y) = \frac{i k e^{i k y}}{4 \pi y} \int_B n \cdot y u(x) \exp(ikx \cdot y) dB(x)$$

Note that this "formula" involves the unknown field $u(x)$, hence it cannot be used immediately to compute the scattered field $u^1(y)$. Also, note that (6.29) is a steady-state ($e^{-i\omega t}$ time dependence) frequency domain formula.

Later on we shall also be interested in the time-domain far-field response. In this case, instead of a time harmonic wave, a pulse will be used as an incident wave. According to Huyghens' principle [25]
every point on the scattering surface B, after being excited by the incident pulse, acts as a secondary source and radiates waves out, giving rise to a complicated scattered wave field. This scattered wave field then propagates out at a finite speed and eventually is detected by a transducer (receiver).

In pulsed wave experiments, the time history of the scattered wave can be displayed on the screen of an oscilloscope and recorded. After obtaining the record, one can always concentrate on analyzing the leading edge response, i.e., the early arriving waves. Since the path connecting the specular point and the receiver is the shortest, the wave emanating from the specular point reaches the receiver first. Thus, all the early arriving waves are emitted from a small neighborhood of the specular point. Accordingly, for the leading edge response the integral in (6.29) can be replaced by an integral over a small cap-like region $B'$ enclosing the specular point.

In the rest of this section, we shall confine ourselves to the pulse-echo case, though the same technique can be used in the more general pitch-catch situation. For pulse-echo, the incident ray striking at the specular point is of normal incidence, and so Eq. (6.22) applies. Using (6.22) in (6.29) and replacing the domain of integration $B$ in (6.29) by $B'$, one obtains

\begin{equation}
(6.30) \quad u^{1}(\mathbf{y},\omega) = u^{1;0}(\mathbf{y},\omega) + u^{1;1}(\mathbf{y},\omega)
\end{equation}

where, for convenience, we have introduced the following definitions:
(6.31) \[ u^{1;0}(y, \omega) = -\frac{A}{2\pi y} \int \frac{i\omega \exp[i\omega(2a^0_x + y)/c]}{n.Y dB(x)} \]

(6.32) \[ u^{1;1}(y, \omega) = -\frac{A(k_1 + k_2)}{4\pi y} \int \frac{\exp[i\omega(2a^0_x + y)/c]}{n.Y dB(x)} \]

Note that the \( \omega \) dependence has been explicitly incorporated into the above two equations, since these equations are actually associated with the particular frequency \( \omega \). The first term \( u^{1;0} \) in (6.30) is essentially the same as the result of the Kirchhoff approximation (see Section 4.4) and the second term \( u^{1;1} \) can be thought of as a correction to the Kirchhoff approximation.

If we agree to use the following definitions for the Fourier transform pair:

(6.33) \[ f(t) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega t} d\omega, \quad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{i\omega t} dt \]

then the incident pulse \( f(t - a^0_x/c) \) can be written as

(6.34) \[ f(t - a^0_x/c) = \int_{-\infty}^{\infty} F(\omega) \exp(ika^0_x - i\omega t) d\omega \]

Now, since (6.30) is the reflected wave due to an incident wave of the form \( A \exp(ika^0_x - i\omega t) \), the reflected wave due to the incident pulse \( f \), by Fourier synthesis, should be
where $u^1(y,\omega)$ is given in (6.30). Substituting (6.31) and (6.32) into (6.30) and the subsequent expression into (6.35) results

\begin{equation}
(6.36) \quad u^1(y,t) = u^{1;0}(y,t) + u^{1;1}(y,t)
\end{equation}

where

\begin{align}
(6.37a) \quad u^{1;0}(y,t) &= \frac{1}{2\pi c} \int_{\mathbb{B}} \left\{ \int_{-\infty}^{\infty} \left( -i\omega \right) F(\omega) e^{-i\omega \xi} \, d\omega \right\} n\cdot Y \, dB(x) \\
(6.37b) \quad u^{1;1}(y,t) &= -\frac{(\kappa_1 + \kappa_2)}{4\pi Y} \int_{\mathbb{B}} \left\{ \int_{-\infty}^{\infty} F(\omega) e^{-i\omega \xi} \, d\omega \right\} n\cdot Y \, dB(x)
\end{align}

The variable $\xi$ in (6.37) is defined by

\begin{equation}
(6.38) \quad \xi = t - \left[ (2a^0 \cdot x + y)/c \right]
\end{equation}

By the definition of Fourier transform, Eqs. (6.37) can be written as

\begin{align}
(6.39a) \quad u^{1;0}(y,t) &= \frac{1}{2\pi c} \int_{\mathbb{B}} \frac{df(\xi)}{d\xi} n\cdot Y \, dB(x) \\
(6.39b) \quad u^{1;1}(y,t) &= -\frac{\kappa_1 + \kappa_2}{4\pi Y} \int_{\mathbb{B}} l(\xi) n\cdot Y \, dB(x)
\end{align}
For convenience, we choose a coordinate system with origin at the specular point (Fig. 6.2) and let the incident wave \( f \) be an ideal impulse, i.e., a delta function. Then \( \mathbf{d} \cdot \mathbf{x} = x_3 \) and \( \mathbf{n} \cdot \mathbf{d} \mathbf{B}(\mathbf{x}) = dS_3 \) (\( dS_3 \) is the projection of \( d\mathbf{B} \) on the \( x_1x_2 \)-plane). By making this choice of coordinates, Eqs. (6.38), (6.39a), and (6.39b) become, respectively,

\[
\xi = t - \left[ \frac{(2x_3 + y)}{c} \right]
\]

\[
u = \frac{1}{2nc} \int_{\mathbf{B}_\xi} \frac{d\xi}{d\mathbf{x}_3} dS_3 \mathbf{d}x_3
\]

\[
u = \frac{-\kappa_1\kappa_2}{4\pi y} \int_{\mathbf{B}_\xi} \delta(\xi) \frac{dS_3}{d\mathbf{x}_3} \mathbf{d}x_3
\]

It proves useful to introduce one more definition:

\[
\tau = t - \left( \frac{y}{c} \right)
\]

Note that \( \tau \) is the time measured from the instant that the wave emanating from the specular point \( (x = 0) \) has just reached the receiver. Thus both \( \tau \) and \( t \) are time variables, the only difference is \( \tau \) lags behind \( t \) by an amount \( \frac{y}{c} \), which is the time required for the wave to travel from the specular point to the receiver. By (6.42), Eq. (6.40) can be written as

\[
\xi = \tau - \left( \frac{2x_3}{c} \right)
\]
Fig. 6.2. A coordinate system with origin at the specular point O
Also, note that from the theory of distributions \[167\],

\begin{equation}
\delta(\xi) = \delta(\tau - (2x_3/c)) = \delta\left(\frac{2}{c} \left[(ct/2) - x_3\right]\right)
\end{equation}

\[= \frac{c}{2} \delta[x_3 - (ct/2)]\]

Now, for any particular instant of time, \(t\) (or \(\tau\)) is a constant. Hence, for any \(t\) (or \(\tau\)), \(\xi\) can be thought of as a function of \(x_3\) alone. Thus, by the chain rule,

\begin{equation}
\frac{d\delta(\xi)}{d\xi} = \frac{d\delta(\xi)}{dx_3} \frac{dx_3}{d\xi} = -(c/2) \frac{d\delta(\xi)}{dx_3}
\end{equation}

Substitution of (6.44) in the right hand side of (6.45) then yields

\begin{equation}
\frac{d\delta(\xi)}{d\xi} = -\frac{c^2}{4} \frac{d}{dx_3} \delta[x_3 - (ct/2)]
\end{equation}

Now we are prepared to evaluate \(u_1;0(\chi, t)\) and \(u_1;1(\chi, t)\) given in (6.41). By (6.44) and the sifting property of the \(\delta\)-function, \(u_1;1(\chi, t)\) can be readily obtained:

\begin{equation}
\frac{1}{8\pi} \frac{dS_3}{dx_3} \bigg|_{x_3 = ct/2} = \frac{c(K_1 + K_2)}{8\pi y}
\end{equation}
However, as can be seen from (6.41a), the evaluation of $u^{1:0}(\chi, t)$ involves the generalized function $d\delta(x - a)/dx$ and is less obvious. But, by the theory of distributions [167], the result of this generalized function on any test function $f(x)$ is $-df(a)/dx$. Hence, by making use of this result and placing (6.46) in (6.41a), we obtain

\[(6.48) \quad u^{1:0}(\chi, t) = \frac{c}{8\pi y} \left[ \frac{dS_3}{dx_3} \right]_{x_3 = ct/2}
\]

Finally, by combining (6.47), (6.48), and (6.36) we find

\[(6.49) \quad u^1(\chi, t) = \frac{c}{8\pi y} \left[ \frac{d^2S_3}{dx_3^2} - (K_1 + K_2) \frac{dS_3}{dx_3} \right]_{x_3 = ct/2}
\]

This is the leading edge approximation for the first order far-field scattered wave. The area function $S_3$ appearing above has a simple geometrical interpretation, it is the cross sectional area of the scatterer intersected by a plane moving in the direction of incident wave and travelling at a speed of $c/2$ (one half of the wave speed). This area $S_3$ is customarily called the silhouette area function in electromagnetic literature [168].

6.3.3 Some remarks

As noted earlier in the paragraph after (6.32), Eq. (6.31) is the same as the result of the Kirchhoff approximation. Thus, if the method of stationary phase [see Eq. (4.32)] had been applied to (6.31), we
would have ended up with a result exactly identical to (4.38), which in our current notation is

\[(6.50) \quad u^{1;0}(y, \omega) = (A e^{iky/2y})(R_1 R_2)^{1/2}\]

If the incident wave is a \(\delta\)-function, the time-domain result derived from (6.50) can be shown to be given by

\[(6.51) \quad u^{1;0}(y, t) = [(R_1 R_2)^{1/2}/2y] \delta(t - y/c)\]

However, in the preceding section, another form of \(u^{1;0}(y, t)\) was derived [cf. (6.48)]. Since these two expressions must agree, we suspect that [by comparing (6.51) with (6.48)]

\[(6.52) \quad \frac{d^2 S_3}{dx_3^2} \bigg|_{x_3 = c\tau/2} = (4\pi/c)(R_1 R_2)^{1/2} \delta(t - y/c)\]

To confirm this conjecture, we first derive a formula for \(S_3(x_3)\), assuming \(x_3\) small. As a first step toward this end, refer to Fig. 6.3. In this figure \(x_3\) is aligned with the inward normal \(n\) of the scattering surface \(B\) at the specular point, and \(x_1\) and \(x_2\) are chosen so that they are coincident with the two principal directions of \(B\). The position vector of any point \(P\) on the surface \(B\) in a small neighborhood of the specular point \((x = 0)\) can be expressed as
Fig. 6.3. Local geometry of the scattering surface near the specular point
(6.53) \[ \mathbf{x} = x_1 \mathbf{x}_1 + x_2 \mathbf{x}_2 + x_3 \mathbf{x}_3(x_1, x_2) \]

where \( \mathbf{x}_k \) is the unit vector corresponding to \( x_k \)-axis. Symbolically, (6.53) can be written as \( \mathbf{x} = \mathbf{x}(x_1, x_2) \). This expression is intended to signify that \( \mathbf{x} \) is a function of the two variables \((x_1, x_2)\). Thus, although \((x_1, x_2)\) are the first two coordinates of the rectangular coordinates \((x_1, x_2, x_3)\) defined in Fig. 6.3, they can also be regarded, in the neighborhood of \( \mathbf{x} = 0 \), as the two surface coordinates (see Appendix A) used to parametrize the surface \( B \). The dual roles of \( x_1 \) and \( x_2 \) should not be confused.

Now since \( \mathbf{x}(x_1, x_2) \) is assumed to be close to 0, both \( x_1 \) and \( x_2 \) are \( \ll 1 \). Hence, \( \mathbf{x}(x_1, x_2) \) can be expanded about 0 in a Taylor series.

(6.54) \[ \mathbf{x}(x_1, x_2) = \mathbf{x}_1 x_1 + \mathbf{x}_2 x_2 + \]

\[ (1/2) \left[ x_{11}(x_1)^2 + 2x_{12}x_1x_2 + x_{22}(x_2)^2 \right] + \ldots \]

where we have used \( \mathbf{x}(0,0) = 0 \), also all the derivatives of \( \mathbf{x} \) are to be evaluated at the specular point 0.

The inner product of \( \mathbf{n}(0,0) \) and \( \mathbf{x}(x_1, x_2) \) is the "vertical" distance \( x_3 \) (Fig. 6.3) from the \( x_1x_2 \)-plane to the point \( \mathbf{x}(x_1, x_2) \). From (6.54), this vertical distance can be computed:

(6.55) \[ x_3(x_1, x_2) = (1/2) \left[ n \cdot x_{11}(x_1)^2 + n \cdot x_{22}(x_2)^2 \right] \]
In obtaining this equation, \( n \cdot x_{,1} = n \cdot x_{,2} = n \cdot x_{,12} = 0 \) have been used.

Now, from differential geometry (Appendix A), we have

\[
(6.56) \quad n \cdot x_{,11} = h_{11} = \kappa_1, \quad n \cdot x_{,22} = h_{22} = \kappa_2
\]

where \( \kappa_1 \) and \( \kappa_2 \) are principal curvatures at \( x = 0 \). Therefore, it follows from Eqs. (6.55) and (6.56) that

\[
(6.57) \quad x_3(x_1, x_2) = \left(\frac{1}{2}\right) \left[ \kappa_1(x_1)^2 + \kappa_2(x_2)^2 \right]
\]

Also, from Fig. 6.3, it is clear that

\[
(6.58) \quad x_1 = p \cos \theta, \quad x_2 = p \sin \theta
\]

where \( p \) is the "radial" distance from the \( x_3 \)-axis to the point \((x_1, x_2)\). Consequently, Eq. (6.57) also can be written as

\[
(6.59) \quad x_3 = \left(\frac{p^2}{2}\right) \left[ \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta \right]
\]

Solving this equation for \( p \), we obtain

\[
(6.60) \quad p(x_3, \theta) = \left[ \frac{2 x_3}{\kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta} \right]^{1/2}
\]
With this result, the silhouette area function $S_3$ now can be calculated:

$$S_3 = \int_0^{2\pi} \frac{1}{2} \rho(\varphi d\theta)$$

Placing (6.60) in the last integral yields

$$S_3 = 2x_3 \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta}$$

This integral can be evaluated exactly, giving

$$S_3 = 2\pi (R_1 R_2)^{1/2} x_3$$

It should be noted that this equation is valid only for $x_3 \ll 1$; furthermore, $x_3$ has to be greater than or equal to zero. Bearing the latter condition in mind, it is more precise to rewrite (6.62) as

$$S_3(x_3) = 2\pi (R_1 R_2)^{1/2} x_3 \mathcal{H}(x_3)$$

where $\mathcal{H}$ is the Heaviside function. After differentiation, we find

$$\frac{dS_3}{dx_3} = 2\pi (R_1 R_2)^{1/2} \mathcal{H}(x_3)$$

(note that $x_3 \delta(x_3) = 0$)
\[ (6.65) \quad \frac{d^2 S_3}{dx_3^2} = 2\pi(R_1 R_2)^{1/2} \delta(x_3) \]

Hence, at \( x_3 = ct/2 \),

\[ (6.66) \quad \left. \frac{d^2 S_3}{dx_3^2} \right|_{x_3 = ct/2} = (4\pi/c)(R_1 R_2)^{1/2} \delta(t - y/c) \]

which verifies what was predicted in (6.52).
7. ELASTODYNAMIC WAVE SCATTERING BY A VOID

This chapter is concerned with the scattering of elastodynamic waves by a smooth void. However, the methods of this chapter can be adapted to solve inclusion scattering problems as well.

7.1 Incident and Reflected Ray Pencils

As an incident wave $u^0(x)$ impinges on the boundary surface $B$ of a void, it interacts with the surface $B$, giving rise to two scattered waves: a P-wave and an S-wave. Since $B$ is assumed to be smooth, there are no edge-diffracted rays as introduced by Keller [117]. Also, the wave fields arising from the grazing incidence of $u^0$ on $B$ are to be ignored in this chapter. With these assumptions, we assume as usual that the incident and scattered waves can be expanded in the following series form:

$$u^p(x) = \exp[i\omega S^p(x)] \sum_{m=0}^{\infty} (i\omega)^{-m} u^p_m(x)$$

where

\begin{align*}
(7.2) \quad p &= \begin{cases} 
0 & \text{for incident wave} \\
1 & \text{for scattered P-wave} \\
2 & \text{for scattered S-wave}
\end{cases}
\end{align*}

In general, there are three rays (associated with the incident and
two scattered waves) passing through a typical observation point $r$. Let us consider one of the three rays. Following Deschamps [163], we shall call this ray an axial ray with respect to the point $r$, and the rays in the immediate neighborhood of the axial ray paraxial rays. The axial ray and its associated paraxial rays form a ray pencil (or ray bundle, ray tube).

Suppose the axial ray being considered is associated with the $p^{th}$ wave [recall Eq. (7.1)]. At the observation point $r$, let us introduce three right-handed orthonormal base vectors $(a^1_p, a^2_p, a^3_p)$. $a^1_p$ and $a^2_p$ are chosen so that they coincide with the two principal directions of the $p^{th}$ wavefront at $r$, and $a^3_p$ is aligned with the $p^{th}$ axial ray. A rectangular Cartesian coordinate system with origin at $r$ can be established by attaching to each of the three unit vectors $a^i_p$ ($i = 1, 2, 3$) an axis $a^i_p$ (Fig. 7.1).

Now consider an arbitrary ray incident on the surface $B$ at a point $O$ and let $O$ be the observation point $r$ mentioned earlier. Shown in Fig. 7.2 is a two dimensional picture of the $p^{th}$ wavefront that happens to pass through the point $O$. The dashed curve in the same figure is a wavefront passing through another point $Q$ on the axial ray. The two wavefronts in Fig. 7.2 are labelled $WF_0$ and $WF_Q$, respectively, for later convenience.

If $(a^1_p, a^2_p, a^3_p)$ is a point on $WF_Q$ and is sufficiently close to the point $Q$, then the distance $z$ (Fig. 7.2) can be approximated by (see Appendix A)
Fig. 7.1. Cartesian coordinate system \( (a_1^p, a_2^p, a_3^p) \) and ray pencil coordinate system \( (\xi_1^p, \xi_2^p, \xi_3^p) \)

Fig. 7.2. Two dimensional view of the \( p^{th} \) wavefront
Here $h_{\alpha\beta}$ is the curvature tensor of $WF_0$ at point $Q$. The phase difference between $WF_0$ and $WF_0$ is $(a_{P,3}^p + z)/c_p$. Hence if $S^P(0,0,0)$ denotes the phase of $WF_0$, the phase of $WF_0$ is given by $S^P(0,0,0) + (a_{P,3}^p + z)/c_p$. That is

$$s^P(0,0,0) = S^P(0,0,0) + (a_{P,3}^p + \frac{1}{2} h_{\alpha\beta} a_{P,\alpha} a_{P,\beta})/c_p$$

In the sequel, we will always choose $a_{P,1}^p$ and $a_{P,2}^p$ so that they coincide with the two principal directions of $WF_0$. According to the results of Section 5.4, the two axes $a_{P,1}^p$ and $a_{P,2}^p$ will be parallel to the two principal directions of $WF_0$, too. As a consequence of this choice, the components of the curvature tensor of $WF_0$ at point $Q$ are

$$h_{11}^P = (R_{11}^P + a_{P,3}^p)^{-1}$$

$$h_{22}^P = (R_{22}^P + a_{P,3}^p)^{-1}$$

$$h_{12}^P = h_{21}^P = 0$$

In (7.5), $R_{11}^P$ and $R_{22}^P$ are the principal radii of $WF_0$ at point $Q$.

At point $(a_{P,1},a_{P,2},a_{P,3})$, let us introduce three local orthonormal base vectors (Fig. 7.1):
In the above three equations, small quantities of second order have been neglected. At point 0 \( a^p_1 = a^p_2 = a^p_3 = 0 \), note that \( \xi^p_i \) coincide with \( a^p_i \) \( (i = 1, 2, 3) \).

The coefficients \( u^p_m \) in Eq. (7.1) can be written in the following component form:

\[
(7.7) \quad u^p_m(a^p_1, a^p_2, a^p_3) = \xi^p_i \ u^p_{mi} \quad \text{(sum over } i = 1, 2, 3)\]

Notice that \( u^p_{m1} \) and \( u^p_{m2} \) are the transverse components and \( u^p_{m3} \) is the longitudinal component with respect to the direction of wave propagation. Substituting (7.6) in (7.7) yields

\[
(7.8) \quad u^p_m(a^p_1, a^p_2, a^p_3) = a^p_1 \left[ u^p_{m1} + \frac{a^p_1}{R^p_1 + a^p_3} u^p_{m3} \right] + a^p_2 \left[ u^p_{m2} + \frac{a^p_2}{R^p_2 + a^p_3} u^p_{m3} \right] + a^p_3 \left[ u^p_{m3} - \frac{a^p_1}{R^p_1 + a^p_3} u^p_{m1} - \frac{a^p_2}{R^p_2 + a^p_3} u^p_{m2} \right]
\]
This is the representation of $u^P_m$ at a point on the paraxial ray in terms of the rectangular base vectors $a^P_i$ ($i = 1, 2, 3$) of the $p$th axial ray. As one shall see in Sections 7.4-7.5, this equation is crucial in the calculation of the first order solution.

7.2 Behavior of Waves at Interface

In order that the phase functions $S^P(x)$ ($p = 0, 1, 2$) be matched at the boundary $B$ of the void, it is necessary that

$$(7.9) \quad S^0(x) = S^1(x) = S^2(x) \quad (x \text{ on } B)$$

Now let us denote the angle of incidence of a particular incident ray by $\theta_0$ and the angles of reflection associated with the reflected $P$ and $S$ waves by $\theta_1$ and $\theta_2$, respectively. Following a similar procedure as described in Section 6.1 [see the paragraph below Eq. (6.2)] one can show, by Eq. (7.9) and the eikonal equations, that

$$(7.10) \quad \sin \theta_0/c_0 = \sin \theta_1/c_1 = \sin \theta_2/c_2$$

which is the familiar form of generalized Snell's law [27].

Next, we proceed to consider the traction free boundary condition. To this end, we consider a particular incident ray striking on the surface $B$ at a point $0$ and set up a Cartesian coordinate system as shown in Fig. 7.3. The $x_1$ and $x_2$ axes are any two mutually perpendicular axes lying on the tangent plane to $B$ at the point of incidence $0$, and the $x_3$ axis is aligned with the unit outward normal to $B$ at $0$. The unit base
Fig. 7.3. Cartesian coordinate system $O-x_1x_2x_3$
vectors corresponding to \( x_1, x_2, \) and \( x_3 \) axes are denoted by \( \bar{x}_1, \bar{x}_2, \) and \( \bar{x}_3, \) respectively.

With respect to this Cartesian coordinate system, the stress tensor associated with the \( p^{th} \) wave is

\[
\sigma_{ij}^p = \lambda \delta_{ij} \bar{u}_k^p \bar{k} + \mu (\bar{u}_{i,j}^p + \bar{u}_{j,i}^p)
\]

Here, \( u_{i}^{p} = u_{i}^{p} \bar{x}_i \) is the \( x_i \)-component of \( u^p \) and \( u_{i,j}^{p} = \partial u_{i}^{p} / \partial x_j \). The corresponding traction vector acting at an infinitesimal area with unit outward normal \( n \) is

\[
T_{i}^{p} = \lambda n_i u_{i,k}^{p} + \mu n_j (u_{i,j}^{p} + u_{j,i}^{p})
\]

Now, we form scalar product of Eq. (7.1) with \( \bar{x}_i \), obtaining

\[
u_{i}^{p} = \exp[i\omega s^p] \sum_{m=0}^{\infty} (i\omega)^{-m} \frac{\partial u_{m}^{p}}{\partial \bar{x}_i}
\]

If we define

\[
u_{m,i}^{p} = \frac{\partial u_{m}^{p}}{\partial \bar{x}_i}
\]

then Eq. (7.13) can be written as

\[
u_{i}^{p} = \exp[i\omega s^p] \sum_{m=0}^{\infty} (i\omega)^{-m} \nu_{m,i}^{p}
\]
Now, before proceeding any further, we wish to take this opportunity to address a few words on some of the notations that have already been used and some that will be used later in the sections to come. In Eq. (7.14), a pair of parentheses was intentionally used to enclose the subscript \(i\), the reason being that \(u^{P}_{m1}\) has been used previously to denote the \(\xi_{1}\)-component (ray pencil coordinate component) of \(u^{P}_{m}\), i.e.,

\[
u^{P}_{m1} = u^{P}_{m} \cdot \xi_{1}\]

Frequently, in what follows we will also encounter quantities such as the \(\xi_{1}\)-component and \(x_{1}\)-component of \(u^{P}_{m-1}\), i.e.,

\[
u^{P}_{m-1} \cdot \xi_{1} \quad \text{and} \quad u^{P}_{m-1} \cdot x_{1}\]

In these cases, it would be confusing to write such quantities as

\[
u^{P}_{m-1 i} \quad \text{and} \quad u^{P}_{m-1(i)}\]

especially for the former one. In order to avoid mixing up indices and possible misunderstanding, we will use a semicolon to separate two adjacent indices. Thus, for example, we will write the above two quantities as

\[
u^{P}_{m-1; (i)} \quad \text{and} \quad u^{P}_{m-1; i}\]
However, for quantities such as

\[ u^p \cdot \xi_2 \quad \text{and} \quad u^p \cdot x_2 \]

we will not write them as

\[ u^p_{0;2} \quad \text{and} \quad u^p_{0;(2)} \]

Instead, we will simply write them as

\[ u^p_{02} \quad \text{and} \quad u^p_{0(2)} \]

since, even without the usage of semicolons, no misunderstanding or confusion on the indices is likely to occur.

In essence, a semicolon is needed only when there is a possibility of mixing up two adjacent indices. When there is no such possibility, no matter what component (\( x_i \)- or \( \xi_i \)-component) is referred to, the semicolon will always be discarded for ease of notations. As one more example on the suppression of semicolons, consider

\[ u^p_m \cdot x_3 \quad \text{and} \quad u^p_m \cdot \xi_3 \]

In keeping with the spirit of our convention, these two quantities will be denoted, respectively, by
Now, back to Eq. (7.15). Substituting (7.15) in Eq. (7.12) yields

\[ T_i^p = \exp[i\omega S^p] \sum_{m=0}^{\infty} (i\omega)^{-m+1} \{ \lambda n_i \left[ s^p, k u^m(k) + u^p_{m-1};(k), k \right] + \mu n_j \left[ s^p, j u^p_m(i) + u^p_{m-1};(i), j + s^p, i u^m(j) + u^p_{m-1};(j), i \right] \} \]

Here \( u^p_{m-1};(i) = u^p_{m-1} \cdot x_i \) and \( u^p_{m-1};(i), j = \partial u^p_{m-1};(i) / \partial x_j \). Similar notations apply to other terms in (7.16). Note in particular that semicolons are used in (7.16) to separate the subscript \( m-1 \) from \( (i), (j), \) and \( (k) \); whereas no semicolons are used to isolate the index \( m \) from indices \( (i), (j), \) and \( (k) \).

By using the traction free condition on \( B \), it is easy to show that, for \( m = 0,1,2,\ldots \),

\[ \sum_{p=0}^{\infty} \{ \lambda n_i \left[ s^p, k u^m(k) + u^p_{m-1};(k), k \right] + \mu n_j \left[ s^p, j u^p_m(i) + u^p_{m-1};(i), j + s^p, i u^m(j) + u^p_{m-1};(j), i \right] \} = 0 \]
This system of equations can be solved recursively to yield \( u_m^{p(i)} \) starting from \( m = 0 \). Knowing \( u_m^{p(i)} \) for all \( m \), one is able to reconstruct the series (7.15).

Now, Eq. (7.17) is valid at every point of \( B \), hence, is certainly valid at the point of incidence 0. At point 0, however, \( n_1 = n_2 = 0, n_3 = 1 \), thus Eq. (7.17) yields the following three equations corresponding to \( i = 1, 2, \) and 3, respectively.

\[
\begin{align*}
(7.18a) \quad & \sum_{p=0}^{2} \left\{ s^{p}_{3} u_{m(1)}^{p} + u_{m-1(1),3}^{p} + s^{p}_{1} u_{m(3)}^{p} + u_{m-1(3),1}^{p} \right\} = 0 \\
(7.18b) \quad & \sum_{p=0}^{2} \left\{ s^{p}_{3} u_{m(2)}^{p} + u_{m-1(2),3}^{p} + s^{p}_{2} u_{m(3)}^{p} + u_{m-1(3),2}^{p} \right\} = 0 \\
(7.18c) \quad & \sum_{p=0}^{2} \left\{ \lambda \left[ s^{p}_{1} u_{m(1)}^{p} + s^{p}_{2} u_{m(2)}^{p} + u_{m-1(1),1}^{p} + u_{m-1(2),2}^{p} \right] + \right. \\
& \quad \left. (\lambda+2\mu) \left[ s^{p}_{3} u_{m(3)}^{p} + u_{m-1(3),3}^{p} \right] \right\} = 0
\end{align*}
\]

The quantities \( u_m^{p(i)} \) and \( u_{m-1(i)}^{p} \) (\( i = 1, 2, 3 \)) in Eq. (7.18) are components of \( u_m^{p} \) and \( u_{m-1}^{p} \) with respect to the Cartesian coordinate system \( (x_1, x_2, x_3) \) defined in Fig. 7.3. So far, the two axes \( x_1 \) and \( x_2 \) were arbitrarily chosen to be any two mutually perpendicular axes lying on the tangent plane to \( B \) at point 0. As we go along, however, it will prove useful to choose \( x_1 \) and \( x_2 \) axes such that they coincide with the two principal directions of \( B \) at point 0. When this choice is made, the three axes \( x_1, x_2, \) and \( x_3 \) will be relabeled as \( a_1, a_2, \) and \( a_3 \); the
reason will become clear later. Also, as we shall see, it is convenient
to express the Cartesian components \( u^P_m_{(i)} \) of \( u^P_m \) with respect to the
\( a_1a_2a_3 \)-system in terms of \( u^P_{mi} \) defined in Eq. (7.7). To this end, we
first introduce a notation:

\[
(7.19) \quad A^P_{jk} = a^P_j \cdot a^P_k
\]

where \( a^P_k \) is the unit base vector corresponding to the \( a^P_k \)-axis. Note
that \( A^P_{jk} \) is just the direction cosine between \( a^P_j \) and \( a^P_k \) axes.

Using (7.8) and (7.19), we obtain

\[
(7.20) \quad u^P_m(k) = A^P_{1k} \left[ u^P_{m1} \cdot \frac{a^P_1}{R^P_1 + a^P_3} \ u^P_{m3} \right] + A^P_{2k} \left[ u^P_{m2} \cdot \frac{a^P_2}{R^P_2 + a^P_3} \ u^P_{m3} \right] \\
+ A^P_{3k} \left[ u^P_{m3} \cdot \frac{a^P_1}{R^P_1 + a^P_3} \ u^P_{m1} - \frac{a^P_2}{R^P_2 + a^P_3} \ u^P_{m2} \right]
\]

Taking partial differentiation of this equation with respect to \( a^P_1 \),
using chain rules, and the following easily proved relation

\[
(7.21) \quad \frac{\partial a^P_{Pj}}{\partial a^P_j} = A^P_{ij}
\]

one obtains, at point 0,

\[
(7.22) \quad \frac{\partial u^P_{m(k)}}{\partial a^P_1} = u^P_{m(k),1} = \left[ \frac{A^P_{11} a^P_{1k}}{R^P_1} + \frac{A^P_{21} a^P_{2k}}{R^P_2} \right] u^P_{m3} - \frac{A^P_{13} a^P_{3k}}{R^P_1} u^P_{m1} -
\]
In Section 7.5 when dealing with the first order solution, we shall focus on the special case of normal incidence only. For this special case, applying Eqs. (7.20) and (7.22) to Eq. (7.18) gives

\[
\frac{A_{n_1}^{P_1} A_{P_2}^{P_3}}{R_2^P} u_{m_2} + \sum_{n=1}^{3} \left\{ A_{n_1}^{P_1} A_{P_2}^{P_3} \frac{a_{m_1}^{P}}{a_{a_1}^{P}} + A_{n_1}^{P_1} A_{P_2}^{P_3} \frac{a_{m_2}^{P}}{a_{a_1}^{P}} + A_{n_1}^{P_1} A_{P_2}^{P_3} \frac{a_{m_3}^{P}}{a_{a_1}^{P}} \right\}
\]

\[
\begin{align*}
(7.23a) & \quad \sum_{p=1}^{2} \frac{u_{m_1}^{P}}{c_p} = \left\{ \frac{u_{m-1;1}^{0}}{R_1^P} - \frac{\partial u_{m-1;1}^{0}}{\partial a_1^{0}} - \frac{u_{m-1;1}^{0}}{c_0} - \frac{\partial u_{m-1;1}^{0}}{\partial a_2^{0}} \right\} \\
& \quad + \sum_{p=1}^{2} \left\{ \frac{u_{m-1;1}^{P}}{R_1^P} - \frac{\partial u_{m-1;1}^{P}}{\partial a_1^{P}} - \frac{\partial u_{m-1;1}^{P}}{\partial a_2^{P}} \right\}
\end{align*}
\]

\[
(7.23b) \quad \sum_{p=1}^{2} \frac{u_{m_2}^{P}}{c_p} = \left\{ - \frac{u_{m-1;2}^{0}}{R_2^P} - \frac{\partial u_{m-1;2}^{0}}{\partial a_2^{0}} + \frac{u_{m-1;2}^{0}}{c_0} + \frac{\partial u_{m-1;2}^{0}}{\partial a_3^{0}} \right\} \\
& \quad + \sum_{p=1}^{2} \left\{ \frac{u_{m-1;2}^{P}}{R_2^P} - \frac{\partial u_{m-1;2}^{P}}{\partial a_2^{P}} - \frac{\partial u_{m-1;2}^{P}}{\partial a_3^{P}} \right\}
\]

\[
(7.23c) \quad \sum_{p=1}^{2} \frac{u_{m_3}^{P}}{c_p} = - \left\{ \lambda \left[ \frac{u_{m-1;3}^{0}}{R_1^P} + \frac{\partial u_{m-1;3}^{0}}{\partial a_1^{0}} + \frac{u_{m-1;3}^{0}}{R_2^P} + \frac{\partial u_{m-1;3}^{0}}{\partial a_2^{0}} \right] \right\} \\
& \quad + (\lambda + 2\mu) \left[ \frac{u_{m_3}^{0}}{c_0} + \frac{\partial u_{m-1;3}^{0}}{\partial a_3^{0}} \right] \right\} \\
& \quad - \sum_{p=1}^{2} \left\{ \lambda \left[ \frac{u_{m-1;3}^{P}}{R_1^P} + \frac{\partial u_{m-1;3}^{P}}{\partial a_1^{P}} + \frac{u_{m-1;3}^{P}}{R_2^P} + \frac{\partial u_{m-1;3}^{P}}{\partial a_2^{P}} \right] \right\}
\]
The above set of equations will be used later in Section 7.5.

7.3 Zeroth Order Solution

As mentioned earlier, for the first order solution we will consider the special case of normal incidence only. For the zeroth order solution, however, we will not impose any restriction on the angle of incidence.

The coordinate system we shall use is shown in Fig. 7.4, which is chosen in a way described below. Let \( \hat{a}_3^0 \) be a unit vector defining the direction of the incident ray striking the surface \( B \) at point 0. The \( x_2 \) axis is chosen so that it lies on the plane of incidence \( E_{\text{inc}} \) and so that the projection of \( \hat{a}_3^0 \) on it is nonnegative (Fig. 7.4). The \( x_3 \) axis is aligned with the unit outward normal of surface \( B \) at point 0. And, finally, the \( x_1 \) axis is chosen so that \( x_1, x_2, \) and \( x_3 \) form a right handed rectangular coordinate system.

For the zeroth order solution (\( m = 0 \)), Eqs. (7.18a–c) become, respectively,

\[
\begin{align*}
(7.24a) & \quad \sum_{p=0}^{2} \left\{ (x_3 \cdot \nabla^p) u_{0(1)} + (x_1 \cdot \nabla^p) u_{0(3)} \right\} = 0 \\
(7.24b) & \quad \sum_{p=0}^{2} \left\{ (x_3 \cdot \nabla^p) u_{0(2)} + (x_2 \cdot \nabla^p) u_{0(3)} \right\} = 0
\end{align*}
\]
Fig. 7.4. Rectangular coordinate system $O-x_1x_2x_3$

Fig. 7.5. Front view of the above figure
In arriving at Eqs. (7.24a-c), $u_P = 0$ and $s^p = x_1 \cdot s^p$ have been used. Now, since

\begin{equation}
\tag{7.25}
s^p = a^p/c_p \quad (p = 0,1,2)
\end{equation}

it follows that (see Fig. 7.5, which is a front view of Fig. 7.4)

\begin{equation}
\tag{7.26a}
x_1 \cdot s^p = 0 \quad (p = 0,1,2)
\end{equation}

\begin{equation}
\tag{7.26b}
x_2 \cdot s^p = \frac{\sin \theta^p}{c_p}, \quad x_3 \cdot s^p = \frac{\cos \theta^p}{c_p} \quad (p = 1,2)
\end{equation}

\begin{equation}
\tag{7.26c}
x_2 \cdot s^0 = \frac{\sin \theta^0}{c_0}, \quad x_3 \cdot s^0 = -\frac{\cos \theta^0}{c_0}
\end{equation}

Making use of (7.26) in (7.24), one can show that

\begin{equation}
\tag{7.27a}
\frac{\cos \theta^1}{c_1} u_0^1(1) + \frac{\cos \theta^2}{c_2} u_0^2(1) = \frac{\cos \theta^0}{c_0} u_0^0(1)
\end{equation}

\begin{equation}
\tag{7.27b}
\frac{\cos \theta^1}{c_1} u_0^1(2) + \frac{\sin \theta^1}{c_1} u_0^1(3) + \frac{\cos \theta^2}{c_2} u_0^2(2) + \frac{\sin \theta^2}{c_2} u_0^2(3)
\end{equation}
\[
\begin{align*}
\cos \theta_0 u_0(1) &= 0, \\
\sin \theta_0 u_0(2) &= \sin \theta_0, \\
\sin \theta_0 u_0(3) &= \sin \theta_0.
\end{align*}
\]

Note that the equations corresponding to the incident wave (p = 0) are not included in (7.28), the reason shall appear apparent later. Making use of Eqs. (7.28) in (7.27) we obtain

\[
\begin{align*}
(7.29a) & \quad \frac{1}{2} \cos \theta_0 u_0(1) \\
(7.29b) & \quad \sin \theta_0 u_0(2) + \gamma \cos \theta_0 u_0(3) = \frac{c_1}{c_0} [\cos \theta_0 u_0(2) - \sin \theta_0 u_0(3)] \\
(7.29c) & \quad \gamma \cos \theta_0 u_0(1) - \sin \theta_0 u_0(3) = -\frac{c_1 c_1}{c_0 c_2} \sin \theta_0 u_0(2) + \gamma \cos \theta_0 u_0(3)
\end{align*}
\]
Here \( \gamma = c_1/c_2 \) is the wave speed ratio. Given any incident wave \( u^0 \), the above three equations can be solved for \( u^1_{03}, u^2_{01}, \) and \( u^2_{02} \) (recall that \( u^1_{01} = u^1_{02} = u^2_{03} = 0 \), see Table 5.1). Also, note that the "out-of-plane" component (SH-component) \( u^2_{01} \) is uncoupled from the two "in-plane" components (SV-components) \( u^1_{03} \) and \( u^2_{02} \). Next, we consider the cases when \( u^0 \) is a time-harmonic plane L- or T-wave.

7.3.1 Incident L-wave

Suppôsé that \( u^0 \) is a time-harmonic plane L-wave propagating in the \( \sigma^0_3 \) direction, i.e.,

\[
(7.30) \quad u^0 = \sigma^0_3 \exp[i \omega_0 \cdot x/c_1]
\]

Note that the L-wave speed \( c_1 \) has been used in this equation. By comparing this equation to (7.1), one sees immediately

\[
(7.31a) \quad S^0 = \sigma^0_3 \cdot x/c_1
\]

\[
(7.31b) \quad u^0_{0(1)} = 0, \quad u^0_{0(2)} = \sin \theta_0, \quad u^0_{0(3)} = -\cos \theta_0
\]

\[
(7.31c) \quad u^0_{m(n)} = 0 \quad \text{for } n = 1,2,3 \text{ and } m = 1,2,\ldots
\]

Using (7.31) in (7.29), we find

\[
2 \frac{c_2}{c_0} \sin \theta_0 u^0_{0(2)} + \frac{c_1 c_1}{c_0 c_2} \cos \theta_0 u^0_{0(3)}
\]
(7.32a) \( u_{01}^2 = 0 \)

(7.32b) \( \sin 2\theta_0 u_{03}^1 + \gamma \cos 2\theta_2 u_{02}^2 = \sin 2\theta_0 \)

(7.32c) \( \gamma \cos 2\theta_2 u_{03}^1 - \sin 2\theta_2 u_{02}^2 = -\gamma + 2\gamma^{-1} \sin^2 \theta_0 \)

Solving Eqs. (7.32b) and (7.32c) simultaneously, one obtains

(7.33) \( u_{03}^1 = \frac{\sin 2\theta_0 \sin 2\theta_2 - \gamma \cos^2 2\theta_2}{\sin 2\theta_0 \sin 2\theta_2 + \gamma \cos^2 2\theta_2} \)

(7.34) \( u_{02}^2 = \frac{2\gamma \sin 2\theta_0 \cos 2\theta_2}{\sin 2\theta_0 \sin 2\theta_2 + \gamma \cos^2 2\theta_2} \)

Observe that the results of (7.32a), (7.33), and (7.34) are precisely the solutions for a time-harmonic incident plane L-wave reflected from an infinite, traction free, flat surface. Later on when considering an incident T-wave, an analogous observation will be made.

7.3.2 Incident T-wave

If \( u^0 \) is a time-harmonic incident plane T-wave, propagating in the \( a_3^0 \) direction, defined by

(7.35) \( u^0 = \vec{d} \exp[i \omega a_3^0 \cdot x/c_2] \)

where \( \vec{d} \) is a unit vector defining the polarization direction of the incident T-wave (Fig. 7.6). Then, from the geometry of Fig. 7.6, it is
Fig. 7.6. An incident T-wave polarized in the $d$-direction
clear that

(7.36a) \[ d = d_{SH} + d_{SV} \]

(7.36b) \[ d_{SH} = x_1, \quad d_{SV} = x_2 \cos \theta_0 + x_3 \sin \theta_0 \]

On substitution of (7.36), Eq. (7.35) becomes

(7.37) \[ u^0 = (x_1 + x_2 \cos \theta_0 + x_3 \sin \theta_0) \exp[i \omega_0 x / c_2] \]

By comparing (7.37) with (7.1) one sees immediately that

(7.38a) \[ S^0 = \frac{x_2}{c_2} \]

(7.38b) \[ u^0_{0(1)} = 1, \quad u^0_{0(2)} = \cos \theta_0, \quad u^0_{0(3)} = \sin \theta_0 \]

(7.38c) \[ u^0_{m(n)} = 0 \quad \text{for } n = 1, 2, 3 \text{ and } m = 1, 2, \ldots \]

Substituting (7.38) into (7.29) yields

(7.39a) \[ u^2_{01} = 1 \]

(7.39b) \[ \sin 2\theta_1 \ u^1_{03} + \gamma \cos 2\theta_2 \ u^2_{01} = \gamma \cos 2\theta_0 \]

(7.39c) \[ \gamma \cos 2\theta_2 \ u^1_{03} - \sin 2\theta_2 \ u^2_{02} = \sin 2\theta_0 \]

Notice that, just like the incident L-wave case, the out-of-plane component \( u^0_{12} \) is not coupled with the other two in-plane components.
Solving Eqs. (7.39b) and (7.39c) simultaneously one obtains

\[ u_{03}^1 = \frac{\gamma \sin 4\theta_0}{\sin 2\theta_0 \sin 2\theta_1 + \gamma^2 \cos^2 2\theta_0} \]

\[ u_{02}^2 = \frac{\gamma^2 \cos^2 2\theta_0 - \sin 2\theta_0 \sin 2\theta_1}{\sin 2\theta_0 \sin 2\theta_1 + \gamma^2 \cos^2 2\theta_0} \]

The above results (7.39a), (7.40), and (7.41) coincide exactly with the solution for a time-harmonic incident T-wave reflected from an infinite traction free flat surface. By recalling the remark following Eq. (7.34) one can conclude that, within the zeroth order approximation, the (curved) scattering surface acts at every point, locally, like a perfectly flat surface, regardless of the type of incident wave. This is usually referred to as the principle of isolated element [11] or the principle of local field character in the literature of ray theory. In essence, this principle states that an incident wave with an arbitrary wave front is reflected or refracted from any point of a curved interface in the same way that a plane wave would be from a plane interface at the same point.

Finally, one should keep in mind that Eqs. (7.32a), (7.34), (7.35) and Eqs. (7.39a), (7.40), (7.41) only give field values for points on the scattering surface B. Given an incident wave (L or T), each point on B corresponds to a certain incident angle \( \theta_0 \); the reflection angles associated with the two reflected rays at that point are then determined by the Snell’s law. With the incident and reflected angles known, the
reflected P and S waves on the surface B can be determined by using the
equations derived in current and the preceding subsection. For points
outside the void, one can use the equations derived in Chapter 5 (see
Table 5.1) to calculate the scattered fields. However, we will not
follow this approach here, but instead will (in Sections 7.5 and 7.6)
use the integral representation of the scattered waves to extend these
values beyond the scattering surface.

7.4 Tangential Derivatives of the Zeroth Order Solution

In the next section (Section 7.5) when we attempt to calculate the
(backscattered) first order solution, we will find that the tangential
derivatives of the zeroth order solution are required. The objective of
the present section is to calculate these derivatives in advance. More
specifically, we shall be calculating the following quantities at the
point of normal incidence:

\[
\frac{\partial u_0^1}{\partial a_1}, \frac{\partial u_0^1}{\partial a_2}, \frac{\partial u_0^2}{\partial a_1}, \frac{\partial u_0^2}{\partial a_2}, \frac{\partial u_0^1}{\partial a_1}, \frac{\partial u_0^2}{\partial a_2}
\]

Since only the case of normal incidence will be considered, our
specific results will be valid for pulse-echo types of scattering
experiments only. As one shall see, this seemingly trivial task turns
out to be not as easy as one might expect. The author believes that it
is the complexity and subtleness involved in this task that has hampered
the emergence of the first order and higher order solutions for general
three dimensional vector wave (elastodynamic wave, in particular)
problems. Lee [28] apparently was the first one who attempted to carry out these complicated calculations. Although he was dealing with electromagnetic wave (vector wave) problems, his idea can be modified to treat elastodynamic wave problems as well.

The starting point for the determination of these tangential derivatives is Eq. (7.17). Using \( u^p_{-1} = 0 \) and setting \( m = 0 \) in (7.17) one obtains

\[
(7.42) \quad \sum_{p=0}^{2} \left\{ \lambda n_i s^p_{-1} u^p_{0(k)} + \mu n_j \left[ s^p_{-1} u^p_{0(i)} + s^p_{-1} u^p_{0(j)} \right] \right\} = 0
\]

In (7.42) the partial derivatives are with respect to the Cartesian coordinates \( a_i \); for example, \( S^p_{,i} = \partial S^P_p / \partial a_i \). Now, since \( S^p_{,j} = a_j \cdot \nabla S^P_p \), Eq. (7.42) can also be written as

\[
(7.43) \quad \sum_{p=0}^{2} v s^p \cdot \left\{ \lambda n_i a_k u^p_{0(k)} + \mu n_j a_j u^p_{0(i)} + \mu a_i n_j u^p_{0(j)} \right\} = 0
\]

\((i = 1, 2, 3 \text{ and sum over } j, k = 1, 2, 3)\)

But

\[
(7.44) \quad a_k u^p_{0(k)} = u^p_{0(i)} , \quad n_j a_j = n_i , \quad n_j u^p_{0(j)} = n \cdot u^p_{0(i)}
\]

thus, on substitution of (7.44), Eq. (7.43) becomes

\[
(7.45) \quad \sum_{p=0}^{2} \left\{ \lambda n_i \left( u^p_{0(i)} \cdot v s^P_p \right) + \mu (n \cdot v s^P_p) u^p_{0(i)} + \mu (a_i \cdot v s^P_p) (n \cdot u^p_{0}) \right\} = 0
\]

\((i = 1, 2, 3)\)
The preceding equation is derived from Eq. (7.17) and is, in fact, completely equivalent to it. Since Eq. (7.17) is valid for every point on the surface B, the same must be true of Eq. (7.45). With this observation in mind, of course, one can infer that Eq. (7.45) is valid in a small "patch" on B encompassing the point of incidence 0. Now, on the small patch the following approximations can be made [see Eqs. (7.6c) and (7.8)]:

\begin{equation}
(7.46a) \quad n = a_1 \frac{a_1}{R_1} + a_2 \frac{a_2}{R_2} + a_3
\end{equation}

\begin{equation}
(7.46b) \quad VSP = \frac{1}{c_p} \left[ \frac{a_1}{R_1 + a_3} + \frac{a_2}{R_2 + a_3} + \frac{a_3}{R_3} \right]
\end{equation}

\begin{equation}
(7.46c) \quad u^0 = \frac{a_1}{R_1 + a_3} u^0_0 + \frac{a_2}{R_2 + a_3} u^0_2 + \frac{a_3}{R_3} u^0_3
\end{equation}

Thus, by Eqs. (7.46c) and (7.19), it follows that

\begin{equation}
(7.47) \quad u^0_{0(i)} = u^0_0 a_1 = A_{11}^p \left[ u^p_{01} + \frac{a_1}{R_1 + a_3} u^p_{03} \right] + A_{21}^p \left[ u^p_{02} + \frac{a_2}{R_2 + a_3} u^p_{03} \right] + A_{31}^p \left[ u^p_{03} - \frac{a_1}{R_1 + a_3} u^p_{01} - \frac{a_2}{R_2 + a_3} u^p_{02} \right]
\end{equation}
Also, by Eq. (7.46), it is easy to verify that

\[(7.48a) \quad u_{0}^{P}, v_{0}^{P} = \frac{1}{c_{p}} \left\{ \frac{a_{1}^{P}}{R_{1}^{P} + a_{3}^{P}} \left[ u_{01}^{P} + \frac{a_{1}^{P}}{R_{1}^{P} + a_{3}^{P}} u_{03}^{P} \right] + \frac{a_{2}^{P}}{R_{2}^{P} + a_{3}^{P}} \left[ u_{02}^{P} + \frac{a_{2}^{P}}{R_{2}^{P} + a_{3}^{P}} u_{03}^{P} \right] + \left[ u_{03}^{P} - \frac{a_{1}^{P}}{R_{1}^{P} + a_{3}^{P}} u_{01}^{P} - \frac{a_{2}^{P}}{R_{2}^{P} + a_{3}^{P}} u_{02}^{P} \right] \right\} \]

\[(7.48b) \quad m^{P} v_{0}^{P} = \frac{1}{c_{p}} \left\{ \frac{a_{1}^{P}}{R_{1}^{P} + a_{3}^{P}} \left[ \frac{A_{11}^{P}}{R_{1}} \frac{a_{1}}{R_{1}} + \frac{A_{12}^{P} a_{2}}{R_{2}} + A_{13}^{P} \right] + \frac{a_{2}^{P}}{R_{2}^{P} + a_{3}^{P}} \left[ \frac{A_{21}^{P}}{R_{1}} \frac{a_{1}}{R_{1}} + \frac{A_{22}^{P} a_{2}}{R_{2}} + A_{23}^{P} \right] + \left[ \frac{A_{31}^{P}}{R_{1}} \frac{a_{1}}{R_{1}} + \frac{A_{32}^{P} a_{2}}{R_{2}} + A_{33}^{P} \right] \right\} \]

\[(7.48c) \quad a_{i}^{P} v_{0}^{P} = \frac{1}{c_{p}} \left\{ \frac{A_{11}^{P}}{R_{1}^{P} + a_{3}^{P}} \frac{a_{1}}{R_{1}} + \frac{A_{21}^{P} a_{2}}{R_{2}^{P} + a_{3}^{P}} + \frac{A_{31}^{P}}{R_{1}^{P} + a_{3}^{P}} \right\}, \quad (i = 1, 2, 3) \]

\[(7.48d) \quad a_{0}^{P} \cdot u_{0}^{P} = \frac{a_{1}^{P}}{R_{1}} \left\{ \frac{A_{11}^{P}}{R_{1}^{P} + a_{3}^{P}} \left[ u_{01}^{P} + \frac{a_{1}^{P}}{R_{1}^{P} + a_{3}^{P}} u_{03}^{P} \right] + \frac{A_{21}^{P} a_{2}}{R_{2}^{P} + a_{3}^{P}} + \frac{A_{31}^{P} a_{2}}{R_{2}^{P} + a_{3}^{P}} u_{03}^{P} \right\} + \frac{a_{2}^{P}}{R_{2}^{P} + a_{3}^{P}} \left[ u_{02}^{P} + \frac{a_{2}^{P}}{R_{2}^{P} + a_{3}^{P}} u_{02}^{P} \right] + \frac{a_{3}^{P}}{R_{3}^{P} + a_{3}^{P}} \left[ u_{03}^{P} - \frac{a_{1}^{P}}{R_{1}^{P} + a_{3}^{P}} u_{01}^{P} - \frac{a_{2}^{P}}{R_{2}^{P} + a_{3}^{P}} u_{02}^{P} \right] \right\} + \frac{a_{3}^{P}}{R_{3}^{P} + a_{3}^{P}} \left[ u_{03}^{P} - \frac{a_{1}^{P}}{R_{1}^{P} + a_{3}^{P}} u_{01}^{P} - \frac{a_{2}^{P}}{R_{2}^{P} + a_{3}^{P}} u_{02}^{P} \right] \right\} \right\} + \frac{a_{3}^{P}}{R_{3}^{P} + a_{3}^{P}} \left[ u_{03}^{P} - \frac{a_{1}^{P}}{R_{1}^{P} + a_{3}^{P}} u_{01}^{P} - \frac{a_{2}^{P}}{R_{2}^{P} + a_{3}^{P}} u_{02}^{P} \right] \right\} \right\} \]
As was mentioned at the beginning of this section, only the normal incidence case will be considered. For this special case (Fig. 7.7),

\[
\frac{a_2}{R_2} \left\{ A_{12} \left[ u_{01}^p + \frac{a_1^p}{R_1 + a_3^p} u_{03}^p \right] + A_{22} \left[ u_{02}^p + \frac{a_2^p}{R_2 + a_3^p} u_{03}^p \right] \right. \\
\left. + A_{32} \left[ u_{03}^p - \frac{a_1^p}{R_1 + a_3^p} u_{01}^p - \frac{a_2^p}{R_2 + a_3^p} u_{02}^p \right] \right\} + \\
\left\{ A_{13} \left[ u_{01}^p + \frac{a_1^p}{R_1 + a_3^p} u_{03}^p \right] + A_{23} \left[ u_{02}^p + \frac{a_2^p}{R_2 + a_3^p} u_{03}^p \right] \right. \\
\left. + A_{33} \left[ u_{03}^p - \frac{a_1^p}{R_1 + a_3^p} u_{01}^p - \frac{a_2^p}{R_2 + a_3^p} u_{02}^p \right] \right\}
\]

(7.49a) \quad A_{ij}^p = \delta_{ij} \quad (p = 1,2)

\begin{align*}
(7.49b) \quad [A_{ij}^0] & = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\end{align*}

These two equations can be used to simplify Eqs. (7.48b,c,d).

Now, here comes the most crucial step in this section. Let the small "patch" mentioned earlier [see the paragraph below Eq. (7.45)] be parametrized by \( a_1 \) and \( a_2 \). Because of this parametrization \( a_1 \) and \( a_2 \), the first two coordinates of the rectangular coordinate system, can also be regarded as the surface coordinates (see Appendix A) for the small patch. The dual roles of \( a_1 \) and \( a_2 \) should never be confused. Now since
Fig. 7.7. A normally incident ray
Eq. (7.45) is valid for every point on the surface B, certainly, one may
differentiate the left hand side of it with respect to the two surface
coordinates \((a_1,a_2)\) and set the resulting expressions equal to zero.
(Since the left hand side of (7.45) is identically zero on the small
patch.) Using this idea, we first set \(i = 1\) in (7.45) and use Eqs.
(7.47) and (7.48) to obtain

\[
\begin{align*}
\left(7.50\right) \quad \sum_{p=0}^{2} & \left[ \frac{\lambda}{c_p} \frac{a_1}{R_1} \left\{ \frac{a_1^p}{R_1^p + a_3^p} \left[ u_{01}^p + \frac{a_1^p}{R_1^p + a_3^p} u_{03}^p \right] + \frac{a_2^p}{R_2^p + a_3^p} \times \\
& \left[ u_{02}^p + \frac{a_2^p}{R_2^p + a_3^p} u_{03}^p \right] \right\} + \left[ \frac{\mu}{c_p} \frac{a_1^p}{R_1^p + a_3^p} \left\{ \frac{a_1^p}{R_1^p + a_3^p} u_{01}^p - \frac{a_1^p}{R_1^p + a_3^p} u_{03}^p \left[ u_{02}^p + \frac{a_2^p}{R_2^p + a_3^p} u_{03}^p \right] + \\
& \frac{a_2^p}{R_2^p + a_3^p} \left[ u_{03}^p - \frac{a_2^p}{R_2^p + a_3^p} u_{01}^p \right] \right\} \right) = 0
\end{align*}
\]

By differentiating this equation with respect to the two surface
coordinates \(a_1\) and \(a_2\), and substituting \(a_1^p = a_2^p = a_3^p = 0\) in the
resulting expressions, one obtains
Furthermore, by setting $i = 2$ and $3$ in Eq. (7.45) successively, one obtains in a similar fashion the following two equations.

\[
\sum_{p=1}^{2} \frac{1}{c_p} \left\{ \frac{\lambda_{1B}}{R_1} u^0_{03} + \mu \left[ \frac{a^0_{01}}{a_\beta} + \frac{2\delta_{1B}}{R_1} u^0_{03} \right] \right\} = 0
\]

\[
- \frac{1}{c_0} \left\{ \frac{\lambda_{1B}}{R_1} u^0_{03} + \mu \left[ \frac{a^0_{01}}{a_\beta} + \frac{2\delta_{1B}}{R_1} u^0_{03} \right] \right\} (\beta = 1, 2)
\]

Since $\beta$ can assume a value of 1 or 2, each of the above three equations (7.51a-c) gives two equations; thus there are a total of six equations. For the special case of normal incidence, the six equations happen to be uncoupled and can be solved independently to yield the six tangential derivatives.

The results corresponding to a plane incident L-wave and a plane
incident T-wave are given below (in Sections 7.4.1 and 7.4.2). Note that for either of the two incident waves, the curvatures associated with the incident wave are identically zero, i.e., \(1/R_1^0 = 1/R_2^0 = 0\), which can be used to simplify the right hand sides of Eqs. (7.51a-c).

### 7.4.1 Incident L-wave

For a time harmonic plane incident L-wave as defined in Eq. (7.30), \(u_{01}^0 = u_{02}^0 = 0\) and \(u_{03}^0 = 1\). By (7.51) the tangential derivatives of the zeroth order solution with respect to the coordinate system shown in Fig. 7.7 are found to be

\[
\begin{align*}
\left. \frac{\partial u_{01}}{\partial a_1} \right|_{a_2} &= \frac{4\gamma}{R_1}, & \left. \frac{\partial u_{01}}{\partial a_2} \right|_{a_1} &= 0, & \left. \frac{\partial u_{02}}{\partial a_1} \right|_{a_2} &= 0, & \left. \frac{\partial u_{02}}{\partial a_2} \right|_{a_1} &= \frac{4\gamma}{R_2}, & \left. \frac{\partial u_{03}}{\partial a_1} \right|_{a_2} &= 0, & \left. \frac{\partial u_{03}}{\partial a_2} \right|_{a_1} &= 0
\end{align*}
\]

where, recall \(\gamma = c_1/c_2\) is the wavespeed ratio. In arriving at the preceding results we have made use the zeroth order solution at point 0: \(u_{01}^0 = -1, \ u_{02}^0 = 0; \ u_{01}^1 = u_{02}^1 = 0, \ u_{03}^1 = 1\). Also, in (7.52) \(R_1\) and \(R_2\) are the two principal radii of curvature of \(B\) at the point of normal incidence 0. By chain rule of differential calculus, it is trivial to show that \(\partial/\partial a_p = \partial/\partial a_\beta (\beta = 1,2; \ p = 1,2)\) at point 0; hence one may also replace (7.52) with

\[
\begin{align*}
\left. \frac{\partial u_{01}}{\partial a_1} \right|_{a_2} &= \frac{4\gamma}{R_1}, & \left. \frac{\partial u_{01}}{\partial a_2} \right|_{a_1} &= 0, & \left. \frac{\partial u_{02}}{\partial a_1} \right|_{a_2} &= 0, & \left. \frac{\partial u_{02}}{\partial a_2} \right|_{a_1} &= \frac{4\gamma}{R_2}, & \left. \frac{\partial u_{03}}{\partial a_1} \right|_{a_2} &= 0, & \left. \frac{\partial u_{03}}{\partial a_2} \right|_{a_1} &= 0
\end{align*}
\]

It is this set of equations, not (7.52), that will be used later to
obtain the first order solution at the point of normal incidence 0.

7.4.2 Incident T-wave

A time-harmonic plane T-wave incident on the surface B is shown in Fig. 7.8, the unit vector \( \mathbf{D} \) in the same figure designates the polarization direction of the incident shear wave. To be specific, suppose that the incident shear wave is defined by

\[
\mathbf{u}^0 = \mathbf{D} \exp(i\omega t \mathbf{x}/c_2)
\]

Clearly, then

\[
\begin{align*}
(7.55a) & & u^0_{01} = -D_1, & & u^0_{02} = D_2, & & u^0_{03} = 0 \\
(7.55b) & & u^0_{mn} = 0 & & (m = 1, 2, \ldots; n = 1, 2, 3)
\end{align*}
\]

Here \( D_1 = D \cdot a_1 \) and \( D_2 = D \cdot a_2 \) are projections of \( \mathbf{D} \) on \( a_1 \) and \( a_2 \) axes, respectively (Fig. 7.8). By Eqs. (7.51) and (7.55) one can show that, at the point of normal incidence 0,

\[
\frac{\partial u^0_{01}}{\partial a_1^2} = \frac{\partial u^0_{02}}{\partial a_2^2} = \frac{\partial u^0_{02}}{\partial a_2^2} = \frac{\partial u^0_{01}}{\partial a_1^2} = 0, \quad \frac{\partial u^0_{03}}{\partial a_1^1} = \frac{4D_1}{\gamma R_1}, \quad \frac{\partial u^0_{03}}{\partial a_2^1} = \frac{4D_2}{\gamma R_2}
\]

This result and (7.53) provide us the values of tangential derivatives at the point of normal incidence, corresponding to planar incident T- and L-waves, respectively. Their importance lies in the fact that they prepare the way for the determination of first order solutions caused by
Fig. 7.8. An incident T-wave polarized in D-direction
the aforementioned two types of incident waves.

To end this section, we wish to point out that the techniques of this section can be extended without any difficulty to calculate the tangential derivatives of higher order solutions \((m = 1, 2, \ldots)\), which are needed in the calculation of solutions beyond the first order \((m = 2, 3, \ldots)\).

### 7.5 First Order Solution at the Point of Normal Incidence

The present section is devoted to the calculation of the first order solutions, caused by either an incident L- or T-wave, at the point of normal incidence. The results obtained, then, as in the scalar wave problem, will be used (in the next section) to obtain the directly backscattered time-domain far-field responses.

To begin with, we first set \(m = 1\) in Eqs. (7.23a-c). Since \(u^0\) is assumed to be a plane wave (L- or T-wave), \(1/R^0_1 = 1/R^0_2 = 0\), \(u^0_{1(i)} = 0\) \((i = 1, 2, 3)\), and the partial derivatives associated with the incident wave \(u^0\) all vanish identically. Thus, Eqs. (7.23a-c) become, respectively,

\[
\begin{align*}
(7.57a) \quad & \sum_{p=1}^{2} \frac{u^p_{11}}{c_p} = \sum_{p=1}^{2} \frac{u^p_{01}}{R^p_1} \left( - \frac{a^p_{03}}{a_1} - \frac{a^p_{01}}{a_3} \right) \\
(7.57b) \quad & \sum_{p=1}^{2} \frac{u^p_{12}}{c_p} = \sum_{p=1}^{2} \frac{u^p_{02}}{R^p_2} \left( - \frac{a^p_{03}}{a_2} - \frac{a^p_{02}}{a_3} \right)
\end{align*}
\]
(7.57c) \[
\sum_{p=1}^{2} \frac{(\lambda+2\mu)u^p_{13}}{c_p} = -\sum_{p=1}^{2} \left\{ \lambda \left[ \frac{u^0_{03}}{R^p_1} + \frac{\partial u^0_{01}}{\partial a^p_1} + \frac{u^0_{03}}{R^p_2} + \frac{\partial u^0_{02}}{\partial a^p_2} \right] + \frac{\partial u^0_{03}}{\partial a^p_3} \right\}
\]

7.5.1 Incident L-wave

For this case, Eqs. (7.57a-c) yield, respectively,

(7.58a) \[
\frac{u^1_{11}}{c_1} + \frac{u^2_{11}}{c_2} = \frac{\partial u^1_{03}}{\partial a^1_3} + \frac{u^2_{01}}{R^2_1} - \frac{\partial u^2_{01}}{\partial a^2_3}
\]

(7.58b) \[
\frac{u^1_{12}}{c_1} + \frac{u^2_{12}}{c_2} = \frac{\partial u^1_{03}}{\partial a^1_3} + \frac{u^2_{02}}{R^2_2} - \frac{\partial u^2_{02}}{\partial a^2_3}
\]

(7.58c) \[
\frac{(\lambda+2\mu)u^1_{13}}{c_1} + \frac{(\lambda+2\mu)u^2_{13}}{c_2} = -\lambda \left[ \frac{u^1_{03}}{R^1_1} + \frac{u^1_{03}}{R^1_2} \right] - (\lambda+2\mu) \frac{\partial u^1_{03}}{\partial a^1_3}
\]

In arriving at the above results we have used the fact that, for the zeroth order solution, the P-wave is purely longitudinal and the S-wave is purely transverse (recall Table. 5.1), i.e.,

(7.59) \[
u^1_{01} = u^1_{02} = u^2_{03} = 0
\]
The tangential derivatives on the right hand sides of Eqs. (7.58) are known by virtue of (7.53). The normal derivatives (viz. \( \partial / \partial a_3 \)) can be evaluated by using the results of Section 5.4. Furthermore, from the zeroth order solution, \( u_{01}^2 = u_{02}^2 = 0, u_{03}^1 = -1 \). Thus the right hand sides of Eqs. (7.58) are known completely. After some straightforward calculations, one has

\[
\begin{align*}
(7.60a) & \quad \frac{u_{11}^1}{c_1} + \frac{u_{11}^2}{c_2} = 0 \\
(7.60b) & \quad \frac{u_{12}^1}{c_1} + \frac{u_{12}^2}{c_2} = 0 \\
(7.60c) & \quad (\lambda + 2\mu) \left[ \frac{u_{13}^1}{c_1} + \frac{u_{13}^2}{c_2} \right] = (\kappa_1 + \kappa_2) \left[ \lambda \left( 1 - \frac{4c_1}{c_2} \right) - 2\mu \right]
\end{align*}
\]

where \( \kappa_1 = 1/R_1 \), \( \kappa_2 = 1/R_2 \) are principal curvatures of surface \( B \) at the point of normal incidence \( 0 \). At a glance, it seems that there are six unknowns whereas only three equations (7.60a,b,c) are available. By recalling the theory of Chapter 5, however, we realize that three out of the six unknowns are minor components, which can be readily determined using Eq. (7.53) and the equations of Table 5.1. The results for these minor components are

\[
(7.61) \quad u_{11}^1 = u_{12}^2 = 0, \quad u_{13}^1 = -4c_1(\kappa_1 + \kappa_2)
\]
Thus by making use of (7.61) in (7.60), one is able to find the remaining three unknowns (three major components):

\[ u_{11}^2 = u_{12}^2 = 0, \quad u_{13}^1 = c_1(\kappa_1 + \kappa_2) \left[ 1 + 8(c_2/c_1) - 4(c_2/c_1)^2 \right] \]

With this result, it then follows from \( u_{p1}^1 = u_{p1n}^{p} w_{n} \) (\( p = 1, 2 \) and sum over \( n = 1, 2, 3 \)) and Eq. (7.61) that

\[
\begin{align*}
(7.63a) & \quad u_1^1 = c_1(\kappa_1 + \kappa_2) \left[ 1 + 8(c_2/c_1) - 4(c_2/c_1)^2 \right] a_3 \\
(7.63b) & \quad u_1^2 = -4c_1(\kappa_1 + \kappa_2) a_3
\end{align*}
\]

Also, from the zeroth order solution, one has

\[
\begin{align*}
(7.64a) & \quad u_0^1 = u_0^1 w_{n} a_{n} = -a_3 \\
(7.64b) & \quad u_0^2 = u_0^2 w_{n} a_{n} = 0
\end{align*}
\]

Hence, from the ray series representation

\[
(7.65) \quad u^p = \exp(i\omega \mathbf{S}^p) \left[ u_0^p + \frac{1}{i\omega} u_1^p + \ldots \right] \quad (p = 1, 2)
\]

and Eqs. (7.63), (7.64), it is not difficult to find
Eqs. (7.66a) and (7.66b) correspond to, respectively, the reflected P- and S-waves at point 0 caused by an incident longitudinal wave defined in Eq. (7.57). Note that, as expected, $u^2$ is not purely transverse (recall the discussion in Chapter 5). The total reflected field $u_*^\text{Ref}$ at point 0 is the sum of $u^1$ and $u^2$, hence, to the first order approximation,

$$u_*^\text{Ref} = \exp(i\omega S^0) \left\{ -1 + \frac{c_1(K_1 + K_2)}{i\omega} \left[ 1 + 8 \left( \frac{c_2}{c_1} \right) - 4 \left( \frac{c_2}{c_1} \right)^2 \right] \right\} a_3$$

Having obtained this result, one may dispense with the cumbersome ray pencil coordinate systems introduced previously and establish a new coordinate system as shown in Fig. 7.9. With respect to this new coordinate system, Eq. (7.67) can be restated as

$$u_*^\text{Ref} = \exp(i\omega \xi/c_1) \left\{ 1 + \frac{c_1(K_1 + K_2)}{i\omega} \left[ 3 - 8 \left( \frac{c_2}{c_1} \right) + 4 \left( \frac{c_2}{c_1} \right)^2 \right] \right\} I$$

The above form for the reflected field $u_*^\text{Ref}$ will prove to be essential later when the time-domain far-field response is considered.
I: unit vector in the incident wave direction

y: observation point in the far-field

|y|: |y|

I: unit vector y/y

Fig. 7.9. A coordinate system with origin at the specular point 0
By an analogous argument employed in Section 6.3, one may assume that Eq. (7.68) is not only valid at the point of normal incidence 0 but also is applicable to a small neighborhood of point 0. This observation, as one has already seen in Section 6.3, plays a crucial role in the determination of the first order solution.

7.5.2 Incident T-wave

For a time-harmonic incident plane T-wave (see Fig. 7.8) as defined in Eq. (7.54), Eqs. (7.57a-c) yield

\[
\begin{align*}
(7.69a) & \quad \frac{u_{11}}{c_1} + \frac{u_{11}}{c_2} = -\frac{\partial u_{03}}{\partial a_1} + \frac{u_{01}}{R_1} - \frac{\partial u_{01}}{\partial a_3} \\
(7.69b) & \quad \frac{u_{12}}{c_1} + \frac{u_{12}}{c_2} = -\frac{\partial u_{03}}{\partial a_2} + \frac{u_{02}}{R_2} - \frac{\partial u_{02}}{\partial a_3} \\
(7.69c) & \quad (\lambda + 2\mu) \left[\frac{u_{13}}{c_1} + \frac{u_{13}}{c_2}\right] = -\lambda \left[\frac{u_{03}}{R_1} + \frac{u_{03}}{R_2}\right] - (\lambda + 2\mu) \frac{\partial u_{03}}{\partial a_1} - \\
& \quad \quad \quad \quad \lambda \left[\frac{\partial u_{01}}{\partial a_2} + \frac{\partial u_{02}}{\partial a_3}\right]
\end{align*}
\]

Here \(R_1^1, R_1^2\) are principal radii of the reflected P-wavefront at the point of reflection 0, and \(R_2^1, R_2^2\) are those of the reflected S-wavefront at the same point; they can be found by using the equations developed in Section 5.4. Other terms on the right hand sides of Eqs. (7.69a-c) can be evaluated as before [see the paragraph following Eq.
(7.59)]. The minor components on the left hand sides of the above three equations can also be evaluated using Table 5.1 and equation (7.56). The results are

\[(7.70) \quad u_{11}^1 = 4c_2D_1/R_1, \quad u_{12}^1 = 4c_2D_2/R_2, \quad u_{13}^2 = 0\]

The three major components left over can then be obtained by solving Eqs. (7.69a-c). They are found to be

\[(7.71a) \quad u_{11}^2 = c_2D_1 \left( \kappa_1 + \kappa_2 \right) \left( 3 - 8 \frac{c_2}{c_1} \right)\]

\[(7.71b) \quad u_{12}^2 = c_2D_2 \left( \kappa_1 + \kappa_2 \right) \left( 3 - 8 \frac{c_2}{c_1} \right)\]

\[(7.71c) \quad u_{13}^1 = 0\]

With the results of (7.70) and (7.71) one can find, by following a procedure similar to that employed in Section 7.5.1, that the reflected field at point 0 is

\[(7.72) \quad u_{\text{Ref}} = \exp(i\omega s^0) \left\{ D + \frac{c_2}{i\omega} \left[ \left( \kappa_2 - \Lambda_T \kappa_1 \right)D_1a_1 + \left( \kappa_1 - \Lambda_T \kappa_2 \right)D_2a_2 \right] \right\}\]

where \(\Lambda_T\) is a constant defined by

\[(7.73) \quad \Lambda_T = 8(c_2/c_1) - 7\]
and

\[(7.74) \quad D_1 = D \cdot a_1, \quad D_2 = D \cdot a_2, \quad S^0 = \frac{\varepsilon_0}{c_2} \cdot x\]

The same remarks stated in the last paragraph of Section 7.5.1 also apply to (7.72).

### 7.6 Time-Domain Far-Field First Order Solution

In this section, the results (7.68) and (7.72) derived in the preceding section will be used to obtain the time-domain far-field first order solution. The approach used below is, in essence, the same as that used previously in Section 6.3.

As in Chapter 6 (Section 6.3) we shall confine ourselves to the directly backscattered leading edge response. Suppose \( y \) is an observation point in the far-field (see Fig. 7.9), then the scattered wave at point \( y \) is, by Eq. (3.30),

\[(7.75) \quad u_m^{sc}(y) = \frac{A_m}{y} \exp(ik_1 y) + \frac{B_m}{y} \exp(ik_2 y)\]

In accordance with the notations of Fig. 7.9, the amplitude vectors \( A_m \) and \( B_m \), from Eqs. (3.28) and (3.29), are given by

\[(7.76) \quad A_m(Y) = Y_{m^1} f_1(k_1)\]
\[(7.77) \quad B_m(Y) = (\delta_{i,m} - Y_{i^1} Y_{m^1}) f_2(k_2)\]
Here $f_{\perp}$ is the $f$-vector defined previously in Eq. (3.29). For a traction free void, which we have assumed and will continue to assume, the second integral in Eq. (3.29) is absent. Also, as in Section 6.3.2, our main interest will be placed on the leading edge response only. Thus, by following the same procedure described in Section 6.3.2 and employing the relationship $\omega = k_{\alpha} c_{\alpha} (\alpha = 1, 2; \text{no sum on } \alpha)$, one obtains from Eq. (3.29)

\begin{equation}
(7.78) \quad f_{\perp}(k_{\alpha}) = -\frac{i\omega}{4\pi c_{\alpha}^3} \int_{B_e} C_{ijkl} Y_i n_k u_1 \exp(-i k_{\alpha} Y \cdot x) \, dB(x), \quad \alpha = 1, 2
\end{equation}

where $u_1$ is the total field on $B_e$, that is

\begin{equation}
(7.79) \quad u_1 = u_1^0 + u_1^\text{Ref} \quad \text{(on } B_e) \quad ,
\end{equation}

Now we consider two cases corresponding to planar incident longitudinal and transverse waves.

7.6.1 Incident L-wave

Following the procedure of Section 6.3.2 we shall use the method of Fourier synthesis to obtain the time-domain response. Accordingly, we shall first determine the frequency-domain response. To this end, let us consider a time-harmonic, longitudinal, incident plane wave defined by

\begin{equation}
(7.80) \quad u_1^0 = I \exp(i \omega \cdot \hat{n}/c_1)
\end{equation}
The reflected wave at the specular point was found to be [see (7.68)]

\[ u_{\text{Ref}} = I \exp(i\omega \mathbf{L} \cdot \mathbf{x}/c_1) \left(1 + \frac{c_1(K_1 + K_2)A_L}{i\omega}\right) \]

where, for brevity, we have defined the constant \( A_L \) by

\[ A_L = 3 - 8(c_2/c_1) + 4(c_2/c_1)^2 \]

As stated in the last paragraph of Section 7.5.1, Eq. (7.81) will be assumed to hold not only at the specular point \( 0 \) (Fig. 7.9) but also at every point on \( B_\varepsilon \). Thus, by substituting (7.80) and (7.81) in (7.79) one obtains the total displacement field on \( B_\varepsilon \):

\[ u_1 = I_1 \exp(i\omega \mathbf{L} \cdot \mathbf{x}/c_1) \left(2 + \frac{c_1(K_1 + K_2)A_L}{i\omega}\right) \]

Placing this result back into the expression (7.78) for the \( f \)-vector, we find for the backscattering case \( (\mathbf{I} = -\mathbf{y}) \) that

\[ f_1(k_\alpha) = \frac{i\omega}{4\pi c_\alpha} \left(2 + \frac{c_1(K_1 + K_2)A_L}{i\omega}\right) \times \int_{B_\varepsilon} C_{ijkl} I_j I_k I_1 \exp\left(i\omega \left[\frac{1}{c_1} + \frac{1}{c_\alpha}\right] \mathbf{L} \cdot \mathbf{x}\right) dB(x) \]
Thus, by Eq. (7.76) and $I = -\mathbf{Y}$,

\begin{equation}
A_m = \frac{i \omega}{4 \pi c_1} \left\{ 2 + \frac{c_1 (K_1 + K_2) A_L}{i \omega} \right\} I_m \times \int_{B \epsilon} C_{ijkl} I_j n_k l I_i \exp(2i \omega \mathbf{L} \cdot \mathbf{x}/c_1) \ dB(x)
\end{equation}

But

\begin{equation}
C_{ijkl} I_j n_k l I_i = (\lambda + 2\mu)(n \mathbf{I})
\end{equation}

so

\begin{equation}
A_m = \frac{\nu_m}{2 \pi c_1} \left\{ \frac{i \omega}{4 \pi} + \frac{(K_1 + K_2) A_L}{4 \pi} \right\} \int_{B \epsilon} \exp(2i \omega \mathbf{L} \cdot \mathbf{x}/c_1) \ dB(x)
\end{equation}

where $dS_3$ is the projection of the differential surface area $dB(x)$ on the $x_1 x_2$-plane, i.e., $dS_3 = \mathbf{Y} \cdot n dB(x)$.

Next, we proceed to consider the determination of $\mathbf{B}$-vector. From Eqs. (7.77) and (7.84),

\begin{equation}
B_m = b \int_{B \epsilon} \left( \delta_{im} - I_m I_m \right) C_{ijkl} I_j n_k l I_i \exp(i \omega \mathbf{L} \cdot \mathbf{x}) dB(x)
\end{equation}
where, for brevity, we have defined

\[(7.89a) \quad b = \frac{i \omega}{4 \pi pc_p} \left\{ \frac{1}{2} + \frac{c_1 (K_1 + K_2)}{i \omega} \right\} \]

\[(7.89b) \quad \Omega = \frac{1}{c_1} + \frac{1}{c_2} \]

By using

\[(7.90) \quad (\delta_{im} - I_i I_m) C_{ijkl} I_j n_k I_l = \mu |n_m - (I \cdot n) I_m| \]

Eq. (7.88) can be expressed in vector form as

\[(7.91) \quad \mathbf{B} = \mathbf{b} \mu \int_{B_\mathbf{c}} [n - (I \cdot n) I] \exp(i \Omega \cdot \mathbf{x}) \, dB(\mathbf{x}) \]

However, from the vector identity [143]: \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \),
the bracketed term in the above equation can be written as

\[(7.92) \quad n - (I \cdot n) I = (I \cdot I)n - (I \cdot n) I = I \times (n \times I) \]

Also, it can be verified that

\[(7.93) \quad I \exp(i \Omega \cdot \mathbf{x}) = \frac{1}{i \Omega} \nabla \left[ \exp(i \Omega \cdot \mathbf{x}) \right] \]
Thus it follows from Eqs. (7.91-7.93) that

\[ B = \frac{b\mu}{i\Omega} \times \int_{B} \left\{ n \times \nabla\left\{ \exp(i\Omega \cdot x) \right\} \right\} dB(x) \]  

(7.94)

By Stoke's theorem [143], however, the above surface integral can be converted into a line integral around the boundary curve \( C_{\varepsilon} \) of the open surface \( B_{\varepsilon} \). Thus, in terms of the resulting line integral, the preceding equation becomes

\[ B = \frac{b\mu}{i\Omega} \times \int_{C_{\varepsilon}} t \exp(i\Omega \cdot x) \, dc \]  

(7.95)

where \( t \) is the unit tangent vector to the boundary curve \( C_{\varepsilon} \) and \( dc \) is the differential arc length along \( C_{\varepsilon} \).

Now, as mentioned earlier, our main interest will be placed on the backscattered leading edge response, i.e., the early arriving waves (see Section 6.3.2). Bearing this in mind, we may interpret \( C_{\varepsilon} \) as the intersection of the scattering surface \( B \) and the planar incident wavefront which just passed the specular point \( x = 0 \). As a consequence, the inner product \( t \cdot x \) has a constant value, say \( d_{\varepsilon} \), around the entire (plane) curve \( C_{\varepsilon} \). Geometrically, this constant \( d_{\varepsilon} \) is the perpendicular distance from the specular point \( O \) to the planar incident wavefront being considered. In view of the above interpretation for \( C_{\varepsilon} \), equation (7.95) may be written as follows:
The line integral in (7.96), however, is zero. Accordingly, the amplitude vector $B$ vanishes. For convenience, let us now collect the results we have obtained so far in the following two equations:

\[
\begin{align*}
(7.97a) & \quad A_m = Y_m \left\{ \frac{i\omega}{2\pi c_1} + \frac{(k_1 + k_2)A_m}{4\pi} \right\} \int_{B_{\epsilon}} \exp(2i\omega \cdot x/c_1) \, dS_3 \\
(7.97b) & \quad B_m = 0
\end{align*}
\]

In Eq. (7.97a), $dS_3$ is the projection of $dB(x)$ on the $x_1x_2$-plane, namely, $dS_3 = Y\cdot n \, dB(x)$. Inserting the above two expressions for $A_m$ and $B_m$ in Eq. (7.75), one obtains the following frequency-domain, far-field scattered wave:

\[
(7.98) \quad u_{Sc}^{\omega}(y, \omega) = Y_m \left\{ \frac{i\omega}{2\pi c_1} + \frac{(k_1 + k_2)A_m}{4\pi} \right\} \int_{B_{\epsilon}} \exp \left\{ i\omega(2\omega \cdot x + y)/c_1 \right\} \, dS_3
\]

Now one is ready to Fourier-synthesize the frequency-domain result to obtain the time-domain response caused by an arbitrary incident pulse. As a first step, suppose that the incident pulse is defined by

\[
(7.99) \quad u^0(x, t) = \mathcal{F}\left\{ t - \frac{I \cdot x}{c_1} \right\} = \mathcal{F}(\omega) \exp\left\{ -i\omega \left( t - \frac{I \cdot x}{c_1} \right) \right\} \, d\omega
\]
Here, \( f \) is an appropriate function specifying the shape of the incident wave form and \( F(\omega) \) is the Fourier transform of the function \( f \) [see Eq. (6.33) for the definition of Fourier transform]. Then, by Fourier synthesis, the time-domain response caused by this incident pulse is

\[
(7.100) \quad u_{m}^{SC}(y,t) = \int_{-\infty}^{\infty} u_{m}^{SC}(y,\omega) F(\omega) \exp(-i\omega t) \, d\omega
\]

In this equation, \( u_{m}^{SC}(y,\omega) \) is the frequency-domain response which has already been given in Eq. (7.98). By following the same line of approach discussed in Section 6.3.2 one can show that

\[
(7.101) \quad u_{m}^{SC}(y,t) = \frac{I_{m}}{2\pi c_{1} y} \int_{B_{\xi}} \frac{df(\xi)}{d\xi} \frac{dS_{3}}{dx_{3}} d\xi - \frac{I_{m}(\kappa_{1} + \kappa_{2}) \lambda_{L}}{4\pi y} \int_{B_{\xi}} f(\xi) \frac{dS_{3}}{dx_{3}} d\xi
\]

where

\[
(7.102) \quad \xi = t - \frac{(21 \cdot x + y)}{c_{1}} = t - \frac{(2x_{3} + y)}{c_{1}}
\]

If, as was assumed in the scalar wave problem, the incident pulse is an ideal impulse, i.e., \( f = \delta \), it can be shown that
\begin{align*}
(7.103) \quad f(\xi) = \delta(\xi) &= \frac{c_1}{2} \delta \left[ x_3 - \frac{c_1 \tau}{2} \right] \\
(7.104) \quad \frac{df(\xi)}{d\xi} = \frac{d\delta(\xi)}{d\xi} &= -\frac{c_1^2}{4} \frac{d}{dx_3} \delta \left[ x_3 - \frac{c_1 \tau}{2} \right]
\end{align*}

where \( \tau \) is defined by

\begin{equation}
(7.105) \quad \tau = t - (y/c_1)
\end{equation}

Notice that Eqs. (7.103) and (7.104) are analogous to Eqs. (6.44) and (6.46). Finally, as in Section 6.3.2, we again appeal to the theory of distributions and obtain from Eqs. (7.101), (7.103), and (7.104) the impulse response. Expressed in terms of the notations of Fig. 7.9, the impulse response is

\begin{equation}
(7.106) \quad u_{sc}^c(y, t) = I \frac{c_1}{8ny} \left[ \frac{d^2 S_3}{dx_3^2} - \Lambda_L (\kappa_1 + \kappa_2) \frac{dS_3}{dx_3} \right] x_3 = c_1 \tau/2
\end{equation}

In the above equation \( S_3 \) is the silhouette area function introduced earlier in Chapter 6. Comparing Eq. (7.106) with Eq. (6.49) one sees immediately the similarity between these two equations, the only distinction being the presence of the constant factor \( \Lambda_L \) in Eq. (7.106).
7.6.2 Incident T-wave

For a time harmonic, transverse, incident plane wave defined by

\( u^0 = D \exp(\imath \omega \overline{\mathbf{I}} \cdot \mathbf{x} / c_2) \)

the total field \( u \) on \( B_e \) can be found by adding (7.107) to (7.72).

Stated in accordance with the notations of Fig. 7.10, \( u \) is given by

\[ u = \exp(\imath \omega \overline{\mathbf{I}} \cdot \mathbf{x} / c_2) \left\{ 2D + \frac{c_2}{\imath \omega} \hat{V} \right\} \]

where, for convenience, we have introduced the definition:

\[ \hat{V} \equiv (\kappa_2 - \Lambda_1 \kappa_1)D_1 \mathbf{x}_1 + (\kappa_1 - Z \kappa_2)D_2 \mathbf{x}_2 \]

Substituting Eq. (7.108) into Eq. (7.78) yields, for the back scattering case (\( \mathbf{I} = -\mathbf{Y} \)),

\[ f_i(k_\alpha) = \frac{\imath \omega}{4 \pi c_{\alpha}^3} \int_{B_e} C_{ijk_l} I_{n_k} J_{n_l} \exp \left\{ \imath \omega \left[ \frac{1}{c_2} + \frac{1}{c_{\alpha}} \right] \overline{\mathbf{I}} \cdot \mathbf{x} \right\} \, dB(x) \]

where \( J_1 \) is a vector defined by

\[ J_1 = 2D_1 + \frac{c_2}{\imath \omega} V_1 \]
Fig. 7.10. An incident T-wave polarized in the D-direction
Notice that this $J$ vector is perpendicular to $I$, i.e.,

(7.112) \[ \mathbf{I} \cdot \mathbf{J} = 0 \]

Combining Eqs. (7.110) and (7.76) and using $Y_m = -I_m$, we obtain

\[
(7.113) \quad A_m = \frac{i \omega}{4 \pi pc^3} \int_{B_\epsilon} C_{ijkl} I_{l} J_{k} J_{m} I_{i} \exp(i \mathbf{Q} \cdot \mathbf{x}) \, dB(x).
\]

where $Q$ is the same as defined in Eq. (7.89b). But, with the aid of Eq. (7.112), one can show that

(7.114) \[ C_{ijkl} I_{l} J_{k} J_{m} I_{i} = \lambda(n \cdot J) I_m \]

Consequently, in vector notation, Eq. (7.113) can be written as

\[
(7.115) \quad A = \frac{i \omega \lambda}{4 \pi pc^3} \int_{B_\epsilon} (n \cdot J) I \exp(i \mathbf{Q} \cdot \mathbf{x}) \, dB(x)
\]

Now since $I \cdot J = 0$, it is clear that

(7.116) \[ (n \cdot J) I = (n \cdot J) I - (I \cdot J) n = J \times (I \times n) \]

where, the last equality follows from the vector triple cross product identity mentioned in Section 7.6.1. By using Eqs. (7.116) and (7.93)
and invoking the Stoke's theorem, the amplitude vector $A$ stated in (7.115) can be expressed in terms of a line integral around the boundary curve $C_\varepsilon$ of $B_\varepsilon$ (cf. Section 7.6.1). The result is

\begin{equation}
A = \frac{-i\omega}{4\pi pc_1^3} \int_{C_\varepsilon} t \exp(ia|x|) \, dc \tag{7.117}
\end{equation}

But, the above line integral has been found to have a zero value in Section 7.6.1. Thus,

\begin{equation}
A_m = 0 \tag{7.118}
\end{equation}

Next, we proceed to determine $B$. From Eqs. (7.77) and (7.110),

\begin{equation}
B_m = \frac{\omega}{4\pi pc_2^3} \int_{B_\varepsilon} c_{ijkl} I_i n_j I_l (\delta_{im} - I_i I_m) \exp(2ia|x|/c_2) \, dB(x) \tag{7.119}
\end{equation}

where use has been made of $Y_m = -I_m$. Also, after some manipulation, it is not difficult to show that

\begin{equation}
c_{ijkl} I_i n_j I_l (\delta_{im} - I_i I_m) = \mu(I \cdot n) I_m \tag{7.120}
\end{equation}

Hence, by inserting this result into Eq. (7.119) and recalling the definition of the vector $J$ [see Eq. (7.111)], Eq. (7.119) gives
Substitution of (7.118) and (7.121) in (7.75) then yields the desired frequency-domain response:

\[
(7.122) \quad u_m^{sc}(y, \omega) = -\left[ \frac{i \omega_m}{2 \pi c_2 y} + \frac{V_m}{4 \pi y} \right] \int_{B_e} \exp \left( \frac{i \omega (2x_3 + y)}{c_2} \right) dS_3
\]

If the incident wave is an ideal transverse impulse, i.e.,

\[
(7.123) \quad u^0 = \delta \left[ t - \frac{r_1 \cdot x}{c_2} \right] \quad (D, I = 0)
\]

then the time-domain response can be found to be

\[
(7.124) \quad u_m^{sc}(y, t) = \frac{c_2}{8 \pi y} \left\{ V_m \frac{d^2 S_3}{dx_3^2} - V_m \frac{dS_3}{dx_3} \right\} \bigg|_{x_3 = c_2 t/2}
\]

By use of the definition (7.109) for the \( V \) vector, the preceding equation can be expressed more explicitly as
(7.125) \[ u^{sc}(y,t) = \frac{c_2}{8\pi y} \left( \left[ \frac{d^2S_3}{dx_3^2} - (\kappa_2 - \Lambda_1\kappa_1) \frac{dS_3}{dx_3} \right] D_1 x_1 + \left[ \frac{d^2S_3}{dx_3^2} - (\kappa_1 - \Lambda_1\kappa_2) \frac{dS_3}{dx_3} \right] D_2 x_2 \right) \bigg|_{x_3 = c_2 \tau/2} \]

Here (see Fig. 7.10),

(7.126) \[ D_1 = D \cdot x_1 \quad D_2 = D \cdot x_2 \quad \tau = t - (y/c_2) \]

Results similar to Eq. (7.125) can be found in the electromagnetic literature [169-174]. However, almost all of the results reported there by various authors were obtained, more or less, based on a method originated from Bennett et al. [169]. Essentially, they started with the space-time integral equation for the currents on the bounded surface \( B \) of a perfectly conducting body. The ordinary method of successive approximations (iterations) in the theory of integral equations was then used to generate a Neumann series. The first term of the Neumann series corresponds to the well known physical optics (or Kirchhoff) approximation, the second term being a correction to it. All terms that follow the second term in the Neumann series were ignored by the aforementioned authors. By using this approach, the electromagnetic counterpart of Eq. (7.125) was obtained.

However, this iterative approach involves a number of ad hoc assumptions [169-174]. And, in fact, our use of ray theory was initiated by the desire of removing such assumptions for the
elastodynamic case.

A few papers using ray techniques, however, did appear in the electromagnetic literature, such as Schensted's [175] and Lee's [28]. But these papers only gave explicit results for simple shape scatterers, such as spheres and axially symmetrical bodies.

In regards to the elastodynamic wave problems, Resende [140], a former graduate student of J. B. Keller in Courant Institute of Mathematical Sciences, seems to be the first one that employed the ray techniques. But he only studied the scattering problems by cylindrical scatterers of infinite lengths, i.e., two dimensional problems. In the mid-1970s Achenbach and his coworkers became involved in this field and made significant contributions to the elastodynamic ray theory for crack scattering problems [12,127-139,176]. For general three dimensional elastodynamic scattering problems, however, no articles using ray techniques appear available.
8. THE INVERSE PROBLEM FOR A VOID

The results of the preceding chapter have some important implications with respect to sizing algorithms that have been previously used and can form the basis for a number of new algorithms.

First, we consider the case of L-wave scattering. From (7.106), it can be seen that the far-field impulse response of a void consists of two contributions, one is proportional to the second derivative of the silhouette (cross sectional) area function and the other is proportional only to the first derivative. The first derivative term, however, contains the coefficient $\Lambda_L$ defined in (7.82). An interesting property of this coefficient is that it is small when the wave speed ratio $c_2/c_1$ is near 1/2, which occurs when the Poisson's ratio $\nu = 1/3$. When $c_2/c_1 = 1/2$, this coefficient $\Lambda_L$ becomes precisely zero. This is important for the following reasons: (i) As shown in Table 8.1, many structural materials used in practice happen to have a Poisson's ratio of approximately 1/3, thus, in many practical situations, the coefficient $\Lambda_L$ can be expected to be relatively small. (ii) For computational convenience, many numerical studies choose $\nu = 1/3$ and, therefore, miss the effects of the first derivative term. (iii) When $\Lambda_L = 0$ the far-field scattered response (caused by the lit side of the scatterer) calculated from Eq. (7.106) coincides with that obtained from the Born approximation.

Let us now expand on consequence (iii) for a moment. As was mentioned previously, the Born approximation is a long-wavelength, weak-scattering theory. It has formed the basis of the familiar 1-D inverse
Table 8.1. Properties of some structural materials

<table>
<thead>
<tr>
<th>materials</th>
<th>$\nu$</th>
<th>$c_2/c_1$</th>
<th>$\Lambda_L$</th>
<th>$\Lambda_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>aluminum</td>
<td>.334</td>
<td>.499</td>
<td>.003</td>
<td>-3.006</td>
</tr>
<tr>
<td>beryllium copper</td>
<td>.285</td>
<td>.548</td>
<td>-.184</td>
<td>-2.613</td>
</tr>
<tr>
<td>carbon steel</td>
<td>.792</td>
<td>.542</td>
<td>-.161</td>
<td>-2.664</td>
</tr>
<tr>
<td>cast iron (gray)</td>
<td>.211</td>
<td>.605</td>
<td>-.377</td>
<td>-2.158</td>
</tr>
<tr>
<td>magnesium</td>
<td>.350</td>
<td>.480</td>
<td>.008</td>
<td>-3.157</td>
</tr>
<tr>
<td>molybdenum</td>
<td>.307</td>
<td>.528</td>
<td>-.108</td>
<td>-2.778</td>
</tr>
<tr>
<td>monel metal</td>
<td>.320</td>
<td>.515</td>
<td>-.057</td>
<td>-2.884</td>
</tr>
<tr>
<td>nickel silver</td>
<td>.322</td>
<td>.512</td>
<td>-.049</td>
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</tr>
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<td>phosphor bronze</td>
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<td>.482</td>
<td>.075</td>
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</tr>
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<td>rubber</td>
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<td>.000</td>
<td>3.000</td>
<td>-7.000</td>
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<tr>
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<td>.530</td>
<td>-.115</td>
<td>-2.762</td>
</tr>
<tr>
<td>titanium</td>
<td>.326</td>
<td>.508</td>
<td>-.033</td>
<td>-2.934</td>
</tr>
</tbody>
</table>

The values given are the average values and depend upon the method of processing and the alloying elements.
Born sizing method and has proved to be useful in sizing weak scatterers as well as strong ones, including voids [70,177]. The success of the 1-D inverse Born method for sizing such strong scatterers as voids has been puzzling, considering its assumptions of the weak scattering limit. Our results, through equation (7.106), provide a rational explanation of why it works so well. To see this, it is necessary to understand the 1-D inverse Born method. Below, we give a brief explanation of this method.

8.1 The Born Approximation for a Void

Consider a convex void subjected to an incident impulsive plane L-wave. In the Born approximation, the directly backscattered L-wave in the far-field can be shown (Appendix B) to be identical to Eq. (7.106) with $A_L$ set to zero, that is,

$$u^{SC}(y, \tau) = \frac{c_1}{8\pi y} \left. \frac{d^2 S_3}{dx_3^2} \right|_{x_3 = c_1 \tau/2}$$

However, in contrast to the physical optics approximation, the area function $S_3(x_3)$ in Eq. (8.1) is now allowed to vary over the entire flaw surface. For a spherical void, for example, the Born approximation [Eq. (8.1)] predicts a scattered response given by Fig. 8.1a. This figure can be compared to the "exact" numerical result obtained by Opsal and Visscher [153] for a spherical void, assuming $\nu = 1/3$ for the host medium. In the Born response we see a leading delta function (the left most spike $A^\delta$ in Fig. 8.1a) followed by a constant step part $A^H$, and
Fig. 8.1. Impulse responses of spherical and ellipsoidal voids
then a second equal magnitude ($A^\delta$) trailing delta function. The numerical response of Opsal and Visscher agrees with both the leading delta function and the initial step value of the Born response. However, unlike the Born response, the leading delta function of the exact response is not followed by a plateau region and trailing delta function, but instead is followed by a trailing creeping wave $A^C$. Here, it should be pointed out that the Born step value only coincides with the exact result because $\nu = 1/3$ (i.e., $A_L = 0$) was selected in Opsal and Visscher's numerical study. In general, the Born approximation can only get $A^\delta$ correct but not $A^H$. If we further compare the Born and exact results with the extended Kirchhoff solution (7.106) for $A_L = 0$ and $A_L \neq 0$ (see Fig. 8.1), we note that, unless $A_L(\kappa_1 + \kappa_2)$ is large, all of the results shown in Fig. 8.1 are in relatively good agreement between the flaw center and the leading delta function. In other words, in spite of its weak-scattering assumption, the Born approximation may be capable of predicting an accurate leading edge response. The condition $A_L(\kappa_1 + \kappa_2)d \ll 1$, where $d$ is a typical flaw dimension, however, is necessary in order to guarantee this.

8.2 1-D Inverse Born Approximation

The Born approximation can be utilized to develop 1-D [85], 2-D [79], as well as 3-D [85] inverse sizing algorithms. Here, we shall only describe the 1-D inverse Born approximation, which has been used to do equivalent flaw sizing in the following fashion. First, the Born response, given by Eq. (8.1), is used to locate the center of the flaw, $C$. The 1-D inverse Born approximation assumes that the flaw center $C$,
as in the spherical void case, is located at the place where the area
function $S_3$ is a maximum. Based upon this assumption, the center $C$ can
be determined by integrating Eq. (8.1) twice [85,177]. Next, by either
a time or frequency domain integration [85,178], the location of the
leading edge delta function (which corresponds to the response when the
incident plane wavefront first strikes the flaw, see Fig. 8.2a) can be
found. Knowing this location for the leading edge delta function and
the flaw center $C$, one is able to determine the perpendicular distance $h$
from center $C$ to the "front face" tangent plane $\Pi$. This sizing
parameter $h$ is the basic output of the 1-D inverse Born approximation.
For spherical flaws, of course, a single $h$ obtained from an arbitrary
pulse-echo experiment determines the flaw's geometry completely. For a
general flaw, however, a single $h$ is not sufficient to describe the
flaw's geometrical features. In this general case, the Born algorithm
has been used to do equivalent flaw sizing in terms of a "best-fit"
ellipsoid. Briefly, this is accomplished by using the 1-D inverse Born
approximation to obtain the distance $h$ at a number of different
directions (Fig. 8.2b). The perpendicular distance $h$ for an arbitrary
ellipsoid, as shown in Appendix C, depends only on the three semi-major
axes and the orientation of the ellipsoid. Thus, one can use a
nonlinear least squares algorithm to obtain an equivalent ellipsoid that
best fits the scattering data [74,179]. This method, in conjunction
with a novel multiviewing transducer system developed at Ames
Laboratory, has been shown to be capable of sizing weak and strong
scatterers, including voids [180].
Fig. 8.2. Perpendicular distance function \( h \) used in equivalent flaw sizing
Why does the 1-D inverse Born approximation work so well on such strong scatterers as voids, even though it is based on a weak scattering theory? Eq. (7.106) and Figs. 8.1a–d provide the answer. Recall that in deriving Eq. (7.106) we used an asymptotic series in inverse power of frequency, which was valid to \(O(1/\omega^2)\) [see Eq. (7.66a), for example]. This means that, in the time domain, our results are valid to \(O(\tau)\), i.e., if we expand \(u_{m}^{SC}(y, \tau)\) for \(\tau\) near the time \(\tau = 0\) when the incident wave first strikes the specular point, we would find

\[
(8.2) \quad u_{m}^{SC}(y, \tau) = A^S \delta(\tau) + A^H H(\tau) + O(\tau H(\tau))
\]

Eq. (7.106), therefore, gets both the delta function and step function coefficients correct. However, Eq. (7.106) shows that if \(A_L (k_1 + k_2)\) is small, as we mentioned previously can be true in many practical situations, then the Born approximation [Eq. (8.1)] also is (approximately) correct for both delta and step function contributions. Since the exact response for a void differs only slightly from that predicted by the Born approximation (compare Figs. 8.1a, b) between the leading delta function and the center C, and it is precisely those values which are used to determine the distance \(h\), we expect the inverse Born approximation can work well in many cases. However, if \(A_L\) is not small or the curvature term is significant, the Born results may be degraded significantly.
8.3 A Modified 1-D Sizing Algorithm

Because Eq. (7.106) is asymptotically correct to $O(t)$ for all values of $\Lambda_L$ and $(\kappa_1 + \kappa_2)$, it is possible that it could form the basis for a more reliable equivalent flaw sizing algorithm than the 1-D inverse Born approximation. Whether or not this is true, of course, depends on numerical and experimental verifications. Here, we will only define the basis for the modified method.

Consider Eq. (7.106) again. For convenience, we repeat this equation below.

$$(8.3) \quad u^{sc}(y, t) = \frac{c_1}{8\eta y} \left[ \frac{d^2 S_3(x_3)}{dx_3^2} - \Lambda_L(\kappa_1 + \kappa_2) \frac{dS_3(x_3)}{dx_3} \right] \bigg|_{x_3 = c_1 T/2}$$

Now suppose that we follow the idea of the 1-D inverse Born approximation and locate when the doubly integrated scattered response is a maximum, or equivalently, the singly integrated response is zero. Then the time interval $T$ between the leading edge response and this maximum can be determined by

$$(8.4) \quad \left. \frac{dS_3(x_3)}{dx_3} - \Lambda_L(\kappa_1 + \kappa_2)S_3(x_3) \right|_{x_3 = c_1 T/2} = 0$$

In Appendix C it is shown that, for a general ellipsoid with semi-major axes $b_1$, $b_2$, $b_3$, the area function $S_3(x_3)$ appearing in the above equation can be expressed as
(8.5) \[ S_3(x_3) = \frac{b_1 b_2 b_3}{h^3} (2h x_3 - x_3^2) H(x_3) \]

Hence, by placing this expression for \( S_3 \) in Eq. (8.4), we find

\[
(8.6) \quad \frac{\Lambda_L (k_1 + k_2) c_1^2}{4} T^2 - \left[ 1 + \frac{\Lambda_L (k_1 + k_2) h}{c_1} \right] c_1 T + 2h = 0
\]

Note that for \( \Lambda_L = 0 \), this equation just gives

\[
(8.7) \quad T = \frac{2h}{c_1}
\]

which is the time as measured from the leading edge to the center of ellipsoid. For \( \Lambda_L \neq 0 \), we obtain instead

\[
(8.8) \quad T = \frac{2h}{c_1} + \left\{ \frac{2 - 2[1 + \frac{\Lambda_L^2 (k_1 + k_2)^2 h^2}{\Lambda_L (k_1 + k_2) c_1}]^{1/2}}{\Lambda_L (k_1 + k_2) c_1} \right\}
\]

Since \( \Lambda_L (k_1 + k_2) \) may be small in some cases, we can also give an approximate expression for \( T \) as

\[
(8.9) \quad T \approx \frac{2h}{c_1} \left\{ 1 - \frac{\Lambda_L (k_1 + k_2) h}{2} \right\}
\]

In either case, if the time between the leading edge delta function and
the maximum of the doubly integrated signal is obtained experimentally, it can be related directly to the ellipsoid parameters, since, as Appendix C shows, we have [Eqs. (12.28), (12.30)]

\[(8.10) \quad h = \left[ b_1^2(\mathbf{I} \cdot \mathbf{X}_1)^2 + b_2^2(\mathbf{I} \cdot \mathbf{X}_2)^2 + b_3^2(\mathbf{I} \cdot \mathbf{X}_3)^2 \right]^{1/2}\]

and

\[(8.11) \quad \kappa_1 + \kappa_2 = \frac{h}{b_1 b_2 b_3} \left[ b_1^2(b_2^2 + b_3^2)(\mathbf{I} \cdot \mathbf{X}_1)^2 + b_2^2(b_1^2 + b_3^2)(\mathbf{I} \cdot \mathbf{X}_2)^2 + b_3^2(b_1^2 + b_2^2)(\mathbf{I} \cdot \mathbf{X}_3)^2 \right]\]

where \( b_1, b_2, b_3 \) are the three semi-major axes of the ellipsoid, \( h \) is the perpendicular distance mentioned previously, \( \mathbf{I} \) is a unit vector along the incident wave direction, and \( \mathbf{X}_i \) (\( i = 1, 2, 3 \)) are unit vectors along the ellipsoidal axes. Eqs. (8.10) and (8.11) show that the relationship between \( T \) and the ellipsoidal parameters is highly nonlinear, regardless of whether we use Eq. (8.8) or (8.9). By using these expressions in a nonlinear least squares fitting procedure, with the time \( T \) measured from different directions, it is possible to obtain a "best fit" ellipsoid similar to what has been done using the Born approximation.
8.4 Zeros of the Scattering Amplitude

Recently, Kogan et al. [181] suggested that it is also possible to do equivalent flaw sizing by using the zeros of the real and imaginary parts of the scattering amplitude. They showed that the predictions of the Born approximation for these zeros are generally in good agreement with corresponding "exact" numerical and experimental results. It should be pointed out, however, that the numerical results they compared the Born approximation with were for $\nu = 1/3$ so that $\Lambda_L = 0$. Below, we will show how a similar set of zeros can be obtained from our Kirchhoff theory and how these zeros can be used to do equivalent flaw sizing.

Consider Eq. (8.3) again. For the case of an ellipsoid, the area function given in Eq. (8.5) is applicable. Thus Eq. (8.3) gives

\[
(u^sc(y, \tau') = I \frac{b_1 b_2 b_3}{4y^2} \left[ 2\delta(\tau' + \frac{2h}{c_1}) - \frac{c_1}{h} U(\tau') + \frac{\Lambda_L (k_1 + k_2)c_1^2}{2h} \tau'U(\tau') \right]
\]

Where $\tau'$ is the time measured from the flaw center [i.e., $\tau' = \tau - 2h/c_1 = t - y/c_1 - 2h/c_1$, see Eq. (7.105)] and $U(\tau')$ is defined as

\[
U(\tau') = \begin{cases} 
1 & \text{for } -2h/c_1 \leq \tau' \leq 0 \\
0 & \text{otherwise}
\end{cases}
\]
A plot of Eq. (8.12) is shown in Fig. 8.1c (which is also true for ellipsoids). The scattering amplitude can be obtained by taking the Fourier transform of Eq. (8.12), the result is

\begin{equation}
2\mu_{sc}(y, \omega) = \frac{I}{4\pi h} \left\{ 2 \exp\left[-\frac{2i\omega h}{c_1}\right] - \frac{2c_1}{\omega h} \exp\left[-\frac{i\omega h}{c_1}\right] \sin\left[\frac{\omega h}{c_1}\right] \right. \\
+ \frac{\Lambda_{1}(\kappa_1 + \kappa_2)c_1^2}{2h} \left[ -\frac{2i\hbar}{c_1\omega} \exp\left[-\frac{2i\omega h}{c_1}\right] + \frac{2i}{\omega^2} \exp\left[-\frac{i\omega h}{c_1}\right] \sin\left[\frac{\omega h}{c_1}\right] \right]\right. \\
\left. \right. \\
\left. \right. \\
= \frac{I}{4\pi h} \left( 2 \exp\left[-\frac{2i\omega h}{c_1}\right] - \frac{2c_1}{\omega h} \exp\left[-\frac{i\omega h}{c_1}\right] \sin\left[\frac{\omega h}{c_1}\right] \right) \\
+ \frac{\Lambda_{1}(\kappa_1 + \kappa_2)c_1^2}{2h} \left[ -\frac{2i\hbar}{c_1\omega} \exp\left[-\frac{2i\omega h}{c_1}\right] + \frac{2i}{\omega^2} \exp\left[-\frac{i\omega h}{c_1}\right] \sin\left[\frac{\omega h}{c_1}\right] \right) \\
\end{equation}

If we examine this expression, we see that at some frequencies the real and imaginary parts have zeros. The real part has zeros if the frequency \( \omega \) satisfies

\begin{equation}
\cos \Xi = \frac{1}{\Xi} \sin \Xi - \frac{\Lambda_{1}(\kappa_1 + \kappa_2)c_1}{2\omega} \left[ 2\sin \Xi + \frac{1}{\Xi} (1 - \cos \Xi) \right] = 0
\end{equation}

where

\begin{equation}
\Xi = 2\omega/c_1
\end{equation}

Similarly, the zeros of the imaginary part are determined by
\begin{equation}
\sin \xi - \frac{1}{\xi} (1 - \cos \xi) + \frac{\Lambda_L (\alpha_1 + \alpha_2)c_1}{2\omega} \left[ 1 - \frac{1}{\xi} \sin \xi \right] = 0
\end{equation}

For \( \Lambda_L = 0 \), these results agree with those previously obtained by Kogan et al. [181]. For \( \Lambda_L \neq 0 \), if the zeros are measured for a number of different scattering directions and the expressions (8.10) and (8.11) are used, a nonlinear least squares procedure could again be employed to estimate the ellipsoidal parameters.

8.5 Use of the Leading Edge Response for Equivalent Flaw Sizing

All of the inverse methods described so far have one common significant limitation—they all require the zero of time be located at a fixed point (usually the flaw center) with respect to the best-fit ellipsoid. For example, in the 1-D inverse Born method and the modified 1-D method discussed in Section 8.3, this zero of time is necessary in order to find the best-fit ellipsoid. Similarly, the expressions for the zeros of the real and imaginary parts of the scattering amplitude are valid only if the zero of time is at the flaw center. Although the location of the zero of time can be done straightforwardly by locating the maximum of the doubly integrated impulse response, this procedure suffers from two limitations: (i) it assumes the validity of the Kirchhoff or Born expressions over the entire lit side of the scattering surface, and (ii) it assumes that the flaw has a center of symmetry. The first assumption, as we have seen, is not strictly true because, for voids, both the Born and Kirchhoff approximations are only asymptotically correct near the specular point. And, certainly, the
second assumption may not be true for many types of real flaws.

Therefore, it would be desirable to have an inverse sizing algorithm that uses only the leading edge response. To this end, we consider the impulse response (8.12) again, but now we shift the zero of time back to the specular point by letting \( \tau = \tau' + 2h/c_1 \) [this \( \tau \) is the same as that defined in Eq. (7.105)]. In terms of \( \tau \), the impulse response is given by

\[
(8.16) \quad u^{sc}(y, \tau) = \frac{b_1 b_2 b_3}{4yh^2} \left\{ 2\delta(\tau) - c_1 \left[ \frac{1}{h} + \Lambda_L(k_1 + k_2) \right] H(\tau) + \frac{c_1^2 \Lambda_L(k_1 + k_2)}{2h} \tau H(\tau) \right\}
\]

Recall from Fig. 8.1c that the leading edge of the flaw response (near \( \tau = 0 \)) consists of a delta function \( (A^\delta) \) and a step function \( (A^H) \) contribution. The amplitude ratio \( R = A^H/A^\delta \), according to equation (8.16), is given by

\[
(8.17a) \quad R = -\frac{c_1}{2} \left[ \frac{1}{h} + \Lambda_L(k_1 + k_2) \right]
\]

where \( h \) and \( (k_1 + k_2) \) can again be expressed in terms of the ellipsoid parameters via Eqs. (8.10) and (8.11). Thus, if a number of measurements for the amplitude ratio \( R \) [designated as \( R_{\text{meas};i} \) \( (i = 1, 2, ..., N) \) in what follows] is available from \( N \) different pulse-echo
directions, then we can form up a function $F$,

\[(8.17b) \quad F = \sum_{i=1}^{N} (R_{\text{meas};i} - R)^2\]

and use a nonlinear least squares approach to minimize this "residual" function $F$ and find the size and orientation of the best equivalent ellipsoid that fits the measured data. The question of how practical is this proposed method can only be answered by numerical and experimental studies. Numerical simulations that have been done for scalar problems [182] show that the most difficult part of this proposed method is likely to be the measurement of $A^H$. This is because when all frequencies are not present in the scattered response, particularly in the low frequency range, then the step function will be distorted severely due to limited bandwidth [183,184]. Thus, it appears certain that some form of low frequency extrapolation [84] is necessary. It should be pointed out, however, that, in order to reliably find a fixed zero of time, the low frequency extrapolation is also required in the other equivalent flaw sizing schemes discussed previously. Therefore, this signal processing requirement is fundamental to all of these sizing methods. An important feature of this proposed leading edge method is that such extrapolated results need not be used at later times in the flaw response where significant modelling errors may be present. Below we summarize some of the advantages and disadvantages of this leading-edge sizing method:
ADVANTAGES:

(1) Determination of the zero of time is not required.
(2) Uses the Kirchhoff model only, where it is exact—no modelling error.
(3) Depends only on "relative" amplitude ratios which are easier to get experimentally than "absolute" amplitude measurements.

DISADVANTAGE:

(1) Requires $A^H - H$ function is very sensitive to the loss of low frequencies in a bandlimited system, so some sort of low frequency extrapolation is necessary.
(2) Still gives only a best-fit ellipsoid, so loses any details about shape other than gross features.
(3) Still requires solution of highly nonlinear optimization problems, solution of which are prone to error.

8.6 Transverse Waves and the Utilization of Polarization

All of the previous discussions in this chapter have centered around Eq. (7.106), which is valid for the case of incident longitudinal waves. Since L waves are the most commonly used types of waves in NDE testing, this focus is warranted. As new transducers (such as eletromagnetic-acoustic transducers (EMATs) [185,186]) develop, however, it is likely that transverse waves of arbitrary polarizations may be more easily produced than is currently possible. This type of development is important because, as we will show, with polarization information present, an inverse scattering method can be developed for voids that is both "exact" (i.e., no model errors are present) and valid for a wide
class of flaw shapes.

For incident T (shear) waves, the far-field scattered waves are given by Eq. (7.125) which can be rewritten in the following equivalent form:

\[
\mathbf{u}^{sc}(x, \tau) = \frac{c_2}{8\pi c_1} \left\{ \frac{d^2 S_3}{dx_3^2} D - \frac{dS_3}{dx_3} \left[ \frac{A_T^1}{\lambda_1^1} \right] D_1 \right\} \mid_{x_3 = c_2 \tau/2}
\]

Equation (8.18) shows that in the zeroth order Kirchhoff approximation the scattered wave is of the same polarization as the incident wave, while the first order approximation contains depolarized terms. This is similar to the depolarization of electromagnetic waves [169,173]. For electromagnetic waves, these depolarization effects are directly proportional to the difference of the principal curvatures. Here, for elastodynamic waves, we see that this is almost true except that now a new elastodynamic constant \( \lambda_1^1 \) [defined in Eq. (7.73)] appears in the above expression. This constant \( \lambda_1^1 \) will be equal to one only if \( c_1 = c_2 \), which can never happen in elastodynamic problems.

Now, recall that the unit vectors \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) in Eq. (8.18) were taken along the principal directions at the specular point. If we choose a fixed coordinate system \( O-X_1X_2X_3 \) such that the \( X_3 \) axis is still normal to the scattering surface at the specular point, but where the horizontal \( X_1 \) and vertical \( X_2 \) axes are at an angle \( \theta \) with respect to the
principal directions, then we can label horizontally polarized shear (SH) waves and vertically polarized (SV) waves as having polarizations along the \( X_1 \) and \( X_2 \) axes, respectively (Fig. 8.3). With respect to the \((X_1, X_2, X_3)\) system, the scattered wave components can be expressed as

\[
\begin{align*}
(8.19a) \quad u_{SV;SV}^{sc} &= \frac{c_2 D}{8 \pi y} \left\{ \frac{d^2 S_3}{dx_3^2} \right\} \left[ (\kappa_2 - \Lambda_1 \kappa_1) \cos^2 \theta + (\kappa_1 - \Lambda_1 \kappa_2) \sin^2 \theta \right] \left[ \frac{dS_3}{dx_3} \right] \bigg|_{x_3 = c_2 \tau/2} \\
(8.19b) \quad u_{SV;SH}^{sc} &= \frac{c_2 D}{8 \pi y} \left\{ (\kappa_1 - \kappa_2)(\Lambda_1 + 1) \sin \theta \cos \theta \right\} \left[ \frac{dS_3}{dx_3} \right] \bigg|_{x_3 = c_2 \tau/2} \\
(8.19c) \quad u_{SH;SH}^{sc} &= \frac{c_2 D}{8 \pi y} \left\{ \frac{d^2 S_3}{dx_3^2} \right\} \left[ (\kappa_2 - \Lambda_1 \kappa_1) \sin^2 \theta + (\kappa_1 - \Lambda_1 \kappa_2) \cos^2 \theta \right] \left[ \frac{dS_3}{dx_3} \right] \bigg|_{x_3 = c_2 \tau/2} \\
(8.19d) \quad u_{SH;SV}^{sc} &= \frac{c_2 D}{8 \pi y} \left\{ (\kappa_1 - \kappa_2)(\Lambda_1 + 1) \sin \theta \cos \theta \right\} \left[ \frac{dS_3}{dx_3} \right] \bigg|_{x_3 = c_2 \tau/2}
\end{align*}
\]

where \( D = |D| \) and \( u_{\alpha; \beta}^{sc} (\alpha, \beta = SV \text{ or } SH) \) is defined as the far-field scattered response of polarization \( \beta \) due to an incident wave of polarization \( \alpha \). Equations (8.19b) and (8.19d) imply that if we measure
Fig. 8.3. Principal coordinate system \((x_1, x_2, x_3)\) with \(x_1\) and \(x_2\) axes directed in the principal directions.
the cross-polarized components for an arbitrary incident polarization, these components will be zero when the incident shear wave is polarized in one of the two principal directions, since in that case \(\sin \theta \cos \theta = 0\). Thus, in this manner the principal directions can be determined and we can henceforth set \(\theta = 0\) in the equations (8.19a,c) to obtain

\[
(8.20a) \quad u_{SV;SV}^{SC} = \frac{c_2 D}{8 \mu y} \left\{ \left( \frac{d^2 S_3}{dx_3^2} \right) - (\kappa_2 - \Lambda_1 \kappa_1) \frac{dS_3}{dx_3} \right\} \bigg|_{x_3 = c_2 \tau / 2}
\]

\[
(8.20b) \quad u_{SH;SH}^{SC} = \frac{c_2 D}{8 \mu y} \left\{ \left( \frac{d^2 S_3}{dx_3^2} \right) - (\kappa_1 - \Lambda_1 \kappa_2) \frac{dS_3}{dx_3} \right\} \bigg|_{x_3 = c_2 \tau / 2}
\]

Similarly, we can write the longitudinal wave component from Eq. (7.106) in this notation as

\[
(8.20c) \quad u_{P;P}^{SC} = \frac{c_1 I}{8 \mu y} \left\{ \left( \frac{d^2 S_3}{dx_3^2} \right) - \Lambda_1 (\kappa_1 + \kappa_2) \frac{dS_3}{dx_3} \right\} \bigg|_{x_3 = c_1 \tau / 2}
\]

Equations (8.20a-c) can be used as the basis for an inverse sizing algorithm that uses only the leading response of the flaw, where these results are exact. To see this, recall Eq. (6.63) where we showed that for \(x_3 \ll l\) (near the specular point) the area function \(S_3\) is given by

\[
(8.21) \quad S_3(x_3) = 2\pi (R_1 R_2)^{1/2} x_3 H(x_3)
\]

If we extend this approximation one order higher, we have
where $\beta$ is an unknown constant. For an ellipsoid, note that $\beta = -1/2h$. For the case of a general surface, however, $\beta$ is a complicated constant that can be solved for and eliminated as will be shown shortly.

Equation (8.22) implies that near $\tau = 0$

\begin{align*}
(8.23a) & \quad \frac{dS_3}{d\xi_3} \bigg|_{x_3 = c_\alpha \tau/2} = 2\pi(R_1 R_2)^{1/2} \left[ 1 + \beta \frac{c_\alpha \tau}{c_\alpha} \right] H(\tau) \\
(8.23b) & \quad \frac{d^2S_3}{dx_3^2} \bigg|_{x_3 = c_\alpha \tau/2} = \frac{4\pi(R_1 R_2)^{1/2}}{c_\alpha} \left[ \delta(\tau) + \beta \frac{c_\alpha H(\tau)}{H(\tau)} \right]
\end{align*}

where $c_\alpha (\alpha = 1, 2)$ are the two wavespeeds. Placing these results into Eqs. (8.20a–c) we find, to $O(H(\tau))$, that near $\tau = 0$

\begin{align*}
(8.24a) & \quad u_{SV;SV}^{SC} = \frac{(R_1 R_2)^{1/2} D}{2y} \left\{ \delta(\tau) + \frac{c_2}{2} \left[ 2\beta - (\kappa_2 - \Lambda_1 \kappa_1) \right] H(\tau) \right\} \\
(8.24b) & \quad u_{SH;SH}^{SC} = \frac{(R_1 R_2)^{1/2} D}{2y} \left\{ \delta(\tau) + \frac{c_2}{2} \left[ 2\beta - (\kappa_1 - \Lambda_1 \kappa_2) \right] H(\tau) \right\} \\
(8.24c) & \quad u_{P;P}^{SC} = \frac{(R_1 R_2)^{1/2} I}{2y} \left\{ \delta(\tau) + \frac{c_1}{2} \left[ 2\beta - \Lambda_2 (\kappa_1 + \kappa_2) \right] H(\tau) \right\}
\end{align*}
Equations (8.24a-c) show that all of these polarizations have, in the leading edge response, a delta function and a step function response. If we follow the notation used here and label the amplitude coefficients of the delta and step terms as $A^\delta_{\alpha;\alpha}$ and $A^H_{\alpha;\alpha}$ ($\alpha = P, SH, SV$), respectively, then the ratios of these amplitudes are just

\[
(8.25a) \quad \frac{A^H_{SV;SV}}{A^\delta_{SV;SV}} = \frac{c_2}{2} \left[ 2\beta - (\kappa_2 - \Lambda_T \kappa_1) \right]
\]

\[
(8.25b) \quad \frac{A^H_{SH;SH}}{A^\delta_{SH;SH}} = \frac{c_2}{2} \left[ 2\beta - (\kappa_1 - \Lambda_T \kappa_2) \right]
\]

\[
(8.25c) \quad \frac{A^H_{P;P}}{A^\delta_{P;P}} = \frac{c_1}{2} \left[ 2\beta - \Lambda_L (\kappa_1 + \kappa_2) \right]
\]

Equations (8.25a-c) are three equations for the three unknowns $\beta$, $\kappa_1$, $\kappa_2$. The determinant $\Delta$ of the above system of equations is just

\[
(8.26) \quad \Delta = 2 \left[ \Lambda_T^2 + 2\Lambda_L (1 + \Lambda_T) - 1 \right]
\]

which, as Appendix B shows, can never be zero (for most structural materials, the Poisson's ratio $\nu = 1/3$ and so $\Delta = 16$). Hence, the system of equations (8.25a-c) is nonsingular and possesses one and only one solution.

Once $\beta$, $\kappa_1$ and $\kappa_2$ are obtained from the solution of Eqs. (8.25a-c),
the sum of the principal radii \((R_1 + R_2)\) can be found from

\[
(8.27) \quad R_1 + R_2 = \frac{K_1 + K_2}{K_1 K_2}
\]

If \((R_1 + R_2)\) are assumed to be known for all pulse-echo directions \((\theta_0, \phi_0)\) in a spherical coordinate system \((r, \theta_0, \phi_0)\) then, as Borden showed [174], the perpendicular distance \(h(\theta_0, \phi_0)\) from a fixed point to the tangent plane where the wavefront first touches the flaw is related to \((R_1 + R_2)\) by the linear partial differential equation:

\[
(8.28) \quad r^2 \nabla^2 h + 2h = -(R_1 + R_2)
\]

Once the distance function \(h(\theta_0, \phi_0)\) is found from the solution of Eq. (8.28), the rectangular Cartesian coordinates \((x, y, z)\) of the flaw surface are given by [174]

\[
(8.29a) \quad x = h \sin\theta_0 \cos\phi_0 + \frac{\partial h}{\partial \theta_0} \cos\theta_0 \cos\phi_0 - \frac{\partial h}{\partial \phi_0} \sin\theta_0 \\
(8.29b) \quad y = h \sin\theta_0 \sin\phi_0 + \frac{\partial h}{\partial \theta_0} \cos\theta_0 \sin\phi_0 + \frac{\partial h}{\partial \phi_0} \cos\theta_0 \\
(8.29c) \quad z = h \cos\theta_0 - \frac{\partial h}{\partial \theta_0} \sin\theta_0
\]
To summarize, this inverse method requires the following steps:

1. By varying the polarization of an incident shear wave, the polarization at which cross-polarization terms vanish must be found. This then determines the principal curvature directions of the flaw surface at a given angle of incidence.

2. Using the knowledge of the principal directions, measurements must be made of the three relative polarization amplitudes \( A_{\alpha}^H / A_{\alpha}^\delta \) (\( \alpha = P, SV, SH \)).

3. Equations (8.25a-c) can then be solved for \( \beta, \kappa_1 \) and \( \kappa_2 \).

4. The sum of the principal radii at the specular point must be found from equation (8.27).

5. Steps (1) to (4) are then repeated for all available angles of incidence.

6. The \( (R_1 + R_2) \) data developed in steps (1) to (5) form the inhomogeneous term in a linear partial differential equation (8.28) that then can be solved for the distance function \( h \) from a fixed point to a plane tangent to the scattering surface.

7. Finally, a knowledge of \( h \) then determines, via equation (8.29), the rectangular coordinates of the scattering surface.

This proposed inversion method is, of course, at this stage still merely a formalism. The solution procedure steps (6) and (7), however, are well known and have been discussed using finite elements [174] and...
separation of variables [187], even for the case when the \((R_1 + R_2)\) data is available only over a finite aperture [188]. The real practical difficulties associated with this method likely lie in the requirement for complete polarization information measurements, particularly in a band limited system. However, to our knowledge, this is the first inversion algorithm proposed for elastic wave flaw scattering that is not limited by strong a priori flaw shape assumptions, or fundamental model errors (such as a weak scattering Born approximation), or by the requirement for absolute amplitude measurements [189]. It is interesting that the proposed method is, in fact, unique to elastodynamics, because in scalar (acoustic) scattering problems the polarization information needed here [see Eqs. (8.25a,b)] is not available; and in electromagnetic problems the longitudinal modes needed [Eq. (8.25c)] are absent. Thus, the complexity of the elastodynamic response may in this case be a significant advantage. Below we list some of the advantages and disadvantages of the sizing method described in this section.

**ADVANTAGES:**

1. Determination of the zero of time is not required.

2. Uses the Kirchhoff model only, where it is exact--no modelling error.

3. Depends only on "relative" amplitude ratios which are easier to get experimentally than "absolute" amplitude measurements.

4. Entire inversion procedure is "linear". This is the first such exact linear solution to inverse problem for elastodynamic waves
that we are aware of.

(5) Discards the ellipsoid model and recovers the details of a scatterer.

DISADVANTAGE:

(1) Requires $A^H - H$ function is very sensitive to the loss of low frequencies in a bandlimited system, so some sort of low frequency extrapolation is necessary.

(2) Requires considerable amount of polarization information which is currently difficult to obtain experimentally.
9. CONCLUSIONS

We have used ray theory in the present work to investigate both scalar and elastodynamic scattering problems. The zeroth and first order time-domain, far-field, leading-edge responses of acoustically hard obstacles and voids were obtained explicitly. These results show that, for both scalar and elastodynamic problems, the zeroth order solution is always proportional to the second derivative of the silhouette area function. They also reveal that the first order solution is proportional to the first derivative of the area function. The proportionality constant in the first order solution is found to contain important curvature information and is, in contrast to the scalar wave case, dependent on the type of problems being solved. For scalar wave problems, this proportionality constant is merely the sum of the two principal curvatures at the specular point. For elastodynamic problems, however, this constant has a slightly more complicated form and involves the two elastodynamic constants $\Lambda_L$ and $\Lambda_T$ introduced in Chapter 7.

In Chapter 8, we pointed out that for a number of important structural materials, the Poisson's ratio $\nu$ is of approximately 1/3 and, therefore, $\Lambda_L$ can be expected to be relatively small in many cases. This observation is important, because it explains why the familiar 1-D inverse Born sizing method works so well for a strong scatterer such as a void. Moreover, it also indicates clearly where the Born method might fail.

From the standpoint of nondestructive evaluation, the results of
Chapter 7 are especially important, because they provide the bases for three alternative equivalent flaw sizing algorithms to the 1-D Born method:

(1) modified 1-D sizing method
(2) correction to zeros of scattering amplitude, and
(3) leading-edge sizing method

The task of determining the zero of time, for a long time, has been a major difficulty in inverting experimental data. In this regard, the leading-edge sizing method is of particular importance.

The scattering response (7.125) of a void caused by an incident T-wave is another important result obtained in Chapter 7. It shows clearly the existence of depolarization effects for transverse elastodynamic waves just as for electromagnetic waves. In Chapter 8, we have indicated how complete polarization data can be used to develop a more exact flaw sizing algorithm that can determine detailed flaw shape information.

In this work, although we have only given explicit results for acoustically hard scatterers and voids, the techniques of Chapters 6 and 7 can, in principle, be extended to treat other types of scatterers, such as elastic inclusions. For the inclusion case, we believe that the ray theory solution can be used to deduce not only the flaw's geometrical features but also some of its material properties. Furthermore, these techniques can also be extended to treat pitch-catch type situations, where the transmitter and receiver are located at different locations.
The elastodynamic ray theory is closely related to the geometrical properties of surfaces, whether the surfaces be real scattering surfaces or reflected/refracted wavefronts. Thus a certain degree of familiarity with these surface properties is a prerequisite to the understanding of ray theory. In order to make this dissertation more readable, some pertinent facts of differential geometry are sketched briefly in this appendix. This appendix is, by no means, intended to be exhaustive but rather only serves to introduce terminologies, notations, and sign conventions that are employed throughout this dissertation. A great deal of classic treatises on this subject, such as McConnell [190], Struik [191], and Stoker [192], exist in the literature. A concise but excellent summary of differential geometry can also be found in Chapter 1, Langhaar [193]. Before exploring the properties of surfaces, which are our main concern, however, some discussion on the theory of space curves is necessary.

10.1 Curves in Space

A space curve is the locus of a point whose coordinates are functions of a single parameter, say \( u \). Let \( x_1, x_2, \) and \( x_3 \) be axes of a rectangular Cartesian coordinate system and let \( \hat{x}_1, \hat{x}_2, \) and \( \hat{x}_3 \) be the unit vectors corresponding to the three coordinate axes. With respect to this coordinate system, a space curve \( C \) can be described by the vector function:
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\[ (10.1) \quad \mathbf{x}(u) = \sum_{k=1}^{3} x_k(u) \quad (\text{sum over } k = 1,2,3) \]

where \( \mathbf{x} \) is the position vector of a generic point on the curve \( C \). The three functions \( x_k(u) \), where \( k = 1,2,3 \), will be assumed to be single-valued, continuous, and differentiable. Also, the range of the parameter \( u \), though not specified explicitly in (10.1), is understood to be an appropriate subset of the real numbers. Since to each value of \( u \) there corresponds one and only one point on \( C \), the parameter \( u \) is usually called a coordinate on \( C \). Alternatively, it is often phrased in the literature that the curve \( C \) is parametrized by the parameter \( u \).

In general there are a variety of ways to parametrize a curve \( C \). If, however, one choose to parametrize the curve \( C \) by an arc-length parameter \( s \) measured along \( C \) from some reference point (i.e., set \( s = 0 \) at this point), then \( d\mathbf{x}/ds \) is a unit vector tangent to \( C \). That is

\[ (10.2) \quad \text{unit tangent vector} = \mathbf{t} = \frac{d\mathbf{x}}{ds} \]

Notice that the unit tangent vector \( \mathbf{t} \) always points in the direction of increasing \( s \).

Now since \( \mathbf{t} \cdot \mathbf{t} = 1 \), differentiation of this equation with respect to \( s \) yields \( \mathbf{t} \cdot \frac{d\mathbf{t}}{ds} = 0 \). This result means that \( d\mathbf{t}/ds \) is orthogonal to the unit tangent vector \( \mathbf{t} \) and, hence, is orthogonal to the curve \( C \). From Fig. 10.1 it is clear that the infinitesimal vector \( d\mathbf{t} \) has the same direction as the principal normal \( \mathbf{n} \) has. Moreover, from Fig. 10.1 one sees immediately that \( |d\mathbf{t}| = d\theta \). Accordingly,
Fig. 10.1. Two successive unit tangent vectors $t$ and $t + dt$.
\[ (10.3) \quad \frac{dt}{ds} = n \, d\theta \]

Thus it follows from this equation and the relationship \( ds = rd\theta \), where \( r \) is the radius of curvature of \( C \) at point \( P \) (see Fig. 10.1), that

\[ (10.4) \quad \frac{dt}{ds} = n \, \frac{d\theta}{ds} = \frac{n}{r} = \kappa \, n \]

In the above, \( \kappa = 1/r \) is the curvature of \( C \) at point \( P \).

The binormal \( b \) is defined by the vector product

\[ (10.5) \quad \text{binormal} = b = \mathbf{t} \times \mathbf{n} \]

Obviously, the three unit vectors \( \mathbf{t} \), \( \mathbf{n} \), and \( b \) form a right-handed orthonormal triad, i.e.,

\[ (10.6) \quad \mathbf{t} = \mathbf{n} \times \mathbf{b}, \quad \mathbf{n} = \mathbf{b} \times \mathbf{t}, \quad \mathbf{b} = \mathbf{t} \times \mathbf{n} \]

Taking the total differential of the third equation in (10.6) and utilizing \( dt \times n = 0 \) [since \( dt = nd\theta \), see (10.3)], one obtains

\[ (10.7) \quad db = t \times dn \]

This result implies that \( db \) is orthogonal to \( t \). Also, since \( db \) is perpendicular to \( b \), the infinitesimal vector \( db \) must be either parallel or anti-parallel to the unit normal \( n \). More precisely, a glance at Fig.
10.2 reveals that \( db \) is, in fact, pointing toward the negative \( n \) direction (i.e., anti-parallel to \( n \)). The torsion \( \tau \) of the curve \( C \) at point \( P \) (Fig. 10.2) is defined by \( \tau = d\phi/ds \), where \( d\phi \) is the angle between two adjacent binormals. Notice that for a plane curve the torsion \( \tau \) is identically zero, since the binormal \( b \) never changes its magnitude and direction. From the geometry of Fig. 10.2 it is not difficult to see that

\[
(10.8) \quad db = -n \ d\phi
\]

Hence

\[
(10.9) \quad db/ds = -n \ d\phi/ds = -\tau \ n
\]

Now, differentiating the second equation in (10.6) with respect to \( s \) one obtains

\[
(10.10) \quad dn/ds = db/ds \times \overrightarrow{t} + b \times dt/ds
\]

Substitution of (10.4) and (10.9) in the preceding equation results

\[
(10.11) \quad dn/ds = (-\tau \ n) \times \overrightarrow{t} + b \times (\kappa \ n) = \tau \ b - \kappa \ t
\]

Equations (10.4), (10.9), and (10.11) are collectively known as the Frenet formulas in the theory of curves. For ease of access, these
Fig. 10.2. Orthonormal triads \((t, n, b)\) and \((t + dt, n + dn, b + db)\)
formulas are summarized below:

\[
(10.12) \quad \frac{dt}{ds} = \kappa \, n, \quad \frac{db}{ds} = -\tau \, n, \quad \frac{dn}{ds} = \tau \, b - \kappa \, t
\]

This set of equations dictates how the three fundamental unit vectors \( t \), \( n \), and \( b \) change their magnitudes and directions as a point moves out along a space curve.

10.2 Surfaces in Space

We begin with some definitions of terminologies. Let \( S \) be a surface embedded in the three dimensional Euclidean space and let \( E \) be a plane containing the unit normal at some point \( P \) on the surface \( S \). A normal section \( C_N \) at the point \( P \) is defined by \( C_N = E \cap S \). Notice that this definition implies that \( C_N \) is a plane curve, not a space curve. Also, notice that an infinite number of normal sections can be defined at any point on the surface \( S \), since there exist infinitely many "sectioning" planes \( E \). For certain choice of normal section \( C_{N(1)} \) at a point \( P \), the radius of curvature of this normal section (remember it is a plane curve) at \( P \) attains a maximum value \( r_1 \), whereas for some other choice of normal section \( C_{N(2)} \) the radius of curvature assumes a minimum value \( r_2 \). These two extreme values \( r_1 \) and \( r_2 \) are called the principal radii of curvature of \( S \) at the point \( P \); their reciprocals \( \kappa_1 = 1/r_1 \) and \( \kappa_2 = 1/r_2 \) are referred to as the principal curvatures. The tangent vectors corresponding to the two normal sections \( C_{N(1)} \) and \( C_{N(2)} \) at point \( P \) define two directions, these two directions are called principal- curvature directions of \( S \) at point \( P \), or more briefly, the
principal directions.

10.2.1 Parametrization of surfaces

A curve can be defined unambiguously by a single parameter as was done in Section 10.1. Analogously, a surface $S$ can be defined uniquely by two suitably chosen parameters, say $a_1$ and $a_2$. More specifically, if $x$ denotes the position vector of a generic point on the surface $S$, then

$$x = x_k x_k (a_1, a_2) \quad \text{(sum over } k = 1, 2, 3)$$

In the above, $x_k (k = 1, 2, 3)$ are unit vectors associated with the rectangular Cartesian coordinate system introduced earlier in Section 10.1. The two parameters $a_1$ and $a_2$ are called surface coordinates, since to each pair of $(a_1, a_2)$ there corresponds one and only one point on the surface $S$.

If $a_2$ is held constant, the vector function $x$ defined in (10.13) depends on $a_1$ only, i.e., $x = x(a_1)$. But this is just the parametric representation of a (space) curve, hence by holding $a_2$ constant and letting $a_1$ be a variable, a space curve on the surface $S$ is traced out. This curve is referred to as an $a_1$-coordinate line. In a similar fashion, an $a_2$-coordinate line can be traced out on the surface $S$ by holding $a_1$ constant. These two families of curves on the surface $S$ constitute a network which can be thought as scribing a (curvilinear) coordinate system on $S$.

Finally, to close this subsection, we remark that $x_{,\alpha} = \partial x / \partial a_\alpha (\alpha = 1, 2)$ is tangent to the $a_\alpha$-coordinate line and that the unit normal $n$ is
defined by \( \mathbf{n} = (x_1 \times x_2) / |x_1 \times x_2| \).

10.2.2 Metric tensor and first fundamental form

As noted in the preceding subsection (Sec. 10.2.1), a surface \( S \) can be defined by \( \mathbf{x} = \mathbf{x}(a_1, a_2) \), where \( \mathbf{x} \) is the position vector of an arbitrary point on \( S \). The distance \( ds \) between two neighboring points on \( S \) is determined by

\[
(10.14) \quad ds^2 = dx \cdot dx
\]

Since \( \mathbf{x} \) is a function of \( a_1 \) and \( a_2 \), \( dx = \mathbf{x}_\alpha da_\alpha \). Hence, Eq. (10.14) can also be written as

\[
(10.15) \quad ds^2 = \mathbf{x}_\alpha \mathbf{x}_\beta da_\alpha da_\beta \quad \text{(sum over } \alpha, \beta = 1, 2) \]

or

\[
(10.16) \quad ds^2 = g_{\alpha \beta} da_\alpha da_\beta
\]

where

\[
(10.17) \quad g_{\alpha \beta} = \mathbf{x}_\alpha \mathbf{x}_\beta
\]

is called the metric tensor of the surface \( S \). Clearly, from Eq. (10.17), the metric tensor \( g_{\alpha \beta} \) is symmetric. Also, since \( ds^2 \) is nonnegative, the quadratic form \( g_{\alpha \beta} da_\alpha da_\beta \) on the right hand side of
(10.16) is positive definite. This quadratic form is called the first fundamental differential quadratic form of the surface S, or more briefly, the first fundamental form. In case that \(a_1\) and \(a_2\)-coordinate lines are orthogonal, which happens frequently in practical problems, Eq. (10.17) yields \(g_{12} = g_{21} = 0\).

10.2.3 Angle between coordinate lines and differential area

Suppose \(dx_{(1)}\) and \(dx_{(2)}\) are two infinitesimal line segments, both emitted from a common point P (see Fig. 10.3) and directed along the \(a_1\) and \(a_2\)-coordinate lines, respectively. Let the two unit vectors tangent to \(a_1\) and \(a_2\)-coordinate lines be denoted, respectively, by \(\mathbf{a}_1\) and \(\mathbf{a}_2\). It then follows from (10.16) that

\[
(10.18) \quad dx_{(1)} = (g_{11})^{1/2} da_1 \mathbf{a}_1, \quad dx_{(2)} = (g_{22})^{1/2} da_2 \mathbf{a}_2
\]

Hence,

\[
(10.19) \quad dx_{(1)} \cdot dx_{(2)} = (g_{11}g_{22})^{1/2} da_1 da_2 \cos \theta
\]

where \(\cos \theta = \mathbf{a}_1 \cdot \mathbf{a}_2\) (see Fig. 10.3). But, since

\[
(10.20) \quad dx_{(1)} = x_1 da_1, \quad dx_{(2)} = x_2 da_2
\]

one also has

\[
(10.21) \quad dx_{(1)} \cdot dx_{(2)} = x_1 x_2 da_1 da_2 = g_{12} da_1 da_2
\]
Note that $|dx(1)| = \sqrt{g_{11}} \, da_1$ \hspace{1cm} $|dx(2)| = \sqrt{g_{22}} \, da_2$

Fig. 10.3. Infinitesimal line elements $dx(1)$ and $dx(2)$
In arriving at the last equality of (10.21), Eq. (10.17) has been used. Thus, by (10.19) and (10.21),

\[(10.22) \quad \cos \theta = \frac{g_{12}}{(g_{11}g_{22})^{1/2}}\]

Also, by utilizing the trigonometry identity \( \sin^2 \theta = 1 - \cos^2 \theta \), one has

\[(10.23) \quad \sin \theta = \left[ \frac{g_{11}g_{22} - g_{12}^2}{g_{11}g_{22}} \right]^{1/2}\]

The differential area \( dS \) of the infinitesimal parallelogram shown in Fig. 10.3 is

\[(10.24) \quad dS = |dx_2(1) \times dx_2(2)| = (g_{11}g_{22})^{1/2}da_1da_2 \sin \theta\]

Hence, upon utilization of (10.23), the foregoing equation becomes

\[(10.25) \quad dS = \left[ g_{11}g_{22} - g_{12}^2 \right]^{1/2}da_1da_2\]

This is the desired expression for the differential area on a surface. Whenever a surface integral is to be evaluated, (10.25) can be used.
10.2.4 Curvature tensor and second fundamental form

Let P be a point on the surface S (see Fig. 10.4). Without loss of

generality, the two surface coordinates \((a_1, a_2)\) of point P may be chosen
to be \((0,0)\). Now, consider a point Q infinitesimally close to the point

P and denote the surface coordinates of Q by \((da_1, da_2)\). The position

vector \(x(da_1, da_2)\) of Q can be expanded in Taylor series about point P
yielding

\[
(10.26) \quad x(da_1, da_2) = x(0,0) + x,\alpha da_\alpha + \frac{1}{2} x,\alpha\beta da_\alpha da_\beta
\]

where small quantities of second and higher orders have been discarded.

Also, the derivative terms in (10.26) are evaluated at point P. Now,
since \(x,\alpha (\alpha = 1 \text{ and } 2)\) lie on the tangent plane to S at P, \(n.x,\alpha = 0\)
where \(n\) is the unit normal to S at point P. Hence, if we define \(H = n.[x(da_1, da_2) - x(0,0)]\), then it follows from (10.26) that

\[
(10.27) \quad H = n.[x(da_1, da_2) - x(0,0)] = \frac{1}{2} n.x,\alpha\beta da_\alpha da_\beta
\]

For reasons that will appear soon, we further define

\[
(10.28) \quad h_{\alpha\beta} = n.x,\alpha\beta
\]

and write (10.27) as
Fig. 10.4. Local geometry of the surface $S$ near a point $P$
Geometrically, $H$ is the perpendicular distance from point $Q$ to the
tangent plane to $S$ at point $P$. This "vertical" distance $H$, obviously,
has nothing to do with how the surface $S$ is parametrized, i.e., $H$ is
independent of the choice of the surface coordinates $(a_1, a_2)$; hence, it
is an invariant. Moreover, since $da_\alpha$ is a contravariant tensor of rank
one (Strictly speaking, in accordance with the conventional tensor
notations, $da_\alpha$ should be written as $da^\alpha$. In our work, however, it is
not really important to distinguish between subscripts and
superscripts.), it follows from (10.29) and the quotient laws of tensor
theory that $h_{\alpha \beta}$ is a covariant tensor of rank two. This tensor $h_{\alpha \beta}$
is known as the curvature tensor and the quadratic form $\frac{1}{2} h_{\alpha \beta} da_\alpha da_\beta$
on the right hand side of (10.29) is called the second fundamental form.
The importance of the curvature tensor $h_{\alpha \beta}$ lies in the fact that $h_{\alpha \beta}$
specifies completely the vertical distance $H$ in the immediate
neighborhood of point $P$. Knowing $H$ everywhere in a small neighborhood
of $P$ on a surface enables one to "graph" the local shape of the surface.

A curve on a surface that everywhere coincides with the principal
curvature direction is called a line of principal curvature. It often
proves useful, notably in actual computations, to parametrize a surface
or a small portion of it in such a way that the two families of
coordinate lines coincide with the lines of principal curvature. Under
this particular situation it can be shown that
Theorem 1:
The lines of principal curvature coincide with the coordinate lines, if and only if,

\[(10.30) \quad h_{12} = h_{21} = 0, \quad \kappa_1 = h_{11}/g_{11}, \quad \kappa_2 = h_{22}/g_{22}.\]

The signs of \(\kappa_1\) and \(\kappa_2\) determined by the above theorem are such that \(\kappa_1\) or \(\kappa_2\) is positive if the corresponding center of curvature lies on the side toward which the unit normal \(n\) is directed. For example, if the normal \(n\) to a sphere of radius \(R\) is directed outward, \(\kappa_1 = \kappa_2 = -1/R\); if \(n\) is directed inward, \(\kappa_1 = \kappa_2 = 1/R\).

Next we give two more theorems, the proofs of which can be found in most texts on differential geometry.

Theorem 2 (Euler theorem):
Let the coordinate lines on a surface \(S\) (or a small portion of \(S\)) be coincident with the lines of principal curvature and let \(\kappa_1\) and \(\kappa_2\) be the two principal curvatures at a point \(P\) on \(S\), then the curvature \(\kappa_\phi\) of a normal section of \(S\) at \(P\), at an angle \(\phi\) counterclockwise from the \(a_1\)-coordinate line (see Fig. 10.5), is given by

\[(10.31a) \quad \kappa_\phi = \kappa_1 \cos^2 \phi + \kappa_2 \sin^2 \phi\]

Theorem 3 (Rodrigues' theorem):
If the coordinate lines coincide with the the lines of principal curvature of a surface \(S\), then the derivatives of the unit normal \(n\) to \(S\) are given by

\[(10.31b) \quad n_{1,1} = -\kappa_1 x_{1,1}, \quad n_{2,1} = -\kappa_2 x_{2,1}\]
Fig. 10.5. "Top" view of a normal section which makes an angle $\phi$ with the $a_1$-coordinate line
Finally, to conclude this section, we give without proof a formula which expresses the second derivative of a position vector in terms of the tangent vectors $\vec{x}_1$ and $\vec{x}_2$ (tangent to the coordinate lines) and the unit normal $\vec{n}$:

\begin{equation}
\vec{x}_{\alpha\beta} = \Gamma_{\alpha\beta}^{\gamma} \vec{x}_{\gamma} + h_{\alpha\beta} \vec{n} \tag{10.32}
\end{equation}

In the above equation $\Gamma_{\alpha\beta}^{\gamma}$ is the Christoffel's symbol of the second kind in tensor theory [160].

10.3 More on Parametrization of Surfaces

As noted before a surface, in general, can be parametrized in a variety of ways. Some parametrizations lead to formidable algebras during the course of analysis; some, however, can simplify considerably one's analysis. Most of the time in ray theories, one is only concerned about an infinitesimal portion of a surface, whether it be a wavefront or a physical surface. In this case, it is often to one's advantage by parametrizing the infinitesimal surface in a way described below.

Suppose $S$ is the infinitesimal surface mentioned above and suppose that $S$ encompasses a point of significance $0$, usually the point of incidence in scattering problems. Choosing the origin at point $0$, we set up a local rectangular Cartesian coordinate system $0-a_1 a_2 a_3$ so that the $a_3$-axis is perpendicular to the surface $S$ at point $0$ (see Fig. 10.6). The plane $a_1 = \text{constant}$ intersects $S$ along a curve which, for obvious reason, will be called an $a_2$-curve. Similarly, the curve on $S$ intersected by the plane $a_2 = \text{constant}$ will be called an $a_1$-curve.
Fig. 10.6 A local cartesian coordinate system with origin at point 0
These two families of curves, like the more general coordinate lines introduced in Sec. 10.2.2, scribe a network on $S$. Suppose now $P$ is a point on $S$, of which the rectangular Cartesian coordinates are $(a_1, a_2, a_3)$. Since $P$ is a point on $S$, clearly, one may also identify $P$ by locating its "address" within the network; for example, if $P$ sits at the intersection of the two curves $a_1 = \alpha_1$ and $a_2 = \alpha_2$, then the address of point $P$ is $(\alpha_1, \alpha_2)$. Therefore, despite that $(a_1, a_2)$ are the first two coordinate components of $P$ with respect to the Cartesian coordinate system, they can also be regarded as the two surface coordinate components. Though exactly the same symbols $(a_1, a_2)$ are used for these two conceptually distinct coordinate systems, the dual roles of $(a_1, a_2)$ should never be confused; the precise meaning associated with $(a_1, a_2)$ should be dictated by the contexts preceding and following it.

Now, suppose $x(a_1, a_2)$ is the position vector of a generic point $P$ on the surface $S$. If the equation for $S$ with respect to the Cartesian coordinate system $0-a_1a_2a_3$ can be expressed by

\[(10.33) \quad a_3 = f(a_1, a_2)\]

Then, it follows from the definition of a position vector, $x = a_\alpha a_\alpha$ (sum over $\alpha = 1, 2, 3$), that

\[(10.34a) \quad x(a_1, a_2) = a_1 a_1 + a_2 a_2 + f(a_1, a_2) a_3\]

\[(10.34b) \quad x, \alpha = a_\alpha + f, \alpha(a_1, a_2) a_3 \quad (\alpha = 1, 2)\]
Notice that, since $f, \alpha = 0$ at point 0,

\[(10.35) \quad x, \alpha = a, \alpha \quad \text{(at point 0)}\]

With the above results at one's disposal, the metric tensor $g_{\alpha\beta}$ and curvature tensor $h_{\alpha\beta}$ can be easily determined by using (10.17) and (10.28), respectively. In particular, at point 0 where the surface coordinates $(a_1, a_2) = (0,0)$ and $f, \alpha = 0$, the results for these two tensors are

\[(10.36a) \quad g_{\alpha\beta} = a, \alpha a, \beta = \delta_{\alpha\beta} \quad \text{(at point 0)}\]

\[(10.36b) \quad h_{\alpha\beta} = f, \alpha \beta \quad \text{(at point 0)}\]

In arriving at (10.36b), it is assumed that the unit normal $n$ to $S$ at point 0 coincides with $a_3$.

Often, in scattering problems, things can be simplified significantly by choosing the Cartesian coordinate system $0-a_1 a_2 a_3$ in such a way that $a_1$ and $a_2$ coincide with the two principal-curvature directions of $S$ at point 0. In the rest of this section, it will be assumed that this particular choice of coordinate system has been made.

With this choice of coordinates, it follows from (10.30) and (10.35a) that $h_{12} = h_{21} = 0$ and the principal curvatures at point 0 are given by
This result (10.37) finds various applications in ray theory and is, in fact, indispensable.

As mentioned in the beginning of this section, the surface $S$ is infinitesimal; hence, if $(a_1, a_2)$ are surface coordinates of some point $P$ on $S$, both $a_1$ and $a_2$ are << 1. Accordingly, one may expand the unit normal $n(a_1, a_2)$ to $S$ at $P$ in Taylor series point $0(0,0)$ obtaining

$$n(a_1, a_2) = n(0,0) + n_{\alpha \alpha} a_\alpha a_\alpha$$

where small quantities of higher orders have been neglected. Also, the derivative terms in (10.38) are evaluated at point $0$. By Rodrigues' theorem and (10.35), it follows from (10.38) that

$$n(a_1, a_2) = n(0,0) - \kappa_1 a_1 a_1 - \kappa_2 a_2 a_2$$

When applying this equation, the sign convention for curvature $\kappa_1$ and $\kappa_2$ should be exercised with particular care. For instance, suppose $S$ is a smooth convex surface as the one shown in Fig. 10.6 and suppose that the unit normal $n$ at $P$ is directed away from the center of curvature corresponding to $P$ (i.e., $n = a_3$), then both $\kappa_1$ and $\kappa_2$ are negative (see the paragraph following Theorem 1). It should be noted that some authors [28,102,164] prefer using "magnitudes" of principal curvatures. Taking the surface shown in Fig. 10.6 for an example and choosing $n = \ldots$
These authors would write (10.39) in their notations as

\[ n(a_1, a_2) = n(0,0) + \kappa_1 a_1 a_1 + \kappa_2 a_2 a_2 \]

This concludes our summary of differential geometry. For details, the readers are referred to the references cited earlier in this appendix.
11. APPENDIX C: DERIVATIONS OF SOME EQUATIONS

11.1 Derivation of Eq. (5.30)

From Eq. (5.29), we have

\[ \frac{\partial^2 u}{\partial t^2} = \exp\{i\omega(S-t)\} \sum_{m=0}^{\infty} (i\omega)^{-m+2} u_m \]

Also, by

\[ \nabla \left\{ \exp\{i\omega(S-t)\} \right\} = i\omega \left\{ \exp\{i\omega(S-t)\} \right\} \nabla S \]

we find

\[ \nabla(\nabla u) = \exp\{i\omega(S-t)\} \sum_{m=0}^{\infty} (i\omega)^{-m} \left\{ (i\omega)^2 (u_m \nabla S) \nabla S + i\omega \left[ (u_m \nabla S + \nabla(u_m \nabla S) \right] + \nabla(u_m \nabla) \right\} \]

Using the vector identity

\[ \nabla^2_{ab} = a \nabla^2_b + 2(\nabla a) b + b(\nabla a) \]

we find, with the aid of Eq. (11.2), that
(11.5) \[ \nabla^2 u = \exp[i\omega(S-t)] \sum_{m=0}^\infty \left( i\omega \right)^m \left\{ (i\omega)^2 (\nabla S \cdot \nabla S) + i\omega (2 \nabla S \cdot \nabla + \nabla^2 S) + \nabla^2 \right\} u_m \]

By substituting Eqs. (11.1), (11.3) and (11.5) into Eq. (5.28) and cancelling the common exponential factor, Eq. (5.30) then follows immediately.

11.2 Derivation of Eq. (8.1)

Consider the scattering of a longitudinal wave by a volumetric flaw. If the incident L-wave is defined by

(11.6) \[ \exp(ik_L x - i\omega t) \]

then from Eqs. (4) and (7) in reference [66], the directly backscattered L-wave is found to be

(11.7) \[ u^{SC}(y, \omega) = \frac{A \exp(ik_L y - i\omega t)}{y} \]

where \( y = |y| \) and \( A \), the scattering amplitude, is given by
Here $\delta \rho$ is the density difference between the flaw and host medium, i.e., $\delta \rho = \rho_{\text{flaw}} - \rho_{\text{host}}$. Similar definitions apply to $\delta \lambda$ and $\delta \mu$.

Also, in Eq. (11.8), $\gamma(x)$ is the characteristic function of the flaw, which is one if $x$ is inside the flaw and is zero outside. The Born approximation assumes that $\delta \rho$, $\delta \lambda$, and $\delta \mu$ are all small quantities. One consequence of this weak scattering assumption is that the bracketed term in Eq. (11.8) can be approximated by

$$
(11.9) \quad \left[ \frac{\delta \rho}{\rho} + \frac{\delta \lambda + 2 \delta \mu}{\lambda + 2 \mu} \right] = 2 \left[ \frac{\delta \rho}{\rho} + \frac{\delta c}{c} \right]
$$

with an error of $O[(\delta \rho)^2, (\delta \lambda)^2, (\delta \mu)^2]$. Using Eq. (11.9) in Eq. (11.8) we have

$$
(11.10) \quad A = \frac{I k_{1}^{2}}{2 \pi} \left[ \frac{\delta \rho}{\rho} + \frac{\delta c}{c} \right] \int_{-\infty}^{\infty} \gamma(x) \exp(2i k_{1} I \cdot x) \, d^{3}x
$$

Strictly speaking, the Born approximation can be applied only when the scatterer is infinitely weak, since it assumes that $\delta \rho$, $\delta \lambda$, and $\delta \mu$ are $\ll 1$. However, experience has shown that it also works well for some strong scatterers, such as a void. We can obtain the case for a void formally by setting
(11.11) \( \delta \rho/\rho = -1 \) and \( \delta c_1/c_1 = -1 \)

in Eq. (11.10) to obtain

\[
\begin{align*}
(11.12) \quad & A = -\frac{\text{Ik}_1^2}{\pi} \int \int \int_{-\infty}^{\infty} \gamma(x) \exp(2\text{i}k_1 x \cdot x) \ d^3x \\
& \text{Eqs. (11.11) follow from the fact that both the density and wavespeed inside the void are zero, since there is no material inside the void.}
\end{align*}
\]

Now suppose, instead of a time harmonic incident wave, the void is subjected to a longitudinally polarized incident impulsive wave defined by

\[
(11.13) \quad I \delta(t - \text{i}x/c_1)
\]

Then, by following a similar procedure described in Section 7.6, we find that the time-domain far-field scattered wave is given by

\[
(11.14) \quad u^{sc}(y, \tau) = \frac{\text{Ic}_1}{8\pi y} \left. \frac{d^2 S_3(x_3)}{dx_3^2} \right|_{x_3 = c_1 \tau/2} \quad \text{with} \quad \tau = t - y/c_1
\]

This agrees with equation (8.1).
11.3 Zeros of $\Delta$

By setting the determinant $\Delta$ given in Eq. (8.26) equal to zero and solving for $\Lambda_T$ in terms of $\Lambda_L$, we find

\[(11.15) \quad \Lambda_T = -\Lambda_L \pm [(\Lambda_L - 1)^2]^{1/2}\]

Regardless of whether $\Lambda_L = 0$ or $\neq 0$, this equation always gives the following two solutions

\[(11.16) \quad \Lambda_T = -\Lambda_L \pm (1 - \Lambda_L) = \begin{cases} 1 - 2\Lambda_L & \text{(case 1)} \\ -1 & \text{(case 2)} \end{cases}\]

These are the only two cases where $\Delta$ can vanish. By using the definitions of $\Lambda_T$ and $\Lambda_L$ [Eqs. (7.73), (7.82)], we obtain for cases 1 and 2, respectively,

\[(11.17) \quad \text{case 1: } \frac{c_2}{c_1} = 1 \pm \sqrt{2}\]
\[(11.18) \quad \text{case 2: } \frac{c_2}{c_1} = 3/4\]

Since the wavespeed ratio can never be negative and $(1 - \sqrt{2}) < 0$, the negative sign in Eq. (11.17) must be discarded. Also the plus sign in Eq. (11.17) yields a $c_2/c_1$ ratio greater than one, which is unphysical. Hence the only possible solution is $c_2/c_1 = 3/4$, which corresponds to $\nu = -1/7$. However, $\nu = -1/7$ is also unphysical, thus we conclude that there is no material that can make the determinant $\Delta$ vanish.
12. APPENDIX C: SOME GEOMETRICAL PROPERTIES OF AN ELLIPSOID

In this appendix we shall investigate some geometrical properties of an arbitrary ellipsoid (designated henceforth as E) pertaining to scattering theory. Specifically, we shall be interested in (i) the cross sectional area of the ellipsoid E intercepted by a plane \( \Pi \), and (ii) the sum of principal curvatures at any point of E. The results obtained here are used in Chapter 8.

To begin with, we establish a rectangular Cartesian coordinate system \( O-X_1X_2X_3 \) with its origin at the center of the ellipsoid E (see Fig. 12.1). The coordinate axes \( X_1, X_2, \) and \( X_3 \) are aligned with the three major axes of E, and the corresponding unit vectors are denoted by \( \hat{X}_1, \hat{X}_2, \) and \( \hat{X}_3 \), respectively. We suppose that the equations for E and \( \Pi \) with respect to this Cartesian coordinate system are

\[(12.1) \quad \text{Ellipsoid E: } \left(\frac{X_1}{b_1}\right)^2 + \left(\frac{X_2}{b_2}\right)^2 + \left(\frac{X_3}{b_3}\right)^2 = 1\]

\[(12.2) \quad \text{Plane } \Pi: \quad \hat{X} \cdot q = \eta \quad \text{with} \quad \hat{X} = X_1\hat{X}_1 + X_2\hat{X}_2 + X_3\hat{X}_3\]

where \( q \) is a unit vector normal to the plane \( \Pi \) and \( \eta \) is an arbitrary constant. Geometrically, \( |\eta| \) is just the perpendicular distance from \( O \) to the plane \( \Pi \).

The cross sectional area \( A \) of the ellipsoid E intercepted by the plane \( \Pi \) can be found by [194]

\[(12.3) \quad A = \iiint_{E} \delta(\hat{X} \cdot q - \eta) \, d^3\hat{X}\]
Fig. 12.1. Coordinate system $0-X_1X_2X_3$
Here the volume integral is extended over the region occupied by the ellipsoid $E$ and $\delta$ is the Dirac delta function. This integral can be evaluated analytically by introducing the transformation:

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} = 
\begin{bmatrix}
b_1 & 0 & 0 \\
0 & b_2 & 0 \\
0 & 0 & b_3
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{bmatrix}
\]

(12.4)

Under this transformation, the ellipsoid $E$ in the $X$-space is "deformed" into a unit sphere in the $\xi$-space:

\[
\xi_1^2 + \xi_2^2 + \xi_3^2 = 1
\]

(12.5)

Furthermore, the inner product $X \cdot q = X_i q_j$ (sum over $i = 1,2,3$) is transformed to

\[
X \cdot q = b_1q_1 \xi_1 + b_2q_2 \xi_2 + b_3q_3 \xi_3
\]

(12.6)

It proves useful to rewrite this equation as

\[
X \cdot q = Q \cdot \xi
\]

(12.7)

where $Q$ is a unit vector and

\[
Q = b_1q_1X_1 + b_2q_2X_2 + b_3q_3X_3
\]

(12.8)
Inserting Eq. (12.7) into Eq. (12.3) and using the transformation (12.4), we find

\[
Q = \left[ (b_1 q_1)^2 + (b_2 q_2)^2 + (b_3 q_3)^2 \right]^{1/2}
\]

Notice that now the integral is extended over the region occupied by the unit sphere defined in Eq. (12.5). This integral has a simple geometrical interpretation, it is the cross sectional area of the unit sphere intercepted by a plane orthogonal to the unit vector \(Q\) and situating at a (perpendicular) distance \(|\eta/Q|\) from the center of the unit sphere (Fig. 12.2). Hence, the above integral can be evaluated simply by inspection, resulting

\[
A = \frac{b_1 b_2 b_3}{Q} \int \int \int_{\text{Sphere}} \delta \left[ Q \cdot \xi - \frac{\eta}{Q} \right] d^3 \xi
\]

The Heaviside function \(H\) is used here to insure that the integral be zero whenever \(1 < |\eta/Q|\). This is necessary, because for \(1 < |\eta/Q|\) the unit sphere and the "cutting" plane has no intersection at all. Placing the preceding result back in Eq. (12.10) gives
Fig. 12.2. A unit sphere intercepted by a plane
The parameter $\eta$ in Eq. (8.12) can assume any value, different $\eta$ merely corresponds to different parallel planes. Among all these $\eta$'s there are two special $\eta$'s (say, $\eta_1$ and $\eta_2$) that correspond to two planes ($\Pi_1$ and $\Pi_2$) tangent to the ellipsoid. Let us consider one of these two tangent planes, $\Pi_1$, that corresponds to $\eta = \eta_1$ (Fig. 12.3). Now, it is of interest to note that the image (in the $\xi$-space) of the tangent plane $\Pi_1$ through the transformation (12.4) is still a plane tangent to the unit sphere (Fig. 12.3). This follows from the fact that (12.4) is a special case of what is called the affine transformation [195], through which a plane remains a plane after transformation. The perpendicular distance $h$ from the center of the ellipsoid $E$ to the tangent plane $\Pi_1$ (Fig. 12.3), by Eq. (12.7), is given by

\begin{equation}
(12.13) \quad h = |\mathbf{\xi} \cdot q| = |q_0 \cdot \mathbf{\xi}|
\end{equation}

But from Fig. 12.3, we have $q_0 \cdot \mathbf{\xi} = -1$. Hence,

\begin{equation}
(12.14) \quad h = Q
\end{equation}

This result shows that the quantity $Q$ defined earlier in Eq. (12.9) is nothing but the perpendicular distance $h$. With this result, it is more convenient to rewrite Eq. (12.12) as

$$
A = \frac{n b_1 b_2 b_3}{Q} \left[ 1 - (\eta/Q)^2 \right] H(\eta/Q + 1) - H(\eta/Q - 1)
$$
Fig. 12.3. Tangent plane $\Pi_1$ and its image
(12.15) \[ A = \frac{nb_1b_2b_3}{h} \left[ 1 - \left( \eta/h \right)^2 \right] \left\{ H(\eta/h + 1) - H(\eta/h - 1) \right\} \]

for \( h \) gives us a more transparent geometrical visualization than \( Q \) does.

In scattering problems the unit vector \( q \) usually corresponds to the incident (plane) wave direction and the plane \( \Pi \) mentioned previously represents the incident wavefront. Moreover, as was discussed in Chapters 6 and 7, for leading edge response all we need to consider is the special case that the cutting plane \( \Pi \) is in the immediate neighborhood of the tangent plane \( \Pi_1 \). Accordingly, for the purpose of determining the leading edge response, we may ignore the second Heaviside function \( H(\eta/h - 1) \) appearing in Eq. (12.15) and write

(12.16) \[ A(\eta) = \frac{nb_1b_2b_3}{h^3} (\eta^2 - h^2)H(\eta + h) \]

where use has been made of \( H(\eta/h + 1) = H(\eta + h) \).

It is important to note that the preceding expression for the area function \( A \) has a coordinate-invariant form, for \( b_1, b_2, b_3 \), and \( h \) are all independent of the choice of coordinate systems. In view of this invariance, we may re-establish a new rectangular coordinate system with origin at the point of tangency (Fig. 12.4). From Fig. 12.4, it is clear that \(-\eta = h - x_3\). Hence, after making a notational change from \( A \) to \( S_3 \) (in keeping with our previous notation adopted in Chapters 6 and 7), the area function given in Eq. (12.16) can be expressed in terms of
Fig. 12.4. A coordinate system with origin at the point of tangency
Comparing this expression to Eq. (6.63) one sees immediately that

\[(12.18) \quad \frac{b_1 b_2 b_3}{h^2} = (R_1 R_2)^{1/2}\]

in which \(R_1\) and \(R_2\) are principal radii of the ellipsoid at the point of tangency. From the above two equations we find that, for \(x_3 \ll 1\),

\[(12.19a) \quad S_3(x_3) = \pi (R_1 R_2)^{1/2} (2x_3 - x_3^2/h) H(x_3)\]

\[(12.19b) \quad \frac{dS_3}{dx_3} = 2\pi (R_1 R_2)^{1/2} (1 - x_3/h) H(x_3)\]

\[(12.19c) \quad \frac{d^2S_3}{dx_3^2} = 2\pi (R_1 R_2)^{1/2} \left[ \delta(x_3) - \frac{H(x_3)}{h} \right]\]

In arriving at Eqs. (12.19b) and (12.19c) we have used the relationships

\[x_3 \delta(x_3) = (x_3)^2 \delta(x_3) = 0\]

in the theory of distributions [167].

In the rest of this appendix, we shall determine the sum of two principal curvatures \((\kappa_1 + \kappa_2)\) of the ellipsoid \(E\) depicted in Fig. 12.1. As a first step toward this end, we parametrize \(E\) by two parameters \(a_1\)
and $a_2$ as follows:

\begin{align}
(12.20a) \quad & X_1 = b_1 \cos a_1 \cos a_2 \\
(12.20b) \quad & X_2 = b_2 \cos a_1 \sin a_2 \\
(12.20c) \quad & X_3 = b_3 \sin a_2
\end{align}

We remark that the graphical meaning of the two parameters (or surface coordinates, see Appendix A) $a_1$ and $a_2$ is not immediately apparent. This, in fact, happens frequently in differential geometry when a surface is parametrized. However, one generally need not care about the meaning of the parameters. Accidentally, as one shall see at the end of this appendix, our final expression for the sum of principal curvatures ($K_1 + K_2$) does not involve the two graphically meaningless parameters ($a_1, a_2$).

By Eqs. (12.20a-c) the position vector $\mathbf{X}$ of any point on $E$, with respect to the coordinate system of Fig. 12.1, can be written as

\begin{align}
(12.20d) \quad & \mathbf{X} = X_1 b_1 \cos a_1 \cos a_2 + X_2 b_2 \cos a_1 \sin a_2 + X_3 b_3 \sin a_2
\end{align}

From differential geometry [192,193], it is known that the mean curvature $M$, which being one half of $K_1 + K_2$, is given by

\begin{align}
(12.21) \quad & M = \frac{g_{11} h_{22} + g_{22} h_{11} - 2 g_{12} h_{12}}{2 |\mathbf{X}_1 \times \mathbf{X}_2|^2}
\end{align}
Here the g's and h's are, respectively, components of the first and second fundamental forms. They are defined in Appendix A. Also, in the preceding equation, $X_1$ and $X_2$ denote the partial differentiations of $X$ with respect to the parameters $a_1$ and $a_2$, respectively. From the definitions of the first and second fundamental forms, we find

\begin{align}
(12.22a) \quad g_{11} &= X_{1,1} \cdot X_{1,1} = b_1^2 \sin^2 a_1 \cos^2 a_2 + b_2^2 \sin^2 a_1 \sin^2 a_2 + b_3^2 \cos^2 a_1 \\
(12.22b) \quad g_{12} &= X_{1,1} \cdot X_{1,2} = (b_1^2 - b_2^2) \sin a_1 \cos a_2 \sin a_2 \cos a_2 \\
(12.22c) \quad g_{22} &= X_{2,2} \cdot X_{1,2} = (b_1^2 \sin^2 a_2 + b_2^2 \cos^2 a_2) \cos^2 a_1 \\
(12.23a) \quad h_{11} &= n \cdot X_{1,1} = h \\
(12.23b) \quad h_{12} &= n \cdot X_{1,2} = 0 \\
(12.23c) \quad h_{22} &= n \cdot X_{2,2} = h \cos^2 a_1
\end{align}

where $h$ is the perpendicular distance mentioned earlier (see Fig. 12.4), and $n$ is the inward unit normal to the ellipsoid $E$. Explicitly, $n$ is given by

\begin{align}
(12.24) \quad n &= - \frac{h}{b_1} \cos a_1 \cos a_2 X_1 - \frac{h}{b_2} \cos a_1 \sin a_2 X_2 - \frac{h}{b_3} \sin a_1 X_3
\end{align}

Moreover, we find that
Placing Eqs. (12.22), (12.23), and (12.25) back in (12.21) and remembering that \( 2M = \kappa_1 + \kappa_2 \), we obtain

\[
(12.26) \quad \kappa_1 + \kappa_2 = \frac{h^3}{b_1 b_2 b_3} \left\{ \frac{b_1^2 (\sin^2 a_1 \cos^2 a_2 + \sin^2 a_2)}{h^2} + \frac{b_2^2 (\sin^2 a_1 \sin^2 a_2 + \cos^2 a_2 + b_3^2 \cos^2 a_1)\} \right\}
\]

This expression can be casted into a coordinate invariant form by making the substitutions:

\[
(12.27) \quad \sin^2 a_1 \to 1 - \cos^2 a_1 \quad \quad \cos^2 a_1 \to 1 - \sin^2 a_1
\]

The result is

\[
(12.28) \quad \kappa_1 + \kappa_2 = \frac{h^3}{b_1 b_2 b_3} \left\{ \left( \frac{b_1^2}{h^2} + \frac{b_2^2}{h^2} + \frac{b_3^2}{h^2} \right) - \frac{b_1^2 n_1}{h^2} - \frac{b_2^2 n_2}{h^2} - \frac{b_3^2 n_3}{h^2} \right\}
\]

or, in a more symmetrical form,
In Eq. (12.27), \( n_1, n_2 \) and \( n_3 \) are projections of the inward unit normal \( n \) on the three coordinate axes shown in Fig. 12.1, i.e., \( n_i = n \cdot x_i \) (\( i = 1, 2, 3 \)). Since \( n \) coincides with the unit vector \( q \) (which usually defines the incident wave direction in scattering problems, see Fig. 12.4), one can also replace \( n \) in Eq. (12.29) by \( q \). The coordinate invariance of the foregoing expression for \( \kappa_1 + \kappa_2 \) can be seen more clearly by recalling Eqs. (12.14) and (12.9):

\[
(12.30) \quad h = \left[ b_1^2(q \cdot x_1)^2 + b_2^2(q \cdot x_2)^2 + b_3^2(q \cdot x_3)^2 \right]^{1/2}
\]

Since this expression for \( h \) is obviously coordinate independent, the coordinate invariance of the expression for \( \kappa_1 + \kappa_2 \) given in Eq. (12.29) follows immediately.

Finally, to conclude this appendix we remark that from differential geometry the Gaussian curvature \( K = \kappa_1 \kappa_2 \) is given by

\[
(12.31) \quad K = \frac{h_{11} h_{22} - h_{12}^2}{|x_{,1} \times x_{,2}|^2}
\]
By substituting (12.23) and (12.25) into this equation and using \( R_1 R_2 = \frac{1}{K} \) we find

\[(12.32) \quad (R_1 R_2)^{1/2} = b_1 b_2 b_3 / h^2\]

This is exactly identical to a previous result [see Eq. (12.18)] for the squared root of \( R_1 R_2 \), which was derived by using another approach.
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