1987

Induction motor analyses including non-sinusoidal excitation

Mohammad El-Hosainy Ismaeel

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Induction motor analyses including non-sinusoidal excitation

Ismaeel, Mohammad El-Hosainy, Ph.D.

Iowa State University, 1987
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Induction motor analyses
including non-sinusoidal excitation

by

Mohammad El-Hosainy Ismaeel

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY
Department: Electrical Engineering and Computer Engineering
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1987
# TABLE OF CONTENTS

## I. INTRODUCTION
- A. Historical Background 1
- B. Problem Formulation 9
- C. Research Objectives 10
- D. Research Outline 11

## II. QUASI-THREE-DIMENSIONAL ANALYSIS OF INDUCTION MOTORS
- A. Introduction 14
- B. Mathematical Model 15
- C. Basic Field Equations
  1. Primary tooth region 20
  2. Secondary tooth region 21
  3. Air-gap region 21
  4. Secondary end-ring constraints 22
  5. Primary overhang region 24
  6. Air overhang region 26
- D. Boundary Conditions
  1. Main regions 29
  2. Overhang regions 31
- E. Solution of the Field System of Equations 32
  1. General considerations 32
  2. Primary tooth region 35
  3. Secondary tooth region 37
  4. Air-gap region 40
  5. Primary overhang region 41
  6. Air overhang region 42
- F. The Arbitrary Constants and the Field Expressions 42
  1. Main regions 42
  2. Overhang regions 54
G. Voltage Equations 56
H. Force and Power Expressions 62
   1. Force expressions 62
   2. Power expressions 64
I. Conclusion 68

III. WAVE IMPEDANCE CONCEPT FOR SOLVING THE FIELD PROBLEM IN INDUCTION MOTORS 69
   A. Introduction 69
   B. Field Analysis 71
   C. Solution of the Field Equations 78
   D. Motor Currents and Performance 86
      1. Current equations 86
      2. Performance equations 87
   E. Conclusion 89

IV. EXTENSION OF THE FIELD ANALYSIS 90
   A. Introduction 90
   B. Mathematical Model 90
   C. Field Analysis 92
   D. Boundary Conditions 95
   E. The Arbitrary Constants and the Field Expressions 96
   F. Other Considerations 99
   G. Conclusion 99

V. NUMERICAL CALCULATIONS AND DISCUSSIONS 101
   A. Introduction 101
   B. Induction Motor Model 101
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>C. Results and Discussion</td>
<td>103</td>
</tr>
<tr>
<td>VI. CONCLUSIONS</td>
<td>120</td>
</tr>
<tr>
<td>VII. BIBLIOGRAPHY</td>
<td>123</td>
</tr>
<tr>
<td>VIII. ACKNOWLEDGMENTS</td>
<td>130</td>
</tr>
<tr>
<td>IX. APPENDIX: FORTRAN PROGRAM FOR CALCULATING THE MOTOR PERFORMANCE</td>
<td>131</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>Figure 2.1a</td>
<td>Part of the developed motor</td>
</tr>
<tr>
<td>Figure 2.1b</td>
<td>The equivalent anisotropic magnetic regions 1, 2 and 3</td>
</tr>
<tr>
<td>Figure 2.2a</td>
<td>The developed secondary, together with its end rings</td>
</tr>
<tr>
<td>Figure 2.2b</td>
<td>The current distribution in an element end ring of length $\delta x$</td>
</tr>
<tr>
<td>Figure 2.3</td>
<td>Sectional view of the induction motor showing regions 1, 2 and 3 (the primary, secondary and air-gap regions), as well as regions 4 and 5 (the primary overhang and air overhang regions)</td>
</tr>
<tr>
<td>Figure 2.4</td>
<td>A sectional view of the induction motor showing the frame of reference, the main dimensions and the governing differential equations in each region</td>
</tr>
<tr>
<td>Figure 3.1a</td>
<td>Part of the developed slotted structure of the induction motor</td>
</tr>
<tr>
<td>Figure 3.1b</td>
<td>The equivalent anisotropic magnetic regions 1, 2 and 3</td>
</tr>
<tr>
<td>Figure 3.2</td>
<td>Element of equivalent circuit of induction motor or transmission line</td>
</tr>
<tr>
<td>Figure 3.3</td>
<td>Transmission line analogy of the different regions of the induction motor and the boundary conditions at the different sections</td>
</tr>
<tr>
<td>Figure 4.1</td>
<td>Sectional view of the induction motor showing the seven regions (1, 2, ..., 7) considered in the analysis</td>
</tr>
<tr>
<td>Figure 5.1</td>
<td>Thrust-speed characteristics for the motor without end rings. Curve A: space and time harmonics are taken into account. Curve B: space and time harmonics are neglected</td>
</tr>
</tbody>
</table>
Figure 5.2. Thrust-speed characteristics for the motor with and without end rings. Curves C and D are for the motor with end rings when harmonics are taken into account and are neglected, respectively. Curves A and B are those of Figure 5.1 for the purpose of comparison.

Figure 5.3. Thrust-speed curve at the low speed range of the motor with end rings. Lower curve: space harmonics only are considered. Upper curve: both space and time harmonics are considered.

Figure 5.4. Thrust-speed characteristics for the motor with different end-ring conductivities. Space and time harmonics are not taken into account. \( (\sigma_E/\sigma) \) is the ratio between the end-ring conductivity and the squirrel cage conductivity.

Figure 5.5. Thrust-speed characteristics for the motor with different end-ring conductivities. Space and time harmonics are taken into account. \( (\sigma_E/\sigma) \) is the ratio between the end-ring conductivity and the squirrel cage conductivity.

Figure 5.6. Thrust-speed characteristics for the motor with different primary tooth-top openings. Space and time harmonics are taken into account.

Figure 5.7. Primary efficiency-speed characteristics for the motor with different end-ring conductivities. Space and time harmonics are taken into account. \( (\sigma_E/\sigma) \) is the ratio between the end-ring conductivity and the squirrel cage conductivity.

Figure 5.8. Primary efficiency-speed characteristics for the motor with different primary tooth-top openings. Space and time harmonics are taken into account.

Figure 5.9. Secondary power factor-speed characteristics for the motor with different end-ring conductivities. Space and time harmonics are taken into account. \( (\sigma_E/\sigma) \) is the ratio between the end-ring conductivity and the squirrel cage conductivity.
Figure 5.10. Secondary power factor-speed characteristics for the motor with different primary tooth-top openings. Space and time harmonics are taken into account
LIST OF TABLES

Table 5.1. Harmonic currents and their percentage values with respect to the fundamental 117

Table 5.2. Harmonic forces and their percentage values with respect to the fundamental 118

Table 5.3. Effect of harmonics on the primary efficiency (space harmonics are taken into account) 119
I. INTRODUCTION

A. Historical Background

The history of the induction motor extends as far back as the 19th century. It started in 1824 with the discovery made by Gambey [from Ref. 79], the instrument-maker of Paris, that a compass needle, when disturbed and set oscillating, comes to rest more quickly when it is in the vicinity of copper than when wood is near it. At that time also, Barlow and Marsh, at Woolwich, had observed the effect on a magnetic needle of rotating it near a sphere of iron [from Ref. 79]. Arago published an account of an experiment with a compass needle within rings of different materials in the 1890s [from Ref. 79]. In this experiment, he pushed the needle aside to about 45° and counted the number of oscillations made by the needle before the swing decreased to 10°. With a ring of wood, the number of oscillations was 145, with a copper ring 66, and with stout copper ring only 33. In 1825, he suspended a compass needle over a rotating copper disc and found that by turning the disc slowly, the needle is deviated out of the magnetic meridian. By rotating the disc fast enough, he found that continuous rotation of the needle could be produced.

The brilliant discovery by Faraday, in 1831, of electromagnetic induction provided the solution to the question of the origin of the forces present in the above experiments of Gambey, Barlow and Marsh, and Arago. Faraday showed that the rotation of the Arago disc was due to induced currents, set up in the disc by the relative motion of the
disc and the compass needle. The magnetic field caused by those induced currents, in turn, interacted with the magnetic needle and caused its motion. From 1831 to 1879, this valuable discovery produced no further results. In June 1879, Walter Baily read a paper [from Ref. 79] before the Physical Society of London on "A Mode of Producing Arago's Rotations." Baily used a fixed electromagnet with four magnet cores joined to a yoke.

The four magnet cores were about four inches long and each was wound with about 150 turns of insulated copper of 2.5 mm diameter. The coils were connected two and two in series, similar to two independent horseshoe magnets, and were diagonally across from one another.

The two circuits were connected separately to a revolving commutator, built from a simple arrangement of springs and contact strips mounted on a piece of wood, with a wire handle by which it was turned. By rotation, the current from two batteries was caused to be reversed alternately in the two circuits, and this gave rise to the following changes in polarity of the four poles.

\[
\begin{array}{ccccccc}
\text{N} & \bullet & \text{N} & \text{N} & \bullet & \text{N} & \text{S} & \text{N} & \text{S} & \bullet \\
\text{S} & \bullet & \text{S} & \text{S} & \bullet & \text{S} & \text{S} & \bullet & \text{S} & \text{N} & \bullet & \text{N}
\end{array}
\]

In this rotating magnetic field, a copper disc was suspended. Baily stated: "The rotation of the disc is due to that of the magnetic field in which it is suspended, and we should expect that if a similar motion
of the field could be produced by any other means, the result would be a similar motion of the disc" [from Ref. 79]. Baily also suggested that if a whole circle of poles was arranged under the disc, successively excited in opposite pairs, the series of impulses would all tend to make the disc revolve in one direction around the axis, and added, "In one extreme case, when the number of electromagnets is infinite, we have the case of a uniform rotation of the magnetic field, such as we obtain by rotating permanent magnets" [from Ref. 79]. It is clear that Baily had grasped the fundamental principle of action of the induction motor, and the motor he exhibited before the Physical Society in 1879 was the first induction motor. But, it needed later important discoveries of methods for producing the revolving field by means of alternating currents to make it the useful machine that it is today.

The next discovery was made by Marcel Deprez in 1883 [from Ref. 79]. He fed alternating current to a coil, which produced an alternating or oscillating field along the OX axis. He supplied another coil, whose magnetic axis made an angle of 90° with the OX axis, with alternating current, whose phase difference was 90° in time from the current in the first coil, and showed that a revolving field of constant amplitude could be produced. The frequency of the two currents was the same. He also showed that if the two currents were of equal frequency, but not of equal amplitude, an elliptically rotating field was produced. The number of turns in each coil was the same.
Professor Ferraris [from Ref. 79] arrived at the same conditions as Baily and Deprez in 1885, and apparently without knowing of the work of either. His paper on "Electrodynamic Rotations Produced by Means of Alternating Currents" was published in 1888. He suggested the method of obtaining currents, differing in phase by nearly 90°, by inserting a resistance in one winding and inductance in the other, thus making the ratio (reactance/resistance) small in one winding and large in the other. This method is largely used for starting single-phase motors.

Next followed the great work of Nicola Tesla between 1887 and 1891 [from Ref. 79]. His research placed the induction motor on a sound foundation. His patents were sold to the Westinghouse Company of America, whose pioneer efforts in this field must be recognized. In that period, however, the only ac supply circuits were single phase, and the frequencies were 133 and 125 Hz. These supply frequencies were obviously unsuitable for the development of the motor.

In 1891, the Electrotechnical Exhibition was held at Frankfort and the three-phase transmission of power was demonstrated. Two turbine-driven three-phase generators were installed at Lauffen, Germany, generating 1400 A at 55 V. The frequency was 40 Hz. Three-phase transformers were installed at each end of the line to raise the voltage to 8000 V at Lauffen, and to reduce it to 65 V at Frankfort. The distance of transmission was 110 miles. This bold experiment demonstrated the feasibility of three-phase transmission of energy.
The load consisted of a 100 hp three-phase motor and also lamps. Several German firms exhibited different types of three-phase induction motors at the Exhibition. One three-phase, 3 bhp motor had the three-phase supply brought into the rotor by three slip-rings, the secondary circuit, being the stator winding, consisted of a closed-circuit winding. Several motors, built by the Oerlikon Company and designed by C. E. L. Brown, were shown. One such motor was a three-phase, 20 bhp motor. It had a distributed stator winding and a squirrel-cage rotor and a small air-gap. The squirrel-cage rotor was the invention of Dolivo-Dobrowolsky, who cooperated with C. E. L. Brown in the design of these motors. This motor of Brown's closely resembled in construction the induction motor of today. From the short account given, it will be realized that much progress was made purely as the result of experiment, and that much theoretical investigation was needed to explain the reactions taking place in the motor, and to show how it could be designed to give the characteristics desired.

To that end, it was necessary to give a lucid theory of alternating currents. Thomas H. Blakesley gave a series of ten brilliant papers in Electrician in 1885 [from Ref. 79]. He discussed, for the first time, alternating current phenomena by means of polar diagrams.

In 1892, F. Bedell and A. C. Crehore [from Ref. 79] published their book, Alternating Currents, in which the polar diagrams were used and applied to the theory of the transformer and the locus of the primary e.m.f. of the current transformer was shown to be a circle. In 1894, Kapp [from Ref. 79] gave a very lucid elementary account of the
phenomena in induction motors. The polar diagram was developed and included the primary resistance and leakage flux.

In 1895, Blondel [from Ref. 79] gave his papers on "Some General Properties of Revolving Magnetic Fields." In these papers, published in Éclairge Electrique, he unfolded the theory of the composition of magnetic fluxes, including leakage fluxes. In 1895, B. A. Behrend [8] also proved that the locus of the primary current of the alternating-current transformer is a circle in the polar diagrams, provided the primary resultant magnetic field is constant. The circle locus of the induction motor was also shown by A. Heyland in 1894 [8]. There has been much controversy about priority in this discovery.

The circle diagram in use today is undoubtedly due to Behrend. Details are described in Behrend's book, the first edition of which was published in 1901 [8]. The second edition was published in 1921 [8]. At that time, the general outline was clearly seen, but there remained many important questions to be answered. The circle diagram clearly demonstrated the need for a small air-gap length, for high power factor, and for a large ideal short-circuit current. All the main characteristics, such as torque, power, current, slip and efficiency are readily determined from the Behrend circle diagram.

Although the relation of dimensions to characteristics was known by about 1900 or so, it remained to determine the effect of harmonics from the distribution of the winding, and from the slots on the performance of the motor. Such harmonics produce noise and vibration and may, by producing saddle backs in the torque
curve, make the machine crawl and prevent it from accelerating to full speed. The question of noise has been investigated by H. Fritz, Kron and Chapman. A large number of papers has been written on this subject [55, 56, 79].

The question of improved efficiency has resulted in improvements in the manufacture of steel laminations, such as through the introduction of silicon into the steel. Brands known as stalloy and super-stalloy have been introduced, and are used in those cases where it is necessary to keep down the iron losses, e.g., in totally enclosed machines, and especially in machines for 400 Hz, such as those used in the automatic pilots in aircraft. The effects of various forms of unbalance, either from unbalanced applied voltage or from the dissymmetry in the rotor circuit, have been studied through the concept of the symmetrical components. This work started by Fortiscue in 1918 [35], and was followed by others.

From time to time, attempts have been made to adapt Maxwell's equations to electrical machines. Perhaps the earliest definitive paper was published by Greedy [25] in 1917, although in earlier papers, Carter [15], Cullen and Barton [26] and Douglas [32] had shown the importance of the applications of the field concept. The next significant contribution was made by Hague, who published a series of papers between 1917 and 1926. These are now available in book form [46]. These authors indicate that the solution to an electrical-machine problem can be obtained by determining the electromagnetic fields in the various regions of the machine.
The contributions from 1899 to the present day are well spread over this entire period. A survey of the literature shows that the field-theory methods of predetermining the performance characteristics of electrical machines can be broadly classified as (a) energy-storage methods, (b) energy-transfer methods, (c) methods derived from analogies between the field theory and circuit theory, and (d) certain special techniques, such as the method of images, conformal transformation and graphical techniques.

The development of static switching devices with high power ratings is leading to their increasing application in the control of electrical machines. Two important applications of this type are in static frequency converters and voltage controllers, both of which are utilized to provide control of the torque and speed characteristics of ac machines. In particular, they are now being used to obtain flexible performance characteristics from robust, but inherently constant-speed cage induction motors. In many of these systems, the output voltage and current waveforms of the static converters are rich in harmonics. These harmonics have detrimental effects on motor performance.

The orders and magnitudes of the current harmonics which are present in the converter output depend on the design of the static converter, and, to a lesser extent, on the type of load. The converter output is usually amenable to analysis by Fourier series.

Some earlier publications [52, 53] on this topic have considered the losses and torques produced in cage induction motors from
distorted supply waveforms. Each of these has indicated, in some detail, that the resultant additional steady-state torque is negligible, although Jain [52] omitted the skin effect in his calculations. In some cases [18, 52, 53], the additional losses have been found to be small, but it has also been shown [19, 20] that if the input waveforms have a high harmonic content, the additional losses may be as high as the losses contributed by the fundamental voltages and currents. This result highlights the need for an accurate assessment of the time-harmonic losses. Fluxes and currents resulting from the input harmonics occur at relatively high frequencies, which means that the skin effect should be included in the $I^2R$ calculations. Core losses due to harmonic main fluxes occur at high frequencies, but the fluxes are highly damped by induced secondary currents. The methods used are based on the conventional induction motor equivalent circuit, but includes modifications to the conventional theory necessary to account for the time harmonic effects [18-20, 52, 53].

Very recently, the problem of harmonic pollution on power systems has led to the detriment of electrical appliances to harmonics, as well as their aging due to harmonics of the power system's voltage [41, 42, 78]. This problem is also treated by using the conventional equivalent circuit of the induction machines.

B. Problem Formulation

The literature search reveals that most of the work in the induction machine theory depends mainly on circuit theory, and field
theory is just used as an aid in obtaining the numbers to go in the lumped circuits. For a long time, generalized machine theory, with its emphasis on circuit ideas, has pushed field theory into the background. The result has been to produce young engineers who are adept at formulation and manipulation of matrix equations, but who have little idea of the physical significance of the circuit parameters [28]. Also, the teaching of electrical machines has emphasized detailed practice. The reaction to this teaching is a feeling that textbook machines are 'energy transducers' bearing no relation to useful machines [28]. It is generally recognized that for a number of the outstanding problems on electrical machines, the solution can only be achieved by field methods. It would be appropriate to adopt field methods to solve the field problem of electrical machines as a boundary value problem. This would give a more exact solution, on one hand, and a deeper insight into the phenomena taking place in these kinds of machines, on the other hand. The development of field techniques for induction motor analysis is one of the objectives of this dissertation.

C. Research Objectives

The purpose of this study is to provide a general analysis for the field problem in induction machines, based on the solution of Maxwell's field equations. The specific objectives of this research work may be summarized as follows:
1. Modeling the electric machine by an appropriate mathematical model, that will accurately represent the actual machine, with the relative ease of applying Maxwell's field equations to it.

2. Formulation of the boundary value problem of the induction machine, and its necessary boundary conditions. The formulation may be in its most general form to permit individual primary loading and to derive balanced loading from it as a special case.

3. Solution of the field equations taking into account both the space and time spectrums that may arise from the machine winding and the electric loading.

4. Development of a simple approach for handling the field problem, as stated above, without sacrificing the modeling implementations of the problem.

5. Illustrate the theory by studying a specific example, using a digital computer, to predict the machine performance under different electric loadings.

D. Research Outline

To achieve the above objectives, this research work has been divided into six chapters and an appendix. The first chapter starts with a brief historical survey to the subject, showing the importance
of using the field analysis for solving the machine problems, as well as getting a reasonable interpretation for the phenomena taking place in the electric machines.

Chapter II is devoted to the development of an electromagnetic theory for the induction motor. The theory is based on a five-region model. The basic field equations in each region have been derived from Maxwell's field equations. The additional constraint field equations that arise from the machine configurations have also been derived. The resulting field equations constitute a system of elliptic partial differential equations. This system, together with its relevant boundary conditions, has been solved from a very general approach, including both the whole space and time spectrums (harmonics). The expressions of force and power are also developed to permit the prediction of motor performance.

Chapter III deals with the development of a simplified field analysis for the induction motor. This analysis is based on the wave impedance concept, analogous to the impedance concept found in the familiar transmission-line theory. To widen the scope of this chapter, a different mathematical model has been chosen to develop this field analysis. The space and time spectrums (harmonics) are also kept during the development. The analysis presented in this chapter represents a generalization and extension to the work of Cullen and Barton [26], Freeman [37, 38], Freeman and Smith [40] and Greig and Freeman [45]. The analysis establishes a general approach capable of
describing any arbitrary electric or magnetic loading, as well as a general method for calculating the machine currents and equivalent impedances. The method given can also be extended to the fault conditions, as well as to single-phase machines.

Chapter IV is devoted to the extension of the solution of the general field problem given in Chapter II, to mathematical models consisting of more regions than those given before. The analysis of this chapter is introduced in a different procedure from that given in Chapter II. It permits a systematic handling of multi-region mathematical models. The purpose of this chapter is to study the influence of these additional regions on the machine performance.

Chapter V presents the results obtained from the application of the theories presented, in the previous chapters, to a specific theoretical induction motor. The computer programs, according to the theories developed, have been coded for this motor. The performance characteristics, as well as the effect of design parameters on the motor performance, have been studied.

Chapter VI is devoted to the conclusions, as well as the basic contribution of this research.
II. QUASI-THREE-DIMENSIONAL ANALYSIS OF INDUCTION MOTORS

A. Introduction

Analysis of electromagnetic machines may be performed by using either the circuit or field approach. The circuit approach when applied directly involves lumped parameter representation, in one form or another. These parameters can be evaluated theoretically by individual field calculations, usually of an approximate character. The electric equivalent circuit has proved to be very useful as a representation for electrical machines in general. However, the field approach is preferable to the circuit approach because it allows more information about conditions in the material that may be subject to electric and magnetic stresses and accompanying nonlinearities. It would be preferable when dealing with rotating machines, in general, to solve the electromagnetic field equations, given the boundary conditions, to obtain the values of electric and magnetic flux densities at all points. This would eliminate many of the approximations found in the electric circuit approach.

Due to the complicated nature of the structure of the induction motor, a direct application of Maxwell's field equations to the different regions of the motor is quite impractical.

A machine model is chosen which substitutes the alternating tooth/slot structure by an anisotropic magnetic mass with equivalent permeabilities. The model, while sacrificing the finer details of construction, yields a faithful global representation of the field
quantities, and permits a thorough direct integration of Maxwell's field equations.

B. Mathematical Model

Figure 2.1a shows part of a developed machine, and Figure 2.1b shows its equivalent anisotropic magnetic regions. An orthogonal stationary system of coordinates is used whereby the x-axis coincides with the lower surface of the secondary slot region, the y-axis points towards the primary region, and the z-axis is parallel to the machine's shaft. This coordinate system forms a right-hand system of axes.

The iron yokes of both the primary and the secondary are assumed to be perfectly permeable, while due to high induction values in the tooth zones, the permeability of the iron in these zones is assumed to be finite with relative permeability $\mu_r$.

The whole tooth area reveals a different magnetic reluctance in the x- and in the y-directions. For the purpose of the following analysis, the original alternating teeth and slots, of any toothed region, may be replaced by a homogeneous anisotropic magnetic mass with different relative permeabilities, $\mu_x$ in the x-direction and $\mu_y$ in the y-direction. The permeabilities of the equivalent anisotropic medium can be easily derived from the series and parallel laws of the reluctance of the magnetic circuit to get

$$\mu_x = \frac{\mu}{1 + (\mu_r - 1) \frac{b_0}{t_s}} \quad (2.1a)$$
Figure 2.1a. Part of the developed motor

Figure 2.1b. The equivalent anisotropic magnetic regions 1, 2 and 3
\[ \mu_y = \mu [1 + (1/\mu_r - 1) \frac{b_o}{\tau_s}] \quad (2.1b) \]

where
\[ b_o = \text{slot breadth}, \]
\[ \tau_s = \text{slot pitch}, \]
\[ \mu_r = \text{relative permeability of the tooth iron, and} \]
\[ \mu = \text{absolute permeability of the tooth iron.} \]

From the electrical point of view, the above hypothetical magnetic mass must be conductive in the z-direction only, so that the model will represent faithfully (although in the form of a continuous distribution) the axially flowing electric currents.

It is clear that the attributed conductivity \( \sigma \) (S/m) should be
\[
\sigma = \frac{b_s}{\tau_s} \gamma_s \bar{\sigma} \quad (2.2)
\]

where \( \gamma_s \) is the slot space factor which is equal to the ratio between the area of the conductors to the area of the slot, and \( \bar{\sigma} \) is the machine conductors' actual conductivity.
C. Basic Field Equations

Maxwell's field equations for stationary conducting medium are

i. $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$

ii. $\nabla \cdot \vec{D} = 0$

iii. $\nabla \times \vec{H} = \vec{J}$

iv. $\nabla \cdot \vec{B} = 0$  \hspace{1cm} (2.3)

To those, usually we add

i. $\nabla \cdot \vec{J} + \partial \rho / \partial t = 0$

ii. $\vec{D} = e\hat{\vec{E}}$

iii. $\vec{B} = \mu \vec{H}$

iv. $\vec{J} = \sigma \hat{\vec{E}}$  \hspace{1cm} (2.4)

In moving media, where $\hat{\vec{E}}$ is induced in a moving medium due to $\vec{B}$ in a stationary medium, only Eq. (2.3.i) should be modified to

$$\nabla \times \vec{E} = -\partial \vec{B}/\partial t + \nabla \times (\vec{B} \times \vec{V})$$  \hspace{1cm} (2.5)

where $\vec{V}$ is the velocity of the moving medium. Other equations of (2.3) and (2.4) remain unchanged.

In our analysis, we shall assume that the electric intensity $\hat{\vec{E}}$ has only a $z$-component $E_z$, and the magnetic induction $\hat{\vec{B}}$ has $x$- and $y$-components, $B_x$ and $B_y$, respectively. Thus,

$$\hat{\vec{E}} = E_z \hat{\vec{z}}$$  \hspace{1cm} (2.6)

$$\hat{\vec{B}} = B_x \hat{\vec{x}} + B_y \hat{\vec{y}}$$  \hspace{1cm} (2.7)
where \( \hat{x}, \hat{y} \) and \( \hat{z} \) are unit vectors along the coordinate axes \( x, y \) and \( z \), respectively.

The relative permeability of the tooth regions has different values in the \( x \) - and \( y \) -directions, as given by Eqs. (2.1a) and (2.1b). Then, the magnetic induction (Eq. 2.7) can be rewritten in the form:

\[
\vec{B} = \mu_x H_x \hat{x} + \mu_y H_y \hat{y}
\]

(2.8)

where \( \mu_x \) and \( \mu_y \) are the permeabilities of the equivalent anisotropic mass to the tooth region.

Making use of the assumptions given by Eqs. (2.6) to (2.8), Maxwell's field equations (2.3) for stationary media will yield:

\[
\frac{\partial E}{\partial y} = -\mu_x \frac{\partial H_x}{\partial t}
\]

(2.9)

\[
\frac{\partial E}{\partial x} = \mu_y \frac{\partial H_y}{\partial t}
\]

(2.10)

\[
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J.
\]

(2.11)

For media moving in the \( x \) -direction with a velocity \( V \), Eqs. (2.9) through (2.11) should be modified by using Eq. (2.5) to take the form:

\[
\frac{\partial E}{\partial y} = -\mu_x \left( \frac{\partial H_x}{\partial t} + V \frac{\partial H_x}{\partial x} \right)
\]

(2.12)

\[
\frac{\partial E}{\partial x} = \mu_y \left( \frac{\partial H_y}{\partial t} + V \frac{\partial H_y}{\partial x} \right)
\]

(2.13)
Equations (2.9) through (2.14) constitute the basic field equations necessary to derive the basic partial differential equations which describe the field phenomena in the different regions of the motor.

In what follows, the indices 1, 2 and 3 will be used for the primary, the secondary and the air-gap quantities, respectively.

1. Primary tooth region

The primary tooth region is defined for values of \( a < y < a_1 \). It is a stationary region, so the set of equations (2.9) through (2.11) will be the one used in this case. To derive the partial differential equation for the electric intensity of this region, the following procedure can be carried out.

Differentiating Eq. (2.9) with respect to \( y \), and introducing the subscript 1 for the field components in this region, we obtain

\[
\frac{\partial^2 E_1}{\partial y^2} = - \mu_{1x} \frac{\partial^2 H_{1x}}{\partial y \partial t} .
\]  

(2.15)

Similarly, on differentiating Eq. (2.10) with respect to \( x_1 \), we obtain

\[
\frac{\partial^2 E_1}{\partial x^2} = \mu_{1y} \frac{\partial^2 H_{1y}}{\partial x \partial t} .
\]  

(2.16)

Finally, differentiating Eq. (2.11) with respect to \( t \), we obtain
If we assume that the field components are continuous to their second partial derivatives, then we can eliminate the partial derivatives of $H_1^x$ and $H_1^y$ using Eqs. (2.15) through (2.17), as well as Eq. (2.11) (with the subscript 1 for this region) to obtain

$$\frac{\partial^2 H_1^y}{\partial t \partial x} - \frac{\partial^2 H_1^x}{\partial t \partial y} = \frac{\partial J_1}{\partial t}. \quad (2.17)$$

2. Secondary tooth region

The secondary tooth region is defined for values of $0 < y < d_2$. This region is a moving region, so the set of Eqs. (2.12) through (2.14) will be applicable to this case. By a procedure similar to that given in case 1 above, the partial differential equation of the electric intensity can be derived. Introducing the subscript 2 for the quantities of this region, the resulting equation will take the form:

$$\frac{1}{\mu_{1y}} \frac{\partial^2 E_1}{\partial x^2} + \frac{1}{\mu_{1x}} \frac{\partial^2 E_1}{\partial y^2} = \frac{\partial J_1}{\partial t}. \quad (2.18)$$

3. Air-gap region

This case can be derived directly, as a special case, either from Eq. (2.18) or (2.19). This is carried out by equating the permeabilities
in the x- and y-directions to that of the free space, and equating the current density to zero. The resulting equation will be Laplace's equation. Introducing the subscript 3 to the quantities in this region we obtain

$$\frac{\partial^2 E_3}{\partial x^2} + \frac{\partial^2 E_3}{\partial y^2} = 0 \quad .$$  (2.20)

Equations (2.18) through (2.20) are the three basic equations, which describe the electric intensities in the secondary tooth, the air-gap and the primary tooth regions of the motor, respectively.

4. Secondary end-ring constraints

Figure 2.2a shows the developed rotor, as seen from the stator, together with its end rings. The squirrel-cage bars are not shown as they are replaced by the anisotropic inductive magnetic mass. Let $\sigma_e$ be the conductivity of the end-ring material, $\sigma_2$ be the conductivity of the anisotropic equivalent mass of the secondary region, $J_2$ be the current density in the secondary region, $I_e$ be the end-ring current, $A_e$ the area of the secondary end ring, and $(2W_2)$ be the secondary width. The secondary is short-circuited; therefore, neglecting the field at the secondary end rings, the line integral of $E_2$ along the contour shown in Figure 2.2a yields

$$E_2 \cdot W_2 - (E_2 + \frac{\partial E_2}{\partial x} \delta x) \cdot W_2 = J_2 \cdot \frac{W_2}{\sigma_2} \delta x \cdot \frac{W_2}{\sigma_2} + 2I_e \frac{\delta x}{\sigma_e A_e} \quad .$$  (2.21)
Figure 2.2a. The developed secondary, together with its end rings

Figure 2.2b. The current distribution in an element end ring of length δx
Simplifying Eq. (2.21), and taking the limit as $\delta x$ tends to zero, we obtain

$$\frac{\partial E_2}{\partial x} = \frac{1}{\sigma_2} \frac{\partial J_2}{\partial x} - \frac{2I_e}{\sigma e A e W_2}.$$  \hspace{1cm} (2.22)

The current continuity at the secondary end rings (see Figure 2.2b) implies that

$$J_2 = \frac{1}{d_2} \frac{\partial I_e}{\partial x}.$$  \hspace{1cm} (2.23)

To eliminate $I_e$ between Eqs. (2.22) and (2.23), we differentiate Eq. (2.22) partially with respect to $x$, and then, eliminate $\partial I_e / \partial x$ between the resulting equation and Eq. (2.23) to obtain

$$\frac{\partial^2 E_2}{\partial x^2} = \frac{1}{\sigma_2} \frac{\partial^2 J_2}{\partial x^2} - \frac{2d_2 J_2}{\sigma e A e W_2}.$$  \hspace{1cm} (2.24)

Eq. (2.24) represents the constraint equation imposed by the secondary end rings on the electric intensity in the secondary region.

5. Primary overhang region

Figure 2.3 shows a sectional view of an induction motor. In order to represent the overhang on both sides of the primary, the secondary has been drawn in its full length, while the primary has been lengthened up to $(a \pi/2)$, where $a \pi$ is equal to one primary-turn's mean
Figure 2.3. Sectional view of the induction motor showing regions 1, 2 and 3 (the primary, secondary and air-gap regions), as well as regions 4 and 5 (the primary overhang and air overhang regions)
length. The primary overhang conductors are normally bent so as to recline on the primary's yoke. This is taken into consideration by continuing the yoke, behind the primary conductors, up to the length \((a_t/2)\). Figure 2.3 shows also the relative permeability of the various regions. The system of coordinates is chosen similar to that used in the previous cases, so that this region is defined for values of \(a < y < a_t\). The current density is assumed to be identical with that of case 1. The differential equation of the electric intensity \(E_4\) in this region is similar to that given by Eq. (2.18) with the sole correction \(\mu_{1x} = \mu_{1y} = \mu_0\); hence,

\[
\frac{\partial^2 E_4}{\partial x^2} + \frac{\partial^2 E_4}{\partial y^2} = \mu_0 \frac{\partial j_1}{\partial t}.
\]  

(2.25)

6. Air overhang region

This is region 5, defined for values of \(0 < y < a\). The differential equation in this air region, which now extends from the primary overhang to the motor's shaft, will be similar to that of case 3, which is Laplace's equation:

\[
\frac{\partial^2 E_5}{\partial x^2} + \frac{\partial^2 E_5}{\partial y^2} = 0.
\]  

(2.26)

The previous results are summarized in Figure 2.4, which shows a sectional view of the motor, and the differential equations which hold in each of its regions.
A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

(1). The Primary tooth Region

\[ \frac{\partial^2 E_1}{\partial x^2} + \frac{\partial^2 E_1}{\partial y^2} = \frac{\partial E_1}{\partial t} \]

(4). The Primary Overhang Region

\[ \frac{\partial^2 E_4}{\partial x^2} + \frac{\partial^2 E_4}{\partial y^2} = \frac{\partial J_1}{\partial t} \]

(3). The Air-gap Region

\[ \frac{\partial^2 E_3}{\partial x^2} + \frac{\partial^2 E_3}{\partial y^2} = 0 \]

(5). The Air Overhang Region

\[ \frac{\partial^2 E_5}{\partial x^2} + \frac{\partial^2 E_5}{\partial y^2} = 0 \]

(2). The Secondary Tooth region

\[ \frac{\partial^2 E_2}{\partial x^2} + \frac{\partial^2 E_2}{\partial y^2} = \frac{\partial E_2}{\partial t} \left( \frac{1}{a_x} + \frac{1}{a_y} \right) J_2 \]

\[ \frac{\partial^2 E_2}{\partial x^2} = \frac{1}{a_x} \frac{\partial^2 E_2}{\partial x^2} - \frac{1}{a_y} \frac{\partial^2 E_2}{\partial y^2} \]

Figure 2.4. A sectional view of the induction motor showing the frame of reference, the main dimensions and the governing differential equations in each region.
D. Boundary Conditions

The field conditions at the boundary between any two regions can be derived by using the integral form of the following two Maxwell equations:

\[ \oint_{L} \mathbf{H} \cdot d\mathbf{L} = \iint_{S} \mathbf{J} \cdot d\mathbf{A} + \oint_{\partial A} \mathbf{D} \cdot d\mathbf{A} \]  
  \( (2.27) \)

\[ \oint_{\partial A} \mathbf{B} \cdot d\mathbf{A} = 0 \]  
  \( (2.28) \)

Applying Eq. (2.27) to any small closed path on the boundary between the two regions yields

\[ H_{x}^{-} = H_{x}^{+} \]  
  \( (2.29) \)

where \( H_{x}^{-} \) and \( H_{x}^{+} \) are the magnetic field intensities just below and above the boundary. Equation (2.29) follows directly from Eq. (2.27) so long as current density and rate of change of electric flux density are finite, and then, the contribution from the right-hand side of Eq. (2.27) will be zero.

Similarly, applying Eq. (2.28) to any closed surface on the boundary between two regions yields

\[ B_{y}^{-} = B_{y}^{+} \]  
  \( (2.30) \)

where \( B_{y}^{-} \) and \( B_{y}^{+} \) are the magnetic flux densities just below and above
Equations (2.29) and (2.30) represent the continuity of the tangential components of the magnetic field intensities and the normal components of the magnetic flux densities, respectively, at the boundary between any two regions.

To express Eqs. (2.29) and (2.30) in terms of the electric field intensity, we differentiate these equations partially with respect to $t$, and make use of Eqs. (2.9) and (2.10) to obtain

\[
\left(\frac{1}{\mu_x} \frac{\partial E}{\partial y}\right)^- = \left(\frac{1}{\mu_x} \frac{\partial E}{\partial y}\right)^+ \quad (2.31)
\]

and

\[
\left(\frac{\partial E}{\partial x}\right)^- = \left(\frac{\partial E}{\partial x}\right)^+ \quad (2.32)
\]

Equations (2.29) through (2.32) are those necessary to obtain the field relations at the boundary between any two regions.

1. Main regions

Referring to Figure 2.3, the main regions are considered to be regions 1, 2 and 3. Now we can apply Eqs. (2.29) through (2.32) to the boundary between these regions. Due to the perfect permeability of the primary and secondary iron, i.e., $\mu = \infty$, the tangential magnetic field $H_x$ must vanish along the planes $y=0$ and $y=a_t$; i.e.,

\[
\text{at } y=0, \quad \frac{\partial E_2}{\partial y} = 0 \quad (2.33)
\]
at \( y = a \),
\[
\frac{\partial E_1}{\partial y} = 0. \tag{2.34}
\]

A direct application of Eqs. (2.31) and (2.32) to the boundary between the air-gap and the primary region yields:

at \( y = a \),
\[
\frac{1}{\mu_{1x}} \frac{\partial E_1}{\partial y} = \frac{1}{\mu_0} \frac{\partial E_3}{\partial y} \tag{2.35}
\]
\[
\frac{\partial E_1}{\partial x} = \frac{\partial E_3}{\partial x}. \tag{2.36}
\]

The same continuity conditions of the tangential magnetic intensity and the normal induction at the boundary (Eqs. 2.29 and 2.30) still hold at \( y = d_2 \), but Eqs. (2.31) and (2.32) no longer hold. The reason is that Faraday's law of induction will take the form given by Eq. (2.5), as the secondary region is a moving medium in this case. The boundary conditions, therefore, between the secondary and the air-gap regions will be expressed in terms of Eqs. (2.29) and (2.30) to obtain

\[
H_{2x} = H_{3x} \tag{2.37}
\]
\[
B_{2y} = B_{3y}. \tag{2.38}
\]

Expressing Eqs. (2.37) and (2.38) in terms of the electric field intensity will be carried out later, using Eq. (2.5), after getting the solution for \( \vec{E} \) and \( \vec{B} \).
2. Overhang regions

Referring to Figure 2.3, we see the overhang regions are the two regions 4 and 5. The condition given by Eqs. (2.31) and (2.32) still holds in this case with the sole modification $\mu_x = \mu_y = \mu_0$. Also, we have assumed the perfect permeability of the primary iron back. Therefore, we can write down the following boundary equations:

\[
\frac{\partial E_4}{\partial y} = 0 \quad (2.39)
\]

\[
\frac{\partial E_4}{\partial y} = \frac{\partial E_5}{\partial y} \quad (2.40)
\]

\[
\frac{\partial E_4}{\partial x} = \frac{\partial E_5}{\partial x} \quad . \quad (2.41)
\]

In addition to these conditions, we consider the distance between the primary overhang and the rotating shaft of the motor is very large. The field created by the overhang winding near the surface of the shaft can then be neglected. This yields the following condition:

\[
\frac{\partial E_5}{\partial y} = 0 \quad (2.42)
\]

According to the boundary conditions given above, we see that the boundaries of the problem are divided into two separate zones (see Figure 2.3). Zone I contains the three regions (1, 2 and 3); and zone II contains the two regions 4 and 5. The common boundaries of
these two zones are the two planes \( z = \pm W_1/2 \). Due to design considerations of the induction motor, there is negligible dependence on the \( z \)-coordinate. Therefore, the two zones are mathematically separate, and we can deal with the system of partial differential equations of each zone, together with its associated boundary conditions, individually.

E. Solution of the Field System of Equations

1. General considerations

Before going into the details of getting the solution of the system of partial differential equations, we shall discuss the outline that governs the functional form of the solution. This form is determined by the imposed nature of the primary current that generates all the electromagnetic phenomena inside the machine. The primary current, \( J_1 \), can be expressed in the form:

\[
J_1 = \sum_{n=1}^{\infty} \sum_{v=-\infty}^{\infty} \tilde{J}_1 e^{j \left( \frac{\nu \pi x + n \omega t}{l} \right)}
\]  

(2.43)

where \( \tilde{J}_1 \) is known as the Fourier coefficient of \( J_1 \), and is exclusively defined in terms of the wavenumber \( \nu \) (also known as the space harmonic order), and the time harmonic order \( n \).

In the case of a double layer, three-phase winding under unbalanced current operations, the expression of \( J_1 \) will be in the form:
\[ \bar{J}_1 = \frac{1}{2\pi} a_n N_p K_w \left[ I_1 + e^{-i \frac{2\pi}{3} \nu} I_2 + e^{-i \frac{4\pi}{3} \nu} I_3 \right] \] (2.44)

where

- \( a_n \) = amplitude of the time harmonic component, equals unity for fundamental,
- \( N_p \) = number of conductors/phase/pole pair,
- \( I_1, I_2, I_3 \) = phase currents,
- \( K_w \) = winding factor,
- \( \tau \) = pole pitch

The general expression of the winding factor, \( K_w \), is given by

\[ K_w = K_b K_c K_d K_0 \] (2.45)

where

\[ K_b = \frac{\sin(\nu m b_s / 2\tau)}{(\nu m b_s / 2\tau)} \] (2.46)

\( K_b \) is known as the slot breadth factor, \( b_s \) being the slot breadth.

\[ K_c = \frac{1}{2} \left( 1 - e^{-i \nu \pi \tau c / \tau} \right) \] (2.47)
$K_c$ is known as the chording factor, $\tau_c$ being the coil span in the same units as the pole pitch $\tau$.

$$K_d = \frac{1 - e^{-j\nu\alpha}}{q(1 - e^{-j\nu\alpha})}; \quad \alpha = \frac{\pi s}{\tau} . \tag{2.48}$$

$K_d$ is known as the phase distribution factor, $\tau_s$ is the slot pitch and $q$ is the number of slots/pole/phase.

$$K_o = \frac{1}{2} (1 - e^{-j\nu\pi}) . \tag{2.49}$$

$K_o$ is known as the odd harmonic factor; this factor is unity for odd values of $\nu$, and is zero for even values of $\nu$.

As mentioned before, the primary current, represented by Eq. (2.43), is the one that is responsible for generating all the electromagnetic phenomena inside the different machine regions. It seems very reasonable to assume that the functional dependence of the generated quantities (the effect) will have the same form as the primary current (the cause).

Therefore, any arbitrary electromagnetic quantity $A(x,y,z,t)$, generated in any of the machine regions, can be represented in the form:

$$A(x,y,z,t) = \sum_{\nu=1}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{A}(\nu,n,y,z) e^{j\left(\frac{\nu\pi}{\tau} x + n\omega t\right)} \tag{2.50}$$

where $\tilde{A}(\nu,n,y,z)$, or simply $\tilde{A}$, is known as the Fourier coefficient (or
the harmonic component) of the function $A(x,y,z,t)$. The expression of $\bar{A}$ will be determined from the solution of the system of partial differential equations, as well as the associated system of boundary conditions.

In the next sections, we shall consider the solution of the system of equations in the five different regions of the machine model under consideration.

2. Primary tooth region

The PDE of the electric field intensity, $E_1$, in the primary tooth region 1, is given by Eq. (2.18). According to the explanation given in the previous section, the solution of $E_1$ can be assumed in the form:

$$E_1 = \sum_{n=1}^{\infty} \sum_{\nu=-\infty}^{\infty} E_1 e^{j\left(\frac{\nu\pi}{\tau} x + \nu \omega t\right)}.$$  \hspace{1cm} (2.51)

The reader will recall $E_1 \equiv E_1 (\nu, n, y, z)$. Equation (2.51), together with Eq. (2.43), should satisfy the PDE of $E_1$ as given by Eq. (2.18). Therefore, on substituting Eqs. (2.51) and (2.43) into Eq. (2.18), we can obtain:

$$\sum_{n=1}^{\infty} \sum_{\nu=-\infty}^{\infty} \left[ \frac{1}{\mu_1 y} \left(\frac{j\nu \pi}{\tau}\right)^2 E_1 + \frac{1}{\mu_1 x} \frac{\partial^2 E_1}{\partial y^2} + jn \omega \frac{1}{\mu_1} \right] e^{j\left(\frac{\nu\pi}{\tau} x + \nu \omega t\right)} = 0.$$  \hspace{1cm} (2.52)

Since Eq. (2.52) is true for any arbitrary values of $x$ and $t$, it should follow that:
The solution of Eq. (2.53) will be the sum of the complementary function and the particular integral. These two solutions can be obtained as shown in the following paragraph.

The complementary function is the solution of Eq. (2.53) with the right-hand side equal to zero. This solution will be in the form

\[ \bar{E}_{11} = A_1 \cosh \eta_1 y + B_1 \sinh \eta_1 y \quad (2.55) \]

where \( A_1 \) and \( A_2 \) are arbitrary constants to be determined from the boundary conditions. The particular integral is any solution that satisfies Eq. (2.53), and is given by

\[ \bar{E}_{12} = \frac{1}{(\nu_1^2 - \eta_1^2)} \left( jn_\omega \mu_{1x} \bar{J}_1 \right) \quad (2.56) \]
where $D^2_y$ stands for the differential operation $\partial^2/\partial y^2$. Since $J_1$ is only a function of $v$ and $n$, and is independent of the $y$-coordinate, the solution of Eq. (2.56) can be obtained by simply substituting zero for $D_y$. The final result can be written in the form

$$E_{12} = -i\alpha_{vn} J_1$$  \hspace{1cm} (2.57a)

where

$$\alpha_{vn} = n\omega \frac{\mu_1}{\eta_1}.$$  \hspace{1cm} (2.57b)

The complete solution will be the sum of Eqs. (2.55) and (2.57).

$$\tilde{E}_1 = A_1 \cosh \eta_1 y + B_1 \sinh \eta_1 y - i\alpha_n J_1.$$  \hspace{1cm} (2.58)

The series expression of $E_1$ can be obtained by substituting Eq. (2.58) into Eq. (2.51). Other field components can be derived from Maxwell's equations (2.9 - 2.11), so long as $E_1$ is obtained and their functional form is defined by Eq. (2.50).

### 3. Secondary tooth region

The system of PDEs in the secondary tooth region is given by Eqs. (2.19) and (2.24). As explained before (see Section E.1), the functional form of both $E_2$ and $J_2$ can be assumed to be in the form
\[ E_2 = \sum_{n=1}^{\infty} \sum_{v=-\infty}^{\infty} \tilde{E}_2 e^{j\left(\frac{v\pi}{\alpha} x + n\omega t\right)} \] (2.59)

\[ J_2 = \sum_{n=1}^{\infty} \sum_{v=-\infty}^{\infty} \tilde{J}_2 e^{j\left(\frac{v\pi}{\alpha} x + n\omega t\right)} \] (2.60)

where both \( \tilde{E}_2 \) and \( \tilde{J}_2 \) are functions of \( v, n \) and the \( y \)-coordinate.

Equations (2.59) and (2.60) should satisfy Eqs. (2.19) and (2.24).

To fulfill this requirement, after substituting Eqs. (2.59) and (2.60) into Eqs. (2.19) and (2.24), we should have

\[ \left(\frac{\partial^2}{\partial y^2} - \eta_2^2\right) \tilde{E}_2 = j\omega \mu_{2x} S_{vn} \tilde{J}_2 \] (2.61)

and

\[ \left(\frac{\nu^2}{\tau}\right) \tilde{E}_2 = \left[\frac{1}{\sigma_2} \left(\frac{\nu^2}{\tau}\right)^2 + \frac{2d_2}{\sigma_2} \right] \tilde{J}_2 \] (2.62)

where

\[ \eta_2^2 = \frac{\mu_{2x}}{\mu_{2y}} \left(\frac{\nu^2}{\tau}\right)^2 \] (2.63)

and \( S_{vn} \) is the motor slip referred to any arbitrary moving wave with space order \( v \) and time order \( n \):

\[ S_{vn} = 1 + \frac{\nu^2}{\omega_0^2} \left(1 + \frac{V}{nV_s/v}\right) \] (2.64)
$S_{vn}$ will be less or greater than one depending on the direction of the motor speed, as well as the sequence of the voltage, i.e., the sign of $\omega$, applied to the motor. It is evident that when both $\nu$ and $n$ equal one, $S_{vn}$ will be the familiar slip of the induction motor.

After eliminating $J_2$ between Eqs. (2.61) and (2.62), the resulting equation will take the form:

$$\left(\frac{\partial^2}{\partial y^2} - \gamma^2\right) \bar{E}_2 = 0$$

(2.65)

where

$$\gamma^2 = \eta_2^2 \left[ 1 + \frac{\nu_2 \gamma_2 \omega S_{vn}}{2 \sigma_{2} A_{2} \omega_{2}} \right]$$

(2.66)

The solution of Eq. (2.65) will be in the form

$$\bar{E}_2 = A_2 \cosh \gamma y + B_2 \sinh \gamma y$$

(2.67)

Once we get $\bar{E}_2$, the field components in the secondary tooth region can be derived from Eqs. (2.59) and (2.60), as well as Maxwell's Eqs. (2.12) through (2.14), noting that the functional form of all the other field components is known to be in the form of Eq. (2.50).
4. Air-gap region

The differential equation of the field intensity in the air-gap region is given by Eq. (2.20). As explained before, the functional form of the electric intensity will be assumed in the form

\[ E_3 = \sum_{n=1}^{\infty} e^{j\left(\frac{n\pi}{\lambda} x + \nu_0 t\right)} \sum_{n=-\infty}^{\infty} E_{3n} e^{j\left(\frac{n\pi}{\lambda} x + \nu_0 t\right)} \]  

(2.68)

This equation should satisfy Eq. (2.20); therefore, this requires

\[ \left(\frac{\partial^2}{\partial y^2} - \eta_3^2\right) E_3 = 0 \]  

(2.69)

where

\[ \eta_3 = \nu_0 / \tau \]  

(2.70)

The solution of Eq. (2.69) will take the form

\[ E_3 = A_3 \cosh \eta_3 y + B_3 \sinh \eta_3 y \]  

(2.71)

The expression of \( E_3 \) can be substituted into Eq. (2.68) to get the field component \( E_3 \). All other field components can be derived, on the assumption that their functional form is the same as in Eq. (2.50) and using Eqs. (2.9) and (2.10).
5. Primary overhang region

The differential equation of the field intensity in this region is that given by Eq. (2.25). Assume the solution in the form

\[ E_4 = \sum_{\eta=1}^{\infty} \sum_{\nu=-\infty}^{\infty} E_4^\nu e^{j(\nu t - \eta x)} \]  \hspace{1cm} (2.72)

Then, in order for Eqs. (2.72) and (2.43) to be a solution for Eq. (2.25), we should have

\[ \left( \frac{\partial^2}{\partial y^2} - \eta_3^2 \right) E_4^\nu = jn\omega_0 \tilde{J}_1 \]  \hspace{1cm} (2.73)

where \( \eta_3 = \nu t/t \), given previously by Eq. (2.70).

The solution of Eq. (2.73) will be composed of the complementary function and the particular integral. Following the procedure given before in Section E.2, the solution can be derived and will take the form

\[ E_4 = A_4 \cosh \eta_3 y + B_4 \sinh \eta_3 y - \frac{jn\omega_0}{\eta_3} \tilde{J}_1 \]  \hspace{1cm} (2.74)

Substitution of Eq. (2.74) into Eq. (2.72) will yield the electric field intensity in region 4. Other field components in region 4 can be derived from \( E_4 \) using Maxwell's equations (2.9) - (2.11), noting that their expression has been assumed to be in the form of Eq. (2.50).
6. Air overhang region

The differential equation of the field intensity in region 5 is given by Eq. (2.26). Assume the solution to be in the form

\[ E_5 = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \hat{E}_5 e^{j \left( \frac{\nu m}{c} x + \nu n \omega t \right)} \]  

(2.75)

Then, Eq. (2.75) will satisfy Eq. (2.26) if

\[ \left( \frac{\partial^2}{\partial y^2} - n_3^2 \right) \hat{E}_5 = 0 \]  

(2.76)

where \( n_3 = \nu m / c \), previously given by Eq. (2.70). The solution of Eq. (2.76) may take the form

\[ \hat{E}_5 = A_5 e^{n_3 y} + B_5 e^{-n_3 y} \]  

(2.79)

Once we have \( \hat{E}_5 \), the electric intensity in region 5 is completely determined, and hence, all the other field components can be derived from it by using Maxwell's equations (2.9) and (2.10).

F. The Arbitrary Constants and the Field Expressions

1. Main regions

Referring to Figure 2.3, the main regions are those of zone 1, which constitute regions 1, 2 and 3. The electric intensity in these regions is given by Eqs. (2.51), (2.59) and (2.68). Since we are
dealing with only single components, then the interesting part for us will be the expressions of $\tilde{E}_1$, $\tilde{E}_2$ and $\tilde{E}_3$ which are given by Eqs. (2.58), (2.67) and (2.71), respectively. These equations contain six arbitrary constants: $A_1$, $B_1$, $A_2$, $B_2$, $A_3$ and $B_3$. Six equations are needed to obtain the values of these six arbitrary constants. The boundary conditions, given by Eqs. (2.33) - (2.38), satisfy this need. According to our assumption given in Section E.1, all the field components have the functional form given by Eq. (2.50). This implies that the partial derivative of the field component with respect to $x$ will be replaced by $(j\nu\pi/t)$, and the partial derivative with respect to time $t$ will be replaced by $jn$. Therefore, we can express the boundary conditions, Eqs. (2.33) - (2.38), in terms of $\tilde{E}_1$, $\tilde{E}_2$ and $\tilde{E}_3$. Summarizing the results, Eqs. (2.33) - (2.36) will yield:

\[
\frac{\partial \tilde{E}_1}{\partial y} \bigg|_{y=a} = 0 \tag{2.78}
\]

\[
\frac{\partial \tilde{E}_2}{\partial y} \bigg|_{y=0} = 0 \tag{2.79}
\]

\[
\left[\frac{1}{\mu_1} \frac{\partial \tilde{E}_1}{\partial y}\right]_{y=a} = \left[\frac{1}{\mu_0} \frac{\partial \tilde{E}_3}{\partial y}\right]_{y=a} \tag{2.80}
\]

\[
[\tilde{E}_1]_{y=a} = [\tilde{E}_3]_{y=a} \tag{2.81}
\]

while Eqs. (2.37) and (2.38) will yield
where \( S_{vn} \) is the motor slip referred to any arbitrary moving wave with space order \( v \) and time order \( n \), as previously given by Eq. (2.64).

Substituting Eq. (2.58) into Eq. (2.78), we obtain

\[
\frac{1}{\mu_2x} \frac{\partial E_2}{\partial y}
\bigg|_{y=d_2} = \frac{S_{vn}}{\mu_0} \frac{\partial E_3}{\partial y}
\bigg|_{y=d_2}
\]  

(2.82)

\[
[E_2]_{y=d_2} = [S_{vn} E_3]_{y=d_2}
\]  

(2.83)

Substituting Eq. (2.67) into Eq. (2.79), we obtain

\[
A_1 \sinh \eta_1 a_t + B_1 \cosh \eta_1 a_t = 0
\]

which yields

\[
B_1 = -A_1 \frac{\sinh \eta_1 a_t}{\cosh \eta_1 a_t}
\]  

(2.84)

Eliminating \( B_1 \) between Eqs. (2.84) and (2.58) yields

\[
\tilde{E}_1 = \frac{A_1}{\cosh \eta_1 a_t} \cosh \eta_1 (a_t - y) - j \alpha_{vn} \tilde{J}_1
\]  

(2.85)

Substituting Eq. (2.67) into Eq. (2.79), we obtain

\[
B_2 = 0
\]  

(2.86)
Then, Eq. (2.67) will be reduced to

$$\frac{E_2}{A_2} = \cosh \gamma y . \quad (2.87)$$

Substituting Eqs. (2.85) and (2.71) into Eq. (2.80), we obtain

$$\frac{1}{\mu_1 x} \cdot \frac{-A_1 \eta_1}{\cosh \eta_1 a} \cdot \sinh \eta_1 d_1 = \frac{\eta_3}{\mu_0} \left[ A_3 \sinh \eta_3 a + B_3 \cosh \eta_3 a \right] \quad (2.88)$$

where $d_1 = a_t - a$. Substituting Eqs. (2.85) and (2.71) into the boundary condition given by Eq. (2.81), we obtain

$$\frac{A_1}{\cosh \eta_1 a} \cdot \sinh \eta_1 d_1 - j \alpha_v \cdot \bar{J}_1 = A_3 \cosh \eta_3 a + B_3 \sinh \eta_3 a . \quad (2.89)$$

Substituting Eqs. (2.87) and (2.71) into the boundary condition given by Eq. (2.82), we obtain

$$\frac{\gamma}{\mu_2 x} A_2 \sinh \gamma y = \frac{\eta_3 S_{vn}}{\mu_0} \left[ A_3 \sinh \eta_3 d_2 + B_3 \cosh \eta_3 d_2 \right] . \quad (2.90)$$

Finally, substituting Eqs. (2.87) and (2.71) into Eq. (2.83), we obtain

$$A_2 \cosh \gamma d_2 = S_{vn} \left[ A_3 \cosh \eta_3 d_2 + B_3 \sinh \eta_3 d_2 \right] . \quad (2.91)$$
Equations (2.88) through (2.91) are sufficient to determine the four arbitrary constants: $A_1$, $A_2$, $A_3$ and $B_3$. To solve these equations, we start by arranging them in the following matrix form:

\[
\begin{bmatrix}
\frac{\mu_0 \eta_1}{\mu_1 \eta_3} \sinh n_1 d_1 & 0 & \sinh n_3 a & \cosh n_3 a \\
-\frac{\cosh n_1 d_1}{\cosh n_1 a_t} & 0 & \cosh n_3 a & \sinh n_3 a \\
0 & -\frac{\mu_0 \gamma}{\mu_2 \eta_3} \sinh \gamma d_2 & S_{\nu n} \sinh n_3 d_2 & S_{\nu n} \cosh n_3 d_2 \\
0 & -\cosh \gamma d_2 & S_{\nu n} \cosh n_3 d_2 & S_{\nu n} \sinh n_3 d_2 \\
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 \\
B_3 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
-j\alpha_{vn} \beta_1 \\
0 \\
0 \\
\end{bmatrix}
\]

\[(2.92)\]

The determinant of the matrix coefficient $D_0$ is given by

\[
D_0 = 
\begin{bmatrix}
\frac{\mu_0 \eta_1}{\mu_1 \eta_3} \sinh n_1 d_1 & 0 & \sinh n_3 a & \cosh n_3 a \\
-\frac{\cosh n_1 d_1}{\cosh n_1 a_t} & 0 & \cosh n_3 a & \sinh n_3 a \\
0 & -\frac{\mu_0 \gamma}{\mu_2 \eta_3} \sinh d_2 & S_{\nu n} \sinh n_3 d_2 & S_{\nu n} \cosh n_3 d_2 \\
0 & -\cosh d_2 & S_{\nu n} \cosh n_3 d_2 & S_{\nu n} \sinh n_3 d_2 \\
\end{bmatrix}
\]
Expanding this determinant along the first column, the final expression of $D_0$ will take the form:

$$D_0 = - S_{vn} \frac{H_c(v)}{\cosh \eta_1 \mu_t}$$  \hspace{1cm} (2.93)$$

where

$$H_c(v) = \cosh \eta_1 d_1 g_1(v) + \frac{\mu_0}{\sqrt{\mu_1\mu_2}} \sinh \eta_1 d_1 g_2(v)$$  \hspace{1cm} (2.94)$$

and

$$g_1(v) = \cosh \gamma_2 \sinh \eta_3 (a-d_2) + \frac{\gamma_0}{\eta_3 \mu_2} \sinh \gamma_2 \cosh \eta_3 (a-d_2)$$  \hspace{1cm} (2.95)$$

$$g_2(v) = \cosh \gamma_2 \cosh \eta_3 (a-d_2) + \frac{\gamma_0}{\eta_3 \mu_2} \sinh \gamma_2 \sinh \eta_3 (a-d_2).$$  \hspace{1cm} (2.96)$$

It should be noted that $H_c(v)$ is also a function of the motor dimensions. It is also evident that for the case of an infinitely thin primary current layer, i.e., $d_1=0$, the second term of Eq. (2.94) will identically vanish, as $\sinh \eta_1 d_1=0$ and $H_c(v)$ are simply given by $g_1(v)$.

The determinant $D_1$ is formed from the coefficient matrix, with the first column replaced by the R.H.S. column of Eq. (2.92):
Expanding this determinant along the first column, it takes the form:

\[
D_1 = - \alpha \psi_n \mathcal{J}_1 \ g_1(\nu) \tag{2.97}
\]

where \( g_1(\nu) \) is given by Eq. (2.95). Similarly, the determinant \( D_2 \) is derived from \( D_0 \) by replacing the second column by the R.H.S. of Eq. (2.92):

\[
D_2 = \begin{vmatrix}
\frac{\mu_0 n_1}{\mu_1 n_3} \frac{\sinh n_1 d_1}{\cosh n_3 a_t} & 0 & \sinh n_3 a & \cosh n_3 a \\
\frac{\cos n_1 d_1}{\cosh n_3 a_t} & -j \alpha \psi_n \mathcal{J}_1 & \cosh n_3 a & \sinh n_3 a \\
0 & 0 & S_{\psi n} \sinh n_3 d_2 & S_{\psi n} \cosh n_3 d_2 \\
0 & 0 & S_{\psi n} \cosh n_3 d_2 & S_{\psi n} \sinh n_3 d_2
\end{vmatrix}
\]
This yields

\[ D_2 = j \frac{\mu_0 n_1}{\mu_1 x_n^3} \cdot S_{\nu n} \alpha_n \cdot \frac{s_i h n_1 d_1}{\cosh n_1 a_t} \]. \hspace{1cm} (2.98)

Similarly,

\[
\begin{bmatrix}
\frac{\mu_0 n_1}{\mu_1 x_n^3} \cdot \text{sinhm}_1 d_1 \\
\frac{\cosh n_1 a_t}{\text{coshm}_1 a_t} \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
-\frac{\gamma_{\mu_0}}{n_3^2 \mu_2 x} \cdot \text{sinh} \gamma d_2 \\
-\text{cosh} \gamma d_2 \\
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
S_{\nu n} \cosh n_3 d_2 \\
S_{\nu n} \sinh n_3 d_2 \\
\end{bmatrix}
\]

which yields:

\[ D_3 = j \frac{\mu_0 n_1}{\mu_1 x_n^3} \cdot \alpha_n \cdot \frac{\text{sinhm}_1 d_1}{\cosh n_1 a_t} \cdot (\text{cosh} \gamma d_2 \cdot \cosh n_3 d_2) \]

\[ - \frac{\gamma_{\mu_0}}{n_3^2 \mu_2 x} \cdot \text{sinh} \gamma d_2 \cdot \sinh n_3 d_2 \]. \hspace{1cm} (2.99)
Finally,

\[
D_4 = \begin{bmatrix}
\frac{\mu_0}{\mu_1 \eta_3} \sinh n_1 d_1 & 0 & \sinh n_3 a & 0 \\
\frac{\sinh n_1 d_1}{\cosh n_1 a_t} & 0 & \cosh n_3 a & -j \alpha_{vn} J_1 \\
0 & -\frac{\gamma_0}{\eta_3 \mu_2 x} \sinh \gamma d_2 & S_{vn} \sinh n_3 d_2 & 0 \\
0 & -\cosh \gamma d_2 & S_{vn} \cosh n_3 d_2 & 0
\end{bmatrix}
\]

which yields:

\[
D_4 = -j \frac{\mu_0}{\mu_1 \eta_3} \alpha_{vn} J_1 S_{vn} \frac{\sinh n_1 d_1}{\cosh n_1 a_t} (\cosh \gamma d_2 \sinh n_3 d_2 \\
- \frac{\gamma_0}{\eta_3 \mu_2 x} \sinh \gamma d_2 \cdot \cosh n_3 d_2) . \tag{2.100}
\]

The arbitrary constants \( A_1, A_2, A_3 \) and \( B_3 \) can be determined using Eqs. (2.93) through (2.100).

The expression of \( A_1 \) will be given by

\[
A_1 = D_1 / D_0 .
\]

Using Eqs. (2.93) and (2.97), we obtain
The expression of $A_2$ is given by

$$A_2 = \frac{D_2}{D_0} .$$

Using Eqs. (2.98) and (2.93), we obtain

$$A_2 = j \left( \frac{\mu_0 n_1}{\mu_1 n_3} \alpha_{vn} S_{vn} \frac{\sinh n_1 d_1}{H_c(v)} \right) J_1 .$$

The expression of $A_3$ is given by

$$A_3 = \frac{D_3}{D_0} .$$

Using Eqs. (2.99) and (2.93), we obtain

$$A_3 = -j \left( \frac{\mu_0 n_1}{\mu_1 n_3} \alpha_{vn} \frac{\sinh n_1 d_1}{H_c(v)} \right) \left( \cosh \gamma d_2 \cosh n_3 d_2 - \frac{\gamma \mu_0}{\eta_3 \mu_2 x} \sinh \gamma d_2 \sinh n_3 d_2 \right) J_1 .$$

Finally, the expression of $B_3$ is given by

$$B_3 = \frac{D_4}{D_0} .$$
Using Eqs. (2.100) and (2.93), we obtain

\[
B_3 = j \frac{\mu_0 \eta_1}{\mu_{1x} \eta_3} \alpha_{vn} \frac{\sinh \eta_1 d_1}{H_c(v)} (\cosh \gamma d_2 \cdot \sinh n_3 d_2)
\]

\[- \frac{\mu_0 \gamma}{\mu_{2x} \eta_3} \sinh \gamma d_2 \cosh n_3 d_2) \tilde{J}_1. \quad (2.104)
\]

Once the arbitrary constants are determined, the electric field intensity in the different regions can be determined. The electric field intensity component, \( \tilde{E}_1 \), in region 1, is obtained by substituting Eq. (2.101) into Eq. (2.85) to obtain:

\[
\tilde{E}_1 = j \alpha_{vn} \frac{g_1(v)}{H_c(v)} \cosh \eta_1 (a_t - y) - 1) \tilde{J}_1 \quad (2.105)
\]

where \( a \leq y \leq a_t \). The electric intensity component, \( \tilde{E}_2 \), in the secondary conductive region, can be obtained by substituting Eq. (2.102) into Eq. (2.87):

\[
\tilde{E}_2 = j \frac{\mu_0 \eta_1}{\mu_{1x} \eta_3} \frac{\sinh \eta_1 d_1}{H_c(v)} \alpha_{vn} S_{vn} \cosh \gamma y \tilde{J}_1 \quad (2.106)
\]

where \( 0 \leq y \leq d_2 \). Finally, the electric field intensity component, \( \tilde{E}_3 \), in the air-gap region, can be obtained by substituting Eqs. (2.103) and (2.104) into Eq. (2.71):
Equations (2.105) through (2.107) give the expressions of the electric intensity components $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ in the primary, the secondary and the air-gap regions, respectively. As a check, we can easily see that $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ satisfy the boundary conditions given by Eqs. (2.78) through (2.83).

Other field expressions can be derived using Eqs. (2.9) - (2.11) for stationary media, and Eqs. (2.12) - (2.14) for moving media. Since the expressions of all the field components should be in the form given by Eq. (2.50), then Eqs. (2.9) and (2.11) can be expressed in terms of the Fourier coefficients (the bar quantities) by

$$\bar{H}_x = \frac{-1}{J \omega \mu_x} \frac{\partial \bar{E}}{\partial y} \quad \text{(2.108)}$$

and

$$\bar{H}_y = \frac{\nu \tau}{\eta \omega \mu_y} \bar{E} \quad \text{(2.109)}$$

Similarly, Eqs. (2.12) and (2.13) will yield

$$\bar{H}_x = \frac{-1}{J \omega S_{\nu h} \mu_x} \frac{\partial \bar{E}}{\partial y} \quad \text{(2.110)}$$
and

\[ \tilde{H}_y = \frac{\nu \pi/\tau}{n \omega} \bar{E} \]  \hspace{1cm} (2.111)

Equations (2.108) through (2.111) are sufficient to determine the magnetic field components \( \tilde{H}_x, \tilde{H}_y \) in the different regions of the motor.

2. Overhang regions

Referring to Figure 2.3, the overhang regions are those of zone II that constitute regions 4 and 5. The electric intensity in these regions is given by Eqs. (2.74) and (2.75). The boundary conditions of these regions are given by Eqs. (2.39) through (2.42). These equations can be expressed in terms of the Fourier coefficients of Eqs. (2.74) and (2.75), which will take the form:

\[ \frac{\partial \tilde{E}_4}{\partial y} \bigg|_{y=a} = 0 \]  \hspace{1cm} (2.112)

\[ \frac{\partial \tilde{E}_4}{\partial y} \bigg|_{y=a} = \frac{\partial \tilde{E}_5}{\partial y} \bigg|_{y=a} \]  \hspace{1cm} (2.113)

\[ \tilde{E}_4 \bigg|_{y=a} = \tilde{E}_5 \bigg|_{y=a} \]  \hspace{1cm} (2.114)

\[ \frac{\partial \tilde{E}_5}{\partial y} \bigg|_{y=-\infty} = 0 \]  \hspace{1cm} (2.115)

Equations (2.74) and (2.77) should satisfy Eqs. (2.112) - (2.115), which provide the necessary equations to determine the arbitrary constants \( A_4, \)
B_4, A_5 and B_5. Equation (2.115) requires

\[ B_5 = 0 \quad (2.116) \]

Then, the expression of \( \bar{E}_5 \) will be given by

\[ \bar{E}_5 = A_5 e^{n_3y} \quad (2.117) \]

Substituting Eqs. (2.74) and (2.117) into Eqs. (2.112) - (2.114), we can get the following equations:

\[ A_4 \sinh n_3 a_t + B_4 \cosh n_3 a_t = 0 \quad (2.118) \]

\[ A_3 \sinh n_3 a + B_4 \cosh n_3 a - A_5 e^{n_3a} = 0 \quad (2.119) \]

\[ A_4 \cosh n_3 a + B_4 \sinh n_3 a - A_5 e^{n_3a} = \frac{j \mu_0 n_3}{n_3} J_1 \quad (2.120) \]

We can follow the same procedure, as given in the previous section, to obtain the arbitrary constants \( A_4, B_4 \) and \( A_5 \). The final expressions will take the following forms:

\[ A_4 = j \frac{\mu_0 n_3 \cosh a_t}{n_3(\cosh d_1 + \sinh d_1)} \quad (2.121) \]

\[ B_4 = - \tanh n_3 a_t A_4 \quad (2.122) \]
Substituting Eqs. (2.121) - (2.123) into Eq. (2.74) and (2.117), the Fourier coefficient of the field components will take the form:

\[ A_5 = -j \frac{\mu_0 n \omega \sinh \eta_3 d_1 e^{-\eta_3}}{\cosh \eta_3 d_1 + \sinh \eta_3 d_1} \hat{J}_1. \]  

(2.123)

Once we have the expressions of the electric intensity in regions 4 and 5, other magnetic components in these two regions can be derived using Eqs. (2.108) and (2.109).

G. Voltage Equations

So far, the field equations have been solved in terms of the primary current of the machine. In the constant voltage drive system, the applied voltages are known quantities, and it is required to get the machine currents in order to determine the machine performance. So, our task now is to write down the voltage equations for the different machine phases, and then solve these equations to get the machine currents.

The voltage induced in the primary winding is due to the electric intensity in both the primary region 1 and the overhang region 4. If we consider only a single harmonic of space order \( v \), and time order \( n \),
acting on the primary winding, then the voltage, $V_{nv}$, induced per phase due to this harmonic component will be given by

$$V_{nv} = -\int_{\text{primary width}} NK_w^* E_1 e^{j \frac{\omega}{2} X_0} dZ - \int_{\text{overhang width}} N K_w^* E_4 e^{j \frac{\omega}{2} X_0} dz$$

where

- $N$ = the total number of conductors per phase of the machine $(P*N_p)$,
- $K_w^*$ = the complex conjugate of the winding factor $K_w$, as defined by Eq. (2.45), and
- $X_0$ = the arbitrary distance measured from the origin to the first coil side of the phase under consideration.

It is evident from Eqs. (2.105) and (2.124) that $E_1$ and $E_4$ are independent of the $z$-coordinate. Then, Eq. (2.126) will take the form

$$V_{nv} = -N K_w^* [W_1 E_1 + W_0 E_4] e^{j \frac{\omega}{2} X_0}$$

where $W_1$ is the primary iron width and $W_0$ is the total overhang width (see Figure 2.3). Summing the voltage induced in the primary winding, due to all the space harmonic components of the electric intensity, the resulting voltage, $V_n$, will be given by

$$V_n = -N \sum_{n=-\infty}^{\infty} K_w^* (W_1 E_1 + W_0 E_4) e^{j \frac{\omega}{2} X_0}$$
Substituting the expressions of $E_1$ and $E_4$, as given by Eqs. (2.105) and (2.124), into Eq. (2.128), the resulting equation can take the form

$$V_n = -N \sum_{\nu=-\infty}^{\infty} G(\nu, y) K_W^* \bar{J}_1 e^{j \frac{\nu \pi}{\tau} x_0}$$ \hspace{1cm} (2.129)

where

$$G(\nu, y) = j W_1 \alpha_{\nu n} \left[ \frac{g_1(\nu)}{H_c(\nu)} \cosh_1 (a_\nu - y) - 1 \right]$$

$$+ j W_0 \frac{\mu_{\nu n} \omega}{n_3} \left[ \frac{\cosh_3 (a_\nu - y)}{\cosh_3 d_1 + \sinh_3 d_1} - 1 \right].$$ \hspace{1cm} (2.130)

Substituting the expression of $\bar{J}_1$, as given by Eq. (2.44), into Eq. (2.129), we obtain

$$V_n = -a_n \frac{NN}{2t} \sum_{\nu=-\infty}^{\infty} G(\nu, y) K_W K_W^* \left[ I_1 + I_2 e^{-j \frac{2\pi}{3} \nu} \right.$$  

$$+ I_3 e^{-j \frac{4\pi}{3} \nu} e^{j \frac{\nu \pi}{\tau} x_0} \right].$$ \hspace{1cm} (2.131)

Equation (2.131) gives the voltage induced per phase of the machine, at any plane $y$, due to all space harmonics $\nu$, and a single time harmonic of order $n$ ($a_n=1$, corresponding to the fundamental harmonic component). It is evident that the voltage given by Eq. (2.131) is a function of $y$, as the electric intensity is varying with the coordinate.
y. The terminal phase voltage will be the average value of Eq. (2.131). This average value is defined by

\[ V_{ph} = \frac{1}{a_1} \int_a^{a_1} v_n \, dy. \tag{2.132} \]

Substituting Eq. (2.131) into Eq. (2.132) and noting that the expression of \( G(v, y) \) is given by Eq. (2.130), the final expression of the average voltage induced per phase will be in the form

\[ V_{ph} = \sum_{y=-\infty}^{\infty} f(v) \left[ I_1 + I_2 e^{-j\frac{2\pi}{3} v} + I_3 e^{-j\frac{4\pi}{3} v} \right] e^{j\frac{\nu \pi}{t}} X_0 \tag{2.133} \]

where

\[ f(v) = j a_n \frac{NN}{2\tau} K_{w} K_{w}^{*} \left[ W_{1} \alpha_{vn} \left( 1 + \frac{g_1(v) \sinh \eta_1 d_1}{H_C(v)} \right) \right] + W_{0} \frac{\mu_0 n \omega}{\eta_3} \left( 1 + \frac{\sinh \eta_1 d_1 / \eta_1 d_1}{\cosh \eta_1 d_1 + \sinh \eta_1 d_1} \right). \tag{2.134} \]

Once we obtain the voltage induced in an arbitrary phase of the machine, the voltage equations for the different phases can be developed as follows.

For Phase I, \( X_0 = 0 \). Let the phase resistance equal \( R_I \) and the applied voltage equal \( V_I \). The applied voltage, \( V_I \), will be balanced with the voltage induced per phase, plus the IR drop in the phase.

Using Eq. (2.133) with \( X_0 = 0 \), as given above, the voltage equation will
take the form
\[ I_1 R_1 + \sum_{\nu=-\infty}^{\infty} f(\nu) [I_1 + I_2 e^{-j\frac{2\pi}{3} \nu} + I_3 e^{-j\frac{4\pi}{3} \nu}] = V_1. \]  (2.135)

For Phase II:
\[ X_0 = 2\pi/3. \]
Let the phase resistance equal \( R_2 \) and the applied voltage equal \( V_1 \). Then, as above, the voltage equation will take the form
\[ I_2 R_2 + \sum_{\nu=-\infty}^{\infty} f(\nu) [I_1 + I_2 e^{-j\frac{2\pi}{3} \nu} + I_3 e^{-j\frac{4\pi}{3} \nu}] e^{\frac{2\pi}{3} \nu} = V_2. \]  (2.136)

For Phase III:
\[ X_0 = 4\pi/3. \]
Let the phase resistance equal \( R_3 \) and the applied voltage equal \( V_3 \). Then, in a similar procedure as given above, the voltage equation will take the form
\[ I_3 R_3 + \sum_{\nu=-\infty}^{\infty} f(\nu) [I_1 + I_2 e^{-j\frac{2\pi}{3} \nu} + I_3 e^{-j\frac{4\pi}{3} \nu}] e^{\frac{4\pi}{3} \nu} = V_3. \]  (2.137)
Equations (2.135) through (2.137) are the voltage equations of the different phases of the machine, required to get the phase currents. Equations (2.135) through (2.137) can be rearranged and written in the following matrix form:

$$
\begin{bmatrix}
R_1 + \sum_{\nu=-\infty}^{\infty} f(\nu) e^{-j\frac{2\pi}{3} \nu} & \sum_{\nu=-\infty}^{\infty} f(\nu) e^{-j\frac{4\pi}{3} \nu} & \sum_{\nu=-\infty}^{\infty} f(\nu) e^{j\frac{2\pi}{3} \nu} \\
\sum_{\nu=-\infty}^{\infty} f(\nu) e^{j\frac{2\pi}{3} \nu} & R_2 + \sum_{\nu=-\infty}^{\infty} f(\nu) e^{-j\frac{2\pi}{3} \nu} & \sum_{\nu=-\infty}^{\infty} f(\nu) e^{j\frac{4\pi}{3} \nu} \\
\sum_{\nu=-\infty}^{\infty} f(\nu) e^{j\frac{4\pi}{3} \nu} & \sum_{\nu=-\infty}^{\infty} f(\nu) e^{j\frac{4\pi}{3} \nu} + \sum_{\nu=-\infty}^{\infty} f(\nu) e^{j\frac{2\pi}{3} \nu} & R_3 + \sum_{\nu=-\infty}^{\infty} f(\nu) \\
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\end{bmatrix}
$$

Equation (2.138) permits the representation of unbalanced voltage operations, as well as fault conditions, as the voltages $V_1$, $V_2$ and $V_3$ are quite arbitrary values. The entries of the coefficient matrix are the self and the mutual impedances of the different phases of the machine. Once we solve Eq. (2.138) for the phase currents, the field components in the different regions can be obtained, and hence, the machine performance can be fully predicted.
H. Force and Power Expressions

1. Force expressions

The force, \( d\vec{F} \), acting on a current carrying conductor, \( I \vec{d}S \), placed in a magnetic field of magnetic induction, \( \vec{B} \), is given in magnitude and direction by the following cross product:

\[
\vec{d}F = I \vec{d}S \times \vec{B} .
\]  

(2.139)

In our case \( I \) is given by \((J_1 \vec{d}x \vec{d}y)\) and is in the z-direction, so that \( \vec{d}S \) will be equal to \( dz \). To get the force component in the x-direction, \( \vec{B} \) will be substituted by \( B_{1y} \) in the primary region 1, and \( B_{4y} \) in the primary overhang region 4. Denoting this force component by \( dF_x \), its expression will take the form

\[
dF_x = -J_1(B_{1y} + B_{4y})dx dy dz .
\]  

(2.140)

The negative sign is introduced to make the force positive in the positive x-direction. It is evident that \( dx dy dz \) is the elementary volume, \( \delta_{vol.} \), occupied by the current. If we denote the total force developed by the motor in the x-direction by \( F_x \), then its value can be derived from the integration of Eq. (2.140) over the whole volume occupied by the primary current, i.e.,

\[
F_x = - \int \int \int_{\text{reg. 1}} J_1 B_{1y} dx dy dz - \int \int \int_{\text{reg. 4}} J_1 B_{4y} dx dy dz .
\]  

(2.141)
Since \( J_1 \), \( B_{1y} \) and \( B_{4y} \) are independent of the \( z \)-coordinate, the integration with respect to the \( z \)-coordinate can be simply replaced by the multiplication by the primary width, \( W_1 \), for the first term, and the multiplication by the overhang width, \( W_o \), for the second term of Eq. (2.141).

Recall that \( J_1 \), as well as the field components, has the form given by Eq. (2.50), where their harmonic components are now fully defined. Therefore, on replacing \( J_1 \), \( B_{1y} \) and \( B_{4y} \) by their series forms and taking the average, first, with respect to time, and then integrating with respect to \( x \) and \( y \), the resulting force expression will take the form

\[
F_x = -\frac{1}{2} (2\pi r) \frac{\tilde{J}_{L} + \tilde{J}_{L}}{\tilde{J}_{L} + \tilde{J}_{L}} (W_1 \tilde{B}_{1y} + W_o \tilde{B}_{4y})^* \tag{2.142}
\]

where

\( \tilde{J}_{L} \) = the linear current density defined by \( (\tilde{J}_{1d_1}) \),

\(* = the conjugate of the complex quantity, and

\( \tilde{B}_{1y}, \tilde{B}_{4y} \) = the average values of the \( y \)-components of the magnetic inductions as given by Eq. (2.109) with \( \tilde{E} \) replacing \( E \) for \( \tilde{B}_{1y} \) and \( \tilde{E} \) replacing \( E \) for \( \tilde{B}_{4y} \), i.e.,

\[
\tilde{B}_{1y} = -j \frac{(\nu_{0}/\tau)}{\nu_{0}} \frac{g_1(\nu)}{H_c(\nu)} \frac{\sinh(n_d_1)}{n_1 d_1} + 1 \tilde{J}_{1} \tag{2.143}
\]

\[
\tilde{B}_{4y} = -j \frac{(\nu_{0}/\tau)}{\nu_{0}} \frac{\sinh(n_d_1)}{\cosh(n_d_1) + \sinh(n_d_1)} + 1 \tag{2.144}
\]
The normal force expressions can be derived in a similar way as above, with $\bar{B}_1^x$ and $\bar{B}_4^x$ substituted for $\bar{B}_1^y$ and $\bar{B}_4^y$ in Eq. (2.142). However, this force is of less importance in the theory of rotating machines.

2. Power expressions

The flow of electromagnetic power through a closed surface is obtained from the surface integral of the time instantaneous quantity

$$\mathbf{\hat{P}} = \mathbf{\hat{E}} \times \mathbf{\hat{H}} \quad (2.145)$$

where $\mathbf{\hat{P}}$ is the Poynting vector. The ingoing power-flux over the surface $S$, assuming that the element of surface area $\mathbf{ds}$ is outward directed, is given by

$$- \iint_S \mathbf{\hat{P}} \cdot \mathbf{ds}. \quad (2.146)$$

Using the above definition as well as the divergence theorem and Maxwell's field equations, we can expand Eq. (2.146) as follows:

$$- \iint_S \mathbf{\hat{P}} \cdot \mathbf{ds} = - \iiint_V \nabla \cdot (\mathbf{\hat{E}} \times \mathbf{\hat{H}}) \, dv$$

$$= \iiint_V (\mathbf{\hat{E}} \cdot \nabla \mathbf{\hat{H}} - \mathbf{\hat{H}} \cdot \nabla \mathbf{\hat{E}}) \, dv$$

$$= \iiint_V [\mathbf{\hat{E}} \cdot (\mathbf{\hat{J}} + \mathbf{\hat{A}}/\mathbf{\hat{t}}) - \mathbf{\hat{H}} \cdot (-\mathbf{\hat{B}}/\mathbf{\hat{t}})] \, dv$$

$$= \iiint_V (\mathbf{\hat{E}} \cdot \mathbf{\hat{A}}/\mathbf{\hat{t}} + \mathbf{\hat{H}} \cdot \mathbf{\hat{B}}/\mathbf{\hat{t}}) \, dv + \iiint_V \mathbf{\hat{E}} \cdot \mathbf{\hat{J}} \, dv$$
The first term on the right-hand side is the time rate of increase of the electromagnetic energy, and the second term is the total dissipated power (\( \mathbf{E} \) and \( \mathbf{J} \) are in the same directions) or generated power (\( \mathbf{E} \) and \( \mathbf{J} \) are in opposite directions) in the system. Equation (2.147) is the Poynting Theorem.

Now, Eq. (2.147) can be applied to our case, but in the opposite sense, viz., the increase in the electromagnetic energy plus the energy dissipated in the system is equal to

\[
- \oint_S P \cdot ds = \frac{\partial}{\partial t} \oiint_S \left( \frac{1}{2} E \cdot D + \frac{1}{2} H \cdot B \right) dV + \oiint_V \mathbf{E} \cdot \mathbf{J} \, dv .
\]

(2.147)

Note that the closed surface \( S \) is composed of the cylindrical surface of length \( 2\pi P \), as well as the circular area closing this cylinder on the sides. According to our assumption regarding the field components,

\[
\mathbf{P} = \mathbf{E} \times \mathbf{H}
\]

\[
= - E_z H_y \mathbf{i} + E_z H_x \mathbf{j} .
\]

(2.149)

The expression of \( ds \) will be:
over the cylindrical surface,

\[
\text{ds} = \text{dxdy} \hat{j}, \quad (2.150)
\]

over the circular sides,

\[
\text{ds} = \pm \text{dxdy} \hat{k}. \quad (2.151)
\]

Using Eqs. (2.149) through (2.151) to form the surface integral given by Eq. (2.148), yields:

\[
\iint_S \vec{P} \cdot \text{ds} = -\iint \mathbf{E}_z \mathbf{H}_x \, \text{dxdy}. \quad (2.152)
\]

Since both \( \mathbf{E}_z \) and \( \mathbf{H}_x \) are independent of the z-coordinate, then the integration with respect to the z-coordinate will be replaced by the multiplication by the field width. The left-hand side of Eq. (2.152) will be the secondary input power, \( P_{\text{sec}} \), if \( \mathbf{E}_z \) and \( \mathbf{H}_x \) are the values of the fields at the boundary between the air gap and the primary region. Thus, we can write

\[
P_{\text{sec}} = -W_1 \int_{y=a}^{2\pi P} [\mathbf{E}_z \mathbf{H}_x] \, \text{dx} - W_0 \int_{y=a}^{2\pi P} [\mathbf{E}_z \mathbf{H}_x] \, \text{dx}. \quad (2.153)
\]
Equation (2.153) is similar to Eq. (2.141) (note that the integration of Eq. (2.141) with respect to z and y means the multiplication by the field width, and replacing the field components B by their average values). Then, we can follow the same procedures given in the previous section to show that the secondary input will be given by

\[ P_{\text{sec}} = -\frac{1}{2} (2\pi \rho) \Re \sum_{n=1}^{\infty} \sum_{\nu=-\infty}^{\infty} (W_1 E_{1z} + W_0 E_{4z}) \bar{J}_L^* \]

(2.154)

where \( E_{1z} \) and \( E_{4z} \) are the harmonic components of the electric intensity at the boundary surface between regions 3 and 1, and between regions 4 and 5.

This secondary power includes all the secondary losses, mechanical losses, stray losses, as well as the mechanical output of the machine. The input power to the motor is simply given by

\[ P_{\text{in}} = \Re \sum_{r=1}^{3} V_r I_r^* \]

(2.155)

where \( V_r \) (r=1,2,3) are the applied voltages to the motor and \( I_r^* \) (r=1,2,3) are the complex conjugate of the motor currents.

Equations (2.142), (2.154) and (2.155) are those required to predict the machine performance, in terms of its design parameters.
I. Conclusion

In this chapter, the electromagnetic theory of induction motors has been developed. The theory is based on a five-region model consisting of primary, secondary, air gap, overhang and air overhang regions. The basic field equations in each region have been derived from Maxwell's field equations. The additional constraint field equations arising from the secondary end rings are also derived. The resulting field equations form a system of elliptic partial differential equations. This system has been solved simultaneously to obtain the field expressions in each individual region. The solution is developed from its most general approach. It includes both the whole space and time harmonics which can act on the motor. The motor force and power expressions have been derived also. This permits the prediction of the motor performance, as well as the determination of the effects of the motor parameters on its performance.
III. WAVE IMPEDANCE CONCEPT FOR SOLVING THE FIELD PROBLEM IN INDUCTION MOTORS

A. Introduction

In the previous chapter, the solution of Maxwell's equation for a five-region mathematical model of an induction motor has been developed. The model replaces the practical toothed structure of the stator and rotor by continuous, though inhomogeneous, regions having appropriately averaged resistivities, and different permeabilities in directions parallel to, and perpendicular to, the air gap. The effect of the end winding is taken into account in the analysis. In general, the analysis given in the previous chapter is lengthy and needs simplification without substantial sacrifice of details.

The purpose of this chapter is to present a simplified electromagnetic theory of the induction motor, using the concept of wave impedance. A different mathematical model has been developed to widen the scope of the work, and to study the effect of more parameters on the motor performance.

For simplicity, the end effects of the motor are neglected in the mathematical model considered, while the tooth top of the motor secondary has been added (see Figures 3.1a and 3.1b). Thus, the new model permits studying the effect of the secondary tooth top, as well as the zigzag leakage flux, on the performance of the induction motor.

Figure 3.1b shows that the stationary frame of reference is chosen to coincide with the lower surface of the secondary slot region. Regions 1 and 2 are the slot and tooth-top regions of the motor.
Figure 3.1a. Part of the developed slotted structure of the induction motor

Figure 3.1b. The equivalent anisotropic magnetic regions 1, 2 and 3
secondary (the rotor). Tooth regions 1 and 2 are moving with the same speed, V, with respect to the stationary frame of reference. The air gap region 3 is a stationary region. The primary current is approximated by an infinitely thin current sheet of linear density, $J_L$. The toothed structure of the secondary part is replaced, as before, by inhomogeneous regions having different permeabilities in the x and y directions as shown. The average conductivity of the secondary region 1 is z-direction, while the conductivity of other regions is zero. The permeabilities of both the secondary and primary iron yokes are considered to be infinitely large.

B. Field Analysis

Following the same steps as given in Chapter II, Section C, Maxwell's equations in region 1 are given by

$$\frac{\partial E_1}{\partial y} = - \mu_1 x \left( \frac{\partial H_{1x}}{\partial t} + v \frac{\partial H_{1x}}{\partial x} \right)$$  \hspace{1cm} (3.1)

$$\frac{\partial E_1}{\partial x} = \mu_1 y \left( \frac{\partial H_{1y}}{\partial t} + v \frac{\partial H_{1y}}{\partial x} \right)$$  \hspace{1cm} (3.2)

$$\frac{\partial H_{1y}}{\partial x} - \frac{\partial H_{1x}}{\partial y} = \sigma_1 E_1 \right) \hspace{1cm} (3.3)$$

Assume all the field components have the functional form previously given by Eq. (2.50), i.e., they vary with x and t as $e^{j(\nu x + \pi t)}$. Therefore, in Eqs. (3.1) - (3.3), we can substitute
$j\omega$ for $\partial/\partial t$, and $j\omega/\partial x$, to get

$$\partial \tilde{E}_1/\partial y = -jS_{vn}\omega \tilde{H}_{1x}$$  \hspace{1cm} (3.4)

$$(j\pi/\mu) \tilde{E}_1 = jS_{vn}\omega \tilde{H}_{1y}$$  \hspace{1cm} (3.5)

$$(j\pi/\mu) \tilde{H}_{1y} - \partial \tilde{H}_{1x}/\partial y = \sigma_1 \tilde{E}_1 .$$  \hspace{1cm} (3.6)

In these equations, $\tilde{E}_1$, $\tilde{H}_{1x}$ and $\tilde{H}_{1y}$ are the Fourier coefficients of $E_1$, $H_{1x}$ and $H_{1y}$, respectively, as explained in Chapter II, Section E.1.

Eliminating $\tilde{H}_{1y}$ between Eqs. (3.5) and (3.6), we get

$$\partial \tilde{H}_{1x}/\partial y = -\left(\sigma_1 + \frac{(\pi/\mu)^2}{3S_{vn}\omega \tilde{H}_{1y}}\right) \tilde{E}_1 .$$  \hspace{1cm} (3.7)

If a solution of Eqs. (3.4) and (3.7) for $\tilde{E}_1$ can be found, the problem is solved, for $\tilde{H}_{1y}$ can easily be found from $\tilde{E}_1$ using Eq. (3.5). Thus, Eqs. (3.4) and (3.7) are the basic equations. Their solution is easily found by making use of the wave impedance concept and employing a transmission-line analog. Corresponding quantities in the analog are most easily seen by setting out the relevant equations side by side, as Booker [12] has done in dealing with other electromagnetic problems.
There is clearly a one-to-one correspondence between the following quantities:

\[ \begin{align*}
\vec{E}_1 & \quad \rightarrow \quad V \\
\vec{H}_{1x} & \quad \rightarrow \quad I \\
jS_{vn}\omega \mu_{1x} & \quad \rightarrow \quad Z_1 \\
\sigma_1 + \frac{(v\mu/\tau)^2}{jS_{vn}\omega \mu_{1y}} & \quad \rightarrow \quad Y_1
\end{align*} \] (3.8)

Since \( Z \) and \( Y \) represent respectively the series impedance and shunt admittance per unit length of the transmission line, we can construct for the induction motor a transmission line equivalent circuit which represents exactly the relationship between \( \vec{E}_1 \) and \( \vec{H}_{1x} \) in region 1. An elementary section of this line is shown in Figure 3.2. The expressions for the characteristic wave impedance \( Z_{01} \) and the propagation factor \( \gamma_1 \) for the \( y \)-direction can be written by analogy:
Figure 3.2. Element of equivalent circuit of induction motor or transmission line
The subscript '1', as stated previously, refers to region 1.

By exactly the same argument, we can set up a transmission-line model for region 2. The characteristic wave impedance $Z_{02}$ and the propagation factor $\gamma_2$ can be obtained from Eqs. (3.9) and (3.10) by replacing $\mu_{1x}$ with $\mu_{2x}$, $\mu_{1y}$ with $\mu_{2y}$, and putting $\sigma_1 = 0$. This gives

$$Z_{02} = j \frac{S_{\nu n} \omega}{(\nu \pi/\tau)} \sqrt{\mu_{2x} \mu_{2y}}$$

$$\gamma_2 = (\nu \pi/\tau) \sqrt{\frac{\mu_{2x}}{\mu_{2y}}}.$$  \hspace{1cm} (3.12)

In order to complete the analogy, we must consider the boundary conditions. Referring to Figure 3.1b at the interface A between regions 1 and 2, both the electric and magnetic intensities must be continuous. This implies that the Fourier coefficients must be equal also, i.e.,
where the superscripts '+' and '-' denote just above and below the interface A. Thus, in the analog, V and I must be continuous at the junction of the two dissimilar transmission lines representing regions 1 and 2. This continuity follows from Kirchhoff's laws.

In region 0, $H_x$ must be zero everywhere since the permeability is infinite. In particular,

$$H_{0x} = 0$$

(3.14)

at the boundary between region 1 and 0. In the analog, we must have $I = 0$ at the corresponding point. The transmission line for region 1 is, therefore, open-circuited at the corresponding end. The open circuit condition can alternatively be obtained by noting that since $\mu = \infty$ in region 0, the associated characteristic wave impedance is infinite, and this is the 'termination' of region 1.

Similarly, we can set up a transmission-line model for region 3, the air gap region. The characteristic wave impedance $Z_{03}$ and the propagation factor $\gamma_3$ can be obtained from Eqs. (3.9) and (3.10) by putting $\mu_{2x} = \mu_{2y} = \mu_0$, $\sigma_1 = 0$ and $S_{\text{un}} = 1$. This gives
Referring to Figure 3.1b, at the interface B, the magnetic induction, as well as the magnetic intensity, must be continuous. This implies that the Fourier coefficients must be equal also:

\[
\begin{align*}
\tilde{B}_y^+ &= \tilde{B}_y^- \\
\tilde{H}_x^+ &= \tilde{H}_x^-
\end{align*}
\]

(3.17)

The first equation of Eq. (3.16) can be expressed in terms of the electric intensity, noting that \( \tilde{B}_y^+ \) lies in the air gap, i.e., a stationary region, and \( \tilde{B}_y^- \) lies in the tooth-top region, i.e., a moving region. Faraday's laws of induction for moving regions are given by Eqs. (3.1) and (3.2), and for stationary regions, the same equations hold with \( \nu = 0 \). Thus, we can use Eq. (3.2) to express the magnetic induction in terms of the electric intensity, and Eq. (3.17) will take the form

\[
\begin{align*}
\tilde{E}_B^+ &= S_{vn} \tilde{E}_B \\
\tilde{H}_x^+ &= \tilde{H}_x^-
\end{align*}
\]

(3.18)
Ampere's circuital law,

$$\oint \mathbf{H} \cdot d\mathbf{S} = \iint_S \mathbf{J} \cdot d\mathbf{A}$$  \hspace{1cm} (3.19)

can be applied at the interface C, between regions 3 and 4. This gives the following relations between the Fourier coefficients:

$$\bar{H}_{cx} = \bar{J}_L$$  \hspace{1cm} (3.20)

where $\bar{J}_L$ represents the Fourier coefficient of the linear current density of the primary winding.

Figure 3.3 shows the transmission-line analog of the interaction regions of the induction motor. The figure also shows the boundary conditions at each junction or interface between the different regions.

C. Solution of the Field Equations

As described in the previous section, the field equations have the transmission line form in each region of the induction motor, with their relevant expressions of the characteristic impedance $Z_0$ and the propagation factor $\gamma$.

In region 1, the field equations, in terms of the Fourier coefficient, are given by
Figure 3.3. Transmission line analogy of the different regions of the induction motor and the boundary conditions at the different sections
\[ \frac{\partial E_1}{\partial y} = -Z_1 \bar{H}_{1x} \]
\[ \frac{\partial \bar{H}_{1x}}{\partial y} = -\gamma_1 \bar{E}_1 \quad (3.21) \]

where \(Z_1\) and \(\gamma_1\) are given by Eq. (3.8). According to the chosen system of coordinates, as shown in Figure 3.1b, the solution of the system given by Eq. (3.21) will take the following matrix form:

\[
\begin{bmatrix}
\bar{E}_A \\
\bar{H}_{Ax}
\end{bmatrix}
=
\begin{bmatrix}
\cosh \gamma_1 d_1 & -Z_{01} \sinh \gamma_1 d_1 \\
-\frac{1}{Z_{01}} \sinh \gamma_1 d_1 & \cosh \gamma_1 d_1
\end{bmatrix}
\begin{bmatrix}
\bar{E}_0^+ \\
\bar{H}_{0x}^+
\end{bmatrix}
\quad (3.22)
\]

where \(Z_{01}\) and \(\gamma_1\) are given by Eqs. (3.9) and (3.10). The subscript \(A\) means the field quantity is at the interface \(A\), and \(0\) has the same meaning at the interface \(0\).

Equation (3.22) can be rewritten in the following abbreviated form:

\[
\begin{bmatrix}
\bar{E}_A \\
\bar{H}_{Ax}
\end{bmatrix}
= D_1
\begin{bmatrix}
\bar{E}_0^+ \\
\bar{H}_{0x}^+
\end{bmatrix}
\quad (3.23)
\]

where \(D_1\) is the transfer matrix of region 1, and is given by
\[
D_1 = \begin{bmatrix}
\cosh \gamma_1 d_1 & -Z_{01} \sinh \gamma_1 d_1 \\
-\frac{1}{Z_{01}} \sinh \gamma_1 d_1 & \cosh \gamma_1 d_1
\end{bmatrix}
\] (3.24)

By exactly the same argument, we can set up the solution of the field quantities in region 2, the tooth-top region, which can be written in the form:

\[
\begin{bmatrix}
E^- \\
H_{Bx}^-
\end{bmatrix} = D_2 \begin{bmatrix}
E^+ \\
H_{Ax}^+
\end{bmatrix}
\]
(3.25)

where the superscripts '-' and '+' denote the values of the field quantities just below and above the interface between the two regions, and \( D_2 \) is the region matrix

\[
D_2 = \begin{bmatrix}
\cosh \gamma_2 d_2 & -Z_{02} \sinh \gamma_2 d_2 \\
-\frac{1}{Z_{02}} \sinh \gamma_2 d_2 & \cosh \gamma_2 d_2
\end{bmatrix}
\] (3.26)

where \( \gamma_2 \) and \( Z_{02} \) are given by Eqs. (3.11) and (3.12). The boundary conditions at the interface A between regions 1 and 2 are given by Eq. (3.13). Then, on eliminating the column matrix on the right-hand
side of Eq. (3.25) using Eq. (3.13), Eq. (3.25) will take the form:

\[
\begin{bmatrix}
  E_B^- \\
  H_{Bx}^-
\end{bmatrix}
= D_2
\begin{bmatrix}
  E_A^- \\
  H_{Ax}^-
\end{bmatrix},
\]  

(3.27)

On eliminating the column matrix on the right-hand side of Eq. (3.27), by using Eq. (3.23), we can write

\[
\begin{bmatrix}
  E_B^- \\
  H_{Bx}^-
\end{bmatrix}
= D_2 D_1
\begin{bmatrix}
  E_0^+ \\
  H_{0x}^+
\end{bmatrix}.
\]  

(3.28)

By exactly the same argument, we can set up the solution for the field quantities in the air gap region 3 as:

\[
\begin{bmatrix}
  E_C^- \\
  H_{Cx}^-
\end{bmatrix}
= D_3
\begin{bmatrix}
  E_B^+ \\
  H_{Bx}^+
\end{bmatrix}.
\]  

(3.29)

where the superscripts '-' and '+' denote the values of the field quantities just below and above the two interfaces C and B, respectively; and \(D_3\) is the region matrix given by
\[ D_3 = \begin{bmatrix} \cosh \gamma_3 d_3 & Z_{03} \sinh \gamma_3 d_3 \\ -\frac{1}{Z_{03}} \sinh \gamma_3 d_3 & \cosh \gamma_3 d_3 \end{bmatrix} \]  

(3.30)

where \( \gamma_3 \) and \( Z_{03} \) are given by Eqs. (3.15) and (3.16).

The boundary conditions at the interface B are given by Eq. (3.18). This equation can be rewritten in the following matrix form:

\[
\begin{bmatrix} E^+ \\ H^+_{Bx} \end{bmatrix}_B = T \begin{bmatrix} E^- \\ H^-_{Bx} \end{bmatrix}
\]  

(3.31)

where the matrix \( T \) is given by

\[
T = \begin{bmatrix} S_{vn} & 0 \\ 0 & 1 \end{bmatrix}
\]  

(3.32)

The matrix can be called 'The Transition Matrix', as it represents the transition relations between the field quantities at an interface separating moving and stationary regions. It is evident that if the moving region becomes stationary, \( S_{vn} = 1 \), then the transition matrix \( T \) becomes the identity matrix \( I \).

Substituting Eq. (3.31) into (3.29) and then, using Eq. (3.28), we get:
In Eq. (3.33), the entries of the region matrices $D_1$, $D_2$, $D_3$ and $T$ are known quantities. Then, Eq. (3.33) defines the two field components at the boundary $C$ in terms of those at the boundary $O$. The two boundary conditions necessary to get any two of the four unknowns, are given by Eqs. (3.14) and (3.20). On substituting Eqs. (3.14) and (3.20) into Eq. (3.33), we can get:

$$
\begin{bmatrix}
\tilde{E}_C^- \\
\tilde{H}^C_x
\end{bmatrix} = D_3 T D_2 D_1 
\begin{bmatrix}
\tilde{E}_0^+ \\
\tilde{H}_0^C_x
\end{bmatrix}.
$$

(3.33)

Let $e_1$, $e_2$, $e_3$ and $e_4$ be the entries of the equivalent matrix formed from the product of the four matrices $D_1$, $D_2$, $D_3$ and $T$; then we can write

$$
\begin{bmatrix}
\tilde{E}_C^- \\
J_L
\end{bmatrix} = 
\begin{bmatrix}
e_1 & e_2 \\
e_3 & e_4
\end{bmatrix} 
\begin{bmatrix}
\tilde{E}_0^+ \\
0
\end{bmatrix}.
$$

(3.34)

The solution of Eq. (3.35) for $\tilde{E}_C^-$ and $\tilde{E}_0^+$ will yield:
The Fourier coefficient of the y-component of the magnetic induction at the interface can be derived from Eq. (3.5), by putting \( S_{\nu n} = 1 \) and replacing the suffix 1 by C to get

\[
\vec{B}_{Cy} = \frac{(\nu \pi / \tau)}{n \omega} \vec{E}_C .
\]  

(3.38)

Eqs. (3.36) - (3.38), as well as Eq. (3.20), constitute the basic equations needed to obtain the field components in any region of the motor.

Once we obtain the Fourier coefficients of the field components, the expressions of these field components can be set up. Their form will be that of Eq. (2.50). Therefore, the field components at the interface C will be

\[
E_C = \sum_{n=1}^{\infty} \sum_{\nu=-\infty}^{\infty} \frac{e_1}{e_3} \vec{J}_L e^{j(\frac{\nu \pi}{\tau} x + n \omega t)}
\]  

(3.39)

\[
B_{Cy} = \sum_{n=1}^{\infty} \sum_{\nu=-\infty}^{\infty} \frac{(\nu \pi / \tau)}{n \omega} \frac{e_1}{e_3} \vec{J}_L e^{j(\frac{\nu \pi}{\tau} x + n \omega t)}
\]  

(3.40)

\[
H_{Cx} = \sum_{n=1}^{\infty} \sum_{\nu=-\infty}^{\infty} \vec{J}_L e^{j(\frac{\nu \pi}{\tau} x + n \omega t)}
\]  

(3.41)
Equations (3.39) and (3.40) are those required to predict the machine performance.

D. Motor Currents and Performance

1. Current equations

By exactly the same argument, as previously given in Chapter II, Section G, the voltage induced per phase of the primary winding due to a certain electric field component having time harmonic order, \( n \), will be given by

\[
V_{ph} = -N W_1 \sum_{n=-\infty}^{\infty} K_n^* \tilde{E}_C e^{j \frac{\nu \pi}{c} x_0}
\]  

(3.42)

where \( x_0 \), \( N \), \( W_1 \) and \( K_n^* \) have the definitions as previously given in Chapter II, Section G. Substituting the expression of \( \tilde{E}_C \), as given by Eq. (3.37), and then eliminating \( \tilde{J}_L \) by using Eq. (2.44) gives the voltage equation for any arbitrary phase, distance \( x_0 \) from the origin:

\[
V_{ph} = \sum_{n=-\infty}^{\infty} f(n) \left[ I_1 e^{j \frac{2\pi}{3} \nu} + I_2 e^{j \frac{4\pi}{3} \nu} + I_3 e^{j \frac{8\pi}{3} \nu} \right] e^{j \frac{\nu \pi}{c} x_0}
\]  

(2.43)

where

\[
f(n) = -\left( \frac{a n}{2 \pi} N W_1 \right) \frac{e_1}{e_3} K_n K_n^* .
\]  

(3.44)
The comparison between Eq. (3.43) and (2.133) shows that both equations are identical with the sole difference in the expressions of \( f(v) \). Thus, the analysis from Eq. (2.133) to Eq. (2.139) will also hold here, and the voltage equations will be given by:

\[
\begin{bmatrix}
R_1 + \sum_{v=-\infty}^{\infty} f(v) e^{j \frac{2\pi}{3} v} & -j \sum_{v=-\infty}^{\infty} f(v) e^{-j \frac{4\pi}{3} v} & I_1 \\
\sum_{v=0}^{\infty} f(v) e^{j \frac{2\pi}{3} v} & R_2 + \sum_{v=-\infty}^{\infty} f(v) e^{-j \frac{4\pi}{3} v} & I_2 \\
\sum_{v=0}^{\infty} f(v) e^{j \frac{2\pi}{3} v} & \sum_{v=-\infty}^{\infty} f(v) e^{-j \frac{4\pi}{3} v} & R_3 + \sum_{v=-\infty}^{\infty} f(v)
\end{bmatrix} = 
\begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}
\]

(3.45)

Equation (3.45) permits the representation of unbalanced voltage operations, as well as fault conditions, as the voltages \( V_1, V_2 \) and \( V_3 \) are arbitrary quantities. The entries of the coefficient matrix are self and the mutual impedances of the different phases of the machine. Once we solve Eq. (3.45) for the phase currents, the field components in the different regions can be determined, and hence, the machine performance can be fully predicted.

2. Performance equations

The analysis of Chapter II, Section H.1 also holds in this case with the sole modification that the overhang length of the primary
winding is neglected, i.e., $W_0 = 0$. Therefore, the tangential force component, $F_x$, as follows from Eq. (2.142), will be given by

$$F_x = -\frac{1}{2} (2\pi W_1) \Re \sum_{n=1}^{\infty} \sum_{\nu=-\infty}^{\infty} B_{Cy} J_L^*$$  \hspace{1cm} (3.46)

where $J_L^*$ is the complex conjugate of the linear current density, and $B_{Cy}$ is the Fourier coefficient of the $y$-component of the magnetic induction at the interface $C$, as given by Eq. (3.38).

Similarly, the analysis of Section H.2 in Chapter II also holds in this case, with $W_0 = 0$. Thus, the secondary input power, $P_{sec}$, can be derived from the Poynting theorem, and the final result will take the form:

$$P_{sec} = -\frac{1}{2} (2\pi PW_1) \Re \sum_{n=1}^{\infty} \sum_{\nu=-\infty}^{\infty} E_C J_L^*$$  \hspace{1cm} (3.47)

where $J_L^*$ is the complex conjugate of the linear current density of the primary winding, and $E_C$ is the Fourier coefficient of the electric intensity at the interface $C$, as given by Eq. (3.37).

Finally, the input power to the motor will be given by:

$$P_{in} = \Re \sum_{r=1}^{3} V_r I_r^*$$  \hspace{1cm} (3.48)

Equations (3.46) - (3.48) are those required to predict the motor performance in terms of its design parameters.
E. Conclusion

In this chapter, a simplified electromagnetic theory of induction motors has been developed. The theory is based on the wave impedance concept, or in other words, on the analogy found between the differential equations of the Fourier coefficients of the field quantities and those of the transmission lines. The analysis presented in this chapter represents a generalization and extension to the work of Cullen and Barton [26], Freeman [37, 38], Freeman and Smith [40] and Greig and Freeman [45]. The analysis establishes a general approach capable of describing any arbitrary electric or magnetic loading, as well as a general method for calculating the machine currents. The analysis can be equally applied to fault conditions, as well as single-phase machines. The expression of force, input power and secondary input power is also derived, which enables the prediction of motor performance.
IV. EXTENSION OF THE FIELD ANALYSIS

A. Introduction

The analysis presented in Chapter II is based on the solution of the field equations in a five-region model. These regions are: the primary slot, the air gap, the secondary slot, the primary overhang and the air overhang. The analysis of Chapter II ignores the important part played by tooth-top and zigzag leakage flux in practical machines. This omission leads to performance equations in which the rotor slot is a principal component of the leakage reactance. The rotor-slot leakage reactance is exceedingly dependent upon the slip. Moreover, the tooth tops in both the primary and secondary members entirely control the flow of the electromagnetic energy from the primary to the secondary side. This effect of the tooth tops can considerably affect the machine performance.

The purpose of this chapter is to extend the field analysis to a more accurate and elaborate model. This model consists of the five regions developed in Chapter II, plus the two regions corresponding to the tooth tops of the primary and secondary members. Another purpose of this chapter is to derive a systematic field analysis, in a way similar to that developed in Chapter III, when the effect of the secondary end ring is taken into account.

B. Mathematical Model

Figure 4.1 shows a sectional view of the mathematical model of a seven-region induction motor. The figure is the same as the model given
Figure 4.1. Sectional view of the induction motor showing the seven regions (1, 2, ..., 7) considered in the analysis.
in Figure 2.3, with two additional regions: the primary and secondary tooth tops. To achieve generality in representation, the thickness of regions 1, 2, ..., 5 are denoted by \(d_1, d_2, \ldots, d_5\), while their interfaces measured from the \(Oxz\) plane are denoted by \(y_1, y_2, \ldots, y_5\). Other dimensions are as shown in the figure. The directional permeabilities are denoted by \(\mu_{ix}\) and \(\mu_{iy}\), where \(i = 1, 2, \ldots, 5\). The overhang regions 6 and 7 are kept without change, as previously given in Chapter II (see Figure 2.3).

C. Field Analysis

The procedure of deriving the basic differential equations and their solutions in the different regions of the motor are exactly the same as previously developed in Chapter II. To avoid repetition, the results of Chapter II will be used directly with their appropriate formulation on the basis of the notation adopted in this chapter. Therefore, according to the notation adopted in Figure 4.1, these equations can be written in the form

\[
\frac{\partial^2 E_1}{\partial x^2} + \frac{\partial^2 E_1}{\partial y^2} = \mu_{1x} \mu_{1y} \left( \frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) J_1
\]  

(4.1)
\[
\frac{\partial^2 E_1}{\partial x^2} = \frac{1}{\sigma_1} \frac{\partial^2 J_1}{\partial x^2} - \frac{2d_1}{\sigma_e A_e W} J_1 \quad (4.2)
\]

where the subscript 1 denotes the quantities of region 1 and the subscript e denotes those of the end ring.

It is clear that the elimination of \( J_1 \) between Eqs. (4.1) and (4.2) will yield a fourth order PDE in \( E_1 \). This PDE represents the field equation in region 1 subjected to the constraint imposed by the end ring of the secondary member.

As explained in Chapter II, the functional form of all the field quantities, as well as the current distribution, should be that given by Eq. (2.50). Therefore, we can express Eqs. (4.1) and (4.2) in terms of their Fourier coefficients (this is simply done by replacing the operations \( \partial/\partial x \) and \( \partial^2/\partial x^2 \) by \( (jv\pi/\tau) \) and \( (jv\pi/\tau)^2 \), respectively), and on eliminating \( \tilde{J}_1 \) between the resulting equations, we can get

\[
\frac{\partial^2 E_1}{\partial y^2} - \eta_1^2 E_1 = 0 \quad (4.3)
\]

where

\[
\eta_1^2 = \frac{\mu_{1x}}{\mu_{1y}} \left( \frac{\nu\pi}{\tau} \right)^2 + j \mu_{1x} S_{1n} \frac{n\omega_1}{\kappa_e} \quad (4.4)
\]

\[
\kappa_e = 1 + \frac{2d_1 \sigma_1/\sigma_e}{A_e W (\nu\pi/\tau)^2} \quad (4.5)
\]
We retain the bar to denote the Fourier coefficient of the field quantity. The solution of Eq. (4.3) will be given by

\[ \tilde{E}_i = A_i \cosh \eta_i y + B_i \sinh \eta_i y. \] (4.6)

**Regions 2, 3 and 4:**

In a similar way, the differential equation of the Fourier coefficient of the field components of regions 2, 3 and 4 can be written in the form

\[ \frac{\partial^2 \tilde{E}_i}{\partial x^2} - \eta_i^2 \tilde{E}_i = 0, \quad i = 2, 3, 4 \] (4.7)

where

\[ \eta_i^2 = \frac{\mu_{ix}}{\mu_{iy}} \left( \frac{\nu_m}{\tau} \right)^2. \] (4.8)

Note that in region 3, \( \mu_{3x} = \mu_{3y} = \mu_0 \). The solution of Eq. (4.8) can be written in the form

\[ \tilde{E}_i = A_i \cosh \eta_i (y-y_{i-1}) + B_i \sinh \eta_i (y-y_{i-1}) \] (4.9)

where \( i = 2, 3, 4 \) as defined above.

**Region 5:**

The differential equation of the Fourier coefficient of the field intensity in this region is given by
\[
\frac{\partial^2 E_5}{\partial x^2} - \eta_5^2 E_5 = j \frac{\mu_{5x}}{\mu_{5y}} \eta_5 \bar{J}_5
\]  
(4.10)

where

\[
\eta_5^2 = \frac{\mu_{5x}}{\mu_{5y}} \left( \frac{\nu \tau}{\tau} \right)^2.
\]  
(4.11)

Note that \( \bar{J}_5 \) is the Fourier coefficient of the primary current, which is a completely defined quantity (see Eq. (2.44)).

The solution of Eq. (4.10) is composed of the complimentary function and the particular integral, and can be written in the form

\[
E_5 = A_5 \cosh \eta_5(y-y_4) + B_5 \sinh \eta_5(y-y_5) - i \frac{\eta_5 \mu_{5x}}{2} \bar{J}_5.
\]  
(4.12)

Equations (4.6), (4.9) and (4.12) constitute the necessary equations to get the solution of the field components in the different regions of the induction motor. These equations contain ten arbitrary constants which can be determined from the boundary conditions.

D. Boundary Conditions

As follows from Eqs. (2.31) and (2.32), the boundary conditions at any interface between any two adjacent regions can be expressed in terms of the Fourier coefficients and are given by
where the '−' and '+' denote the value of the field components, or its derivative, just above and below the interface under consideration.

E. The Arbitrary Constants and the Field Expressions

Consider the $i$th and the $(i+1)$th regions to be any two adjacent regions. Equations (4.13i) and (4.13ii) can be applied to the interface between these two adjacent regions. The relation between the arbitrary constants of the field expressions can be written in the form

$$
\begin{pmatrix}
A_{i+1} \\
B_{i+1}
\end{pmatrix}
= D_i
\begin{pmatrix}
A_i \\
B_i
\end{pmatrix},
\quad i = 1, 2, 3
$$

(4.14)

where

$$
D_i = \frac{S_{uni+1}}{S_{uni}}
\begin{pmatrix}
1 & 0 & \cosh \eta_i d_i & \sinh \eta_i d_i \\
0 & \frac{\eta_i/\mu_i x}{\eta_{i+1}/\mu_{i+1} x} & \sinh \eta_i d_i & \cosh \eta_i d_i
\end{pmatrix}.
$$

(4.15)
The advantage of using Eq. (4.14) for the first four regions is that it permits expressing the arbitrary constants of region 4 in terms of those of region 1. This is carried out by letting \( i \) take the values of 1, 2 and 3, and substituting the resulting matrix equation iteratively to get

\[
\begin{pmatrix}
A_4 \\
B_4
\end{pmatrix} = D_3 D_2 D_1
\begin{pmatrix}
A_1 \\
B_1
\end{pmatrix} . \tag{4.16}
\]

In region 5, at \( y = y_4 \), the boundary conditions will lead to the following matrix equation:

\[
\begin{pmatrix}
A_5 \\
B_5
\end{pmatrix} = D_4
\begin{pmatrix}
A_4 \\
B_4
\end{pmatrix} + \begin{pmatrix}
\frac{js_{\gamma n5^n\omega_5\bar{J}_5}}{\eta_5^2} \\
0
\end{pmatrix} \tag{4.17}
\]

where \( D_4 \) has the same definition as given by Eq. (4.15), with \( i = 4 \). Clearly, the slip \( s_{\gamma n5} = 1 \), as region 5 is a stationary region, i.e., \( v = 0 \) (see Eq. 2.64).

Eliminating \( (A_4 B_4) \) between Eqs. (4.16) and (4.17), we can write

\[
\begin{pmatrix}
A_5 \\
B_5
\end{pmatrix} = \begin{pmatrix}
e_1 & e_2 \\
e_3 & e_4
\end{pmatrix}
\begin{pmatrix}
A_1 \\
B_1
\end{pmatrix} + \begin{pmatrix}
\frac{j\omega \mu_5 \bar{J}_5}{\eta_5^2} \\
0
\end{pmatrix} \tag{4.18}
\]

where

\[
\begin{pmatrix}
e_1 & e_2 \\
e_3 & e_4
\end{pmatrix} = D_4 D_3 D_2 D_1 . \tag{4.19}
\]
Evidently, $e_1$, $e_2$, $e_3$ and $e_4$ are the entries of the equivalent matrix formed from the product of the matrices of the first four regions. Equation (4.18) shows that the field expressions in regions 1 and 5, Eqs. (4.6) and (4.12), can be expressed by the same arbitrary constants $A_1$ and $B_1$. The two additional boundary conditions required to get the values of $A_1$ and $B_1$ are:

$$\begin{align*}
\text{at } y = 0, & \quad \frac{\partial E}{\partial y} = 0 \\
\text{at } y = y_5, & \quad \frac{\partial E}{\partial y} = 0.
\end{align*}$$

(4.20)

Equation (4.20) together with Eqs. (4.18), (4.6) and (4.12) can be solved to get the values of the arbitrary constants $A_5$ and $B_5$. Their expressions will be given by

$$\begin{align*}
A_5 &= j \frac{n \omega \mu_0}{n_5} \cdot \frac{e_3}{e_3 + e_1 \tanh n_5 d_5} J_5 \\
B_5 &= -j \frac{n \omega \mu_0}{n_5} \cdot \frac{e_3 \tanh n_5 d_5}{e_3 + e_1 \tanh n_5 d_5} J_5.
\end{align*}$$

(4.21)

(4.22)

Once we get the expressions of the arbitrary constants $A_5$ and $B_5$, the field expression in region 5, Eq. (4.12), is fully determined. The magnetic intensity, as well as the magnetic induction, can be derived from the expression of the electric intensity by using the relations given by Eqs. (2.9) and (2.10).
F. Other Considerations

The above analysis gives the solution of the field problem in the five main regions (1, 2, ..., 5) of the induction motor. The solution of the field problem in the other overhang regions 6 and 7 is exactly similar to that previously developed in Chapter II (see Sections C.5, C.6, D.2, E.5 and F.2), with the sole modification of the number of regions. The voltage equations, as well as the force and power expressions, are exactly similar to those developed in Chapter II (see Sections G and H). Therefore, the analysis will not be repeated here and the results of those sections will be used directly.

G. Conclusion

In this chapter, the field analysis is developed for a more elaborate mathematical model. The model consists of seven regions that can represent faithfully the induction motor with its actual regions. The purpose of this chapter, in addition to extending the regions of the mathematical model, is to develop a systematic analysis capable of handling larger region models with relative mathematical ease. The approach developed in this chapter permits the determination of the fields in the required region only, without the need to determine the fields in the other regions. This differs from the analysis developed in Chapter II. This is a short cut procedure and avoids the solution of ten simultaneous algebraic equations to get the the arbitrary constants for the solution of the differential equations. The method developed here is a combination of the methods developed in
Chapters II and III. It is similar to that developed in Chapter II as it deals with the differential equations of a single field component in all the regions of the mathematical model. It is also similar to that developed in Chapter III as it deals with the matrices of the different regions, permitting the elimination of the arbitrary constants of the regions not involved directly in getting the machine performance.
V. NUMERICAL CALCULATIONS AND DISCUSSIONS

A. Introduction

In order to achieve a reasonable demonstration of the newly developed analysis of the previous chapters, the performance characteristics of an induction motor, designed for this purpose, will be studied.

The performance characteristics of the given model have been calculated under the normal, constant voltage drive system. Some considerations have also been given to varying some of the motor parameters to obtain their effects on the motor performance. Both the space and time harmonics have been taken into account in the calculations. Harmonic time effects have been studied individually, as well as globally.

A computer program has been developed for this purpose, and is shown in the Appendix.

B. Induction Motor Model

The induction motor designed for the purpose of this investigation is assumed to have the following specifications.

**Primary side:**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slot depth</td>
<td>10 mm</td>
</tr>
<tr>
<td>Slot width</td>
<td>12 mm</td>
</tr>
<tr>
<td>Tooth top thickness</td>
<td>2 mm (4 mm)</td>
</tr>
<tr>
<td>Tooth top opening</td>
<td>12 mm (8, 4, 1 mm)</td>
</tr>
</tbody>
</table>
Slot pitch 24 mm
Pole pitch 216 mm
Core length 200.6 mm
Relative permeability of core iron 500
No. of slots/pole/phase 3
Coil pitch 1-8
No. of poles 4
No. of conductors/slot 40
Resistance/phase 0.5 ohm
Slot factor 0.7
Overhang length (both sides) 50 mm (0, 100 mm)
No. of phases 3
Frequency 60 Hz

Type of winding:
3 phase, double layer, star connected.

Voltage:
3 phase, balanced system, 220V (rms).

Time harmonics:
Fundamental 100%
Third harmonic 9%
Fifth and higher harmonics (9.0/harmonic order) %
Space harmonics:
-11, -5, 1, 7, 13 (or fundamental only).

Secondary side:
- Slot depth: 20 mm (15, 10 mm)
- Slot width: 12 mm
- Tooth top thickness: 2 mm (4 mm)
- Tooth top opening: 10 mm (semi-closed)
- Slot pitch: 24 mm
- Conductivity of squirrel cage conductors (AL.): 3.45 x 10^7 s/m
- Secondary core length: 200.2 mm
- End ring width (if considered): 5 mm
- Conductivity of end ring (Cu): 1.148 x 10^6 s/m
  (Other conductivities have been also considered.)
- Relative permeability of secondary iron: 500 mm

Air gap length:
- 2 mm

The numbers in parentheses have been investigated too.

C. Results and Discussion

The following paragraphs describe the basic results as concluded from the study of the given model using the analyses developed in the previous chapter.
Figures 5.1 and 5.2 show the thrust-speed characteristics of the motor when the end ring effects are neglected and when they are taken into account. Curves A and C show the characteristics when both the space and time harmonics, in the sense indicated in Section B above, are taken into account. Curves B and D are the characteristics when harmonics are neglected. The figures clearly show that the effects of harmonics are pronounced at the low speed range.

Figure 5.3 shows the exact shape of the irregularity in the thrust-speed curve in the low speed range. The figure shows almost two coincident curves, the lower curve is drawn when the space harmonics are acting only in the air gap, while the time harmonics are neglected. The upper curve is drawn when both the space and time harmonics are acting together on the motor. It is evident that the effect of the time harmonics is almost negligible. Both curves of Figure 5.3 are drawn for a motor with end rings.

Figures 5.4 and 5.5 show the effect of the conductivity of the end rings of the squirrel cage structure on the thrust-speed characteristics. Figure 5.4 is drawn when both space and time harmonics are neglected, while Figure 5.5 shows the characteristics when the effects of these harmonics are taken into account. Clearly, the performance of the motor is considerably improved when the end-ring conductivity is increased. This suggests a motor design with end-ring conductivity higher than that of the squirrel cage structure.

Figure 5.6 shows the thrust-speed characteristics for the given model with different values of primary tooth top openings. Both space
Figure 5.1. Thrust-speed characteristics for the motor without end rings. Curve A: space and time harmonics are taken into account. Curve B: space and time harmonics are neglected. (Without end rings means that the secondary core length is assumed to be infinite)
Figure 5.2. Thrust-speed characteristics for the motor with and without end rings. Curves C and D are for the motor with end rings when harmonics are taken into account and are neglected, respectively. Curves A and B are those of Figure 5.1 for the purpose of comparison. (Without end rings means that the secondary core length is assumed to be infinite)
Figure 5.3. Thrust-speed curve at the low speed range for the motor with end rings. Lower curve: space harmonics only are considered. Upper curve: both space and time harmonics are considered
Figure 5.4. Thrust-speed characteristics for the motor with different end-ring conductivities. Space and time harmonics are not taken into account. $(\sigma_E/\sigma)$ is the ratio between the end-ring conductivity and the squirrel cage conductivity.
Figure 5.5. Thrust-speed characteristics for the motor with different end-ring conductivities. Space and time harmonics are taken into account. $\sigma_{e}/\sigma$ is the ratio between the end-ring conductivity and the squirrel cage conductivity.
Figure 5.6. Thrust-speed characteristics for the motor with different primary tooth-top openings. Space and time harmonics are taken into account.
and time harmonics are taken into account. The figure shows that the pattern of the thrust-speed curves does not change as the tooth-top opening gets narrower, but the thrust magnitude is distinctly reduced with the reduction of the tooth-top opening. The figure clearly shows that the tooth-top opening can behave as a gate that controls the flow of the electromagnetic energy from the primary side to the air gap of the motor. It is evident from Figure 5.6 that the usual replacement of the motor winding by a single current sheet irrespective of the dimension of the tooth top of the electrical machine is questionable.

Figure 5.7 shows the effect of the conductivity of the end rings on the efficiency-speed characteristics of the given motor. Space and time harmonics are taken into account in the calculation of Figure 5.6, as well as for all of the efficiency and power factor curves. The figure shows clearly that the lower the conductivity of the end ring, the poorer is the performance of the motor in the working speed range.

Figure 5.8 shows the effect of the primary tooth-top opening on the primary efficiency of the motor. Clearly, as the primary tooth opening gets narrower, the motor performance is degraded.

Figure 5.9 shows the effect of the conductivity of the end rings on the power factor of the secondary input power. The figure shows that as the conductivity of the end rings is increased, the power factor is improved in the low speed range, but decreases in the high speed range.

Figure 5.10 shows the effect of the primary tooth-top opening on the power factor of the secondary input power. Clearly, the power
Figure 5.7. Primary efficiency-speed characteristics for the motor with different end-ring conductivities. Space and time harmonics are taken into account. \( \sigma_e/\sigma \) is the ratio between the end-ring conductivity and the squirrel cage conductivity.
Figure 5.8. Primary efficiency-speed characteristics for the motor with different primary tooth-top openings. Space and time harmonics are taken into account.
Figure 5.9. Secondary power factor-speed characteristics for the motor with different end-ring conductivities. Space and time harmonics are taken into account. \((\sigma_e/\sigma)\) is the ratio between the end-ring conductivity and the squirrel cage conductivity.
Figure 5.10. Secondary power factor-speed characteristics for the motor with different primary tooth-top openings. Space and time harmonics are taken into account.
factor is degraded as the tooth-top opening is decreased.

Table 5.1 shows that the values of the currents drawn by the motor due to time harmonics in the voltage system are almost constant over the whole speed range (about 4.8% decrease from standstill to the synchronous speed). For the relatively high distortion in the given voltage, the values of the harmonic currents are slightly larger than 3.5% in most of the speed range. The value can reach 12% in the working speed range. The effect of these harmonic currents on the thrust force is very small, as shown in Table 5.2. The harmonic currents decrease the primary efficiency of the motor by about 1% in the working speed range, as shown in Table 5.3.
Table 5.1. Harmonic currents and their percentage values with respect to the fundamental\(^a\)

<table>
<thead>
<tr>
<th>Per unit speed</th>
<th>Fundamental current A</th>
<th>3rd harmonic current A</th>
<th>%</th>
<th>5th harmonic current A</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>20.9380</td>
<td>0.7455</td>
<td>3.5603</td>
<td>0.0971</td>
<td>0.4638</td>
</tr>
<tr>
<td>0.05</td>
<td>20.7586</td>
<td>0.7436</td>
<td>3.5819</td>
<td>0.0970</td>
<td>0.4671</td>
</tr>
<tr>
<td>0.10</td>
<td>20.5818</td>
<td>0.7416</td>
<td>3.6032</td>
<td>0.0968</td>
<td>0.4704</td>
</tr>
<tr>
<td>0.15</td>
<td>20.3897</td>
<td>0.7396</td>
<td>3.6274</td>
<td>0.0967</td>
<td>0.4741</td>
</tr>
<tr>
<td>0.20</td>
<td>20.2214</td>
<td>0.7373</td>
<td>3.6463</td>
<td>0.0965</td>
<td>0.4772</td>
</tr>
<tr>
<td>0.25</td>
<td>19.9701</td>
<td>0.7355</td>
<td>3.6831</td>
<td>0.0963</td>
<td>0.4824</td>
</tr>
<tr>
<td>0.30</td>
<td>19.6774</td>
<td>0.7349</td>
<td>3.7347</td>
<td>0.0962</td>
<td>0.4888</td>
</tr>
<tr>
<td>0.35</td>
<td>19.3375</td>
<td>0.7332</td>
<td>3.7918</td>
<td>0.0960</td>
<td>0.4963</td>
</tr>
<tr>
<td>0.40</td>
<td>18.9394</td>
<td>0.7309</td>
<td>3.8592</td>
<td>0.0958</td>
<td>0.5058</td>
</tr>
<tr>
<td>0.45</td>
<td>18.4690</td>
<td>0.7297</td>
<td>3.9511</td>
<td>0.0958</td>
<td>0.5190</td>
</tr>
<tr>
<td>0.50</td>
<td>17.9084</td>
<td>0.7292</td>
<td>4.0720</td>
<td>0.0958</td>
<td>0.5347</td>
</tr>
<tr>
<td>0.55</td>
<td>17.2347</td>
<td>0.7276</td>
<td>4.2215</td>
<td>0.0856</td>
<td>0.5549</td>
</tr>
<tr>
<td>0.60</td>
<td>16.4188</td>
<td>0.7258</td>
<td>4.4204</td>
<td>0.0955</td>
<td>0.5817</td>
</tr>
<tr>
<td>0.65</td>
<td>15.4242</td>
<td>0.7240</td>
<td>4.6936</td>
<td>0.0954</td>
<td>0.6182</td>
</tr>
<tr>
<td>0.70</td>
<td>14.2063</td>
<td>0.7221</td>
<td>5.0826</td>
<td>0.0950</td>
<td>0.6689</td>
</tr>
<tr>
<td>0.75</td>
<td>12.7141</td>
<td>0.7201</td>
<td>5.6638</td>
<td>0.0952</td>
<td>0.7484</td>
</tr>
<tr>
<td>0.80</td>
<td>10.8945</td>
<td>0.7181</td>
<td>6.5915</td>
<td>0.0951</td>
<td>0.8730</td>
</tr>
<tr>
<td>0.85</td>
<td>8.7036</td>
<td>0.7161</td>
<td>8.2274</td>
<td>0.0950</td>
<td>1.0912</td>
</tr>
<tr>
<td>0.90</td>
<td>6.1259</td>
<td>0.7140</td>
<td>11.6554</td>
<td>0.0948</td>
<td>1.5480</td>
</tr>
<tr>
<td>0.95</td>
<td>3.2026</td>
<td>0.7119</td>
<td>22.2280</td>
<td>0.0947</td>
<td>2.9563</td>
</tr>
<tr>
<td>1.00</td>
<td>0.3761</td>
<td>0.7097</td>
<td>188.7042</td>
<td>0.0945</td>
<td>25.1333</td>
</tr>
</tbody>
</table>

\(^a\)Fundamental voltage = 220.0 V (rms), 3rd harmonic voltage = 19.8 V (rms), and 5th harmonic voltage = 3.96 V (rms).
Table 5.2. Harmonic forces and their percentage values with respect to the fundamental

<table>
<thead>
<tr>
<th>Per unit speed</th>
<th>Fundamental force N</th>
<th>3rd harmonic force N</th>
<th>%</th>
<th>5th harmonic force N</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>310.8450</td>
<td>0.3442</td>
<td>0.1107</td>
<td>0.0054</td>
<td>0.0017</td>
</tr>
<tr>
<td>0.05</td>
<td>310.1975</td>
<td>0.3438</td>
<td>0.1108</td>
<td>0.0054</td>
<td>0.0017</td>
</tr>
<tr>
<td>0.10</td>
<td>305.9543</td>
<td>0.3435</td>
<td>0.1123</td>
<td>0.0054</td>
<td>0.0018</td>
</tr>
<tr>
<td>0.15</td>
<td>299.7314</td>
<td>0.3433</td>
<td>0.1146</td>
<td>0.0054</td>
<td>0.0018</td>
</tr>
<tr>
<td>0.20</td>
<td>290.3235</td>
<td>0.3443</td>
<td>0.1186</td>
<td>0.0054</td>
<td>0.0019</td>
</tr>
<tr>
<td>0.25</td>
<td>284.6108</td>
<td>0.3434</td>
<td>0.1207</td>
<td>0.0054</td>
<td>0.0019</td>
</tr>
<tr>
<td>0.30</td>
<td>278.7537</td>
<td>0.3361</td>
<td>0.1206</td>
<td>0.0054</td>
<td>0.0019</td>
</tr>
<tr>
<td>0.35</td>
<td>272.4097</td>
<td>0.3333</td>
<td>0.1224</td>
<td>0.0054</td>
<td>0.0020</td>
</tr>
<tr>
<td>0.40</td>
<td>265.4331</td>
<td>0.3223</td>
<td>0.1252</td>
<td>0.0054</td>
<td>0.0021</td>
</tr>
<tr>
<td>0.45</td>
<td>257.6414</td>
<td>0.3295</td>
<td>0.1279</td>
<td>0.0053</td>
<td>0.0021</td>
</tr>
<tr>
<td>0.50</td>
<td>248.8066</td>
<td>0.3243</td>
<td>0.1304</td>
<td>0.0053</td>
<td>0.0021</td>
</tr>
<tr>
<td>0.55</td>
<td>238.6402</td>
<td>0.3219</td>
<td>0.1349</td>
<td>0.0052</td>
<td>0.0022</td>
</tr>
<tr>
<td>0.60</td>
<td>226.7850</td>
<td>0.3198</td>
<td>0.1410</td>
<td>0.0052</td>
<td>0.0023</td>
</tr>
<tr>
<td>0.65</td>
<td>212.7924</td>
<td>0.3178</td>
<td>0.1494</td>
<td>0.0052</td>
<td>0.0024</td>
</tr>
<tr>
<td>0.70</td>
<td>196.1072</td>
<td>0.3160</td>
<td>0.1611</td>
<td>0.0052</td>
<td>0.0027</td>
</tr>
<tr>
<td>0.75</td>
<td>176.0505</td>
<td>0.3142</td>
<td>0.1785</td>
<td>0.0051</td>
<td>0.0029</td>
</tr>
<tr>
<td>0.80</td>
<td>151.8320</td>
<td>0.3125</td>
<td>0.2058</td>
<td>0.0051</td>
<td>0.0033</td>
</tr>
<tr>
<td>0.85</td>
<td>122.6192</td>
<td>0.3109</td>
<td>0.2535</td>
<td>0.0050</td>
<td>0.0041</td>
</tr>
<tr>
<td>0.90</td>
<td>87.7399</td>
<td>0.3093</td>
<td>0.3525</td>
<td>0.0050</td>
<td>0.0057</td>
</tr>
<tr>
<td>0.95</td>
<td>47.1066</td>
<td>0.3077</td>
<td>0.6532</td>
<td>0.0050</td>
<td>0.0106</td>
</tr>
<tr>
<td>1.00</td>
<td>5.5126</td>
<td>0.3062</td>
<td>5.5547</td>
<td>0.0050</td>
<td>0.0901</td>
</tr>
</tbody>
</table>

Fundamental voltage = 220.0 V (rms), 3rd harmonic voltage = 19.8 V (rms), and 5th harmonic voltage = 3.96 V (rms).
Table 5.3. Effect of harmonics on the primary efficiency (space harmonics are taken into account)\(^a\)

<table>
<thead>
<tr>
<th>Per unit speed</th>
<th>Prim. eff. (no time harmonics)</th>
<th>Prim. eff. (with time harmonics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.05</td>
<td>0.023261</td>
<td>0.023212</td>
</tr>
<tr>
<td>0.10</td>
<td>0.044861</td>
<td>0.044767</td>
</tr>
<tr>
<td>0.15</td>
<td>0.066207</td>
<td>0.066070</td>
</tr>
<tr>
<td>0.20</td>
<td>0.090217</td>
<td>0.090030</td>
</tr>
<tr>
<td>0.25</td>
<td>0.116902</td>
<td>0.116649</td>
</tr>
<tr>
<td>0.30</td>
<td>0.145158</td>
<td>0.144837</td>
</tr>
<tr>
<td>0.35</td>
<td>0.175602</td>
<td>0.175209</td>
</tr>
<tr>
<td>0.40</td>
<td>0.208743</td>
<td>0.208269</td>
</tr>
<tr>
<td>0.45</td>
<td>0.245065</td>
<td>0.244484</td>
</tr>
<tr>
<td>0.50</td>
<td>0.285111</td>
<td>0.284419</td>
</tr>
<tr>
<td>0.55</td>
<td>0.329515</td>
<td>0.328688</td>
</tr>
<tr>
<td>0.60</td>
<td>0.379033</td>
<td>0.378041</td>
</tr>
<tr>
<td>0.65</td>
<td>0.434579</td>
<td>0.433375</td>
</tr>
<tr>
<td>0.70</td>
<td>0.497259</td>
<td>0.495768</td>
</tr>
<tr>
<td>0.75</td>
<td>0.568390</td>
<td>0.566485</td>
</tr>
<tr>
<td>0.80</td>
<td>0.649505</td>
<td>0.646950</td>
</tr>
<tr>
<td>0.85</td>
<td>0.742171</td>
<td>0.738470</td>
</tr>
<tr>
<td>0.90</td>
<td>0.846966</td>
<td>0.840842</td>
</tr>
<tr>
<td>0.95</td>
<td>0.955545</td>
<td>0.942120</td>
</tr>
<tr>
<td>1.00</td>
<td>-0.337778</td>
<td>-0.245724</td>
</tr>
</tbody>
</table>

\(^a\)Fundamental voltage = 220.0 V (rms), 3rd harmonic voltage = 19.8 V (rms), and 5th harmonic voltage = 3.96 V (rms).
VI. CONCLUSIONS

Comprehensive and rigorous electromagnetic field analyses for induction motors have been presented in this dissertation. These analyses may be used to predict motor performance and to enhance the methods of motor design. The analyses presented are based on the solution of a system of partial differential equations that faithfully express the field phenomena in the different regions of the induction motor. The following paragraphs describe the principal contributions of this dissertation.

1. The field analyses are presented from a very general point of view that can be used to get the field structure in the different zones of the machine under any electric and magnetic loading designs. The approach used in these analyses can be equally applied to single- and two-phase machines, as well as to the theories of linear machines. It also can be applied to the magnetohydrodynamics area as in the case of liquid metal induction pumps.

2. A short cut method for the field analysis has been developed to determine the fields in only the required regions of the motor. This method greatly simplifies the analytical expression used to describe the field quantities, as well as the computer work required to get these values.

3. A generalization of the transmission-line concept for solving the field problem has been introduced, for the first time, in a way that permits the inclusion of the field constraints imposed by the
4. The analyses presented suggest new design features and parameters to improve the motor performance. The following paragraphs describe some of these examples.

i. The tooth-top opening of the slotted structure plays a dominant role in controlling the transfer of electromagnetic energy from the primary side to the secondary side. This energy is greatly reduced by the decrease of the slot opening and will be zero for a closed slot structure with highly permeable iron.

ii. The end ring of the squirrel cage structure can greatly influence the performance of the induction motor. This dissertation presents, for the first time, the detailed equations necessary to get the optimal machine performance by properly designing the end ring of the rotor structure.

iii. The effect of the space harmonics is limited to the low speed range, so its effect on the machine performance can be neglected in the operating range of the machine (in the low slip range).

iv. Time harmonic voltage (9% third harmonic and 1.8% fifth) has a very small effect on the torque-speed characteristic. This amount of time harmonic voltage reduces the overall efficiency of the motor by about 1% in the working speed range.
5. This dissertation suggests a reformulation of some of the basic ideas used in the electromagnetic theory of electrical machine analysis. An example is the use of an equivalent current sheet for the primary windings. This practice may not allow a proper understanding of the phenomena taking place in the machines.
VII. BIBLIOGRAPHY


44. Graham, Q. "Dead Points in Squirrel-Cage Motors." AIEE Transactions 59 (1940): 637-642.


VIII. ACKNOWLEDGMENTS

The author wishes to express his gratitude to his co-major professors, Dr. Robert E. Post and Dr. A. L. Day, for their helpful discussions and assistance during the research work.

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The author wishes to express his gratitude to his wife, Samia Zaki, for her understanding and patience throughout this research work. God bless them all.
IX. APPENDIX: FORTRAN PROGRAM FOR CALCULATING THE MOTOR PERFORMANCE

THIS PROGRAM COMPUTES THE PERFORMANCE OF AN INDUCTION MOTOR.
EIGHT-REGION MODEL
SECONDARY SLOT, SECONDARY TOOTH TOP, AIR GAP, PRIMARY TOOTH TOP AND PRIMARY CURRENT REGION
SECONDARY END RING IS TAKEN INTO ACCOUNT
SEE SUBROUTINE RGMTRX. BOTH THE PRIMARY AS WELL AS THE PRIMARY OVERHANG REGIONS ARE MODELED BY FINITE THICKNESS REGIONS. THE UPPER AND THE LOWER REGIONS ADJACENT TO THESE REGIONS ARE EITHER OF FINITE OR INFINITE PERMEABILITIES. ZOE IS CALCULATED FROM THE E-H APPROACH, THEN CHANGED TO B-H APPROACH, SEE SUB. RGMTRX.

COMPLEX CIH(3),FTH, FNH, PSH, PINH
COMPLEX VLTG(3), ENT(3), Z(3,3), CI(3), FT, FN, PS, PINC
REAL D(4), R(3)

THE FOLLOWING ARE THE DRAWING PARAMETERS:
COMPLEX CIDR(3,25)
REAL FDR(25), FDNR(25), PSDR(25), PFPDR(25), EFSDR(25), EFPDR(25), * PFSDR(25), PFPDR(25), SPRDR(25)

COMMON NRG, DP, BO, BS, BOS, TAUS, TAUSS, SLTF, W2, WI, WH, WE
S, PRMO, PRMR, EM, P, Q, TAU, U, CNDPP, HZ, PI, SEG, SEGZ, OMG, VS
COMMON /MPIMFP/ SSPH, NSPH, DSPH

MOTOR DESIGN PARAMETERS:

space harmonic, DSPH : difference bet. two successive space
harmonics, NTH : no. of time harmonics, STH : starting time
harmonic, DTH : difference bet. two successive time harmonics.

READING THE MOTOR PARTICULARS:

READ (12,*) NRG
READ (12,*) (D(I), I=1,4), DP
READ (12,*) BO, BS, BOS, BSS, TAUS, TAUSS, SLTF
READ (12,*) W2, WI, WH, WE, PRMR, EM, P, Q, U
READ (12,*) CNDPP, (R(I), I=1,3), HZ

TAU = EM*Q*TAUS
SEG = 0.574E+08 * 0.6
SEGE = 2.0 * SEG
PI = 4.0*ATAN(1.0)
PRMO = 4.0*PI*1.0E-07
OMG = 2.0 * PI * HZ
VS = 2.0 * TAU * HZ

Specifications of space and time harmonics, time Harmonics are
only in Main Program. For balanced operations, DSPH = 2.0*EM.

SSPH = -11.0
NSPH = 13
DSPH = 2.0
STH = 1.0
NTH = 3
DTH = 2.0

No. of data points, NDTP:

NDTP = 20

THE WRITING STATEMENTS OF THE MOTOR DESIGN PARAMETERS

WRITE (6,19)
19 FORMAT (/10X,'**************************************************','/)
WRITE (6,11)
11 FORMAT (/17X,'MOTOR DESIGN PARAMETERS (ROTDRW3, Y-CONN) :'/)
WRITE (6,12)
12 FORMAT (/5X,'NRG, D(1), D(2), D(3), D(4), DP, BO, BS, BOS :'/)
WRITE (6,13) NRG, (D(I), I=1,4), DP, BO, BS, BOS
13 FORMAT (18,8F12.5)
WRITE (6,14)
14 FORMAT (/5X,'BSS, TAUS, TAUSS, SLTF, W2, WI, WH, PRMR, EM :'/)
WRITE (6,18) BSS, TAUS, TAUSS, SLTF, W2, WI, WH, PRMR, EM
WRITE (6,15)
15 FORMAT (/5X,'U, TAU, CNDPP, R(I) : I=1,3 , HZ, SEG, SEGE :'/)
WRITE (6,185) U, TAU, CNDPP, (R(I), I=1,3), HZ, SEG, SEGE
18 FORMAT (9F12.5)
FORMAT (7F11.4,2E14.4)
WRITE (6,186)
FORMAT (/5X,'SSPH, NSPH, DSPH, STH, NTH, DTH and (NDTP) : '/)
WRITE (6,187) SSPH,NSPH,DSPH,STH,NTH,DTH,NDTP
FORMAT (F12.4,I8,2F12.4,I8,F12.4,5X,'NDTP =',I6)
WRITE (6,19)

C
SPR = 0.0
DSPR = -0.1
DO 90 II = 1,NDTP
C
IF (ABS(SPR).GT.0.7) DSPR = -0.05
IF (ABS(SPR).GT.0.8) DSPR = -0.025
IF (ABS(SPR).EQ.1.0) SPR = 1.005 * SPR
V = SPR * VS
C
THE LINE VOLTAGES (Y-CONN.), GIVEN HERE IS -ive PHASE SEQUENCE,
C
VLTG(1) = (0.0,220.0)
VLTG(2) = (0.0,220.0) * (-0.500000, 0.866025)
VLTG(3) = (0.0,220.0) * (-0.500000,-0.866025)
C
Initialization the parameters :
DO 17 I = 1,3
CITH(I) = (0.0,0.0)
17 CONTINUE
FTH = (0.0,0.0)
FNH = (0.0,0.0)
PSH = (0.0,0.0)
PINH = (0.0,0.0)
C
EN = STH
DO 35 KK = 1,NTH
C
IF (KK.EQ.1) GO TO 25
IF (KK.EQ.2) HRAT = 0.09
IF (KK.EQ.3) HRAT = 1.0 / EN
IF (KK.GE.4) HRAT = (EN-DTH) / EN
C
DO 23 I = 1,3
VLTG(I) = HRAT * VLTG(I)
23 CONTINUE
25 CONTINUE
C
CALL IMPEDN(EN,R,D,V,Z)
C
Formation of the R.H. entries of the VLTG-CURR Matrix
DO 51 I = 1, 3
ENT(I) = VLTG(I)
IF (I.EQ.3) ENT(I) = (0.0,0.0)
CONTINUE

CALL CURRNT(Z,ENT,CI,JJ)
JJ is a control integer which stops the program, if the value
of the determinant is zero as coming from the subroutine CURRNT
IF (JJ.EQ.0) GO TO 200

CALL FCENP(EN,V,D,CI,FT,FN,PS)
PINC = (0.0,0.0)
DO 29 I = 1, 3
  PINC = PINC + VLTG(I) * CONJG(CI(I)) / SQRT(3.0)
CONTINUE
PINH = PINH + PINC

DO 27 I = 1, 3
  CIH(I) = CIH(I) + CI(I)
  CIDR(I,II) = CIH(I)
CONTINUE
FTH = FTH + FT
FNH = FNH + FN
PSH = PSH + PS

EN = EN + DTH
CONTINUE

EFF = \(-V \times \text{REAL}(FTH) / \text{REAL}(PSH)\)
PPFC = \text{REAL}(PSH) / \text{CABS}(PSH)

TEFF = \(-V \times \text{REAL}(FTH) / \text{REAL}(PINH)\)
TPFC = \text{REAL}(PINH) / \text{CABS}(PINH)

FDR(II) = \text{REAL}(FTH)
FNDR(II) = \text{REAL}(FNH)
PSDR(II) = \text{REAL}(PSH)
PPDR(II) = \text{REAL}(PINH)
EFSDR(II) = EFF
EFPPDR(II) = TEFF
PPSDR(II) = PPFC
PPPPDR(II) = TPFC
SPRDR(II) = \text{ABS}(SPR)

SPR = SPR + DSPR
CONTINUE
THE WRITING STATEMENTS OF THE RESULTS

```
WRITE (6,96)
96 FORMAT (///5X,'SPR-LINE CURRENTS :'/)  
DO 97 I = 1,NDTP  
WRITE (6,98) SPRDR(I),(CIDR(J,I),J=1,3)
97 CONTINUE  
98 FORMAT (1F10.4,' : ',6E14.6)

WRITE (6,101)
101 FORMAT(///5X,'SPR-TANG. FORCE CHARACTERISTICS'/)  
DO 102 I = 1,NDTP  
WRITE (6,150) SPRDR(I),FDR(I)
102 CONTINUE

WRITE (6,103)
103 FORMAT (///5X,'SPR-NORM. FORCE CHARACTERISTICS'/)  
DO 104 I = 1,NDTP  
WRITE(6,150) SPRDR(I),FNDR(I)
104 CONTINUE

WRITE (6,105)
105 FORMAT (///5X,'SPR-SEC. POWER CHARACTERISTICS'/)  
DO 106 I = 1,NDTP  
WRITE (6,150) SPRDR(I),PSDR(I)
106 CONTINUE

WRITE (6,107)
107 FORMAT (///5X,'SPR-PRIM. INP. POWER CHARACTERISTICS'/)  
DO 108 I = 1,NDTP  
WRITE (6,150) SPRDR(I),PPDR(I)
108 CONTINUE

WRITE (6,109)
109 FORMAT (///5X,'SPR-SEC. EFFICIENCY CHARACTERICS'/)  
DO 110 I = 1,NDTP  
WRITE (6,150) SPRDR(I),EFSDR(I)
110 CONTINUE

WRITE (6,111)
111 FORMAT (///5X,'SPR-PRIM. EFFICIENCY CHARACTERICS'/)  
DO 112 I = 1,NDTP  
WRITE (6,150) SPRDR(I),EFPDR(I)
112 CONTINUE

WRITE (6,113)
113 FORMAT (///5X,'SPR-SEC. P.F. CHARACTERISTICS'/)
```
DO 114 I = 1,NDTP
  WRITE (6,150) SPRDR(I),PFSDR(I)
CONTINUE

C
WRITE (6,115)
115 FORMAT (///5X,'SPR-PRM. P.P. CHARACTERISTICS',/) 
DO 116 I = 1,NDTP
  WRITE (6,150) SPRDR(I),PFPDR(I)
CONTINUE
C
150 FORMAT (2F15.6)
C
200 CONTINUE
C
STOP
END
C
Q *****************************************************************
C THE 3 MAIN SUBROUTINES : IMPEDN, CURRNT AND FRCENP
C *****************************************************************
C
SUBROUTINE IMPEDN(EN,R,D,V,Z)
C
C THIS SUBROUTINE DETERMINES THE 9 ENTRIES OP THE MATRIX [Z], GIVEN 
C BY THE EQN. I * Z = VLTG, AT A CERTAIN SPEED V.
C
REAL R(3),D(4)
COMPLEX Z(3,3),ZOS,ZOH,ZM(3,3),ZOE,FCI(3),FCH(3),CMF,
$  EKW,EKWM,C1,C2
C
COMMON NRG,DP,BO,BS,BOS,BSS,TAUS,TAUSS,SLTF,W2,WH,WE
$,PRM0,PRMR,EM,P,Q,TAU,U,CNDRP,HZ,PI,SEG,SEG,OMG,VS
COMMON /IMNFRC/  PRMX,PRMY,ZOS,ZOH
COMMON /MPIMFP/ SSPH,NSPH,DSPH
C
Primary region parameters and boundaries : 

PRM = PRM0 * PRMR
PRMX = PRM/(1.0-(PRMR-1.0)*BS/TAUS)
PRMY = PRM*(1.0 + (1.0/PRMR - 1.0) * BS/TAUS)
ZOS = (0.0,1.0E+20)
ZOH = (0.0,0.01) * PRMO
C
ZOE : input impedance at the lower prim. interf. (sec. rgn.)
ZOH : input impedance at the lower prim. interf. (overhang rgn.)
ZOS : input impedance at the upper prim. interf. (back rgn.)
DO 30 J = 1,3
DO 20 I = 1,3
ZM(I,J) = (0.0,0.0)
20 CONTINUE
30 CONTINUE

CTR = - P/(2.0*TAU) * CNDPP*CNDPP
RN = SSPH
N1 = NSPH
JJ = 1
DO 40 I = 1,N1

Primary iron region:
CALL RGMTRX(RN,EN,D,V,ZOE)
PENF = 1.0
CALL FIELDC(RN,EN,PRM0,PRM0,Z0H,Z0S,OENF,FCH)

Overhang region with iron back (Z0S is very large):
OENF = 1.0
CALL FIELDC(RN,EN,PRM0,PRM0,Z0H,Z0S,OENF,FCH)
CALL WDGFA(RN,EKW,EKWM)

CMF = CTR * EKW * CONJG(EKW) * (WI*FCI(3) + WH*FCH(3))

THE INDUCED VOLTAGE REVERSES SIGN AS (V .GT. VS)
IF (ABS(V).GT.ABS(VS)) CMF = - CMF

THE FOLLOWING STATEMENTS NEEDED FOR THE ACCURACY FOR SMALL RESIS
IRN = RN
IF (IRN.EQ.1) THEN
C1 = (-0.500000, -0.866025)
C2 = (-0.500000, 0.866025)
ELSE
C1 = CEXP ((0.0,-1.0)*2.0*PI*RN/3.0)
C2 = CEXP ((0.0,-1.0)*4.0*PI*RN/3.0)
END IF

Formation of the machine self and mutual impedances
ZM(1,1) = ZM(1,1) + CMF
ZM(1,2) = ZM(1,2) + CMF*C1
ZM(1,3) = ZM(1,3) + CMF*C2
ZM(2,1) = ZM(2,1) + CMF/C1
ZM(2,2) = ZM(1,1)
ZM(2,3) = ZM(2,3) + CMF*C1
**SUBROUTINE CURRENT**

**SUBROUTINE CURRENT:**  
This subroutine solves a 3x3 system of machine impedance equations. The input is the machine impedance array `Z(3,3)` and the applied voltage `VLTG(3)`, and the output is the machine phase/delta or line/wye currents.  
`JJ`: A control integer = 0, when the matrix rank < 3.

**Complex**: `Z(3,3), VLTG(3), CI(3), DET`  
**Integer**: `JJ`  

**Det** =  
$Z(1,1)*Z(2,2)*Z(3,3) + Z(1,2)*Z(2,1)*Z(3,3) - Z(3,1)*Z(2,2)*Z(1,3)$  
$+ Z(1,3)*Z(2,1)*Z(3,2) - Z(3,1)*Z(2,3)*Z(1,2)$  

Need to check the matrix rank if `DET = 0`, the rank < 3.
IF (CABS(DET).LE.7.0E-6) THEN
  JJ = 0
  WRITE (6,100)
  RETURN
END IF

C
CID) = VLTG(1) * ( Z(2,2)*Z(3,3) - Z(2,3)*Z(3,2) )
$ + VLTG(2) * ( Z(1,3)*Z(3,3) - Z(1,2)*Z(3,3) )
$ + VLTG(3) * ( Z(1,2)*Z(2,3) - Z(1,3)*Z(2,2) )
CI(1) = CI(1)/DET

C
CI(2) = VLTG(1) * ( Z(2,3)*Z(3,1) - Z(2,1)*Z(3,3) )
$ + VLTG(2) * ( Z(1,1)*Z(3,3) - Z(1,3)*Z(3,1) )
$ + VLTG(3) * ( Z(1,3)*Z(2,1) - Z(1,1)*Z(2,3) )
CI(2) = CI(2)/DET

C
CI(3) = VLTG(1) * ( Z(2,1)*Z(3,2) - Z(2,2)*Z(3,1) )
$ + VLTG(2) * ( Z(1,2)*Z(3,1) - Z(1,1)*Z(3,2) )
$ + VLTG(3) * ( Z(1,1)*Z(2,2) - Z(1,2)*Z(2,1) )
CI(3) = CI(3)/DET

RETURN

C
100 FORMAT (' SOLUTION NOT FEASIBLE. MATRIX RANK < 3',
$ ' THE DETERMINANT OF THE IMPED. MAT. = ZERO'//)
END

C
*******************************************************************
C
SUBROUTINE FRCENP(EN,V,D,CI,FT,FN,PW)
C
C THIS SUBROUTINE COMPUTES THE TANGENTIAL FORCE FT (&FX), THE
C NORMAL FORCE, FN, AND THE SECONDARY INPUT POWER PW AT A CERTAIN
C SPEED V. THE INPUT ARE THE SPEED AND THE PHASE CURRENTS I(1),
C I(2) AND I(3), AS WELL AS THE REGION THICKNESSES D(1), ..., D(4).
C
COMPLEX CI(3), FT, FN, PW, Z0S, Z0H, EKW, EKWM, C1, C2, CJ, CNJ
$ , Z0E, FCI(3), FCH(3)
REAL D(4)

C
COMMON NRG, DP, BO, BS, BOS, BSS, TAUS, TAUS, SLTF, W2, WI, WH, WE
$ , PRM0, PRHR, EM, P, Q, TAU, U, CNDPP, HZ, PI, SEG, SEGE, OMG, VS
COMMON /IMFREC/ PRMX, PRMY, Z0S, Z0H
COMMON /MPIMFP/ SSPH, NSPH, DSPH

C
OMGN = EN * OMG

C
FT = (0.0,0.0)
FN = (0.0,0.0)
PW = (0.0,0.0)
CTC = 1.0/(2.0*TAU)
CTF = - 2.0 * TAU * P

RN = SSPH
N1 = NSPH
DO 40 I = 1,N1

Current calculations:
CALL WDGFAC(RN,EKW,EKWM)

THE FOLLOWING STATEMENTS NEEDED FOR THE ACCURACY FOR SMALL RESIS.
IRN = RN
IF (IRN.EQ.1) THEN
   CI = (-0.500000, -0.866025)
   C2 = (-0.500000, 0.866025)
ELSE
   CI = CEXP ((0.0,-1.0)*2.0*PI*RN/3.0)
   C2 = CEXP ((0.0,-1.0)*4.0*PI*RN/3.0)
END IF

CJ = CTC * CNDPP*EKW * (CI+1)+CI(2)*CI+CI(3)*C2
CNJ = CONJG(CJ)

The field coefficients of the iron region
CALL RGMTRX(RN,EN,D,V,ZOE)
PENF =1.0
CALL FIELD(C(RN,EN,PRMX,PRMY,ZOE,ZOS,PENF,FCI)

The field coefficients of the overhang region
OENF = 1.0
CALL FIELD(RN,EN,PRMO,PRMO,ZOH,ZOS,OENF,FCH)

The Force and Power Calculations:
FT = FT + CTF * (FCI(1)*WI + FCH(1)*WH) * CJ * CNJ
FN = FN + CTF * PRMO *(FCI(2)*WI + FCH(2)*WH) * CJ * CNJ
PW = PW + CTF * (FCI(3)*WI + FCH(3)*WH) * CJ * CNJ

RN = RN + DSPH
40 CONTINUE

RETURN
END
**THE AUXILIARY SUBROUTINES : RGMTRX, FUNCTS, FIELDCC AND WDGFAC**

**SUBROUTINE RGMTRX(RN, EN, D, V, ZOE)**

This subroutine computes the four entries \( \text{ENT}(1), \text{ENT}(2), \text{ENT}(3), \text{ENT}(4) \) as well as the equivalent region impedance \( ZOE \) of the \( N \) different regions at a certain speed \( V \), wave number (space harmonic order) \( RN \) and time harmonic order \( EN \). The subroutine uses the entries of \( E - H \) solution approach.

```fortran
REAL D(4), PRMX(4), PRMY(4), SEGZ(4)
COMPLEX ZOE, GM(4), Z00, Z, Y, Z0(4), ZA, CCH, CSH, A(4), *
* B(4), BNT(4), DT(4,4)
```

```fortran
DATA Z0O/(0.0,1.0E+20)/
COMMON NRG, DP, BO, BS, BSS, TAUS, TAUSS, SLTF, W2, WI, WH, WE
$, PRMO, PRMR, EM, P, Q, TAU, U, CNDPP, HZ, PI, SEG, SEGE, OMG, VS
```

```fortran
PRM = PRMO * PRMR
PRMX(1) = PRM/(1.0+(PRMR-1.0)*BSS/TAUSS)
PRMY(1) = PRM*(1.0 + (1.0/PRMR - 1.0) * BSS/TAUSS)
PRMX(2) = PRM/(1.0+(PRMR-1.0)*BOS/TAUSS)
PRMY(2) = PRM*(1.0 + (1.0/PRMR - 1.0) * BOS/TAUSS)
PRMX(3) = PRM0
PRMY(3) = PRM0
PRMX(4) = PRM/(1.0+(PRMR-1.0)*BO/TAUS)
PRMY(4) = PRM*(1.0 + (1.0/PRMR - 1.0) * BO/TAUS)
```

```fortran
SEGZ(1) = SEG*(BSS/TAUSS)*SLTF
SEGZ(2) = 0.0
SEGZ(3) = 0.0
SEGZ(4) = 0.0
```

**MODEL : III**

- \( \text{AREA} = 0.05 * 0.02 \)
- \( \text{SEGR} = \text{SEG} / \text{SEGE} \)
- \( \text{ENRF} = 1.0 + 2.0*\text{D}(1)*\text{SEGR}/(\text{WI}*_\text{AREA}*(\text{RN}*_\text{PI}/\text{TAU})*(\text{RN}*_\text{PI}/\text{TAU})) \)
- \( \text{SEGZ}(1) = \text{SEGZ}(1) / \text{ENRF} \)

Calculations of the propagation factors and the characteristic impedances of the \( N \) regions.

- \( \text{OMGN} = \text{EN} * \text{OMG} \)
- \( \text{SLIP} = 1.0 + (\text{RN}*_\text{PI}/\text{TAU})*\text{V}/\text{OMGN} \)
- \( \text{RNT} = \text{RN}*_\text{PI}/\text{TAU} \)
DO 15 I = 1,NRG
C
IF (I.LT.3) OMGR = OMGN * SLIP
IF (I.GE.3) OMGR = OMGN
C
Where 3 is the air gap region (regions 3 or more are stationary)
C
Z = (0.0,1.0) * OMGR * PRMX(I)
Y = SEGZ(I) + RNT * RNT / ((0.0,1.0) * OMGR * PRMY(I))
Z0(I) = CSQRT(Z/Y)
GM(I) = CSQRT(Z*Y)
15 CONTINUE
C
Formation of the entries of the four matrices
DO 17 I = 1,NRG
ZA = GM(I) * D(I)
CALL FUNCTS(ZA,CCH,CSH)
DT(I,1) = CCH
IF (I.EQ.3) DT(I,1) = DT(I,1) / SLIP
DT(I,2) = - Z0(I) * CSH
DT(I,3) = - 1.0/Z0(I) * CSH
IF (I.EQ.3) DT(I,3) = DT(I,3) / SLIP
DT(I,4) = CCH
17 CONTINUE
C
Calculation of the matrix product
DO 20 J =1,4
A(J) = DT(1,J)
20 CONTINUE
C
DO 50 I = 2,NRG
C
ENT(1) = DT(I,1)*A(1) + DT(I,2)*A(3)
ENT(2) = DT(I,1)*A(2) + DT(I,2)*A(4)
ENT(3) = DT(I,3)*A(1) + DT(I,4)*A(3)
ENT(4) = DT(I,3)*A(2) + DT(I,4)*A(4)
C
DO 40 J =1,4
A(J) = ENT(J)
40 CONTINUE
C
50 CONTINUE
C
BETA = OMGN / RNT
ZOE = ( SLIP*ENT(1) - ENT(2) / (BETA*Z00) ) / ( SLIP*BETA*ENT(3) - ENT(4) / Z00 )
C
RETURN
END
SUBROUTINE FUNCTS(Z, CCH, CSH)
C
THIS SUBROUTINES DETERMINES THE EXPRESSIONS OF COMPLES COSH AND
SINH:
COMPLEX Z, CCH, CSH, EX
EX = CEXP(Z)
CCH = 0.5 * (EX + 1.0/EX)
CSH = 0.5 * (EX - 1.0/EX)
RETURN
END

SUBROUTINE FIELDC(RN, EN, RMX, RMY, ZOE, ZOS, PENF, FC)
C
This subroutine is used to determine the FIELD COEFFICIENTS,
F(1), FC(2) and FC(3) in the primary region, which are given
by the expressions
Bav = FC(1)*Jlin, Hav = FC(2)*Jlin and Eav = FC(3)*Jlin
C
COMMON NRG, DP, BO, BS, BOS, BSS, TAUS, TAUSS, SLTF, W2, WI, WH, WE
$, PRM0, PRMR, EM, P, Q, TAU, U, CNDPP, HZ, PI, SEG, SEGE, OMG, VS
C
COMPLEX ZOE, ZOS, FC3, ZP, ZOP, DLT, CNU, G1, G2
C
OMGN = EN * OMG
GMP = SQRT(RMX/RMY) * RN*PI/TAU
GMD = GMP * DP
ZP = (0.0,1.0) * RMX * RN*PI/TAU
ZOP = (0.0,1.0) * SQRT(RMX*RMY)
DLT = (COSH(GMD)/ZOP + SINH(GMD)/ZOS) /ZOE -
       (SINH(GMD)/ZOP + COSH(GMD)/ZOS) /ZOP
C
SHR = SINH(GMD)/GMD
CHR = (COSH(GMD) - 1.0) /GMD
CNU = SHR + (ZOP/ZOS) * CHR
G1 = CNU / (ZOP*DLT)
G2 = -CNU / (ZOE*DLT)
C
FC(1) = G1*SHR + G2*CHR + ZOP* ( (SHR-1.0)/GMD )
FC(2) = - (G1*CHR + ZOP*(CHR/GMD) + G2*SHR) /ZOP
C
PENF, primary end factor, is used when the end effect of the
overhang winding is taken into consideration,
E = OMG B / (PENF * RN * PI / TAU).
FC(3) = OMGN / (PENF*RN*PI/TAU) * FC(1)

RETURN
END

******************************************************************************

SUBROUTINE WDGPAC(RN, EKW, EKWM)

THIS SUBROUTINE, WINDING FACTOR, COMPUTES THE FOURIER TRANSFORM
THE FOURIER COEFFICIENT OF A DOUBLE LAYER SINGLE-PHASE WINDING,
AS WELL AS THE M-PHASE DOUBLE LAYER WINDING IN CASE OF BALANCED
OPERATIONS.

KB : TOOTH BREADTH FACTOR (REAL).
KC : COIL-PITCH FACTOR.
KD : PHASE-DISTRIBUTION FACTOR.
KO : ODD-HARMONIC FACTOR.
KP : POLE-NUMBER FACTOR.
KM : PHASE-NUMBER FACTOR, FOR BALANCED M-PHASE OPERATIONS.
EKW : THE EQUIVALENT SINGLE-PHASE WINDING FACTOR.
EKWM : THE EQUIVALENT M-PHASE WINDING FACTOR (BALANCED OPERATION).

REAL KB
COMPLEX EKW, EKWM, KC, KD, KO, KP, KM

COMMON NRG, DP, BO, BS, BOS, BSS, TAUS, TAUSS, SLTF, W2, WI, WH, WE,
$PRMO, PRMR, EM, P, Q, TAU, U, CNDPP, HZ, PI, SEG, SEGE, OMG, VS

TAUC = TAU - U*TAUS
ALF = PI*TAUS/TAU
IF (RN.EQ.1.0) RN = RN + 0.001

KB = SIN(RN*PI*BS/(2.0*TAU))/(RN*PI*BS/(2.0*TAU))
KC = (1.0 - CEXP((0.0,-1.0)*RN*PI*TAUC/TAU)) / 2.0
KD = (1.0 - CEXP((0.0,-1.0)*Q*RN*ALF))/
     / (Q*(1.0 - CEXP((0.0,-1.0)*RN*ALF)))
KO = 0.5*(1.0 - CEXP((0.0,-1.0)*RN*PI))

KP = (1.0,0.0)
KM = (1.0 - CEXP((0.0,-1.0)*PI*(RN-1.0)))/
     / (EM*(1.0 - CEXP((0.0,-1.0)*PI*(RN-1.0)/EM)))

EKWM = EKW*KM

RETURN
END

******************************************************************************