Comparative studies of kriging, multiquadric-biharmonic and other methods for solving mineral resources problems

Supachai Sirayanone
Iowa State University

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Comparative studies of kriging, multiquadric-biharmonic and other methods for solving mineral resources problems

Sirayanone, Supachai, Ph.D.

Iowa State University, 1988
Comparative studies of kriging, multiquadric-biharmonic and other methods for solving mineral resources problems

by

Supachai Sirayanone

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CHAPTER 1. INTRODUCTION AND REVIEW OF PREVIOUS WORK

Introduction

Both multiquadric-biharmonic and kriging methods are interpolation schemes that fit the data points of the observed values exactly. If one looks at the "surface" of both methods, they look alike; but only when one begins to look "inside" will the differences be revealed--theoretically and practically. One of the purposes of this dissertation is to determine the differences and the similarities of both interpolants. Selected literature will be discussed in Chapter 1. The theories of both methods will be discussed in Chapter 2. Some selected case studies from the author's work during his residency at Iowa State University will be discussed in the following chapters. Conjectures, ideas, and new theories are openly discussed and presented. Programs are documented in the appendices.

The multiquadric method was discovered and named by Dr. Rolland L. Hardy, Professor of Civil and Construction Engineering at Iowa State University in the 1960s (Hardy, 1971b). Hardy originally used this method to represent topography in the field of surveying but recognized its usefulness for other purposes. Due to the efficiency of multiquadric equations, its ease of application, and its versatility, many researchers in various fields have reported on its applications. Hardy and others have begun to expand the theories that can explain and support the favorable properties of the multiquadric method that have been discovered
in application (Franke, 1979; Micchelli, 1986; Kansa, 1987). Franke (1979) suggested that there was a need for a more elegant explanation of the excellent results from the multiquadric method. Now it is known that multiquadric equations satisfy the biharmonic differential equation (Hardy, 1982a, b) and they are an optimal solution in a certain Hilbert space via construction of a reproducing kernel (Madych and Nelson, 1988).

The kriging method

The kriging method was developed by Matheron in the 1960s in the so-called "French School" (David, 1977). This method was originally used for mining. The name Krige is credited to D.G. Krige, a South African statistician who provided some of his mine data collection for testing this method. The kriging method has been used extensively in European mining, but not so much in the United States. Although geostatistics is the discipline that kriging belongs to, this type of statistics is completely different from the classical statistics in American universities (David, 1977). The theory of regionalized variables is the basis for the kriging theory, with the assumption of second order stationarity or at least the quasi-stationarity.

The multiquadric-biharmonic method is physically deterministic, while kriging involves a stochastic process. The kriging method includes preprocessing procedures for computing discrete semivariograms and models leading to continuity, while the multiquadric-biharmonic uses
a predetermined kernel function with a rigorous physical meaning that is continuous at the outset.

**General Purpose of Research**

The first objective of this research was to study thoroughly and independently the approximation and prediction techniques known as "kriging". This was followed by experimenting with variations of these techniques to improve their use for mining and mineral resources related problems. Although kriging was well-known for mining applications in European universities and the international literature, there was little or nothing known about applications of the multiquadric-biharmonic method to mineral resource problems. I had studied the multiquadric-biharmonic method in a graduate level course and knew that both methods can be applied to the same mineral resources-related problem. The multiquadric-biharmonic method was known to be more efficient and more economical than similar methods ("least squares collocation" and "optimal interpolation") in other applications. There was a need to discover new methods to improve the techniques applied to mining. Hence, the second objective of this research was to make comparative studies of kriging and a wide variety of the most promising of the approximation and prediction techniques, especially the multiquadric method. There was a need to document the relative quality of the tested methods.
Problems to be Solved

Most problems solved by approximation and prediction techniques in mining and mineral resources involve a sophisticated form of surface and/or subsurface mapping from a minimum of sample points (for reasons of economy). More than geometric considerations are involved. Depending on the sophistication of the data collection at sample points, several ordinate values may exist at the same point. For example, the depth of the sample, the percentage of ore, and the percentage of undesirable impurities, or other parameters can provide an almost endless variety of mapping fluctuations in the same volume. The basic problem to be solved is to find the most efficient approximation and prediction technique for converting discrete data samples into useable continuous mapping (with finite discontinuities) for the desired parameters in each mining and mineral resources problem.

Both kriging and the multiquadric-biharmonic methods will be used with the same sets of data, thus providing a variety of solutions to typical problems in the evaluation of mining and mineral resources. A lot of computer programming and considerable computer time were used in this research, involving a large number of typical problems. Relative computer costs of both methods are compared as well as their accuracy in results for each case. It is already known that the most accurate prediction methods can save data collection costs by reducing the number of required sample points to achieve the same accuracy as the less
sophisticated methods. Hence, this study will provide documentary evidence of both accuracy and economy of both methods.

Literature Review

Multiquadric-biharmonic

The multiquadric method was discovered by Dr. Rolland L. Hardy in 1968, which he called multiquadric equations in 1971 (Hardy, 1971a, b). The original application was to represent topographic surfaces. Later, many other applications were reported (Hardy, 1975). This method has also been used by many investigators, i.e., Schut, 1974; Lee, Lynn, and Shaw, 1974; Krohn, 1976; Temfli and Makarovic, 1979; Franke, 1979; Groten, 1981; Franke, 1982; Holdahl, 1983; Baram, 1984; Schiro and Williams, 1984; Rashad, 1985; Franke, 1986; Micchelli, 1986; Vittal, Rajagopal, McCormack, Movali and Fouad, 1986; Sandwell, 1987; and Kansa, 1987. The biharmonic nature of the most used kernel in the multiquadric method was reported by Hardy (1982a, b), thus justifying the title of this section. Recently, the multiquadric method (without the biharmonic explanation) appeared in at least one textbook on curve and surface fitting (Lancaster and Salkauskas, 1986).

Hardy (1976) listed the following applications of multiquadric equations:

1. least squares prediction of topography and bathymetry (from discrete height or depth information);
2. terrain correction and isostatic gravity anomaly reduction;
3. least squares prediction problems involving continuous interpolations of data within each of the following categories;
   a. gravity anomalies (from discrete gravity survey data),
   b. geoidal undulations (from discrete satellite altimetry points or profiles),
   c. deflections of the vertical (from discrete astrogeodetic data), and
   d. vertical gradient anomalies (from discrete gravity measurements on towers);

4. least squares prediction of deflections of the vertical, using any or all data listed in Item 3 above (hybrid data);

5. least squares prediction of vertical gradient anomalies, using any or all data listed in Item 3 above (hybrid data); and

6. least squares prediction of external properties of the earth's potential field (gravity anomalies, equipotential undulation), using any or all data listed in Item 3 above.

Hardy (1976) also suggested that multiquadric equations can be used in certain aspects of dynamic satellite geodesy. It is also applicable to problems in geology, geophysics, surveying, photogrammetry, cartography, remote sensing, and other fields of science, technology, and engineering because iso-ordinate contours are needed for many phenomena outside of geodesy.

The following will be the summarization of the development of theories and applications of the multiquadric-biharmonic method.
Brooks (1970) tested the conic multiquadric equation of topography for eight geomorphologically diversified areas. Data points were selected for two modes of operation, significant points on a surface and a grid pattern. Four hyperboloid multiquadric equations were also tested to show their justification as one of the many possibilities for computing multiquadric surfaces. Deriving from the general form of multiquadric surfaces

\[ \sum_{j=1}^{n} c_j [q(x_j, y_j, x, y)] = z \]  

Brooks represented the summation of a series of circular hyperboloids in two sheets by:

\[ \sum_{j=1}^{n} c_j [(x_j - x)^2 + (y_j - y)^2 + \delta_j^2] = z \]  

and the summation of a series of hyperboloid sections or profiles in xz plane by:

\[ \sum_{j=1}^{n} c_j [(x_j - x)^2 + \delta_j^2] = z \]

Brooks stated that we do not have to make any assumptions about the geometry of the surface, so the topographic surface will be represented by fitting the data points exactly. Brooks used the United States National Map Accuracy Standards as the criterion for evaluating the accuracy of his work in both the horizontal accuracy and the vertical accuracy. From one test area he found that the accuracy of a multiquadric surface can be increased by increasing the number of data points.
The lack of data points in the critical points such as the ridge and the drainage area cause poor defining of those areas. In general, multi-quadric equations represent the topography of those test areas very accurately. By using the conic multi-quadric surfaces, Brooks concluded that the multi-quadric surface in this form can efficiently represent a surface of topography. The form of this equation is a series of right circular cones which are reflected in the general circular shape of the hilltops, particularly at the higher elevations. The type of topography, rather than the contour interval density, determines the density and distribution of data points for multi-quadric analysis. The more complicated the topography, the greater the data point density and distribution needed to represent a topographic surface by multi-quadric analysis. The significant point mode has distinct advantages over the grid point mode, especially in the area of the highest point of elevation of a hilltop, the exact shape of a stream bed or drainage pattern, depressions, saddles, etc. The grid mode of operation has the distinct advantage of not requiring any special skills in the selection of data points. The choice of the mode of operation depends largely on the intended use for the multi-quadric surface and the qualifications of the personnel's knowledge of topography. By using the hyperboloid multi-quadric surfaces, Brooks concluded that the hyperboloids with $\delta = 0.001$ and $\delta = 0.01$ gave very good results. The magnitude of the coefficient influences the flatness or sharpness of the slope change of the surface that occurs at the vertex of each cone or the nose of
each hyperboloid. The smoothness and shape of the transition between data points are controlled principally by the characteristics of the multiquadric surface used in the summation. The hyperboloid surfaces show a trend of representing low elevations and smooth-slope hill areas very well. However, the tops of hills are better represented by the cone and the sharp-nosed hyperboloid.

Hardy (1972a) and Cain (1971) studied a variety of multiquadric series, which included connected straight line segments, hyperbolas, reciprocal hyperbolas, trigonometric curves, circular cones, and hyperboloids. Cain also extended the principle of multiquadric equations to develop a harmonic multiquadric series in three dimensions. Linear systems of multiquadric equations were analyzed to determine their adaptability to short matrix inversion schemes. More explicitly, Cain's list of possible equations included the following (1971):

\[
(\Delta r) = R + \sum_{j=1}^{n} c_j \left( \sin^2 [\theta_j - \theta] + \sin^2 [(\lambda_j - \lambda) / 2] + \delta \right)^{\frac{1}{2}}
\]  

where \( \theta \) and \( \lambda \) are the spheroidal coordinates on a spheroidal reference surface. Conceptually, \( (\Delta r) \) is the observed variation of an irregular surface with respect to a sphere or spheroid as a reference surface.

\[
Z = \sum_{j=1}^{n} c_j |X - X_j|
\]

for the multiquadric curve formed by the summation of a series of connected line segments.
\[ Z = \sum_{j=1}^{n} c_j |X^2 - X_j^2| \quad (6) \]

for the multiquadric curve formed by the substitution of \( X = X^2 \) and \( X_j = X_j^2 \) into the multiquadric series of connected line segments.

\[ Z = \sum_{j=1}^{n} c_j (|X - X_j|)^{3/2} \quad (7) \]

for the multiquadric curve formed by the summation of a kernels, interpretable as distance to the 1.5 power.

\[ Z = \sum_{j=1}^{n} c_j [(X - X_j)^2 + \delta]^{-\frac{1}{2}} \quad (8) \]

for the multiquadric curve formed by the summation of a series of reciprocal hyperbolas.

\[ Z = \sum_{j=1}^{n} c_j |\sin(\theta - \theta_j)| \quad (9) \]

for the multiquadric equation representing the summation of a series of single sine functions.

\[ Z = \sum_{j=1}^{n} c_j \{\sin^2 (\theta - \theta_j) + \delta\}^{\frac{1}{2}} \quad (10) \]

for the multiquadric equation of sines containing the arbitrary constant which results in a multiquadric curve having no slope discontinuities.

\[ Z = \sum_{j=1}^{n} c_j |1 - \cos(\theta - \theta_j)| \quad (11) \]
for the multiquadric equation of a series formed by the summation of cosines.

\[
Z = \sum_{j=1}^{n} c_j \left[ \sin^2 (\theta - \theta_j) \right]
\]  

(12)

for the multiquadric equation of sine squared.

For the multiquadric equation for a series of circular hyperboloids we have

\[
Z = \sum_{j=1}^{n} c_j \left[ (X - X_j)^2 + (Y - Y_j)^2 + \delta \right]^{\frac{1}{2}}
\]  

(13)

By setting \( \delta = 0 \), in Eq. (13) we have

\[
Z = \sum_{j=1}^{n} c_j \left[ (X - X_j)^2 + (Y - Y_j)^2 \right]^{\frac{1}{2}}
\]  

(14)

which is the multiquadric equation of circular cones.

A more complicated version of Eq. (4) was also presented.

\[
(\Delta r) = \sum_{j=1}^{n} c_j \left[ (\sin \theta \cos \lambda - \sin \theta_j \cos \lambda_j)^2 + (\sin \theta \sin \lambda \sin \theta_j \sin \lambda_j)^2 + (\cos \theta - \cos \theta_j)^2 + \delta \right]^{\frac{1}{2}}
\]  

(15)

In addition, Cain developed a multiquadric harmonic series he proposed for use in physical geodesy. It was the equivalent (for \( \delta = 0 \)) to a point mass anomaly model. His equation was

\[
T = \sum_{j=1}^{n} c_j \left[ (|X - X_j| + \delta)^2 + (|Y - Y_j| + \delta)^2 + (|Z - Z_j| + \delta)^2 \right]^{-\frac{1}{2}}
\]  

(16)
where $T$ is the disturbing potential on a closed reference surface, e.g. a sphere $R$ where $R^2 = X^2 + Y^2 + Z^2$. The point mass anomalies located at $X_j, Y_j, Z_j$ are equivalently located on a sphere $r$ such that $r^2 = X_j^2 + Y_j^2 + Z_j^2$. Cain's equivalent expression in spherical coordinates was

$$T = \sum_{j=1}^{n} c_j \left[ (|r \sin \theta \cos \lambda_j - r_j \sin \theta_j \cos \lambda_j| + \delta)^2 \right]^{-\frac{1}{2}}$$

Cain also referred to the work of Hardy and Secker (1970). The direct solution for the coefficients, $c_1, c_2, c_3, \ldots, c_n$ for the multiquadric equation of a series of connected line segments is

$$c_j = \frac{1}{|X_j - X_{j-1}|} \left[ \frac{Z_{j+1} - Z_j}{|X_{j+1} - X_j|} - \frac{Z_j - Z_{j-1}}{|X_j - X_{j-1}|} \right]$$

where $j = 2, 3, 4, \ldots, n-1$.

In the case for $j = 1$,

$$c_1 = \left\{ \frac{1}{|X_1 - X_n|} \right\} \left\{ Z_n - [c_2|X_2 - X_n| + c_3|X_3 - X_n| + \ldots + c_{n-1}|X_{n-1} - X_n|] \right\}$$

For $j = n$,

$$c_n = \left\{ \frac{1}{|X_n - X_1|} \right\} \left\{ Z_1 - [c_2|X_2 - X_1| + c_3|X_3 - X_1| + \ldots + c_{n-1}|X_{n-1} - X_1|] \right\}$$
Cain has shown generally that multiquadric equations can be developed and evaluated using quadric sections (except the parabola), the quadric surfaces (except the paraboloids), and many quadric-like surfaces involving trigonometric functions. Moreover, transformations can be made which express ordinates with respect to closed spherical or spheroidal surfaces. Thus multiquadric equations can be written that are truly global in extent. Such equations may provide an efficient means of representing global irregular surfaces such as the world's topography (Hardy, 1972c, 1975).

Hardy (1971b) presented the osculatory modes of the multiquadric equation. This mode is formed from quantitative observations of minimum slope. The partial derivative of z with respect to x and y in Eq. (13) are determined and set equal to zero. This gives

\[ \sum_{j=1}^{n} c_j \left[ (x_j - x)^2 + (y_j - y)^2 \right]^{-1/2} (x_j - x) = 0 \]  
(21)

and

\[ \sum_{j=1}^{n} c_j \left[ (x_j - x)^2 + (y_j - y)^2 \right]^{-1/2} (y_j - y) = 0 \]  
(22)

By using Eqs. (21) and (22), a set of slope observation equations can be formed from some of the data points used to form the system of equations represented by Eq. (14). The appropriate coordinates are the horizontal positions of the points that are hilltops or depressions inside the boundary, and highs and lows on the x, y boundaries of the area being mapped. The additional equations thus formed can
be combined with the system in Eq. (14) to give an overdetermined system for adjustment by least squares.

The major purpose of the zero-slope observation equations is to minimize the possible horizontal and vertical displacement of maximum and minimum points in the multiquadric equation of topography when $\delta$ in the equations is large enough to cause this difficulty. One possible osculating mode involves combining multiquadric summation with an ordinary polynomial series. This combination not only can collocate the surface coordinates with the data point coordinates, but can also cause surface tangents to coincide at specified points. This is collocation or contact of a higher order. For example, when adding an expression for a polynomial series to Eq. (8), one obtains

$$\sum_{j=1}^{n} c_j \left[ (x_j - x)^2 + \delta^2 \right]^{1/2} + \sum_{i=1}^{m} k_i x^i = z$$  \hspace{1cm} (23)

Differentiate Eq. (23) and set it equal to zero. Hence,

$$- \sum_{j=1}^{n} c_j \left[ (x_j - x)^2 + \delta^2 \right]^{1/2} (x_j - x) + \sum_{i=1}^{m} i k_i x^{i-1} = 0$$  \hspace{1cm} (24)

The equation formed from Eqs. (23) and (24) is then combined into a system of $(n + m)$ equations with $(n + m)$ unknowns. After the equations are solved for the unique value of $c_j$ and $k_i$, these coefficients are substituted into Eq. (23) to form an osculating hyperbolic-profile mode. The principle can be extended to other quadrics and to three-dimensional cases. The resulting surface profile is a multiquadric solution in which the effects of a normally low degree polynomial are
superimposed to reduce the slopes to zero at the desired maximums and minimums. In this paper, Hardy used fictitious topography containing high and low-point drainage junctions, etc., to investigate the feasibility of multiquadric analysis. He also applied the multiquadric equations to a topographic model from Krumbein (1966), which was hand contoured from a part of a U.S. Geological Survey quadrangle map of McClure, Pennsylvania. He concluded that the most obvious deficiency in the multiquadric surface resulted mainly from poor elevation choices for representing the hills and the saddles between them. The quadrics, other than the cone, tend to displace the maximums and minimums unless an osculating mode is used. The multiquadric series which is neither a power series nor a harmonic series, can closely approximate topography with relatively few data points. In this experiment, multiquadric series appear to have a higher level of efficiency than the classical series approximation.

By using points of known gravity anomalies and horizontal gradients, Woodbury (1971) applied the osculating mode of the multiquadric equations to provide a more accurate prediction of the gravity anomaly field. The multiquadric series was combined with a polynomial series, thus taking the form of Eqs. (23) and (24). He noted that if $x_j = x$, the arbitrary constant $c$ cannot be taken as zero or be negative, or the solution becomes undefined (Eq. (24)). Also, in the polynomial term of Eq. (24), when $i = 1$, $x$ cannot equal zero or the result will again be undefined. Therefore, if Eq. (24) is to be used, a positive, non-zero
\( \delta \) must be chosen and, if \( x = 0 \) is a slope data point, then all \( x \) values must be translated a constant amount to eliminate the \( x = 0 \) term. The choice of \( \delta \) will affect the shape of the profile and its choice is not truly arbitrary, but it should be determined empirically for the area under investigation. The osculating mode of the multiquadric was also applied to estimate the three-dimensional anomaly field.

The extended form of Eqs. (23) and (24) into the third dimension is

\[
Z = \sum_{j=1}^{n} C_j \left[ (x - x_j)^2 + (y - y_j)^2 + \delta \right]^{\frac{1}{2}} + \sum_{i=1}^{m} A_i x^i + \sum_{k=1}^{k} B_k y^k \ldots
\]  

(25)

The gradient or slope conditions can again be added by differentiating:

\[
\frac{\partial Z}{\partial x} = \sum_{j=1}^{n} C_j \frac{(x - x_j)}{\left[ (x - x_j)^2 + (y - y_j)^2 + \delta \right]^{\frac{3}{2}}} + \sum_{i=1}^{m} i A_i x^{i-1}
\]  

(26)

and likewise

\[
\frac{\partial Z}{\partial y} = \sum_{j=1}^{n} C_j \frac{(y - y_j)}{\left[ (x - x_j)^2 + (y - y_j)^2 + \delta \right]^{\frac{3}{2}}} + \sum_{k=1}^{k} k B_k y^{k-1}
\]  

(27)

The appropriate equation, (26) or (27), would be used if the gradient for one direction is used. When the gradient in both directions is known, both Eq. (26) and Eq. (27) could be used. This results in \( n + m + k \) equations with \( n + m + k \) unknowns. Woodbury concluded that the use of gradient data at points of known anomaly provided a better estimate than the estimates made without gradients. On the other hand,
it was found that using gradients at prediction points of unknown anomaly generally worsened the prediction. This method tended to emphasize the effect of the gradients, especially when higher order polynomial terms were included. This resulted in extrapolating the gradient effects far beyond the point where they represented the surface trend.

Hardy (1975) developed another version of the osculating mode for multiquadric equations by substituting a linear combination of derivatives of the original multiquadric equations for the polynomial series in Eq. (25) and also extended the principle to surfaces as well as profiles.

For this purpose the following notation was used

\[ Q_j = [(X - X_j)^2 + (Y - Y_j)^2 + \delta]^\frac{1}{2} \]

\[ Q_{ij} = [(X_i - X_j)^2 + (Y_i - Y_j)^2 + \delta]^\frac{1}{2} \]

\[ \frac{\partial Q_j}{\partial X} = \frac{(X - X_j)}{[(X - X_j)^2 + (Y - Y_j)^2 + \delta]^\frac{1}{2}} \]

\[ \frac{\partial Q_{ij}}{\partial X} = \frac{(X_i - X_j)}{[(X_i - X_j)^2 + (Y_i - Y_j)^2 + \delta]^\frac{1}{2}} \]

Then the complete multiquadric system of equations to provide osculation through the first partial derivatives of \( Q_j \) in \( X \) and \( Y \) is as follows

\[ \sum_{j=1}^{n} A_j Q_{ij} + \sum_{j=1}^{n} B_j \frac{\partial Q_{ij}}{\partial X} + \sum_{j=1}^{n} C_j \frac{\partial Q_{ij}}{\partial Y} = T_i \quad i = 1, 2, \ldots, n \]
\begin{align}
\sum_{j=1}^{n} A_j \frac{\partial Q_{ij}}{\partial X} + \sum_{j=1}^{n} B_j \frac{\partial^2 Q_{ij}}{\partial X^2} + \sum_{j=1}^{n} C_j \frac{\partial^2 Q_{ij}}{\partial X \partial Y} &= \left( \frac{\partial T}{\partial X} \right)_i \\
\sum_{j=1}^{n} A_j \frac{\partial Q_{ij}}{\partial Y} + \sum_{j=1}^{n} B_j \frac{\partial^2 Q_{ij}}{\partial Y \partial X} + \sum_{j=1}^{n} C_j \frac{\partial^2 Q_{ij}}{\partial Y^2} &= \left( \frac{\partial T}{\partial Y} \right)_i 
\end{align}

Note the measured data is \( T_i \), \( \left( \frac{\partial T}{\partial X} \right)_i \) and \( \left( \frac{\partial T}{\partial Y} \right)_i \), i.e. \( T \) and the rates that \( T \) is changing in the positive \( X \) and \( Y \) directions respectively.

Each set of \( n \) equations involves a linear combination of linear combinations. The first set of \( n \) equations involves the original multiquadric equation for \( T \) with the kernel \( Q_{ij} \). Two other linear combinations with kernels \( \frac{\partial Q_{ij}}{\partial X} \) and \( \frac{\partial Q_{ij}}{\partial Y} \) are added. The derivatives of this first linear combination of linear combinations with respect to \( X \) leads to the second set of \( n \) equations, and similarly with respect to \( Y \) leads to the third set of \( n \) equations. When all terms are present, this combined symmetric system of \( 3n \) equations with \( 3n \) coefficients can be solved uniquely for a surface that will not only fit all observed values \( T_i \) but will fit the observed values of the derivatives \( \left( \frac{\partial T}{\partial X} \right)_i \) and \( \left( \frac{\partial T}{\partial Y} \right)_i \) exactly. This principle provides an extremely powerful osculatory surface fit of much higher order than ordinary collocation.

When no gradients are provided in the data the system above reduces to
\[
\sum_{j=1}^{n} A_j Q_{ij} \quad i = 1, 2, \ldots, n
\] (29)

Hardy (1972b, 1976) has shown that the undetermined multiquadric coefficients of an analytical topographic surface can be determined from aerial photographs using the projective transformation equations of analytical photogrammetry. A linear system of multiquadric equations is combined with linearized projective transformation equations as used for an absolutely oriented stereopair in a standard space intersection problem. This method was checked with a few data points, but it is not known whether it is practical for a large number of points, without modification. It has been shown, theoretically, that analytical photogrammetry can solve both parts of the general problem of photogrammetry. If this theory is reduced to practice, it is probable that it will be combined with automatic image-matching of photo-coordinates, thus resulting in a procedure that could be called an automated, analytical stereocompilation.

Shaw and Lynn (1972) compared multiquadric analysis with the bicubic spline function as a method of representing areal rainfall. The statistical analysis showed a slightly better fit with the bicubic spline than with multiquadric equations. Hardy (1976) commented that the kernel of the multiquadric equation used was a cone, i.e., \( \delta \) in (2) was taken as zero. Thus, it is not certain that either the optimum kernel or the optimum \( \delta \) in the kernel was used to form the system of multiquadric equations for use in the comparison. Another point brought
out by Shaw and Lynn (1972) and Lee, Lynn, and Shaw (1974) is that the bicubic spline function theory used was limited to grid data. In the rainfall analysis, the data points and the rain gauge sites are usually placed irregularly over the area of interest. For such applications, the multiquadric technique shows great promise.

Lee, Lynn, and Shaw (1974) used the multiquadric surfaces constructed from a hyperboloid of two sheets, and cones, for the representation of the spatial distribution of rainfall. They concluded that both the multiple hyperboloid and multiple cone surfaces may be fitted in a collocation mode by solving a simple system of linear equations. The multiple hyperboloid surface is smoother than the equivalent multiple cone surface and may give a better representation of the true rainfall surface, especially when the data points are few. The characteristics of the multiple hyperboloid surface depend upon the pre-selection of a suitable $\delta$ in Eq. (2). Based on evidence from two of their test surfaces, the optimum value of this parameter is related to half the scale of the horizontal coordinates. Although the exact value is not critical, computational difficulties may occur if the $\delta$ is set too high. The multiple hyperboloid surface may give better estimates of the areal rainfall for simple storms. However, the estimates for more irregular rainfall patterns are very similar to those obtained from the multiple cone surface. They preferred to use the multiple cone surface for estimation of areal rainfall, since it is simple, efficient, and entirely objective. The multiple hyperboloid surface may be a
useful alternative if a detailed representation of the rainfall surface is required, since the hyperboloids tend to emphasize the peaks and hollows.

Hardy and Göpfert (1975), Hardy (1976), and Goodrow (1974) used a harmonic form of the multiquadric series on the problem of representing world geoids given observed geoidal undulations as input data. The series was successfully applied to two existing world geoids in both a significant point mode and a regularly spaced or gridded input data mode. This study showed that the multiquadric harmonic series of the form

\[ N = \sum_{j=1}^{n} c_j [R^2 - 2Rr_j (\cos \theta \cos \theta_j + \sin \theta \sin \theta_j \cos (\lambda - \lambda_j)) + r_j^2]^{-\frac{3}{2}} \]

(30)
can be used effectively to represent world geoids. In Eq. (30), \( R \) is the earth's mean radius, \( r < R \) is the radius of the sphere in which the point mass anomalies are located, \( \theta \) is the spherical polar distance, and \( \lambda \) is the longitude with respect to the discrete nodal coordinates \( \theta_j, \lambda_j \). Goodrow concluded that the fit improved as the number of data points or nodes used in the formation of the series increased. More regularly spaced or gridded data are required to form a good fit than well-chosen, significantly located data. The multiquadric harmonic series may prove to be superior to the traditional spherical or elliptical harmonics for many applications.
Hardy (1976) and Goodrow (1974) referred to the work of O'Hayre (1973) when using a multiquadric series to transform digitized imagery into picture functions by using sparse arrays of pictorial elements. O'Hayre used the two-dimensional conic form to create a series for each scan line of a picture. Using this series, the coefficient $c_j$ can easily be determined by using Eqs. (18) to (20), which were developed by Cain. O'Hayre found that he could exactly reproduce a picture in eight gray levels using 50% of the data and a close approximation could be obtained with 25% of the available data. For O'Hayre's gross gray level test, multiquadric analysis compared favorably with the better known Fourier and Hadamard transformations. These data compressions could be useful in satellite photography, remote sensing, closed circuit TV, picture-phone, or many other applications involving transmission of large quantities of image information.

Goodrow (1974), referring to the work by Brown (1973) in using a harmonic form of the multiquadric, applied to regional geoid determination by using a short arc reduction of satellite altimetry. Goodrow points out that, unlike a model based on spherical harmonics, the multiquadric is as well-suited to regional applications as to global applications and has the virtue of great simplicity. He suggests that the multiquadric model be given a careful consideration in future investigations of the short arc method.

Krohn (1976) developed a digital computer method of making gravity station terrain corrections that uses a linear system of multiquadric
equations. This system is fitted to the points defined by square topo­
graphic compartments and the point defined by the station itself to
give a mathematically described surface. He found that the multiquadric
equation method is potentially more accurate than a hand chart method
for near-station terrain corrections in a theoretical example. Field
examples in an area of rugged topography show that the multiquadric
equation gives values that compare favorably to Hammer's chart values
and is more efficient than the other computer methods available at
the time. Krohn pointed out explicitly in his discussion that this
project has shown the multiquadric equation technique to be reliable
and effective. The multiquadric equation method will, in general,
require more computer time than the other methods in his work, but
this is compensated for by an increase in accuracy and flexibility.

Hardy (1976, 1977) discussed the similarities and the dissimilar­
ities of the multiquadric and the covariance functions. The multiquadric
kernels are based on geometric and physical considerations rather than
stochastic processes, as is the case of covariance kernels. Thus,
the procedure of determining and fitting empirical covariances to select
an analytical covariance function is unnecessary in multiquadric analy­
sis. Topography, gravity anomalies, and other phenomena are not sta­
tionary in the sense of stationary random functions, which is the heart
of justifying for least squares prediction with covariance functions.
Moreover, Hardy (1977) has shown that comparative studies indicate
superior interpolation or computational characteristics for multiquadric
functions over covariance functions in applications involving topography and gravity anomalies.

In this same paper, Hardy referred to Yaglom (1962) for a definition of the geometric properties of a correlation function which are as follows

\[ c(0) > 0 \quad (31) \]
\[ c(-s) = c(+s) \quad (32) \]
\[ |c(s)| \leq c(0) \quad (33) \]

Specification (31) requires the correlation at the origin of coordinates to be positive. Specification (32) requires the correlation function to be an even function, i.e., symmetric with respect to the origin. Specification (33) requires the correlation at any distance \( s \) cannot be greater than the correlation at the origin. In other words, the correlation at the origin is at least one of the maxima, if not the only maximum of the function. The multiquadric kernel functions, e.g., the cone and hyperboloids, are even functions as in (32). But, obviously these multiquadric kernels do not satisfy the important specifications (31) and (33) simultaneously for covariance functions. Moreover, Hardy (1977) commented that the "bounded" cone and "bounded" hyperboloid fail to have non-negative Fourier transforms. Non-negative Fourier transforms are an indirect requirement for the famous Wiener-Khintchine relations of correlation functions and power spectra. Therefore, a
negative, or partially negative Fourier transform of an assumed covariance function, is proof that the assumed covariance function is incorrect. Yaglom (1962) used these criteria (Eq. (31)-(33)) repeatedly in example problems to determine whether or not a specified analytical function qualified as a covariance function. Thus, there is no direct theoretical relationship between the quadric kernels mentioned above to true covariance functions.

However, Hardy (1986) pointed out the striking geometric similarity of some semivariograms to a basic point source biharmonic disturbing potential. In some physical applications he even suggested that the construction of a semivariogram may be a rough stochastic estimate of a multiquadric kernel function. It is easy to see that \((1 - \frac{1}{x^2})\), i.e. one minus the reciprocal distance (harmonic MQ) for a source at a proper distance from the evaluation surface provides a good illustration of a semivariogram with a sill. Furthermore the MQ-B kernel, namely the distance, provides a good illustration of a kriging linear model without a sill. However, the shape of the MQ-B kernel is always one, whereas the linear model in kriging provides various estimates of the slope.

An important difference between the multiquadric and covariance function approach including kriging as a special case, is that the choice of a covariance kernel (or semivariogram kernel) is often (but not always) based on the computation of a discrete covariance or semivariogram sequence variously termed "empirical" or "apparent" covariance
or "experimental" semivariogram. After an apparent covariance or semi-
variogram is computed, an analytical function resembling the "apparent"
covariance or semivariogram is usually chosen as the kernel. An exact
fit of an "apparent" covariance with a single analytical kernel is
seldom, if ever, possible. This, in itself, leads to a preliminary
least squares problem. In other words, an analytical covariance kernel
function or semivariogram is frequently selected by means of least
squares fit. In the multiquadric approach, this preliminary procedure
is not relevant.

Hardy (1977) also listed the applications of multiquadric functions
in photogrammetry and remote sensing as follows:

(1) lens distortion corrections,
(2) film deformation corrections,
(3) prediction of corrections to pass points in strip and block
    triangulation,
(4) digital terrain model (DTM) contouring and profiling,
(5) camera and reseau calibration,
(6) geometric correction of radar imagery,
(7) geometric correction of panoramic camera imagery, and
(8) image function processing and analysis.

Image processing and analysis has, in itself, the potential for
a breakdown into many subapplications. Among these are

(1) pattern recognition,
(2) boundary location (or more generally, gradient analysis),
(3) density slicing, color coding, and B&W to color conversion,
(4) image function compression,
(5) image function reconstruction (expansion),
(6) correlation of perspective image functions, and
(7) orthoprojection of perspective image functions.

Of these subapplications, Hardy experimented with items (4) and (5).
In a sense, items (4) and (5) involve an inverse or transform a rela-
tionship with respect to each other. It is possible to describe an
application and the corresponding process by beginning with either
(4) or (5) as the problem and ending with either (5) or (4), respec-
tively, as being essential to the solution.

In the same paper (Hardy, 1977) the multiquadric method was applied
to the picture elements of the image of Lincoln from the cover of the
15 June, 1973 issue of Science magazine. This experiment suggested
a new application, an extension of the image function reconstruction
concept. Some computational processes, such as the multiquadric equa-
tions, are capable of restoring relative continuity to an enlarged
step function in a logical manner. This brings out "predicted" details
of continuity that were never directly recorded in the original image.

Franke (1979, 1982) included multiquadric equations in testing
approximately 30 different methods of constructing a smooth bivariate
function (at least continuous first partial derivative), \( F(x, y) \), which
takes on certain prescribed values, \( F(x_k, y_k) = f_k, \ k = 1, \ldots, n \).
The points \( (x_k, y_k) \) are not assumed to satisfy any particular conditions
as to spacing or density, hence, the term "scattered." This basic set of test problems consisted of six different test functions. The multiquadric equations were the hyperboloid Eq. (13), and the reciprocal multiquadric

$$ G_k (x, y) = ((x - x_k)^2 + (y - y_k)^2 + r^2)^{-\frac{1}{2}} $$  \hspace{1cm} (34)

The $\delta$ in Eq. (13) or $r$ in Eq. (34) was computed from

$$ r = \frac{NPPR}{10} R $$  \hspace{1cm} (35)

$$ R = \frac{D}{\sqrt{N}} $$  \hspace{1cm} (36)

where $D$ is the diameter of the point set $\{(x_k, y_k)\}$. A nominal value of 25 was assumed for NPPR by Franke (1979). $N$ is the number of data points. Franke concluded that the multiquadric method, Eq. (13), proposed by Hardy (1971a) performs very well. The plots show that the method produces very smooth and pleasing surfaces. Franke's (1979) deviation tables show the method is consistent and one of the most accurate. "Reciprocal multiquadrics", as the basis functions, also worked quite well. The surfaces were again seen to be very smooth. The basis functions resemble the rotated Gaussian, but generally perform much more reliably than the Gaussian. The multiquadric method is consistently best or nearly best in terms of accuracy, and always results in visually pleasant surfaces. However, Franke (1979) stated that the method had no apparent mathematical basis to explain its efficacy,
and this led him to a certain degree of skepticism. However, he recom-
mended the use of either the multiquadric method or the thin plane
splines as the best of the global basis function methods considered.
The reciprocal multiquadric method, which was pointed out to be a point
mass model (Hardy, 1975), has some potentially bad effects. One effect
includes using too small a value for $r$, which will give poor results.
There seems to be no reason to use the reciprocal multiquadric method
rather than the multiquadric method.

Hardy (1981) summarized the research work done by others in 1979,
which included comparative studies of multiquadric equations and other
methods of interpolation and prediction. Hardy brought to attention
the importance of Franke's (1979) results in the geometric similarities
between the thin plate spline (TPS) of Duchon (1975) and the original
multiquadric (MQ) method by Hardy (1971a, b). Hardy further stated
that the TPS and MQ functions are in the same class, i.e., they satisfy
a biharmonic differential equation in two and three variables. He
also made a brief comment on a possible reason why the MQ method gave
generally better results than Duchon's TPS method as defined by Franke
(1979). Duchon's method involves the direct application of externally
concentrated forces at the surface of a reference plane; there are
no body forces. The MQ method uses body forces induced by anomalous
gravitation; there are no concentrated external forces. Hence, the
MQ biharmonic function is generally a smoother function than Duchon's
Hein and Lenze (1979), Technical University Darmstadt, studied the accuracy and economy of eight well-known interpolation and prediction methods applied to higher surveying. The multiquadric method was given the highest rating of all methods tested. A sample terrain representation from the German Basic Map (DGK5) was used as a test model for all methods (linear interpolation in triangles, correlation procedures, first degree polynomial approximation, second degree polynomial approximation, two-dimensional spline interpolation, linear prediction with region-wise determination of covariance functions, linear prediction with point-wise determination of covariance functions, and multiquadric interpolation). Each approximation method was used to predict an interpolation grid of 841 points (29 x 29) from a set of 640 irregularly spaced topographic features selected as control points. In general, all methods were optimized by partitioning into "computation units" that would meet accuracy standards for the map. Hein and Lenze (1979) mentioned that matrix inversion was not necessary for MQ systems of equations, however, it seemed to be necessary for most other methods. Apparently a very simple simultaneous equation solution for linear, symmetric, and MQ systems of equations was developed. Many of the methods used, e.g., covariance methods, interpolation in triangles, and spline interpolation, required a great deal of preprocessing that is not required by MQ. Hein and Lenze (1979) concluded that the MQ
method is absolutely the fastest of all considered procedures, very simple and easy to program with no inversion necessary, and gives a very smooth shape of contours. The procedure is the best possible adaptation for routine applications in all areas of geodesy, and provides a good agreement with linear prediction. Hardy (1981) also introduced the results of his work on the mathematical theory of multiquadric equations and found that they satisfy the biharmonic differential equation. He believed that they were optimal in some Hilbert space as proven by Duchon (1975) for thin plate splines, which were also biharmonic. It was confirmed by Madych and Nelson (1988) with a formal mathematical proof that multiquadric is an optimal solution in a certain Hilbert space via construction of a reproducing kernel.

Holdahl (1983) applied multiquadric analysis to interpolate (surface fitting) the vertical crustal motion rates near Palmdale, California. The procedure has been described by Holdahl and Hardy (1979) and Hardy (1978). The selected quadric form for modeling vertical crustal motion was chosen to be a hyperboloid, Eq. (15). The unknowns in the adjustment are the heights and the coefficients $c_1, c_2, \ldots, c_n$, which are used to calculate the uplift rates. Holdahl assumed that if there was no motion in the study area, all the coefficients should be zero. Only five of the 35 MQ coefficients were significant, being twice as large as their standard deviations. Two of those coefficients corresponded to nodal points which are just within the bounds of the subsidence area in the study area. Other significant coefficients correspond
to the nodal points that appear related to the surrounding high topography.

Schiro and Williams (1984) reported on techniques for applying multiquadric interpolants to data sets involving a large number of irregularly-spaced two-dimensional data. The general technique involved building a data structure to partition the domain of the data into a set of smaller divisions called cells. Any adjacent cells whose data are similar are then adaptively combined into one group. After grouping the cells, multiquadric interpolants are then iteratively applied to each cell group until the maximum error between the resulting data model and all original data are within a user specified tolerance. The method was used on five representative sets of hydrographic data, consisting of over 12,000 data points, and two sets of data computed from common, mathematically-defined surfaces in the literature. Their results showed that the conic quadrics out performed hyperbolic and reciprocal hyperbolic quadrics for hydrographic data. Their method is obviously feasible and well tested for modeling large numbers of irregularly-spaced data. The multiquadric model approximated the original mathematical surfaces very well, even in areas in which original sample data were not present. This approximation was done using 50 percent of the original sample data. The tests showed that the method can produce good results and be reasonably efficient, even with rather strict error criteria. There were no indications of either ill-condi-
tioning or, singularities during the software development for surface analysis.

Schiro and Williams (1984) also referred to the work of Pickrell (1979) in using the hyperbolic, reciprocal hyperbolic, and conic multi-quadric surfaces on hydrographic data, where the conic quadric performed the best. Pickrell found that if $\delta \neq 0$ in the hyperbolic model, then the surface is scale-dependent. For hydrographic and other types of data, this is not acceptable. Therefore, the conic quadric was recommended when scale dependency is considered important. In Pickrell's study, the majority of computation was done with conics.

Schiro and Williams (1984) modified the equation for $Z$ in each multiquadric model to ensure that constant data would result in a constant as the only model term. The equation was modified to

$$Z = \sum_{j=1}^{n} c_j \left[ (x - x_j)^2 + (y - y_j)^2 \right]^{-\frac{1}{2}} + Z_{\text{mean}} \quad (37)$$

This provides the ability to reproduce constants and still preserve the symmetry property of the resulting matrix, thereby allowing the employment of an algorithm for the solution of the system of equations which exploits the symmetry and is efficient with regard to storage space and execution time. Note that $(Z - Z_{\text{mean}})$ defines the fitted ordinates as a set of statistically determined anomaly values. Later Hardy and Nelson, (1986) showed that if $\sum c_j = 0$ there is a constant to be subtracted from the raw $Z$ values. This constant may be near
the mean, but it is not necessarily so. This new constant satisfies
the mathematics and physics for the region.

Lancaster and Salkauskas (1986) included Hardy's multiquadric
method in their textbook as one type of interpolant. As the basic
function, they used the cone in which the vertex is rounded off or
the circular hyperboloids as stated by Hardy. Lancaster and Salkauskas
commented that the graph of the model was very smooth, and perhaps
the surface derived from it should not be called a surface spline.
They depended exclusively on the original paper by Hardy (1971a) and
did not use later references such as Hardy and Nelson (1986). They
also observed that the multiquadric method can give good results as
an interpolation process. By using \( \delta = 0.6 \), the desired smoothness
of the interpolant is attained. The variation of \( \delta \) can cause a dramatic
effect on the surface produced. The ill-conditioning of the matrix,
involved in the process of computing the MQ coefficients, is likely
to arise when \( \delta \) is larger compared with the spacing of the data points.

Vittal, Rajagopal, McCormack, Movali and Fouad (1986) reported
a project done at Iowa State University using computer graphics as
a tool to investigate the shape of the potential energy surfaces for
a multimachine power system. The goal was to use any graphics package
available as a creative and challenging method to generate information
concerning the shape of a function of \( n \) - variables (specifically the
potential energy), which is difficult to conceptualize. The problem
was alleviated by plotting the potential energy with respect to two
variables at a time, thus providing a three-dimensional projection of (n+1) dimensional surfaces. Due to a very limited set of data for a three-dimensional plot, the problem was to calculate the potential energy at more points in order to enhance the quality of the plots. Using the classical method this step can be computationally burdensome. Based on the graphics package and a literature search in the areas of topography and multidimensional interpolation, they adopted a method suggested in Hardy (1971a). The data obtained by the multiquadric equations were then plotted using AGRAPH, a graphics routine (Read, 1985). The above process helped gain a better understanding of power systems.

Kansa (1987, 1988) applied a multiquadric method to solve the computational fluid dynamics problem. From his point of view, the multiquadric is a true scattered data, grid-free scheme for representing surfaces and bodies, unlike finite difference and finite element schemes. The multiquadric method is continuously differentiable, integrable, and capable of representing functions with steep gradients with a very high accuracy. Kansa (1987, 1988) stated that their modified multiquadric scheme was an excellent method for not only very accurate interpolation, but also for partial derivative estimates. He presented the results in which multiquadric was used in dynamic problems such as in hyperbolic partial differential equations and higher order arbitrary Lagrangian Eulerian (ALE) rezoning. The surfaces could be approximated to a very high degree of accuracy by permitting $\delta^2$ to vary with basis function numbers. The value of $\delta^2$ controlled the local shape of the
basis function. Large $\delta^2$ values gave rise to flat sheet-like basis functions, whereas intermediate $\delta^2$ values gave rise to bowl-like basis functions, and small $\delta^2$ values gave rise to rounded, narrow cone-like basis functions. By adding and subtracting a diverse collection of different shaped basis functions, they attained very accurate results.

Hardy (1982b) and Hardy and Nelson (1986), showed that multiquadric is an appropriate approximation to biharmonic representation and reciprocal multiquadric is an appropriate approximation to a harmonic representation of the disturbing potential. The multiquadric-biharmonic representations and approximations have advantages over other methods, since data points do not need to be separated from source points.

Hardy and Nelson (1986) gave a physical reason as to why multiquadric-biharmonic performs so well. The multiquadric-biharmonic is an alternative form for representing a disturbing potential which is biharmonic in nature and can be used for evaluations at points collocated with sources. This is unlike the harmonic form of the functions representing the disturbing potential which has singularities at the source point and cannot be easily evaluated at or near sources which induce the potential. Although the multiquadric-biharmonic basis functions are independently unbounded at infinity, they have shown that if the sum of the multiquadric coefficients is zero the sum of multiquadric terms approximating the potential vanishes at infinity as does the actual potential itself.
Micchelli (1986) proved that the multiquadric interpolation is always solvable for distinct data. He showed that multiquadric coefficient matrix of rank n has one positive real eigenvalue and n-1 negative real eigenvalues.

By using the infinite Taylor series expansion, Kansa (1987) advanced an explanation of why multiquadric works so well. The multiquadric Taylor series expansion is an infinite order expansion of all even terms of the distance. Unlike the finite polynomial expansions, the multiquadric expansion is an infinite order multi-variant polynomial expansion in terms of a finite number of data points. Because the multiquadric constant, $\sigma^2$ can vary many orders of magnitude, the effective expansion is a very high order up to the remaining terms which have become truncated in a finite precision computer. Furthermore, the multiquadric expansions contain contributions from not only the direct terms, but all cross product contributions. Since the $\delta_j$ terms differ vastly by orders of magnitude, the contributions range from locally constant to very high order multi-variate polynomial expansions. Kansa (1987) concludes that the multiquadric is considered to be a very high order scheme which does not require fine zoning for high accuracy, while low order polynomial-based methods require very fine resolution in steep gradient regions in order to reduce the truncation error.
Kriging

The kriging method was developed by G. Matheron (1965) and was named for a South African statistician, D. G. Krige. The kriging or geostatistical method was originally developed for mining. Later, it was applied to many other fields such as water resources, agronomy, meteorology, agricultural engineering, soil science, etc.

The following will summarize the development of theories and applications of the kriging or geostatistics method. Matheron (1963) gives the history of geostatistical development. This technique is concerned with the study of the distribution in space of useful values in mineral resources evaluation, such as grade, thickness, or accumulation. Matheron takes into account the space characteristics of mineralization, while the traditional methods in mining failed to express an important characteristic of mineralizations, which is their variability or dispersion. Matheron also stated that Sichel (1952) used to believe that he was applying classical statistics. However, the developed methods differed more and more from classical statistics, and were adjusted spontaneously to the objective. Hence, geostatisticians started elaborating their own methods and their own mathematical formalism, which is nothing more than abstract formulation and a systematization of secular mining experience. This formalism has inherited from its statistical origin a language in which one still speaks of variance and covariance, however they are used in a new and different content. This similarity in vocabulary is somewhat deceiving.
Another important discussion by Matheron (1965) is that at the end of a protracted evolution the geostatistical theory had to admit that it was facing a natural phenomenon distributed in space, instead of random occurrences. Therefore, its methods are supposed to approximate mathematical physics and, more specifically, harmonic analysis.

At the beginning of the geostatistical theory, Matheron found the inability of common statistics to take into account the spatial aspect of the phenomenon, which is precisely its most important feature. A block of ore is mined only once and there is no possibility of repeating the test indefinitely. When the grade of a sample is involved, which may be a groove sample of a given size, for example, the result is exactly the same because the grade of a groove located in a point with coordinates \((x, y)\) is unique and can be determined. Two neighboring samples are certainly not independent.

As a result of controversy involving Dr. D. G. Krige and Prof. E. H. T. Whitten, Matheron's opinion was that the kriging procedure was valid for stationary or intrinsic random functions, whereas the polynomial interpolation procedure should be applied only in specific cases (Matheron, 1967). Matheron showed in his study that fallacious evidence for a real trend may occur as a result of purely random cumulative effects. Local fluctuations are generally meaningless and should be eliminated. Only regional trends are of interest. Each value of \(f(x)\) may be represented as the sum:
\[ f(x) = m(x) + e(x) \]  \hspace{1cm} (38)

where \( m(x) \) is a very regular and continuous function expressing the trend, and \( e(x) \) is a meaningless random fluctuation to be eliminated. Eq. (38) is often useful in physics or information theory, when an interesting phenomenon or message is altered by a noise. Naturally, a distinction between message and noise requires a serious theoretical background to be asserted a priori. When such a theoretical background is missing, as is the general case in geology, the distinction between \( m(x) \) and \( e(x) \) expresses nothing but a conceptual illusion. For example, if one is given fourteen experimental points, it is always possible to find a 13-degree polynomial fitting them exactly. However, it is universally admitted that such perfection is a fallacy; when increasing the degree for polynomial interpolation, tremendous and meaningless fluctuations are always appearing between interpolation points and, as a result, a pure artifact is obtained. However, at the same time the best estimator for ore grade \( f(x_i) \) at a sampled point, \( x_i \), remains \( f(x_i) \) because it is a known true value. The question here is that if we use the polynomial fit, how can one trade-off between fitting the data points exactly and the meaningless fluctuation between data points? In geology, the so-called "trend" \( m(x) \) generally has exactly the same stochastical character as noise itself. A distinction between \( m(x) \) and \( e(x) \) is only a matter of scales. Such a distinction does
not appear to be founded on a criterion that could be stated in a ra-
tional language.

David (1974) discussed the concept of "estimation variance" and the concept of "extension". His point of view is if we give a block the grade of its samples, how much of an error will we commit if we extend the grade of a sample v to predict the grade of volume V? He called this type of prediction scheme "extension." David also found and emphasized that in mining there is an unusual variance-volume rela-
tionship, where the variance of the block is inversely proportional to the volume. The reason that we must be concerned with the variance-
volume relationship is because of the mining problem of calculating the number and size of production units to give a stable production, daily fluctuations, weekly fluctuations, secular fluctuations, etc. Matheron believes the variance of a point in a block, depends only on the size, shape, and orientation of the point and the block, and not on the particular location of the block in the deposit (David, 1974). Based on this idea, Matheron formulated the theory of regional-
ized variables, and called this concept the intrinsic character of the deposit. This can be represented rigorously, but would lead us to consider the deposit as a realization of a random function, not stationary itself, but with order two stationary increments. This means that the deposit is homogenous, maybe with a linear trend and an anisotropy which can be deleted by some geometric transformation.
Bubenicek and Haas (1969) applied the geostatistic method to the Lorraine deposit of minette iron ore in France to proceed to new iron ore reserves. They found the method to be cumbersome. It did not afford appreciable gain in accuracy but had the advantage of avoiding systematic errors. Bubenicek and Haas also listed the qualitative characteristic of regionalized variables that conventional statistical methods were incapable of expressing as follows:

1. Localization: This means the values of a regionalized variable only hold good in a geometric field, e.g., the deposit. Moreover, the variable is closely linked to mining support, e.g., drilling, cutting, etc.

2. Continuity: The regionalized variable is an "on-average" continuity, which is not exactly the same as mathematical continuity. This means that if the \( f(x_i) \) approaches \( f(x_{i+h}) \), only the average value of \( [f(x_i) - f(x_{i+h})]^2 \) will approach zero. If the "on-average" continuity does not exist, this is called the "nugget effect."

3. Anisotropies: The regionalization may be anisotropic when there are one or more directions that the values change more rapidly than the others.

4. Transition phenomena: The structures of regionalized variables often consist of superimposed lenses, e.g., in sedimentary formations. These structures form a network of discontinuity at the edges of the lenses. This is called the "transition phenomena."
Olea (1974) discussed the main characteristic of universal kriging as the intensive use of the correlation that exists between spatially "nonindependent" random variables, which resemble the stochastic process. He defined the regionalized variable in a different way. It is any numerical function with a spatial distribution that varies from one place to another with apparent continuity but whose changes cannot be represented by any workable function. Blais and Carlier (1968) and Matheron (1970) also used this definition. This definition characterizes many variables that describe natural phenomena such as the ore content of a mineralized body or the production per acre of farmland.

A basic assumption in the theory is that a regionalized variable is a random variate that is generated by some probability density function (Matheron, 1969). The spatial setting of each sample is an integral aspect of a regionalized variable, thus the replicated experiments cannot be run. Therefore, we will never be able to determine the probability density function practically. It only theoretically exists.

In order to overcome this limitation in the availability of the probability density function of the regionalized variable, the stationarity assumption is assumed. This provides a basis for a statistical inference. This constraint is similar to the ergodicity assumption (theory of large numbers) in the stochastic processes. The other important characteristic of the regionalized variable is that spatial correlation of the neighboring points is higher than the spatial correlation of the points farther apart.
Olea (1974) also listed the information that one can get from the semivariogram, 1) the size of the zone of influence around a sample, 2) the isotropic nature of the variable, and 3) the continuity of the variable through the space. In his experiment, Olea applied the kriging technique to the Lansing group, which is a sequence of interbedded limestones and shales of Pennsylvanian age widespread in the subsurface of central and western Kansas. The contouring program (SURFACE 2) of the Kansas Geological Survey was used to contour the grid of kriged values. Olea's comment on the usage of the contour is that contouring is a technique necessary to represent surfaces imbedded in a three-dimensional space as a single defined function with two-dimensional isopleths. Olea (1974) indicates that all other contemporary estimation methods in producing the regular grid node values to supply a contouring program are philosophically unsatisfactory, mainly because they follow rules that are arbitrary and empirical. On the other hand, universal kriging is the optimum method because it is unbiased, an exact interpolation procedure, and has a minimum estimation variance. However, the estimation variance is not an indicator of the effectiveness of the universal kriging technique, but rather depends upon the quality of sampling. A contour map produced by using universal kriging is approximately three times as expensive as one obtained by using simple empirical estimation procedures. Universal kriging is a complex statistical method not recommended for preliminary examinations.
David (1976) commented that kriging still puzzles most of the mining people. In 2-dimensional problems, it is only found to have a marginal advantage in the quality of the estimate, and is a definite drawback in computer cost. David also stated that "The next point which should be made crystal clear too is that one cannot hope to run an efficient kriging program without having a deep understanding of all geostatistical concepts, mining problems, and computer problems."

David listed the basic structure of a kriging program as follows:

1. Data: a file of samples with their grades and coordinates, a file of blocks to be estimated from the first file, a variogram equation.

2. Processing step for each block: The sample file is searched for samples having an influence on the block, the covariances between these samples should be computed, the covariance of these samples and the block should be computed. These covariances should be arranged in a linear system form, the linear system of equation should be solved, the solution is the set of weights by the grades of samples retained, the precision on the grade estimation is computed.

Item (2) above must be repeated for each block. He stated that the program for the above logic is very easy to write, but making it efficient is a totally different story, e.g., the variation of computer types and computer tricks. The searching step for neighboring points, the computation of covariances, and the solving of a linear system of equations are the computer's time consuming steps.
There is no indication of the total number of samples to be used in the kriging system in a given block (David, 1976). David asserts that the common belief that one sample, further apart from the block than the range in the same direction, has zero influence on the outcome is simply wrong. Even if negligible, the influence of these samples is not zero and sometimes many negligible weights may add up to cause a significant change in grade estimation. Rather than computing one weight for each available sample, one may want to pool together all samples which are far apart from the block and use the estimator:

\[ Z^*(x_i) = \sum_{i=1}^{n-1} \lambda_i x_i + \lambda_n \bar{x} \]

where \( \bar{x} \) is the average of all available samples. The domain of validity of the variogram equation is used to decide the number of data points to be used to compute \( \bar{x} \).

David (1976) also gave a remarkable conclusion from the computational point of view that people doing geostatistics have long incorrectly believed about handling symmetrical matrices. The only advantage of the symmetrical matrix is storage space, which is a marginal cost in the computer usage. It was found that the particular techniques used to solve the symmetric matrix consume more time than one that was used to solve the full matrix.

David (1976) stated that the linear estimation is deceptively robust. A major change in the coefficients of the linear combination enhances only a relatively small change in its variance. Taking the
difficulty in accurately defining the variogram parameter into account, one should be happy about the predicted results rather than wasting time seeking the best set of weights. This leads us to the problem of selecting the best variogram. The possible way to do this is to take each data point and estimate it by kriging using different variogram models, compute the residual sum of squares, and select the model that minimizes the residual sum of squares.

David (1976) also discussed regular grid sample versus irregular grid sampling. In regular grid sampling, the computation is limited to some geometry. The same coefficients can be applied for all similar geometry. However, this is the rare case.

Delfiner (1976) discussed the risks of bias when using classical linear kriging to estimate the non-linear function of data.

Chiles (1976) applied kriging to contour mapping. He found that kriging gave an optimal map by fitting data points exactly, and providing associated estimate of error based on the empirical semivariogram and with an assumption of the error distribution. However, the kriged map is smoother than the actual map because of the estimator. Although it is optimum, it cannot restitute details that have not been surveyed. Chiles defined the conditional simulation in the geostatistical method as a realization of a random function with these two characteristics: 1) the map has the same covariance as the phenomenon under study, and 2) each data point is fitted exactly. A conditional simulation is
not reality but only a version of reality. The various conditional simulations depict possible aspects of the unknown true map.

Rutledge (1976) discussed the major reason for a lack of interest in geostatistics in his country, namely a stereotyped approach to ore reserve estimation and block grade estimation in Australia. Australia's approach appears to be a cook-book approach. This negates use of a geostatistical method before it can be studied and made routine, i.e. provide a new cook book. Geostatistics must be thoroughly understood before it is used significantly. To acquire this understanding calls for some hard thinking and time. Another major reason for the neglect of geostatistics is the absence of widespread facilities for learning about the geostatistical models and for applying this knowledge in a straightforward way.

Rutledge (1976) listed the "benefits" of the geostatistical models as follows:
1. The variogram behavior infers the structure of a region.
2. An area estimation of the ore body can be made.
3. Estimation and extension variance can be computed.
4. The kriging technique gives the estimation of grade at the unsampled points.
5. A kriging analysis can be a decision-making tool about the best place for drilling.
6. Point kriging enables the user to generate an unbiased, minimum-variance contour map of an area.
7. Kriging procedures can be used to provide estimates of the grades of mining blocks for detailed day-to-day planning.

8. A conditional simulation is available to produce an uncertainty map of the ore body.

9. A library of a priori variograms of the various types of ore bodies can be built.

Rutledge (1976) also listed the "costs" of the geostatistical approach as follows:

1. It may be difficult to get enough data.

2. It may be difficult to fit an experimental variogram. Fitting a model to an experimental variogram appears to be largely a matter of experience.

3. It may be difficult to decide whether drift is present and needs to be taken in consideration. The use of universal kriging is a matter of experience.

4. A geostatistical evaluation may be lengthy and time consuming.

Jousselin and Haas (1976) described their application of the geostatistics method to acquire the knowledge of subsurfaces in prospecting for oil fields with seismographic data. The principle advantage of kriging other other methods, from their point of view, was that kriging provides a formula for determining the error of the estimation.

Sinclair and Deraisme (1976) applied the geostatistical method to the 2-dimensional data of the Little Chief copper deposit in Whitehorse Copper Mines, Yukon, Canada. The kriging of thickness and assay
data provided unbiased grade and tonnage estimates of slopes, pillars, and sills. The ore body of interest was defined by an irregular grid of horizontal drill holes. For each drill hole, the data base consisted of 2-dimensional coordinates, thickness of the mineralized intersection, and the average Cu grade. Among other conclusions, they determined that in this particular theory a 100 x 100 foot drilling grid produces grade estimates with a standard deviation of about 10%. They considered the geostatistical estimation to have been applied successfully.

Agterberg (1976) explored methods for the statistical analysis of geological features on the pyritic massive sulphide deposits of a volcanic exhalative origin. He concluded that the mineral deposits are rare events because they differ in a number of aspects from all other deposits. Every mineral deposit can be regarded as a unique event. Grouping deposits for statistical analysis requires consultations with geologists who know the deposits, and consideration of existing metallogenic theory regarding the origin on the deposits. The major difficulties of a more mathematically-oriented approach to regional geological problems are lack of suitable geomathematical techniques and also a lack of well-established geological "facts" which can be used as a foundation for method development.

Akima (1975) severely criticized the universal kriging paper by Olea (1974). It is clear that a single paper concerning many new or different ideas is not easily understood, at best. It is good to look at Akima's ideas. Akima (1975) stated that the universal kriging con-
tains some problems based on the presentation by Olea. He questions the claim that kriging is an optimal technique. The word "kriging" is not entered in the unabridged Webster's Third New International Dictionary, nor in most standard dictionaries. Also kriging has a restriction of stationarity restriction, which Akima believes is a very significant restriction based on references to his own work involving non-stationary data. The kriging technique has high computational costs. It has a complicated computational procedure including determination of optimum weight by Lagrange's multiplier method that must be carried out for each output grid point. The drift, \( m(x) \), defined by the equation (Olea, 1974)

\[
m(x) = a_0 + m'(x) = a_0 + \sum_{i=1}^{n} b_i f_i(x)
\]

cannot represent the trend because the estimated coefficients are determined at each point so that \( b_i \) varies from point to point. The estimation variance of \( Z^*(x_0) - Z(x_0) \), calculated using the optimum weights \( \lambda_j \), is physically meaningless or not readily understood. The method of drift searching is not unique. The final solution of drift is selected from among all acceptable possibilities on the basis of convenience. The question arises whether such a selection can be made objectively and automatically with a prescribed algorithm. Much of this latter criterion appears justifiable to this author.

Olea (1975) responded to Akima's criticism as follows. The equations in the universal kriging system must be solved as many times
as the number of points to be estimated only if the data configuration is the irregular grid system. The kriging method contrasts to the least squares or similar methods in that those methods estimate the single set of coefficients of a single function. Then, these coefficients are used to describe an infinite number of points on an entire surface. Universal kriging is optimum in the sense that it is unbiased, is an exact interpolation procedure, and the estimation variance is minimum. Optimality is viewed in the theoretical sense that the universal kriging provides the best estimate without consideration of how expensive it might be to obtain this estimate. Universal kriging always provides estimates of the regionalized variable, providing the stationarity restriction criteria are met. The question of whether the surface is continuous is meaningless because the universal kriging system of equations does not yield the equation of a surface, but rather estimates values of the regionalized variable at a point. Obtaining coincidence between the theoretical and the experimental semivariogram is more an art than a science. Starting with four or five points and simple drifts is the safest way. The result of a search for the drift and the intrinsic functions is not unique, as in the problem of finding roots which usually have more than one distinct solution. There are always several combinations of drift and semivariogram expressions that may be equally satisfactory. Therefore, experimentation should continue until a collection of solutions is obtained. The final solu-
tions are selected from among all acceptable possibilities on the basis of convenience.

Perhaps more than any other papers up to 1975, this dialogue by Olea and Akima is typical of the controversial attitudes concerning kriging that are continuing to appear in the international literature, particularly the journal Mathematical Geology. See the journals of Mathematical Geology, Nos. 1, 5, and 7, Vol. 18 (1986) and Nos. 2, 4, 5, and 6, Vol. 19 (1987). The authors of this collection of suggested reading for understanding the controversy concerning kriging are A. G. Journel, D. G. Krige, G. Matheron, D. E. Myers, G. M. Philip, J. Serra, R. M. Srivastava, and D. F. Watson. Many of these authors have been referenced in my review of literature concerning kriging, but none of the above articles involved in the controversy have been referenced.

Stanley (1977) applied kriging to the ore reserve estimation at the Henderson mine, Empire, Colorado. The Henderson mine contains a molybdenite deposit. The geology consists of a classic subvolcanic rhyolite porphyry sequence with quartz-molybdenite mineralization occurring in an extensively fractured granite stock complex. The ore body was sampled by means of underground drill holes which create an east-west fan pattern spaced roughly 60 meters apart in the north-south direction. The variograms were computed for different subsets of samples. This allowed a regularization of 15 degrees on either side of the samples to determine the order to incorporate more samples into the computation.
The variogram showed pure noise. The new variograms were computed on a larger scale in expectation that they would exhibit the existence of a macro-structure as opposed to the random characteristics of the smaller scale mineralization. The spherical model was fitted to the new variogram to give a theoretical variogram. Then it was used for the kriging system to estimate the interest regionalized variables of the deposit.

Stanley (1977) suggested the method and an equation to compensate for anisotropy in the vertical direction compared to the range of influence in the horizontal direction. The ratio of the range of influence in the vertical variogram and the horizontal variogram was applied as the factor to the Z coordinate in the variogram function. For example, the spherical model can be modified to

\[ \gamma(h) = C_0 + C\left(\frac{3h}{2a} - \frac{1}{2} \frac{h^3}{a^3}\right) \]  

where \( h = \sqrt{(X_i - X_j)^2 + (Y_i - Y_j)^2 + \frac{r_v}{r_h} (Z_i - Z_j)^2} \). The \( r_v \) and \( r_h \) in Eq. (41) are the ranges of influence in the vertical and horizontal variogram, respectively.

Clark and White (1977) applied the simple kriging to a complex base metal sulphide (Pb/Zn) deposit. They commented that modern mining technology has increased in complexity. Thus, it becomes increasingly more important that the mining engineer is able to evaluate quantitatively the implications of the efficiency and economics of the proposed exploitation methods. The kriging estimation technique predicts the
grade at any given point or volume within a deposit. However, the actual grade mined during the production will always differ from the predicted values.

Vallee, Belisle, and David (1977) applied the geostatistical method to estimate the metal contained within a given volume of copper orebody found in Quebec. A variogram was first calculated along single diamond drill holes. Then, variograms of adjacent drill holes were grouped together to calculate an average variogram of the grades. The average variogram allowed them to establish the basic parameters of mineralization, such as nugget effect, sill, and range of influence. Although the method of defining these parameters may be crude and disputable, they concluded the estimation of reserve was acceptable and robust. The meaning of robust is the estimation is unsensitive to variogram change.

Lallement (1977) used the geostatistical method to investigate an ore body in France. He interpreted the variogram to give the determination of the best drilling pattern. For instance, if the variograms computed by the previous data show that an anisotropy exists between the two directions of the ore body, this means that the drill pattern must be rectangular but not a square. Another example, if the ore body is completely unknown, the exploration project should be planned into two steps, the first step with a larger grid pattern than the second one. Then, the geostatistical method can be applied to the first step data to determine the best pattern for the second drilling.
Delhomme (1979) applied the geostatistical method to characterize the uncertainty of the transmissivity field of an aquifer and analyzing its effect on predicted head values. He discussed the properties of regionalized variables, e.g., continuity or differentiability depend on the working scale. For instance, the smooth behavior can be found on a microscopic scale, while the fluctuating behavior is observed on a macroscopic scale and vice versa. In other words, those properties are more or less the illusion caused by the decision of the researcher in selecting the scale, rather than the true characteristic of the phenomena. Delhomme (1979) concluded that the kriging method takes into account the effects of both sparse sampling and spatial variability in the study of the uncertainty about the transmissivity field of the aquifer. However, the present approach is still incomplete and several limitations are found. For example, some imprecision exists in fitting the model to the experimental semivariogram. In the practical problems, the number of sample points can be so small that the direct inference of the semivariogram would be impossible.

Gambolati and Volpi (1979) tried to fill the theoretical gap between the kriging technique and other commonly used interpolators from a hydrological point or view. They discussed, in detail, the useless trend or drift concept and also the impossibility of proving the real existence of trend. The following describes their opinion. Gambolati and Volpi stated that the kriging technique has been very seldom applied elsewhere to analyze hydrologic data. The kriging technique does not
need to be developed in a stochastic context. This method may be viewed as a classical weighted linear interpolator satisfying the same prerequisites which underly, for instance, the Lagrange and the least squares interpolation. An ad hoc criterion for deriving the individual weight is the only basic difference between the kriging and the other methods. The deterministic approach in kriging was developed also by Matheron (Matheron, 1962). Instead of following the stochastic approach, Gambarlati and Volpi (1979) analyzed kriging theory and the variogram in the light of deterministic approaches. They considered the selection of a final theoretical variogram as partially arbitrary at best. Additional arbitrariness is also involved in the choice of the main trend, unless some extra information, seldom available, was given. In their comparison between kriging and the traditional interpolator, they stated that kriging is recognized, in principle, superior to the traditional interpolation schemes because of the illusion that the theoretical variogram adds information concerning the regionalized variable to the information objectively contained in the available data. However, in the real world application, this extra amount of information is beyond our scope because the regionalized variable is usually known only in the observation points. Hence, any assessment of the semivariogram still relies exclusively on the data obtained. In their experience, the empirical semivariogram is usually highly irregular and is meaningful only for the small values of the lagging distance. Substantial modifications can occur in the selection of the final semivariogram, since
we never know the relationship between the hypothetical true semivario-
gram and the best theoretical semivariogram. Thus, a subjective arbi-
trary component is always likely to be present in the semivariogram.

By the above reasoning, there is no proof that the accuracy of
a model based on the kriging technique should be greater than some
traditional interpolators. The choice of a number of arbitrary basic
functions, e.g., a polynomial function in Lagrange interpolation and
the selection of an operative semivariogram play similar roles. In
this respect, Gambolati and Volpi maintain that the estimates of a
good manual interpolator cannot be described as less reliable than
the predictions given by the stochastic approach.

Gambolati and Volpi (1979) highlighted use of the main trend as
not only supported by theory in the stochastic approach, but as also
supposed to improve the accuracy of the estimates in real world applica-
tions. The equation of the regionalized variables with the trend gen-
erally takes the form

\[ z(x) = \sum_{k=0}^{1} a_k f_k(x) + g(x) \]  \hspace{1cm} (42)

where the \( f(x) \) are known basic functions. The coefficients \( a_k \) and
\( g(x) \) are assumed to be unknowns. This equation, (42), can hardly have
a practical meaning. It would require a priori knowledge of the basic
functions \( f_k(x) \) and nothing can provide such a knowledge. Any choice
concerning the number and the shape of the \( f_k(x) \) would be an arbitrary
choice. The variogram is provided by the \( g(x) \), but we do not have
any record of $g(x)$ which is the random part of the observed data. If we decided to use Eq. (42), we would be forced to evaluate a preliminary semivariogram with the observed data. This will substantially distort the hypothetical true semivariogram. Besides, Eq. (42) is exactly the same as the equation in the deterministic approach. Therefore, the kriging technique is also subjected to the arbitrariness of the basic function as in the deterministic approach in traditional interpolation schemes. The use of a main trend will impact the accuracy of the estimate, i.e., the magnitude of the estimation error, in a way which is not predictable a priori and the reliability of the results will not generally be improved. They finally recommended that if the observation network is dense enough in relation to the regularity of a possible known trend, the trend should not be used at all since the choice of trend is not supported by a clear physically based conviction.

Hughes and Lettenmaier (1980) gave a valuable remark in their research concerning hydrologic projects. They stated that neither the network design procedure nor the use of kriging as in interpolation offers much of an advantage over other techniques when the process is highly irregular or is the large nugget effect. In this case, a smoother interpolator, which is not forced to pass through the data, may produce more useful results.

Aboufirassi and Marino (1983) applied the kriging technique to the elevation of the water table in Morocco. The spherical model and Gaussian model were used to fit the semivariogram. They concluded
that the spherical and Gaussian models performed equally well where there was a high density of data. Their performances differed at the area where the data were lacking. They prefer the map produced by the spherical model to the one produced by the Gaussian model, since it provides a more regular gradient of the water table.

Davis and Culhane (1983) discussed the problems in the application of kriging to contouring when using the moving or local neighborhood kriging. It has some very serious drawbacks. The basic idea behind this technique is to select only a small number of points in the proximity of the point to be estimated to set up the kriging system. Maps produced by this method exhibit spurious behavior such as discontinuities in curvature and rapid changes in areas where there is no data control. From the volumetrics point of view, these idiosyncrasies are not significant. However, since the whole purpose of a contour map is to provide a visualization to aid data interpretation, appearance of such spurious features can sufficiently detract from the overall appearance of the map to lead the geologist to seriously question the validity of the interpolation method. They concluded that for certain types of problems, kriging may not be the most appropriate estimator. For example, in the subsurface structural interpretations, the smoothing induced by the kriging process may filter out some essential information.

Jones (1983) discussed the problems in using geostatistics for petroleum application as follows. The geologists typically complain about "visual flaws" in kriged maps. Geostatistics is widely used
in the mining industry, but seldom used in the petroleum industry. A common problem in the presence of a nugget effect is that the map does not honor the data points. The kriging gives maps that are considered overly smooth and that do not extend projections into areas with little or no control. Petroleum studies rarely have as much information as minerals applications, hence generation of a variogram and further analysis is usually difficult. In addition, geostatistics has a reputation for being complicated and difficult to understand. Certain aspects of petroleum application require techniques that are not yet developed in geostatistics. Presently, geostatistics may be applied to detect trends in structure or internal properties of reservoirs by means of the semivariogram.

Soulie (1983) applied the geostatistical method to the penetration resistance for alluvial deposits and shear strength for lacustrine and marine days for the quality control of the compaction of earthen dams. He concluded that the second order model, i.e., mean and covariance can represent some geotechnical parameters, e.g., hydraulic head or porewater pressure that are considered as the regionalized variables. However, these models cannot be chosen arbitrarily because the regionalized variables have to verify some field equations. More research is needed for this type of problem.

Diamond and Armstrong (1984) noted that attention has largely been centered upon robust estimation of the experimental variogram and tended to ignore the possible sensitivity of further procedures,
e.g., kriging, upon the choice of the variogram model itself and the sensitivity to the configuration of experimental data points. Matheron (1978) stated, in effect, that if data were sent to a number n of geostatisticians, each of whom independently fitted distinct variogram models from this one data set, there would be n different solutions. Nevertheless, predictive estimates using diverse models produced almost equivalent results.
CHAPTER 2. OVERVIEW OF MULTIQUDRATIC-BIHARMONIC AND KRIGING THEORIES

Multiquadric-Biharmonic Theory

Unrestricted by any assumption of stationarity of the data, and founded on a principle of mathematical physics, the multiquadric-biharmonic method provides a true scattered data and a grid-free scheme for representing quantities on surfaces and in bodies (Franke, 1979; Micchelli, 1986; Kansa, 1987; Hardy, 1986). Under the umbrella of its physical meaning related to the disturbing potential, the multiquadric-biharmonic satisfies the equilibrium and the minimum energy concept with a mathematical condition that the sum of multiquadric coefficients should be zero (Hardy and Nelson, 1986; Micchelli, 1986; Franke, 1987).

One may categorize the phenomena in the universe into two groups: implicitly observable and explicitly observable. The height of topography on land being continuously and directly observable provides the observer with a common sense opportunity to associate data collection with maximum and minimum values; breaks in slope, etc.; it is called an explicitly observable phenomenon. On the other hand, deep ocean bottom topography, not being directly and continuously observable, is in another group. Certainly phenomena such as minerals under the ground and phosphorous pollution content in lakes are not amenable to the observer's control of data collection that is related directly to location of maximum and minimum values, and other significant features. The existence of such conditions provide us with no other choice
than indirect observation, which categorizes the data collection as being associated with implicitly observable phenomena. In general, the grid sample technique is preferred to the scattered sample technique for the implicitly observable phenomena, unless there is some clue about the location of the maximum and minimum values of the quantity under observation. The scattered sample technique or the significant mode technique is preferred to the grid sample technique in the explicitly observable phenomena. This takes advantage of this characteristic of the quantity under observation in order to be able to give the closest prediction. In sampling the explicitly observable phenomena, it is more or less art and applied science, rather than pure science. One must rely on experience and thinking processes in order to get the closest replication of that phenomena, e.g., the shape of topography. On the other hand, in the implicitly observable phenomena, a suitable logic is needed to estimate spacing of grids to be sampled. This depends on the physical characteristics of the phenomena. The greater the number of anomalies which are inherent in the phenomena, the greater is the number of grid nodes required to approximate the phenomena satisfactorily.

To be on the safe side, two steps of experimental design are generally suggested, particularly for mining applications. First, the preliminary grid nodes covering the whole region of interest should be sampled, with additional scattered sample points within this region. Then, the multiquadric-biharmonic is applied to the grid samples and
used to predict the rest of the scattered sample points. The error between the observed values of those scattered points (the test set) and the predicted values are computed. Second, the number of additional grid nodes can be computed if needed. This additional number of grid nodes to be observed should theoretically be a function of the errors and the existing density of the preliminary number of grid nodes. The exact form of this function should be based on the justification and the experience of the observer. This process could be repeated until the error of prediction is within a user limit or within some standard prescribed by an authoritative document such as National Map Accuracy Standards (Brooks, 1970).

The multiquadric procedure

The multiquadric procedure generally includes the setting up of \( n \) or \( (n + 1) \) multiquadric equations with \( n \) or \( (n + 1) \) unknown multiquadric coefficients. These coefficients, after obtaining their solution, are used to predict unsampled points in the area of interest. These provide for the production of isarithmic maps and the positional block variation as needed.

To explain the multiquadric process mathematically, let \( F(x, y, z) \) be a function of a cartesian coordinate system in \( \mathbb{R}^3 \) space that explains the magnitude of the phenomenon under observation. The multiquadric coefficients of point \( j \) are \( \alpha_j \) and \( \rho_j \) where \( j = 1, \ldots, n \). The cartesian coordinates of the data points are \( x_j, y_j, \) and \( z_j \). The
cartesian coordinates of the points $x$, $y$, $z$ whose magnitude is the function to be computed, may be either the data point or the unsampled point. Then, the multiquadric-harmonic system of equations in $R^3$ space can be written as:

$$F(x, y, z) = \sum_{j=1}^{n} \alpha_j \left[ (x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2 + \delta^2 \right]^{-\frac{1}{2}}$$  (43)

Similarly, the multiquadric system of equations in $R^2$ space is:

$$F(x, y) = \sum_{j=1}^{n} \alpha_j \left[ (x - x_j)^2 + (y - y_j)^2 + \delta^2 \right]^{-\frac{1}{2}}$$  (44)

Using the same definition of the variables, the multiquadric-biharmonic system of equations in $R^3$ space can be written as:

$$F(x, y, z) = \sum_{j=1}^{n} \rho_j \left[ (x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2 + \delta^2 \right]^{\frac{1}{2}}$$  (45)

Similarly, the multiquadric for $R^2$ space in this context is

$$F(x, y) = \sum_{j=1}^{n} \rho_j \left[ (x - x_j)^2 + (y - y_j)^2 + \delta^2 \right]^{\frac{1}{2}}$$  (46)

Note for comparative purposes that the multiquadric-harmonic equations in (43) and (44) differ from the multiquadric-biharmonic equations in (45) and (46) only by the algebraic sign of the exponent, i.e. plus or minus 0.5, associated with kernel functions in the two cases. In the first pair (for $\delta^2 = 0$) the kernel function is the Euclidean reciprocal distance and in the second pair (for $\delta^2 = 0$) the kernel function is simply the Euclidean distance. Please note carefully that the kernel function (or reciprocal kernel) is the distance (or reciprocal distance).
The kernel function is not a function of the distance, as in kriging and other schemes. In general, as to the magnitude $\delta^2$, if there is no rigid experimental, mathematical, geometric or physical basis, including relativity, time, etc., then $\delta^2$ should be set equal to zero. The geometrical meaning of setting $\delta$ or $\delta^2$ equal to zero is that the sources (or causes) of variation in the MQ-B are to be collocated with the location of the observations or measurements made in one, two, or three dimensional spaces.

The physical mean of the $\alpha_j$'s in applications involving gravitational potential anomalies, for example, is that the $\alpha_j$'s are point values (approximations) of the density anomaly function in small unit volumes within the Earth. Essentially, this involves a point mass concept for which $\delta$ cannot be zero because of singularities. It turns out in this case that there is a geometric and mathematical reason for a choice of $\delta \neq 0$, but there is no rigorous way of choosing it. Generally this is done empirically based on data spacing.

This leads practically to favoring the multiquadric-biharmonic over the multiquadric-harmonic. In the multiquadric-biharmonic method the $\rho_j$'s, applied again to gravitational potential anomalies, could be defined as point values of one half the Laplacian of the density anomaly function in a layer on the Earth's surface which, from Chasle's theorem (Heiskanen and Moritz, 1967), can be substituted for the internal density anomaly function inside the Earth. In this case it makes sense, conceptually, to collocate surface layer biharmonic sources $\rho_j$ with
surface measurements of potential. This does not exclude the prospect, however, of measuring potential or other quantities inside of volumes and performing three dimensional mapping without singularities. More of this will be described later.

A precise physical meaning of a biharmonic source in a geological setting is rather more obscure than the gravitational potential in a geodetic setting. But one can begin to generalize the concept of a biharmonic source for various applications that are less definite than gravitational potential. Potential is a form of work and energy, and this suggests that MQ-B is a prediction scheme based on a physical transition in one, two, or three dimensions from data point to data point when there is a bent spline, a plate deformation, or a volume distortion of least resistance (minimum energy). Essentially there is no physical or geometric nonsense going on between data points. This seems to be a more or less natural state of the forces at work in the past, present, and future. In geological terms it is certainly within the realm of imagination or speculation that ore deposits exist where they are and in the quantities surveyed, because the forces of nature in the distant past established a favorable collection of causes, that permitted ore deposits to occur and be distributed in accord with paths of least resistance. If so, it may be believable that generalized biharmonic sources exist to explain all of this in a form of mathematical physics as an alternative to the frequently troublesome problem arising from the concept of stochastic processes or random functions.
Solving multiquadric system of equations

The next step in the multiquadric process is to solve for the multiquadric coefficients.

If one is theoretically or physically convinced that the phenomenon under investigation obeys the minimum energy concept or is in a state of equilibrium after a past expenditure of energy, then the sum of multiquadric coefficients should be zero. This constraint leads to the result that the sum of the multiquadric functions will vanish at infinity which corresponds to potential theory. This constraint can be written as:

\[ \sum_{j=1}^{n} \alpha_j = 0 \]  
\[ \sum_{j=1}^{n} \rho_j = 0 \]  

(47)  
(48)

In order to solve the multiquadric system of equations, one generally does so in a matrix form. Here the multiquadric-biharmonic system of equations, Eq. (45), with the additional constraint, Eq. (48) is replaced by matrix notation

\[
\begin{bmatrix}
Q_{11} & \cdots & Q_{1n} & 1 \\
\vdots & & \vdots & 1 \\
Q_{n1} & \cdots & Q_{nn} & 1 \\
1 & \cdots & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\rho_1 \\
\vdots \\
\rho_n \\
\rho_0
\end{bmatrix}
= 
\begin{bmatrix}
T_1 \\
\vdots \\
T_n \\
0
\end{bmatrix}
\]  

(49)
or in the short form

\[
[Q] \begin{bmatrix} \rho \end{bmatrix} = \begin{bmatrix} T \end{bmatrix} \tag{50}
\]

where

\[
Q_{ij} = [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{\frac{1}{2}}
\]

\[
T_j = F(x_j, y_j, z_j)
\]

Then, the solution of \((n+1)\) simultaneous equations can be solved by

\[
\begin{bmatrix} \rho \end{bmatrix} = [Q]^{-1} \begin{bmatrix} T \end{bmatrix} \tag{51}
\]

provided that \([Q]\) is an invertible or a non-singular matrix.

The alternative method of solving Eq. (50) can be done using the least squares solution form, which is the expression that is well-known in the linear algebra,

\[
\begin{bmatrix} \rho \end{bmatrix} = [Q^TQ]^{-1} [Q^T T] \tag{52}
\]

Equation (52) may be regarded as the limiting least squares case.

Instead of getting a least squares solution that minimizes only the sum of squares error as in the classical least squares method, the least squares error in the limiting least squares case always equals zero, meaning that the residuals at the data points are zero or equivalently the system fits data points exactly. Theoretically the weighted least squares method can be used to solve the multiquadric system of equations. The resulting system is:
\[ \vec{\rho} = [Q^TWQ]^{-1} [QTWT] \] (53)

where \([W]\) is the weight matrix. But practically this is not useful since the system will still fit the data exactly.

Hence the multiquadric system of equations is almost never solved in the weighted least squares form. It could be done in least squares smoothing which is described later. In a well-planned sampling design, non-unit weights are not necessary. At each data point, the number of observations, the quality of the instrument used to measure the data value, the ability of the observer, and the environmental conditions should be controlled by any scientific or experimental method to be theoretically and practically equivalent. For instance, if it were necessary to change the observers that have unequal ability, both observers are to be calibrated to find the standard error of their ability to operate the observing instrument. Then, the number of observations that will be done by both observers cannot be equal, but adjusted to give the same observation quality. The number of observations of each observer will be the function of the standard error of their ability as mentioned earlier.

**Multiquadric prediction step**

The multiquadric system uses the multiquadric coefficients to predict the unsampled points in the region of interest. This can be done by the following equation in the matrix form.
\[ [T_p] = [Q_p] [\rho] \] (54)

where \([T_p]\) is the column vector of the predicted values of \(F(x_p, y_p, z_p)\), \([Q_p]\) is the \(k \times n\) rectangular matrix with \(k\) rows of the number of unsampled points and \(n\) columns of the number of data points, the \(Q_{pj}\) element of \(Q_p\) matrix is defined as the same multiquadric kernel function used to compute the multiquadric coefficient, e.g., \(Q_{pj} = [(x_p - x_j)^2 + (y_p - y_j)^2 + (z_p - z_j)^2]^\frac{1}{2}\) for the multiquadric-biharmonic kernel function in \(R^3\) space, and \([\rho]\) is the column vector of multiquadric coefficients as defined previously.

**The isarithmic map**

The multiquadric isarithmic map represents the field of interest and is superimposed with isarithmic lines or surfaces. (Preferably the map meets the U.S. national map specifications.) These lines or surfaces must be interpolated from the multiquadric predicted values in the appropriate plane or volume grid of the multiquadric system. The appropriate resolution or spacing of the grid points is determined by observing the degree of smoothness, or the lack of it, in the contour lines drawn by the computer. The computer graphics contouring routine must be a routine that does not override the multiquadric predicted value by any implicitly mathematical function used in that routine. In other words, the computer graphics contouring routine must draw the isarithmic lines or surfaces by the linear interpolation technique.
which is equivalent to or better than hand drawing. Generally, no smoothing, transformation, or filtering techniques are required or helpful in producing the multiquadric isarithmic map. If the isarithmic lines or surfaces change abruptly on the multiquadric isarithmic map, those changes should be the exact interpolations which come from the multiquadric process and no other. In order to get a better aesthetic plot of the multiquadric isarithmic map, it may be necessary to increase the number of predicted points. This will improve the aesthetics of the map and preserve the accuracy of the multiquadric process at the same time.

**Structural analysis**

Here multiquadric structural analysis corresponds somewhat with so-called structural analysis in kriging where the focus is upon interpretation of the semivariogram with its associated assumption of stationarity. The multiquadric method is not directly concerned with either semivariograms or stationarity. A semivariogram is not constructed and stationarity is not assumed. A mapping scientist must look at and interpret the total output of his mapping system, namely the map. Advances in the mapping sciences associated with computers and advanced systems in space and elsewhere should make the mapping scientist a full fledged partner in interdisciplinary teams involved with environmental and resource problems. On the other hand, it goes without saying, practically, that the mapping scientist will not progress to his greatest
potential for good if he does not cooperate with and request advice from geologists, geophysicists, hydrologists, atmospheric scientists and other experts in a greater variety of scientific or engineering fields. This ensures that data interpretation will follow the typical interpretation techniques of the particular field of interest. This also prevents unnecessary introduction of nonsense terminology that may be misinterpreted by a scientist not in his or her field of specialization.

The positional block variance

In a mathematical sense we can compute the variation \( V_D \) of a mapped quantity within a volume, roughly analogous to variance in statistics, presuming we know the function \( f(X, Y, Z) \) represents the distribution of the quantity mapped, i.e. formally

\[
V_D = \frac{1}{\text{vol}_D} \iiint (f(X, Y, Z) - \bar{f}(X, Y, Z))^2 \, dXdYdZ \tag{55}
\]

in which \( V \) represents variation and subscript \( D \) suggests that the mapping applies to a complete mineral deposit, as contrasted with a subregion or block of the deposit.

In mapping a deposit it is customary to sample the deposit, assay the samples, and to presumably know the XYZ coordinates of the sample points. Then we can solve for the multiquadric coefficients and express \( f(X, Y, Z) \) continuously by the approximation
\[ f(X, Y, Z) = \sum_{j=1}^{n} \alpha_j Q(X, Y, Z; X_j, Y_j, Z_j) \]

More particularly for mapping purposes we select a set of prediction points \( p \), equally spaced in the deposit by means of a volume grid—these at a distance small enough (high resolution) to give smooth contour surfaces within the volume, and also to produce computer graphics in the form of a stereo model. Notice that \( j = 1, 2, \ldots, n \) is the number of originally observed sample points, while \( p = 1, 2, \ldots, m \) with \( m > n \) is the number of prediction (evaluation) points used as a basis for an accurate map. In another sense the points \( p \) are a resampling of the continuous function developed from a finite set of observed samples, either regularly spaced or scattered depending on the observing philosophy, which can be expanded into a continuous function by MQ-B equations. Hence a large number of resample points are represented by

\[ f(X_p, Y_p, Z_p) = \sum_{j=1}^{n} \alpha_j \left[ (X_p - X_j)^2 + (Y_p - Y_j)^2 + (Z_p - Z_j)^2 \right]^{\frac{3}{2}} \]

\( p = 1, 2, \ldots, m \) \hspace{1cm} (56)

and

\[ \bar{f}(X_p, Y_p, Z_p) = \frac{\sum_{p=1}^{m} f(X_p, Y_p, Z_p)}{m} = \text{mean grade} \]

i.e. the average of the quantities resulting from the mapping of the deposit with \( m \) resamples within the deposit. For practical purposes Eq. (55) reduces to
where the quantity within the outer parentheses in the numerator compares the ore grade at each point of the resampling with the mean grade of the resampled points.

In a completely analogous manner, the variation within any block B which is a subregion of the deposit D can be computed from

\[
V_D = \frac{\sum_{p=1}^{m} \left( \sum_{j=1}^{n} a_Q p_j - \left( \sum_{p=1}^{m} \sum_{j=1}^{n} a_Q p_j \right)/m \right)^2}{m^D}
\]

Practically, this means that the mining engineer can, after mapping by MQ-B, choose any size block anywhere in the deposit, and from the known values of ore grade at the resample points, compute the mean grade of the block and mean square variation of the ore grade in the block. This procedure is expected to be of much greater value to the mill operator than the block variance and variance of the deposit in kriging (refer to Table 11 and the accompanying discussion). Note that the MQ-B block variation varies with position as well as size of the block, whereas in kriging assumed stationarity is constrained to the variance being independent of the position in the deposit.

For that reason we call this MQ-B procedure "deposit variation" and
"positional block variation", contrasted with the "deposit variance" and "block variance" in kriging.

There is a great similarity between kriged maps with MQ-B, (especially the linear and spherical models; see Figs. 23 and 24). From a visual inspection of these two maps it is evident that kriged maps do not, in fact, support an assumption of stationarity of stochastic process in the mapped area. That is to say it is obvious that blocks containing a great deal of "contour activity" in one area of the map actually have more block variation or variance than the same size blocks in a region of less "contour activity". This is a paradox that defies explanation by kriging alone. A reasonable explanation can occur if we accept the notion of an explanation by mathematical physics involving MQ-B.

**Least squares smoothing with the multiquadric method**

Hardy (1977) discussed least squares smoothing with the multiquadric method as follows. The basic hypothesis of multiquadric analysis is that any smooth mathematical surface and any smooth arbitrary (mathematically undefined) surface may be approximated to any desired degree of exactness by the summation of a wide variety of regular, mathematically defined surfaces, particular quadric forms. The quadric forms are not only the simplest, but also the most efficient in converging on irregular surfaces. The more or less standard form for the multiquadric method is to fit data points exactly. It is assumed, as is common in precise mapping sciences, that blunders have been eliminated and
that what is left in the data set are most probable values in some sense, even the case of a single blunder-free observation provided by a skilled observer with precise equipment. A data set that is genuinely unreliable should generally be rejected by some specified criteria before it is formatted for use in an interpolation or prediction scheme.

Nevertheless, there are occasions when smoothing is justified, or equivalently there is a need to reduce the size of large systems of simultaneous equations (Schiro and Williams, 1984). In any case, we will consider here the matter of least squares prediction (in a smoothing sense) in which the problem involves m data points, m > (n + 1), where n is the number of source or nodal points for functions centered at x_j, y_j and the extra equation is \( \Sigma p_j = 0 \). Since the choice of the n functions may otherwise be arbitrary, let us discuss some ways of minimizing this arbitrariness. A skilled observer of a graphical display of the location of data points with their associated ordinate values can quickly find points that will least affect the mapping outcome adversely. An ordinate, for example, that is located near a straight line connecting ordinates at points on both sides of it is nearly useless data. Computers can be used to thin out the number of data points based on geometric considerations of this type. See O'Hayre (1973) and Schiro and Williams (1984). Now if we use multiquadric matrix notation to represent a system of m observation equations with (n + 1) unknowns we have
in which \( Q, P, \) and \( T \) were defined earlier, and \( V_j, j=1, \ldots, m \) are the discrepancies of the predicted values from the true or observed values at the \( m \) data points, or more simply

\[
\begin{bmatrix}
Q \\
P \\
T
\end{bmatrix}
= \begin{bmatrix}
V \\
\end{bmatrix}
\]  

(59)

From this, the unit weighted least squares solution for the column vector of coefficients is

\[
\begin{bmatrix}
P
\end{bmatrix}
= \begin{bmatrix}
Q^T Q
\end{bmatrix}^{-1} \begin{bmatrix}
Q^T T
\end{bmatrix}
\]  

(60)

With the known coefficients, \( \begin{bmatrix}
P
\end{bmatrix} \), the prediction of the variables \( \begin{bmatrix}
T_p
\end{bmatrix} \), at the coordinates \( x_p, y_p \) are

\[
\begin{bmatrix}
T_p
\end{bmatrix}
= \begin{bmatrix}
Q_p
\end{bmatrix} \begin{bmatrix}
P
\end{bmatrix}
\]  

(62)

where \( j = 1, \ldots, n \).

For analysis of the prediction error at data points, we can determine \( \sum V^2 \) with

\[
\begin{bmatrix}
V^T V
\end{bmatrix}
= \begin{bmatrix}
\begin{bmatrix}
Q \\
P \\
T
\end{bmatrix}
= \begin{bmatrix}
V
\end{bmatrix}
\end{bmatrix}^T \begin{bmatrix}
\begin{bmatrix}
Q \\
P \\
T
\end{bmatrix}
= \begin{bmatrix}
V
\end{bmatrix}
\end{bmatrix}
\]  

(63)
This procedure corresponds roughly to what is called least squares filtering in some applications, involving random error or noise, e.g., radio signals, etc.

**Multiquadric-harmonic potential theory**

Hardy (1976, 1981, 1982b) defined Eq. (43) as the reciprocal multiquadric model. It is actually a point mass anomaly model for disturbing potential, \( T_p = \frac{G}{v_{ol}} \delta q \delta q \). The multiquadric kernel function is a continuous reciprocal distance function in three variables. For computational convenience, Hardy (1982b) located point mass anomalies at a constant depth \( \delta = z_j \). Then, all measurements of \( T(x, y, z) \) can be made on the xy plane at \( z = 0 \). The \( (z - z_j) \) becomes the \( \delta \) in Eq. (43). The multiquadric process, as explained earlier, can be applied to give the required solutions.

**Multiquadric-biharmonic potential theory and Hardy & Nelson's equation**

Hardy (1976, 1981, 1982b, 1986) and Hardy and Nelson (1986) discussed the multiquadric-biharmonic as follows. The mathematical expression for the biharmonic function, \( U \), in three variables is:

\[
\nabla^2(\nabla^2u) = \nabla^2u = \frac{\partial^4u}{\partial x^4} + \frac{\partial^4u}{\partial y^4} + \frac{\partial^4u}{\partial z^4} + \frac{2\partial^4u}{\partial x^2\partial y^2} + \frac{2\partial^4u}{\partial y^2\partial z^2} + \frac{2\partial^4u}{\partial z^2\partial x^2} = 0 \quad (64)
\]

\[
\nabla^4 \left( \sum_{j=1}^{n} \rho_j Q_j \right) = 0 \quad (65)
\]

The distance function, \([ (x - x_j)^2 + (y - y_j)^2 + (z - z_j)^2 ]^{1/2} \), will satisfy the biharmonic differential equation in three variables, Eq.
(64). Thus a linear combination of all distance functions used in the multiquadric approximation is biharmonic.

Hardy and Nelson (1986) provided an alternative form to the classical gravitational disturbing potential integral equation at a point outside a material body. The classical form is

$$T_p = G \int \int \int \frac{\delta q \cdot \rho q}{r^2} \, dV_q$$  \hspace{1cm} (66)$$

and the alternative form is

$$T_p = G \int \int \int \frac{\rho q \cdot \rho q}{r^2} \, dV_q$$ \hspace{1cm} (67)$$

where $\rho q = \frac{1}{2} \nabla^2 \delta q$.

Inside the volume, the Poisson's equation is,

$$\nabla^2 T_q = -4\pi G \delta q$$ \hspace{1cm} (68)$$

which, when rearranged to define $\delta q$, is

$$\delta q = \frac{\nabla^2 T_q}{-4\pi G}$$ \hspace{1cm} (69)$$

Hence,

$$\rho q = \frac{1}{2} \nabla^2 \delta q = \frac{1}{2} \nabla^2 \left( \frac{\nabla^2 T_q}{-4\pi G} \right) = \frac{\nabla^4 T_q}{-8\pi G}$$ \hspace{1cm} (70)$$

which is called Hardy and Nelson's equation.

Substitution for $\rho q$ in Eq. (67), we obtain

$$T_p = G \int \int \int \frac{\nabla^4 T_q}{-8\pi G} \cdot \frac{\rho q}{r^2} \, dV_q$$ \hspace{1cm} (71)$$
The integral in Eq. (67) is solved, if we know or find the solution for the product of some constants and the double Laplacian or biharmonic operator on \( T \) inside the volume. See Eq. (71). For that reason it is proper to call the disturbing potential in Eq. (67) a disturbing biharmonic potential, even though the results of the formal integration of Eq. (67) are identical with the results of integrating the classical integral equation, Eq. (66), through the same material body, if it were possible to do the integration in both cases. Note again the classical integral equation, Eq. (66) is an improper integral for points \( p \) inside the body, while Eq. (67) is an absolutely convergent proper integral for points \( p \), both inside and outside the body. Seldom useable at present by formal integration, this advantageous property nevertheless carries over into finite numerical integration of the integral equation in Eq. (67), as compared with finite numerical integration of the integral in Eq. (66). The numerical approximation of a solution of Eq. (66) is a finite summation called a point mass model. This takes the form of Eq. (43), in which the unknown coefficients \( \alpha_j \), representing point mass anomalies, can be found from the multiquadric process. In this process, obviously a point mass source and a data point cannot coincide because when any multiquadric kernel (distance) equals zero, \( \alpha_j \rightarrow \infty \). This confirms the practical difficulty in integrating Eq. (66) for points inside the body, because the summation in Eq. (43) becomes the formal integral in Eq. (66), as the number of \( \alpha_j \)'s in the body becomes indefinitely large in a uniform, non-duplicative fashion.
The finite numerical approximation and solution of Eq. (67) is provided by a summation called the multiquadric-biharmonic method, Eq. (45). The unknown $\rho_j$'s represent point values (biharmonic sources) of the biharmonic operator applied on the internal disturbing potential $T_j$, i.e., $\rho_j = (\nabla^4 T_j)/(-8\pi G)$. These $\rho_j$'s can be solved by the numerical multiquadric method. This numerical method becomes the formal integral in Eq. (67), as the number of the biharmonic sources, $\rho_j$, inside the body becomes indefinitely large in a uniform, non-duplicative fashion. The favorable property of an absolutely convergent proper integral in Eq. (67) is easily confirmed by Eq. (45). When point $i$ coincides with any internal point $j$, then the distance, $\varepsilon_{ij}$, becomes zero, and hence, the product $\rho_j$ with the multiquadric kernel function ($\kappa_{ij}$) equals zero. This informs us that a biharmonic source at any $q$ inside the body, which coincides with an evaluation point $p$ does not contribute to disturbing potential at that point $p$ inside the body. This is accounted for by the fact that the distance $\varepsilon_{pq} = 0$, not because the biharmonic source at $p$ is zero. In other words, the disturbing potential at point $p$ inside a material body is caused only by the sum of all products of the biharmonic sources and the non-zero distances. For the exceptional case of distance $\varepsilon_{ij} = 0$, which produces a null effect at the point where $i = j$, the non-zero biharmonic source at point $j$ is $((\nabla^4 T_j)/(-8\pi G))$, which is consistent with Poisson's equation.
Hardy and Nelson proved the identity of Eqs. (66) and (67) mathematically by Green's second identity as follows.

\[
\iiint_V (U(v^2V) - V(v^2U)) \, dv = \iint_S (U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n}) \, ds \tag{72}
\]

Let

\[U = \delta(x_q, y_q, z_q), \quad V = \frac{1}{2} \lambda_{pq}\]

so that

\[v^2U = v^2\delta(x_q, y_q, z_q), \quad v^2V = \lambda_{pq}^{-1}\]

Eq. (72) becomes

\[
\iint_V \frac{\delta_q}{\lambda_{pq}} \, dv_q - \iint_V \frac{v^2\delta_q}{2} \lambda_{pq} \, dv_q = \iint_V (\delta_q \frac{3(\frac{1}{2}\lambda_{pq})}{\partial n}) - \frac{1}{2} \lambda_{pq} \frac{3\delta_q}{\partial n} \, ds \tag{73}
\]

In the surface integral term of Eq. (73) at an infinitesimal distance outside the sphere of fixed radius R, the density and radial derivative of the density vanish. The non-zero quantity, \(\frac{1}{2}\lambda_{pq}\), and the radial derivative of \(\frac{1}{2}\lambda_{pq}\) exist for point p, an infinitesimal outside the sphere, while the variable r of point p decreases to the fixed radius of the sphere R in a limiting process. Hence, the right hand side of Eq. (73) is zero, and we get
\[
\iint \frac{\delta q}{V} y_{pq} \, dvq = \iiint \frac{\nu^2 \delta q}{2} \, z_{pq} \, dvq
\] (74)

which shows the equivalence of Eqs. (66) and (67) if we let \( \rho_j = \frac{\nu^2 \delta q}{2} \).

**Kriging Theory**

There are many kriging schemes, and users are still trying to change or simplify the theory. However, most schemes seem to follow the general trend that will be overviewed in this section. The word "kriging" itself, by some authors, refers to the step of estimating the values at given spatial locations (David, 1977; Henley, 1981). The kriging techniques are all related, and are refined versions of the weighted moving average technique used by Krige (Krige and Ueckerman, 1963). Examples of these techniques are simple point or punctual kriging, block kriging, lognormal kriging, disjunctive kriging, co-kriging, universal kriging, etc.

The theory of regionalized variables, as formalized by Matheron in 1965, is considered to be the basis of the approach in geostatistics (Matheron, 1965; David, 1974, 1977; Journel and Huijbregts, 1978). This specific theory and a different terminology were developed to use in very specific problems which occurred in the mining industry. Among other peculiarities, grade values are much less continuous than most of the biological or the ecological data considered by biometricians. Besides this, mines are truly three-dimensional bodies and samples are not "points."
The function will be called a regionalized variable, \( z(x) \), which is the value of the function at point \( x \) in space \( \mathbb{R}^3 \). In other words, if a variable is distributed in space, it is said to be "regionalized." The theory considers such a function \( z(x) \) where \( x \) is a point or a vector of \( \mathbb{R}^n \) as one realization of a random function (RF) \( Z(x) \). This means a perfectly well-defined, unique numerical value is turned into a realization of a random process. The characteristics of the RF \( Z(x) \) are used to estimate the unknown points. A regionalized variable possesses two characteristics, a random and a structure aspect that can be represented by a mathematical function (Journel and Huijbregts, 1978).

The regionalized variables \( z(x) \) and \( z(x_j) \) between points \( x_i \) and \( x_j \) apart at some distance, are correlated by its spatial structure. Geostatistical analysis quantifies the random components and spatial components by means of second-order moment about the expectation \( m(x) \) of the regionalized variable (RV) \( Z(x) \), i.e.,

\[
\text{Var} \{Z(x)\} = E[(Z(x) - m(x))^2]
\]

(75)

If any phenomenon varies from place to place in such a way that it can be considered as a random function, (e.g., ore grade, ore thickness, etc.), then semivariograms may be constructed to describe the spatial variability of that phenomenon. By using the characteristics of spatial variability, the prediction of the regionalized variable of that phenomena at any other points in the same domain can be done by the technique called kriging.
Constructing the semivariogram

After sampling the realizations of a random function, the first step in the geostatistics process is to construct the semivariogram of point value or apparent or experimental semivariogram. The semivariogram is one-half of a generalization of mean-square successive differences to the spatial distribution of the variables (Koch and Link, 1971). If \( N \) is the number of observations, \( g_i \) is the order of the observation. The semivariogram of point value at an arbitrary lag interval \( h \) is

\[
\gamma^*(h) = \frac{1}{2(N-h)} \sum_{i=1}^{N-h} (g_i - g_{i+h})^2
\]

\( i = 1, 2, \ldots, N \)

\( h = 1, 2, 3, \ldots, < N. \)

Royle (1980) suggested that the maximum of \( h \) is \( N/2 \). This is because the variable \( N \) decreases with increasing distance \( h \), which in turn reduces the accuracy of the semivariogram. David (1977) had considered the same matter earlier, and had explained it in terms of the fluctuation of the local variogram. David found that for small values of \( h \), the relative variance is proportional to \( h/L \), where \( L \) is the length over which the variogram is computed. When \( h \) becomes equal to \( L/2 \), the relative variance becomes 1. Hence, beyond the point where \( h = L/2 \) there is no correlation between the local variogram \( \gamma^*(h) \).
and the theoretical variogram $\gamma(h)$, meaning the continuous function fitted to the discrete semivariogram.

Using the semivariogram as a means of explaining the spatial variability is justified by the assumption of the stationarity (Journel and Huijbregts, 1978; David, 1977). The theory of a regionalized variable assumes no knowledge of the mathematical relation in that region of interest. The expectation of each regionalized variable at every point in that region is assumed to be the same. The correlation between the RVs is dependent on the geometry of the sampling points only, regardless of the position. This is called the stationarity assumption.

The stationarity assumption can be categorized into two classes: second-order stationarity and quasi-stationarity. A random function is of second-order stationarity if (a) the mathematical expectation of the regionalized variable exists and does not depend on the support point $x$, so that $E(Z(x)) = m$ for all $x$, (b) the spatial covariance of the regionalized variable $Z(x)$ is the same all over the field of interest, and (c) for each pair of regionalized variables $Z(x)$ and $Z(x + h)$, the semivariogram $\gamma(h)$ exists and depends on the separation distance $h$ only. Quasi-stationarity is the same, except it does necessarily satisfy (b) for practical purposes. This is because the semivariogram is used within the neighborhood of estimation only. Thus quasi-stationarity only is needed. This means that the assumption of stationarity is merely needed for limited distances, not for the whole region of interest.
Before discussing the characteristics and the interpretations of the shape of the semivariogram by the geo-statistician, it should be noted that many authors shorten the word "semivariogram" to "variogram." One should first check whether an author means semivariogram, $\gamma$, or variogram $2\gamma$ (Royle, 1980). Also, the word "semivariance" is equally used by some authors to mean "semivariogram." The equivalent term for "variogram" is "variance," which is the variance used in the spatial correlation and is not equivalent to the variance which is used in conventional statistics. In general, the semivariogram is an increasing function of distance $h$. This implies that on the average the difference of the regionalized variables increases with the distance or the lagging of the correlation (Matheron, 1963). A regionalized variable, which has the spatially dependent or the spatial correlation, will show this spatial structure on the semivariogram curve. Due to lagging of the correlation beyond a certain distance, the slope of the semivariogram, $d\gamma(h)/dh$ is progressively reduced from positive value and approaches zero. The first point on the curve from the origin where the slope is equal to zero is the point where the spatial correlation vanishes. The distance, measured along the abscissa to the point starting at the origin, is called the range of influence. In other words, it is the average of the maximum distance where the regionalized variables in the area of interest have the correlation among each other. The distance, measured along the ordinate to the point beginning with the origin, is called sill. The sill is theoretically equal to the
variance of the regionalized variables (Royle, 1980). The sill can be divided into two components: (a) the random variance or nugget variance, which is the distance from the origin to \( \gamma(h) \)-intercept, and (b) the spatial variance, which is the distance from the \( \gamma(h) \)-intercept to the sill (see Figure 1). The random variance accounts for the random variations and the residual influence of all variabilities where their ranges are smaller than the distance of observation or the distance that is shorter than the first lag distance used in computing the semivariogram. Although the theoretical semivariogram curve should intercept the \( y \) or \( \gamma(h) \) axis at the origin, some phenomena may cause the curve to intercept above the origin. The mineralization that may occur as nuggets or blobs is an example of these phenomena. This leads to the expression of the nugget effect \( E \) (Royle, 1980):

\[
E = \frac{\text{Random component of the variance}}{\text{Spatial component of the variance}}
\]  

(77)

The large value of nugget variance reveals that the variability present is mainly random and cannot be accounted for in the spatial structure of the regionalized variable in the field of interest. The researcher needs to decide from his experience whether he wants to resample to determine if the large value of nugget variance is a real characteristic of that phenomena or not.

David (1977), Journel and Huijbregts (1978), Royle (1980), and others categorized the general shape of the semivariogram and its behavior at the origin into four basic types: (1) continuous, (2) linear,
Figure 1. A typical semivariogram
(3) nugget effect, and (4) random (see Figure 2). The continuous type has a parabolic trend at the origin and represents a regionalized variable with high continuity, such as a bed-thickness. The linear type is characterized by an oblique tangent at the origin, and represents a variable which has an "average" continuity (Matheron, 1963). The nugget effect type reveals a discontinuity at the origin and represents the nugget variance as mentioned earlier. The random type is a limit case corresponding to the classical notion of the pure random variable. Between the continuous type and the random type appears a range of intermediates--the study of which is the proper objective of geostatistics. When the range exists, the semivariogram is said to be a transition type phenomenon. It is said to be a non-transition type if there is no range of influence. In the non-transition type, the semivariogram increases continuously with distance, which indicates that either a sill is truly absent in the spatial structure or the longest distance between two observation points was still within the range of influence. The non-transition type semivariogram can show linear, concave, or convex curve or combinations of these (see Figure 3). Highly convex semivariograms are believed to reveal "drift" or "trend" in the spatial structure of a regionalized variable. The linear model, \( \gamma(h) = \rho h \), and the generalized linear model, \( \gamma(h) = \rho h^\lambda \) \( 0 < \lambda < 2 \), are the most frequently used as the semivariogram with no sill.

In some phenomena, the slope, \( d\gamma(h)/dh \), or the semivariogram curve produces a negative value after approaching zero at the sill, and then
Figure 2. The basic shapes of a semivariogram and its behavior at origin (Matheron, 1963)
Figure 3. Semivariogram models without sills
reverses to take positive values again like the cycle of sine or cosine curve. This is called a "hole effect." An example is the semivariogram that represents the alternation of two different types of ore, where the sampling direction cuts across both ore types. A deposit of this type caused an anisotropic condition.

The special case of the semivariogram shape is the discontinuous curve (see Figure 4). This shows that the difference in regionalized variable values produces a sharp change at some point with respect to an increase in distance. The spatial structure of the shorter distance is different than the spatial structure of the longer distance. This shape is generally found in a vertical semivariogram that represents the different depositional strata, which is the stratum that is deposited at different geologic times.

A nested structure is a phenomenon that results from several processes being superimposed on each other (David, 1977; Journel and Huijbregts, 1978). Phenomena of this type induce the complex structures of the semivariogram, especially if these phenomena operated at different scales. Journel and Huijbregts (1978) represented this structure by the sum of a number of variograms, each characterizing the variability at a particular scale:

$$\gamma(h) = \gamma_0(h) + \gamma_1(h) + \gamma_2(h) + \ldots + \gamma_i(h)$$  

(81)
Figure 4. The discontinuous semivariogram
\( i = \text{number of processes}. \)

More details on the nested structure can be found in the references cited.

It is worthwhile to discuss the correlogram based on the title of this dissertation. There is a mathematical relationship between the semivariogram and the covariance function in the least squares collocation scheme (Heiskanen and Mortiz, 1967; Agterberg, 1974; David, 1977; Journel and Huijbregts, 1978). Journel and Huijbregts (1978) and Agterberg (1974) defined it in the form \( c(h) = c(0) - \gamma(h) \), where \( c(h) \) is the covariance function, \( c(0) \) is the covariance function at zero lag distance or variance. Journel and Huijbregts also defined it in the form \( \rho(h) = 1 - \frac{\gamma(h)}{c(0)} \), where \( \rho(h) \) is called a correlogram.

The correlogram is another type of tool for the analysis of the spatial structure of the regionalized variable. This will not be investigated in this dissertation. The point I want to make here is that working with the semivariogram or covariance in the kriging process is equivalent. The semivariogram is more important to the geo-statistician for the spatial structure analysis.

**Fitting the semivariogram by the mathematical model**

Fitting the semivariogram to the model is the next step in the preprocessing procedure after obtaining the semivariogram of the point values. The appropriate mathematical model is chosen by the researcher to fit the semivariogram point values by any of the numerical methods,
preferably the least squares technique, iterative method, or regression analysis. This will put the semivariogram of the point value in the operational form. Supposingly, this should enable the geo-statistician to do the structural analysis anywhere in the region of interest. However, this process implicitly creates an error during the least squares fitting and it depends upon the justification and an experience of the researcher as to which mathematical model he should choose. Definitely, the justification must come from the physical knowledge and common sense. If the variogram of the point value shows a very erratic curve, one would have to reevaluate the experimental design scheme, choose a new lagging distance (h), or use different semivariograms for the different part of the region. Again, these processes are based on the experience and knowledge of the researcher. The structural analysis of semivariogram is the most important part of geostatistics and can be studied as an independent subject. The reliability of the semivariogram is based solely on the reliability of the data.

Journel and Huijbregts (1978) remarked that the kriging system has a unique solution if and only if the covariance matrix is a strictly positive definite and, thus, has a strictly positive determinant. In order to meet this requirement, none of the data points can coincide with one another, or the determinant of the covariance matrix will be zero. This condition also insures that the kriging variance will be a non-negative value. Therefore, any mathematical model that satisfies this condition can be used in the kriging procedure.
Mathematical models for fitting to the semivariogram (Bubenicek and Haas, 1969; Agterberg, 1974; David, 1977; Royle, 1980; Henley, 1981)

The following models are generally used by geostatisticians:

1. Model with sill
   1.1. The spherical model:
   \[ \gamma(h) = c \left[ \frac{3}{2} \frac{h}{a} - \frac{1}{2} \left( \frac{h^3}{a^3} \right) \right] + c_0 \quad h \leq a \]
   \[ \gamma(h) = c + c_0 \]
   \[ \gamma(0) = 0 \]

   1.2. The exponential model:
   \[ \gamma(h) = c_0 + c \left[ 1 - \exp \left( - \frac{h}{a} \right) \right] \]

   1.3. The Gaussian model:
   \[ \gamma(h) = c_0 + c \left[ 1 - \exp \left( - \frac{h^2}{a^2} \right) \right] \]

2. Model without sill:
   2.1. The Logarithmic model or De Wijsian model
   \[ \gamma(h) = c_0 + c \ln(h) \]

   2.2. Generalized linear model
   \[ \gamma(h) = c_0 + ch^\lambda \quad \forall \ 0 < \lambda < 2 \]
3. Other models:

3.1. The hole effect model

\[ \gamma(h) = c(1 - \frac{\sin ah}{ah}) \] 

(84)

These models are designed for the isotropic spatial structure case. However, they are used in the anisotropic spatial structure case by adding the mathematical conditions to define isotropic angles. Then the proper scale factors are applied to increase or decrease the range of influence and modify the shape of the semivariogram at particular direction.

**Kriging, the Interpolation Scheme**

**Punctual kriging or simple point kriging**

A kriging scheme is used to predict the distribution of the regionalized variable in the domain \( \mathbb{R}^n \) at any required position. The kriging weights or coefficients are applied to the sampled points by using the "best linear unbiased estimator" (BLUE). This estimation takes into account the spatial structure among the data points, and the spatial structure between the predicted point and the data points used by the semivariogram. The kriging system uses a linear combination of data at a set of sample points to predict a value at predicted points in the form
\[ z(x_p) = \sum_{i=1}^{n} \lambda_i z(x_i) \quad i = 1, \ldots, n \]  
(85)

where \( n \) = number of data points.

The weights, \( \gamma_i \), are obtained by using the minimum variance condition (Journel and Huijbregts, 1978):

\[ E \{ (z^*(x_p) - z(x_i))^2 \} = \text{minimum} \]  
(86)

which can be written in the form of semivariogram as:

\[ E \{ (z^*(x_p) - z(x_i))^2 \} = - \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \gamma(x_i, x_j) + 2 \sum_{j=1}^{n} \lambda_j \gamma(x_i, x_p) \]  
(87)

where \( \gamma(x_i, x_j) \) is the semivariogram between the data point \( x_i \) and \( x_j \) and \( \gamma(x_i, x_p) \) is the semivariogram between the data point \( x_i \) and the predicted point \( x_p \).

The condition that the sum of the weights equal to 1 is added to ensure the unbiasedness of the estimation in the form (David, 1977):

\[ \sum_{i=1}^{n} \lambda_i = 1 \]  
(88)

By using the Lagrange multiplier, \( \mu \), well-known in the least squares prediction technique, the system of \( (n+1) \) linear equations can be solved simultaneously to give the weights. This system of equations is called the "kriging system."
\[
\sum_{j=1}^{n} \lambda_{pj} y_{ij} + \mu = y_{pj} \tag{89}
\]

\[
\sum_{j=1}^{n} \lambda_{pj} = 1
\]

which can be written in the matrix form

\[
\begin{bmatrix}
  \gamma_{11} & \cdots & \gamma_{1n} & 1 \\
  \vdots & \ddots & \vdots & \vdots \\
  \gamma_{n1} & \cdots & \gamma_{nn} & 1 \\
  1 & \cdots & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  \lambda_{p1} \\
  \vdots \\
  \lambda_{pn} \\
  \mu
\end{bmatrix}
= 
\begin{bmatrix}
  \gamma_{p1} \\
  \vdots \\
  \gamma_{pn} \\
  1
\end{bmatrix}
\tag{90}
\]

Then, \([\lambda_{pj}]\) is solved by:

\[
\begin{bmatrix}
  \lambda_{p1} \\
  \vdots \\
  \lambda_{pn} \\
  \mu
\end{bmatrix}
= 
\begin{bmatrix}
  \gamma_{11} & \cdots & \gamma_{1n} & 1 \\
  \vdots & \ddots & \vdots & \vdots \\
  \gamma_{n1} & \cdots & \gamma_{nn} & 1 \\
  1 & \cdots & 1 & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
  \gamma_{p1} \\
  \vdots \\
  \gamma_{pn} \\
  1
\end{bmatrix}
\tag{91}
\]

Hence, the regionalized variable at point \(p\), \(z^*(x_p)\), is then predicted by

\[
\begin{bmatrix}
  z(x_1) \\
  \vdots \\
  \hat{z}(x_n) \\
  0
\end{bmatrix}
= \begin{bmatrix}
  \lambda_{p1} & \cdots & \lambda_{pn} & \mu
\end{bmatrix}
\begin{bmatrix}
  \gamma_{p1} \\
  \vdots \\
  \gamma_{pn} \\
  1
\end{bmatrix}
\tag{92}
\]

The estimation variance or kriging variance is computed by:

\[
S^2 = \begin{bmatrix}
  \gamma_{p1} & \cdots & \gamma_{pn} & 1
\end{bmatrix}
\begin{bmatrix}
  \lambda_{p1} \\
  \vdots \\
  \lambda_{pn} \\
  \mu
\end{bmatrix}
\tag{93}
\]
The weights in the kriging system depend only on the geometry of the data points, the predicted points, and their spatial structure, but not on the magnitude of error of data (Journel and Huijbregts, 1978). The weights associated with the data points that are closest to the predicted data point have a greater value than the weights for prediction points farther away. The data points that lie between the furthest data point and the predicted data point will reduce the weight that would otherwise be applied to the furthest data point. The kriging system will fit the data points value exactly.

Block kriging is the same as simple kriging except that mean block grade values are substituted for point grade values in the construction of semivariograms, $\gamma(h)$ for point kriging and $\overline{\gamma}(h)$ for block kriging (Henley, 1981). $\overline{\gamma}(h)$ expresses the variance between the data point or observation and the block to be estimated. This variance will, in general, be less than the variance between the point at the center of the respective volumes, because of the average effect.

Simple kriging, whether of point or block values, obeys the assumption of second order stationarity (David, 1977). There is no regional trend nor drift in the regionalized variable values and the semivariogram is independent of location. Henley (1981) indicated that although the kriging estimator will be the best linear unbiased estimator, it will not reproduce the true values and will not give a model surface with properties which even closely resemble those of the "true" surface.
<table>
<thead>
<tr>
<th>DISTRIBUTION</th>
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<th>Simple known (e.g. lognormal)</th>
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<td>Disjunctive kriging</td>
<td>?</td>
<td>?</td>
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<td></td>
<td>Simple kriging (point or block)</td>
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<td>Generalised</td>
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</tbody>
</table>

**Figure 5.** Kriging methods vs. the spatial characteristics (Henley, 1981)
Other types of kriging schemes

Henley (1981) summarized available kriging methods against the spatial conditions for which the kriging methods should be used (Figure 5). The rest of this section will be devoted to a rough overview of other kriging systems and discussing the associated geostatistical theories because they are not comparable to the multiquadric-biharmonic method in the sense used in this dissertation.

Lognormal kriging (David, 1977; Henley, 1981) Lognormal kriging is applied to a non-normal distribution of ore grade. The ore grade may have a high positive or negative skew. The ideal approach is to transform the observed ore grades by the following equation:

\[ y_i = \log(z(x_i) + a) \]  

where \( a \) is an arbitrary constant to optimize the fitness of the transformed values, \( y_i \), to the normal distribution. The semivariogram is then constructed by using the transformed values instead of the original observed values, \( z(x_i) \). The kriging system will estimate the predicted values in terms of logarithms. The inverse transformation is applied to the predicted values, \( y^*(x_p) \), to obtain the predicted ore grade, \( z^*(x_p) \). The \( y^*(x_p) \) value will not be a linear estimated value, but will be a weighted geometric mean (if \( a \) in Eq. (94) is zero), for instance.
This may be a good point estimator, but the method will not minimize the estimation variance.

**Disjunctive kriging** (David, 1977; Henley, 1981) Disjunctive kriging is a practical method that is closest to the best ideal theoretical estimator known as the "condition of expectation." This is because the information that is necessary to determine the conditional expectation is never available in a real situation. Theoretically, the conditional expectation of the predicted regionalized variable, $y^*(x_p)$, which is the best possible estimator, will be obtained if the real distribution of $n+1$ regionalized variables $z^*(x_p)$ and $z(x_1), \ldots, z(x_n)$ is precisely known.

In order to approach the ideal situation, the Gaussian distribution with stationarity is assumed to be applied to the regionalized variables by transforming the original data with some function. In disjunctive kriging, a set of polynomial functions is applied to the transformation process. The regionalized variables are univariate normally distributed for each $z(x_i)$ value, and bivariate normally distributed for every pair, $z(x_i)$ and $z(x_j)$. Theoretically, the disjunctive kriging provides a better estimation than simple linear kriging, but not as good as the conditional expectation. Using disjunctive kriging is more restricted to the assumption than using simple kriging. There is no indication that the stationarity constraint might be relaxed and the computational
cost is higher than the simple kriging. The estimator obtained from this method is no longer the linear estimator.

**Universal kriging (Journel and Huijbregts, 1978; Henley, 1981)**

Universal kriging will be used if the "trend" or "drift" appears in the random function $Z(x)$. This can be observed by the rapid increase of the semivariogram value over the square of lagging distance ($h$). The universal kriging attempts to avoid the onerous requirements of the simple kriging assumptions, which are rarely found in nature. The so-called "universal kriging" is not universal, either in theory or in application. The new assumption for universal kriging is that the expectation or mean of the regionalized variables is not constant, but follows a trend over the region of interest. A polynomial of proper degree and order is generally used to represent this trend, as has been used in the polynomial trend surface analysis. The system of equations in universal kriging is merely the combination of the simple linear kriging system and the polynomial trend surface analysis system. Thus, there will be $n + 1 + k$ unknowns in the universal kriging system of equations for each of the predicted points, $x_p$. The $n$ is the number of weights for $n$ data point values, one (1) is the Lagrange multiplier, and $k$ is the number of coefficients for the polynomial trend.

The coefficients of the polynomial trend are unknowns that must be solved before solving for the kriging coefficients, both of which must be done to complete the process called universal kriging. This
is a specialized procedure. Otherwise, there will be no way to compensate for the actual drift in the semivariogram curve, since every \( \{z(x_i) - z(x_j)\} \) term in the semivariogram formula will contain the random part and an unknown contribution from the deterministic drift. David (1977) concluded that "there is no known direct solution; one can only make a set of assumptions and try to verify them by trial and error." The assumed semivariogram curve is used, and an attempt is made to estimate a random part from an approximated drift with a subjectively chosen neighborhood size or range. When a self-consistent solution is obtained, it is assumed correct and used in the universal kriging estimation. Webster and Burgess (1980) concluded that in rare cases the benefits of universal kriging are to provide greater accuracy in the estimation and to give a smaller kriging variance when nugget variance is negligible. In all other cases in their study, simple kriging was more appropriate because of its simplicity, time savings, and computation cost.

**Cokriging** (David, 1977; Journel and Huijbregts, 1978) This is the last type of kriging that will be discussed. Cokriging is employed when there is a need to estimate more than one regionalized variable. This situation could happen if one type of the regionalized variable is well sampled, and the other regionalized variable is not. A cross-semivariogram is constructed between two or more types of a regionalized variable, under the assumption that both have the same
spatial structural characteristics. Theoretically, the cross-semivariogram will reveal the spatial correlation between two or more variables. This procedure is supposed to give an accurate estimation of a regionalized variable, although it is not well sampled. The system of equations for cokriging is very similar to simple kriging.

**Block variance and estimation variance** (David, 1977; Journel and Huijbregts, 1978)

The block variance and estimation variance will be discussed only to compare the idea of block variance and estimation variance in the geostatistic definition and the block variation in the multiquadric-biharmonic definition. "Block variance" is synonymous with "spatial block variance." "Kriging estimation variance" is a combination of the classical statistical "variance" and additional spatial properties. These terms are clarified here to avoid confusion in communication that may occur to the reader during the rest of this section.

The geostatisticians want to determine these variances because they want to provide knowledge of the estimation reliability of the regionalized variable, and also to provide knowledge of the spatial variability of the regionalized variable to the mining engineer. The mining engineer wants to know the "spatial" variance of blocks of a given size rather than the "spatial" point variance because the mill is fed by the block of ore rather than the "piece" of ore. By using the "spatial" block variance, the mining engineer can forecast the
variability of stopes, truck loads, or blasts, which are related to the regularity of a mill feed. In other words, the geostatisticians have to establish or predict the "spatial" variance-volume relationship in order to supply information to the mining engineer.

The estimation variance

The error of estimation in the kriging method is a function of the similarity which exists between the value of a regionalized variable at one point and the value of the regionalized variable at another point, some distance away in the same region of interest (David, 1977). Thus, the geostatistics error of estimation for a given geometrical pattern depends only on the pattern and not on the particular location.

Practically, the geostatistical assumption and the semivariogram are used as a means to compute estimation variance of the point. This is because the true grade is never known for applying the classical statistical formula (David, 1977).

Similar to the "spatial" estimation variance of the point, the "spatial" estimation variance of the block can be ideally achieved by comparing the unknown true grade of the block and the predicted grade of the block. Again, the geostatistician prefers to compute this value by means of the semivariogram. The following equation will show the relationship between the semivariogram and the spatial estimation variance. Let the $z(V_i)$ be true unknown grades of the blocks
\( V_i \), and the \( z*(V_i) \) are the linear combinations \( z*(V_i) = \sum_{j=1}^{n} \lambda_{ij} z(x_j) \) of the known sample grades at location \( x_j \) (\( j = 1, \ldots, n \)). Then:

\[
\begin{align*}
\text{Var} (\varepsilon_i) &= \text{Var} [z(V_i) - z*(V_i)] \\
\text{Var} (\varepsilon_i) &= \text{Var} [z(V_i) - 2 \text{cov} [z(V_i), z*(V_i)] + \text{Var} [z*(v_i)] \\
\text{Var} (\varepsilon_i) &= \text{Var} [z(v_i)] - 2 \sum_j \lambda_{ij} \text{cov} [z(V_i), z(x_j)] + \sum_i \sum_j \lambda_i \lambda_j \text{cov} [z(x_i), z(x_j)] (95)
\end{align*}
\]

If we considered each variance and covariance separately, and compare Eq. (95) with the semivariogram formula, Eq. (76), it can be seen that they should be computed from the semivariogram (David, 1977). For more details on this point, consult the references cited.

The block variance

There are three categories of "spatial" block variance: the spatial variance of the blocks in the region of interest, the spatial variance within the block, and the spatial covariance of the grade of a block and the grade of a sample. The geostatistician makes a distinction between the spatial variance of the block and the point because it is found by experience that the block mined has a smaller variance than the samples (David, 1977). This idea is not familiar to American schools as David stated, in effect: "We will also introduce you to the model which we will later use and which we will simply call geostatistics, since this name appeared in France around 1960 to designate
this very particular subject which has nothing in common with what the American school called geostatistics." Only the spatial variance within the block will be overviewed here.

The spatial variance within the block is the variability of the grade of the block. It depends on the average difference in grade which exists between any two points inside the block (David, 1977). Since the difference between the grades at two points is expressed by the semivariogram, it should be able to compute the spatial variance within the block from the variogram. Again, the spatial variance of this type is not dependent on the location of the block, but on the shape or geometry of the block, because of the onerous requirements of the stationarity hypothesis.

Only the method of computing the spatial variance of the block in the region of interest will be presented here because of its comparability to the positional block variation of the multiquadric-biharmonic method. The main target of geostatistics is to be able to predict and plot the spatial block variance of the block in the region of interest against the size of the block. This is called the variance-volume relationship (David, 1977; Sabin, 1984). In mining practice, the volumes to be considered are usually regular blocks, thus a few models of semivariogram are sufficient to describe the spatial correlations and charts can be produced to facilitate the computation of the spatial block variance and other parameters (David, 1977; Journel and Huijbregts, 1978). The key theory used in computing these quantities is Krige's
relationship that "the variance of the grade of a block within a deposit
is equal to the variance of the grade of a point within a deposit minus
the variance of the grade of a point in the block" (David, 1977).
The spatial block variance is computed according to the shape and size
of the block. Based solely on the geostatistician and mining engineer,
the purpose of the spatial block variance is mainly to give the mining
engineer an idea as to the distribution of the regionalized variable
in which he or she is interested. The shape of the block used in comput­
ing the spatial block variance may be not the same as the shape of
the block that is actually mined. The idea that the mining engineer
gets from the computed or theoretical spatial block variance will allow
a reasonable estimation of the distribution of the mined block. As
has been said, "knowing only the infinitesimal piece of information
is better than not knowing at all." Here again, the theoretical spatial
block variance may have nothing to do with the real distribution of
the regionalized variable under consideration, but at least it has
a rigorous theoretical basis, which is the most believable information
that can be obtained from the data.

The following is a practical example of computing the spatial
block variance by the preproduced F-chart (Sabin, 1984). Suppose we
want to compute the spatial block variance of a small volume (v) with
dimensions (b x h x l). One dimension, h, of the small volume is kept
constant at 10 units. The small volume moves within a large volume
with dimension (b x H x L), which is assumed to be (10 x a x a) unit^3
where "a" is a range of influence of the spatial structure of the region­
alized variable. The problem is to compute the spatial block variance
of the small volume given b = 10 units, while l is increased at 10
units at a time from 10 units to 100 units. Assuming that the spherical
model, Eq. (79) was fitted to the semivariogram of the point value
with C = 18.53 and a = 1066.98.

To solve this problem, equation (Sabin, 1984)

$$\sigma^2(V/V) = C[F\left(\frac{H}{a}, \frac{1}{a}\right) - F\left(\frac{b}{a}, \frac{1}{a}\right)] \tag{96}$$

is used. In this particular case, two preproduced charts are needed,
namely a 2-dimensional, F-function and a 1-dimensional, F-function,
see Figures 6 and 7 (Royle, 1977; Journel and Huijbregts, 1978).

The solution, for instance, l = 10 units is simply computed by:

$$\sigma^2(V/V) = 18.53 \left[F\left(\frac{1066.98}{1066.98}, \frac{10}{1066.98}\right) - F\left(\frac{10}{1066.98}, \frac{10}{1066.98}\right)\right]$$

$$= 18.53 \left[F(1, 1) - F(0.0093, 0.0093)\right]$$

By using both charts, the F(1, 1) and F(0.0093, 0.0093) are read as
0.66 and 0.469, respectively. Hence,

$$\sigma^2(V/V) = 18.53 (0.66 - 0.00469) = 12.14 \text{ unit}^2$$

Therefore, the spatial block variance of a small volume with dimensions
(10 x 10 x 10) unit$^3$ moving in the large volume with dimensions (10
x 1066.98 x 1066.98) unit$^3$ equals 12.14 unit$^2$. The same computational
Figure 6. Two-dimensional, F-function, $F(L/a)$: variance of samples in block of sizes. $H$ and $L$
Figure 7. One-dimensional, F-function, $F(L/a)$: variance of samples on line of length $L$, for values of $L/a$ less than 0.2, $F(L/a)$ equals 0.5 $L/a$
method and the same charts are used for $d$ equals 20, 30, ..., 100 units, respectively. The variance-volume relationship diagram may be plotted. The mining engineer can use this diagram as a guide line for further decisions related to the mining processes. More details of this topic and discussion about $F$-functions may be found in the references cited.

Discussion of the Multiquadric-Biharmonic and Kriging Theories

It should be clear by now that the multiquadric method is very different from the kriging method—both theoretically and practically. Hardy (1971a) discovered the originally empirical multiquadric method in 1968. Later it was announced that the multiquadric method is explainable in terms of harmonic and biharmonic theory (Hardy, 1982). There are excellent reasons for preferring the biharmonic form but the theory is sometimes considered to be difficult. Fortunately the theory is only used to explain why the method works in a basic sense. It does not interfere with putting the method into practice. The basic multiquadric-biharmonic method is as easy to use now (1988) as it was in 1971, before its foundation in mathematical physics was established. The new mathematical foundation for MQ-B justifies the use of classical mathematical methods such as differential and integral calculus, differential equations, numerical analysis, linear algebra, and vector spaces to broaden and enhance the application of the multiquadric method (Hardy, 1975; Hardy and Nelson, 1986; Kansa, 1987, 1988; Madych and Nelson, 1988; Micchelli, 1987). On the other hand, the kriging method, the
mainstay of the geostatistical method, was discovered and evolved on the basis of assumptions and concepts of spatial statistics from the work of Matheron in the 1960s (Matheron, 1965). Kriging is involved with the concept of stochastic processes, and obeys the random function theory associated with Matheron's definition of a regionalized variable. Stochastic process theory is as difficult if not more difficult for practitioners to understand than the potential theory associated with the multiquadric-biharmonic method. Unfortunately for applications of kriging its assumption of stationarity enters in an obtrusive way into every significant step of progress, and into every attempt to provide improved variations of the kriging method. A controversy continues between theoreticians who seem to have no way of achieving flexibility or broadening the scope of their activity without tearing down the foundation of kriging, and practitioners who understand enough about it to insist that the stationarity concept is contrary to nature.

In this discussion I now assume that the multiquadric-biharmonic and the kriging techniques are comparable, in some sense, on a point by point basis. The multiquadric-biharmonic is computationally equivalent to point or punctual kriging in part. Both methods attempt to give the best prediction to the unsampled points in the region of interest. The multiquadric-biharmonic method does not require a preprocessing step, while the kriging method requires long preprocessing steps based on the assumption of the second-order stationarity of the regionalized variable. Based, in part, on the teaching experience of this author
of a course (CE 508) in the department of Civil and Construction Engineering at Iowa State University, the students suffer from the long and experience-dependent preprocessing process. The sampling of the regionalized variable often provides an empirical semivariogram that is not fitted well by any available mathematical model. This leads to reestablishment of the lag distance (h), and truncating a part of the semivariogram when it is known to be unmanageable. The worst case is to replan the experimental design if the semivariogram accidentally shows the pure nugget effect curve. Both methods give an unbiased estimation (proven in Chapter 4), while the multiquadric-biharmonic does not claim to be the best linear unbiased estimation or the optimum technique.

Hardy (1988) limits the kernel function of the multiquadric biharmonic to its distance, while the kriging technique uses the function of distance as its kernel function in the normal case or the function of the angular distance in the hole effect case. The kriging coefficients or the weights that are applied to the data point values are definitely not the same as the multiquadric-biharmonic coefficients. This difference even confused the author. The sum of the multiquadric coefficients are set equal to zero to satisfy the state of equilibrium and the minimum energy concept in the physical theory. The sum of the kriging weights are set equal to one to give an unbiased estimation in the statistical theory. I discovered that the multiquadric-biharmonic gives an unbiased estimator automatically if the user
temporarily assumes a statistical basis. During the literature search, there was generally no complaint about the solutions provided by the multiquadric-biharmonic method when applied to various fields of study. On the other hand, kriging is frequently criticized for the additional assumptions needed to implement practical applications of stochastic processes and statistical theory. Theoretically, a general algorithm can be written for each of the multiquadric-biharmonic and the kriging methods. The logic of the general algorithm is that the program will accept any type of data. In the kriging method, this can be accomplished by including all mathematical models that may be used to fit the semivariogram in the general program. The justification of which mathematical model will be properly used to explain the spatial structure of the regionalized variable can be done by fitting all models to the semivariogram of point values. The discrepancies of the fitted curves are computed, then the standard error for each model is computed by the statistical formula:

\[ S = \sqrt{\frac{\sum V^2}{n-1}} \]  

(97)

The model that has the minimum standard error is chosen to estimate the regionalized variable of the unsampled points. However, one should be aware of the problems that one may encounter with this program. For instance, the mathematical model may not fit the entire range of the semivariogram of point values. One may have to define the size of the neighborhood which is significant for semivariogram and to trun-
cate the point values further apart. The drift or trend may be unavoidably contained in the semivariogram of point values, and only a trial and error method can handle the situation.

In the multiquadric-biharmonic method which is restricted to the distance only as the kernel function, one need not be concerned about failure cases. The kernel is justified in advance by mathematical physics (Hardy and Nelson, 1986), and the coefficient matrix is always invertible (Micchelli, 1987). In the original osculating mode \((n + m + k)\) simultaneous equations with \((n + m + k)\) unknowns were solved automatically (Woodbury, 1971). In a later method of osculation involving multiquadric derivatives as well as the basic function, unique solutions were obtained with \(3n\) unknowns (Hardy, 1975).

One already existing kriging program prepared with the general algorithm concept was 2,336 lines long in Fortran 77 computer language. The author and some of his colleagues succeeded in executing this program for case studies. The problems mentioned earlier were encountered, especially when the trend was superimposed on the semivariogram of point values. The author then avoided the use of this program by writing another kriging program that reduced the cost of computation and was more straightforward. This was done by not having the computer program select a proper mathematical program. A proper model was then selected by human interpretation rather than by computer logic. This reduced the time and the cost of processing at least 50%.
A multiquadric-biharmonic program was written in Fortran 77 computer language for both the main program and the subroutine with 140 lines. The subroutine is experimentally usable with any phenomena that obeys the diffusion process (Read, 1988). An additional multiquadric-biharmonic program for solving boundary value problems involving differential equations contains only 432 lines. Programs of this type cannot be developed within the scope of kriging theory.

The multiquadric-biharmonic kernel function is very close to the linear model of the kriging method in which the semivariogram is said to be the non-transition type. The slope of the multiquadric-biharmonic kernel function is restricted to unity, while the slope of the linear model varies with the slope of the semivariogram of point values. The y-intercept of the multiquadric-biharmonic is always equal to zero, while the y-intercept of the linear model varies with the amount of random variance which accounts for the random variations, and the residual influence of all variabilities of which their ranges are smaller than the distance of observation or the distance that is shorter than the first lag distance used in computing the semivariogram.

Another similarity exists between the original osculating mode of the multiquadric-biharmonic method and the universal kriging version of kriging as a general expression. Both methods add a polynomial of proper degree and order for a particular purpose. In multiquadric-biharmonic analysis the polynomial is used in conjunction with zero derivatives at maxima and minima to prevent undesirable horizontal
and vertical displacement of the maximum and minimum points. In universal kriging the polynomial is used as a basis for estimating trend, which is not relevant in MQ-B theory.

The positional block variation of the multiquadric-biharmonic method is closely related to the spatial variance of the blocks in the region of interest in the kriging method. Kriging block variance and MQ-B block variation are used to compute the variability of the particular size of the block in the region of interest. The multiquadric-biharmonic positional block variation gives the variation related to the position of the block in that region, while the spatial block variance gives the variance which does not depend on the location due to the assumption of the second-order stationarity. The multiquadric-biharmonic positional block variation is not based on the assumption of the distribution of the regionalized variable, and is not subjected to the approximation of using the semivariogram.

The spatial estimation variance of the kriging method is loosely comparable to the cross validation error for either the multiquadric-biharmonic method or the kriging method. The cross validation error is not based on the statistical assumption of the distribution of the error of the regionalized variable, e.g., normal or Gaussian distribution, etc. (Journel and Huijbregts, 1978). The cross validation error is intuitively and experimentally a more reliable way to represent estimation error than the spatial estimation error that is provided
by a semivariogram. Experimentally, the kriging spatial estimation variance is generally underestimated, and leads investigators into believing that their predictions are better than they actually are (Hardy, 1988; Liebe, 1988).

Cross-validation is a statistical technique which is independent of the MQ-B and the kriging techniques. It is done by pretending to drop a data point out of the data set one at a time, sequentially, to include all data points except those defining the convex hull. Then the MQ-B and kriging processes will be applied to predict at each dropped data point. The root mean square of the predicted-minus-observed errors is called the cross validation error. It will be used to improve kriging spatial estimation variances and cross validation on the same problem. It will also be used to compare MQ-B and kriging predictions.

This chapter concludes with the schematic diagrams representing the steps in kriging's process (Fig. 8), the steps in the multiquadric-biharmonic's process (Fig. 9), and a table comparing both methods (Table 1).
Figure 8. Schematic diagram of the kriging or geostatistic processing step
Figure 9. Schematic diagram of the multiquadric-biharmonic (MQ-B) processing step
<table>
<thead>
<tr>
<th>Table 1. Comparison of multiquadric-biharmonic and kriging methods</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Multiquadric-biharmonic (MQ-B)</strong></td>
</tr>
<tr>
<td>1. Simple MQ-B (without variation)</td>
</tr>
<tr>
<td>2. a. Original osculating mode MQ-B (included polynomials)</td>
</tr>
<tr>
<td>b. Second osculating mode (MQ-B) (includes derivatives)</td>
</tr>
<tr>
<td>3. Deterministic process, does not assume stationarity</td>
</tr>
<tr>
<td>4. (Not applicable - NA)</td>
</tr>
<tr>
<td>5. NA</td>
</tr>
<tr>
<td>6. Always using the MQ-B kernel function, unique mathematical model</td>
</tr>
<tr>
<td>7. Distance is the kernel function</td>
</tr>
<tr>
<td>8. Unbiased estimation</td>
</tr>
<tr>
<td>9. NA</td>
</tr>
<tr>
<td>10. MQ-B coefficients, sum equal to zero</td>
</tr>
<tr>
<td>11. Satisfies physical theory of equilibrium and minimum energy concept</td>
</tr>
<tr>
<td>Multiquadric-biharmonic (MQ-B)</td>
</tr>
<tr>
<td>-------------------------------</td>
</tr>
<tr>
<td>12. Obey the diffusion law</td>
</tr>
<tr>
<td>13. Generally no criticism</td>
</tr>
<tr>
<td>14. Shorter algorithm (140 lines)</td>
</tr>
<tr>
<td>15. Solve for n + 1 + m + k simultaneous equations in the original version of an osculating model</td>
</tr>
<tr>
<td>16. NA</td>
</tr>
<tr>
<td>17. Not heavily dependent on experience of the user</td>
</tr>
<tr>
<td>18. No human needs to interrupt the process of computation</td>
</tr>
<tr>
<td>20. Slope of MQ-B kernel at data points is unity (in one and two dimensions)</td>
</tr>
<tr>
<td>21. Structural analysis by the isarithmic map</td>
</tr>
</tbody>
</table>
Table 1. Continued

<table>
<thead>
<tr>
<th>Multiquadric-biharmonic (MQ-B)</th>
<th>Kriging (K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>22. No so-called nugget effect</td>
<td>22. y-intercept of kernel function varies with the magnitude of spatial random variance (nugget effect)</td>
</tr>
<tr>
<td>23. MQ-B matrix always invertible and is conditionally positive definite automatically</td>
<td>23. K-matrix always invertible, but the positive definite characteristic is not automatic</td>
</tr>
<tr>
<td>24. Add polynomial function to the original osculating mode. Not necessary for the refined osculating mode</td>
<td>24. Generally add polynomial function to universal K for evaluating trend</td>
</tr>
<tr>
<td>25. Do not claim to be universal</td>
<td>25. Not universal as the name implies</td>
</tr>
<tr>
<td>26. Use osculating mode to minimize the displacement at maximum and minimum data points; use another version to control geometric slopes at data points</td>
<td>26. NA to universal K</td>
</tr>
<tr>
<td>27. Positional block variation varies with the location of the block and is not subjected to a stationarity assumption; continuous function version in lieu of classical discrete statistical equations</td>
<td>27. Spatial block variance of the blocks in the region of interest does not vary with the location of the block, subject to a second order stationarity assumption, and is approximated by the semi-variogram</td>
</tr>
<tr>
<td>Multiquadric-biharmonic (MQ-B)</td>
<td>Kriging (K)</td>
</tr>
<tr>
<td>-------------------------------</td>
<td>-------------</td>
</tr>
<tr>
<td>28. Cross validation error estimation for the isarithmic map</td>
<td>28. Spatial estimation variance, based on the assumption of Gaussian distribution of the error, approximated by the semivariogram; should be replaced by cross validation for the isarithmic map</td>
</tr>
<tr>
<td>29. Cross validation error is more appropriate than the estimation variance</td>
<td>29. Estimation variance leads investigators into the belief that their results are more accurate than is actually the case</td>
</tr>
<tr>
<td>30. Is able to use 3-dimensional data with a small increase in amount of time and effort</td>
<td>30. Is able to use 3-dimensional data with a much greater amount of time and effort</td>
</tr>
</tbody>
</table>
CHAPTER 3. CASE STUDIES AND DISCUSSION OF RESULTS

In Chapter 3, selected case studies, which have significant contribution values, are presented. The results of the case studies, based on multiquadric-biharmonic and kriging theory, and the opinions of this author, are discussed.

Temperature in Colorado

Description of data

Temperature data in Colorado were provided by the National Oceanic and Atmospheric Administration (NOAA). Temperature data from 21 observation stations was taken on Aug. 1, 1986 at 17:15. The stations are geographically fixed. The latitude and longitude of the stations are given, with the height of the topography. The range was from the lowest (4,550 feet) to the highest (11,500 feet). The justification for using these data was the immobility of the observation stations, the difference in the topographic height of the stations, and the scattering of the observing stations. The data, in computer-ready format to input in the program to change the latitude and longitude of the state plane coordinates, are provided in Appendix B. The dimensions of the experimental area are 142 x 173 kilometers, with the width of the area running east-west and the length running north-south. The map of the experimental area is shown in Fig. 10.
Figure 10. Map of the experimental area of the temperature and wind velocity in Colorado
Purpose of study

Due to the scattered data characteristic, the fixed observing stations, and the height difference at the data points, the data can be used as a good model in experimentation. Twenty-one observations is considered to be a small number in the interpolation scheme. The height difference due to the mountainous terrain makes it possible to observe the effect of height on the prediction. The main purpose of studying these data was to apply both the multiquadric-biharmonic and the kriging methods to the observed temperatures. Then, the temperature of unsampled points can be predicted in a grid pattern. The isarithmic map of the temperature was plotted by the graphics package.

Methods of application

The FORTRAN computer program was written to transform the latitude and longitude to the state plane coordinates of Colorado's northern zone (Appendix B). The method of transformation was based on the U.S. Coast and Geodetic Survey (Claire, 1968).

The cone model of multiquadric-biharmonic in $\mathbb{R}^2$ space,

$$z = \sum_{j=1}^{n} c_j [x_j - x]^2 + (y_j - y)^2]^\frac{1}{2}$$

was used. The multiquadric system of equations as explained in Chapter 2 was formed with the constraint that the sum of coefficients is equal to zero. Then, this system of equations was solved for the multiquadric coefficients. These coefficients were then used to predict the tempera-
tured at unsampled points in a grid pattern by the multiquadric method (Chapter 2). The kriging method was applied to the same data set using a modified technique to produce a semivariogram. Two dimensional distances were computed and then rotated into a one dimensional array of distances, all with respect to the same origin. The observed value of the temperature at the end of each distance vector was treated as an ordinate at the corresponding abscissia in one dimension. An appropriate interval of distance, corresponding to what is called the lag interval (interval of time) in time series analysis, was chosen by trial and error. This provides a basis for spacing of point values to give a reasonable shape to the experimental semivariogram (Chapter 2) (Fig. 11). The chosen interval was 60,000 feet for computing point values, in effect representing the average temperature in the neighborhood of the distance corresponding to multiples of the interval. Hence the spherical model was chosen to fit the experimental semivariogram by the least squares method. The parameters of the spherical model were then used in the kriging system to predict the temperature of unsampled points in a grid pattern.

Conceptually, in a geometric visualization, the one dimensional semivariogram was rotated about its origin to produce a two dimensional semivariogram surface. Note that this contrasts considerably with the notion in ordinary kriging that irregular data must otherwise be transformed by some other method to get a grid in advance. This has been one of the main problems in kriging (Cressie, 1984). The computer
Figure 11. Plot of semivariogram of point values against the theoretical semivariogram of temperature data.
graphics package "Contr 2" (which is installed in the Iowa State University computer system) was used to produce the isarithmic maps from the grid values that were produced from both techniques.

The stochastic error prediction formula (based on Heiskanen and Moritz, 1967) is

\[ m^2_p = \gamma(0) - \gamma_{ij}(\gamma_{ij})^{-1}(\gamma_{pj})^T \]  \hspace{1cm} (98)

This and cross validation was used to compute the prediction error of both the multiquadric-biharmonic and kriging methods.

The multiquadric-biharmonic program with the temperature input data in computer-ready format is provided in Appendix C. The kriging preprocessing programs, the interpolation program, and the temperature input data in the computer-ready format are provided in Appendix D.

Discussion

Both the multiquadric-biharmonic and the kriging method can handle two-dimensional temperature data. The multiquadric-biharmonic method can use any small collection of data at known coordinates in space and be formatted for reading into the computer to give the solution. On the other hand, a small collection of irregularly spaced data is not adequate for defining a semivariogram and is not suitable for the usual semivariogram computation formula, which is available in the geostatistical references. The semivariogram formula, Eq. (76), requires evenly spaced data, which was not available in this case study. In
order to make a regular grid-spaced data available for this case study, one must move the observation station, reconstructing the observation building, etc., which is obviously impossible or impractical at best.

As mentioned earlier, Dr. Rolland L. Hardy at Iowa State University suggested a study to find a method of computing the semivariograms for kriging purposes that is not dependent on regularly spaced data. Other problems of the semivariogram construction are computation costs, man hours and justification in using the proper semivariogram model. By using previous researcher experience, the theoretical semivariogram models were narrowed to a few standard models as mentioned in Chapter 2. To predecide the semivariogram model based on the type of data is very risky and unsupported from a theoretical point of view. Thus, it is very important that the researcher be able to interrupt the computer routine to look at the experimental semivariogram first. Then, the appropriate model can be chosen to represent the experimental semivariogram. It was found that whatever model was used to fit the experimental semivariogram, the prediction results were not greatly different. The data points fitted exactly. The shape of the isarithmic map was almost always the same. Each isarithmic map represented one variation of the true map of the regionalized variable. Especially in the local area in which the separation of the data point was closer than the range of influence of the spatial structure, the isarithmic maps were identical or were undetectably different. The greatest impact when changing the theoretical semivariogram model was mainly on the error
prediction from the stochastic formula. Again, this is another obvious cause of illusion about the prediction error by this formula.

The average cost of computation of the multiquadric-biharmonic is four times less than the kriging's cost. See total cost, Item 15, in Table 4. This is dependent on the number of data points. The more data points, the greater the preprocessing cost. To produce an isarithmic map from the multiquadric-biharmonic costs approximately $4 to $6 for 30 to 40 data points, which produce 1600 to 2000 predicted grid points. The kriging method using the same number of data points costs between $16 and $24. Again, there are a lot of variations in the cost. The cost of computation is also dependent upon computer knowledge and how the program is written. The standard program generally tends to cost more than the program written for handling a particular job. There was no experimentation on the microcomputer, due to the time constraints of this author. However, a perceived problem on the microcomputer is that computation time will increase tremendously. The good inversing matrix subroutine that was written for mainframe must be modified for the microcomputer, etc. More research in this area is suggested.

There is a direct correlation between the resemblance of the isarithmic maps produced by both methods to the localization of the experimental area (Figs. 12 and 13). In the localized area that has enough density of the data points the shape of the isarithmic maps are identical. The identity of these maps is reduced as the number of data points
Figure 12. Isarithmic map of temperature in Colorado produced by the multiquadric-biharmonic method.
Figure 13. Isarithmic map of temperature in Colorado produced by the kriging method
in the particular localization decreases. The geometrical configuration of the data points also affects the similarity of the shape of the isarithmic map. For example, in the area surrounded by the data points, the shape of the isarithmic map will be identical or closely similar. Conversely, in the local area that has sparse data points, the shape of the isarithmic maps are less similar or completely different. Especially in the extrapolated areas, the magnitude of the predicted temperature by the kriging's spherical model increases faster at the local areas that are close to the high magnitude data points than the multiquadric-biharmonic prediction. It decreases faster at the local areas that are close to the low magnitude data points than the multiquadric biharmonic prediction. The multiquadric-biharmonic predicts and gives a smoother isarithmic map in both localized areas.

In plotting the isarithmic map, there are two important problems to be considered. One is the computer graphics plotting subroutine and the other is the number of grid nodes predicted by the interpolation scheme. The computer graphics plotting routine must be chosen so the routine itself does not use any other mathematical model built-in to smoothing, interpolating, or adjusting the isarithmic lines that are plotted. If this is the case, distortion will result, which is another illusion for the user. This causes the isarithmic map users to believe that their map looks good with aesthetic value, while it has, in reality, lost its technical value. Care must be taken because many of the computer graphics routines are written in this way.
The aesthetic value of the isarithmic map should be gained from the number of grid nodes that are predicted by the favored or accuracy proven interpolant scheme, rather than the illusive smoothing scheme of the computer graphics routine mentioned above. The closer the grid nodes predicted by the accuracy proven interpolant scheme, the better looking and smoother the isarithmic map will be.

Some practical aspects of constructing the experimental semivariogram are worthwhile to note here. As stated earlier, constructing experimental semivariograms for the irregularly spaced data is completely out of the scope of geostatistical theory. This author, however, has derived some acceptable techniques to overcome this limitation of the geostatistical theory. This accomplishment is proven by the close similarity of the isarithmic map produced by the multiquadric-biharmonic and the kriging method. The semivariogram constructing techniques will be amplified here for future use.

The first technique begins with the concept of computing all possible distances between data points, as mentioned earlier. Then, the distance vectors are sorted from the minimum value to the maximum value. Ordinates at each end of each distance vector are used one time in the computation. By justification of the researcher, an appropriate interval is selected and a single representative ordinate in each interval is determined by averaging or weighted averaging. This process is repeated until the semivariogram that explains the spatial structure of the regionalized variable is achieved.
The second technique is the graphics method, which can be handled easier than the first technique. This method, however, is traded off by the accuracy of the experimental semivariogram. After the distance vectors are sorted from the minimum to maximum magnitude. The distance vector and the magnitude of the regionalized variable assigned to that distance are plotted. The distance is the abscissa and the magnitude of the regionalized variable is the ordinate. The result will be the variation of the ordinate in the plot. Then a true lag distance is chosen from the shape of the graph. Again, this process is continued until the satisfied experimental semivariogram is achieved.

Another problem of the experimental semivariogram is that it is not generally smooth enough to be justified for the theoretical mathematical model. The experimental semivariogram of the first few lag distances may be justified for one mathematical model, while the rest may be justified for another model or unjustified for any mathematical model. However, based on the kriging literature reviewed, researchers tend to accept this difficulty and use only the first few values of an experimental semivariogram. This author also accepts their suggestions in order to reach and compare the results from both methods.

Consider the results from cross validation error (Table 2). The total error by the multiquadric-biharmonic method is less than the total error from the kriging geostatistical method. The minimum positional error of the multiquadric-biharmonic method is 0.090°F, while the minimum positional error of the kriging method is 0.147°F. The
Table 2. Comparison of the cross validation error of the multiquadric-biharmonic and kriging methods of fitting temperatures in Colorado at all data points

<table>
<thead>
<tr>
<th>Point</th>
<th>Multiquadric error in temperature</th>
<th>Kriging error in temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ARV</td>
<td>-0.346</td>
<td>0.506</td>
</tr>
<tr>
<td>2. AVR</td>
<td>2.809</td>
<td>1.497</td>
</tr>
<tr>
<td>3. BOU</td>
<td>-3.196</td>
<td>-2.321</td>
</tr>
<tr>
<td>4. BGD</td>
<td>-0.090</td>
<td>0.661</td>
</tr>
<tr>
<td>5. BRT</td>
<td>1.746</td>
<td>0.312</td>
</tr>
<tr>
<td>6. BYE</td>
<td>-2.825</td>
<td>-5.426</td>
</tr>
<tr>
<td>7. ELB</td>
<td>-1.693</td>
<td>-5.910</td>
</tr>
<tr>
<td>8. ERI</td>
<td>0.806</td>
<td>0.936</td>
</tr>
<tr>
<td>9. EPK</td>
<td>-5.968</td>
<td>-2.681</td>
</tr>
<tr>
<td>10. FOR</td>
<td>-0.912</td>
<td>-1.421</td>
</tr>
<tr>
<td>11. FTM</td>
<td>4.485</td>
<td>1.472</td>
</tr>
<tr>
<td>12. GLY</td>
<td>-1.095</td>
<td>-1.352</td>
</tr>
<tr>
<td>13. ISG</td>
<td>10.859</td>
<td>13.028</td>
</tr>
<tr>
<td>14. KNB</td>
<td>-0.961</td>
<td>-1.822</td>
</tr>
<tr>
<td>15. LTN</td>
<td>-6.651</td>
<td>-6.229</td>
</tr>
<tr>
<td>16. LGM</td>
<td>-1.030</td>
<td>-0.798</td>
</tr>
<tr>
<td>17. LVE</td>
<td>0.635</td>
<td>0.147</td>
</tr>
<tr>
<td>18. NUN</td>
<td>2.723</td>
<td>2.280</td>
</tr>
<tr>
<td>19. PTL</td>
<td>-3.367</td>
<td>-0.966</td>
</tr>
<tr>
<td>20. ROL</td>
<td>-3.867</td>
<td>-3.441</td>
</tr>
<tr>
<td>21. WRD</td>
<td>6.898</td>
<td>8.245</td>
</tr>
</tbody>
</table>

Mean Square Error ±3.936 ±4.284

Temperature in °F
Observation on August 1, 1986, 17:15 p.m.
Table 3. Comparison of estimated temperature error with formula from kriging vs. actual error by cross validation in the convex hull

<table>
<thead>
<tr>
<th>Point missing</th>
<th>Error by formula</th>
<th>Error by cross validation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ARV</td>
<td>0.847</td>
<td>0.506</td>
</tr>
<tr>
<td>2. AVR</td>
<td>0.815</td>
<td>1.497</td>
</tr>
<tr>
<td>3. BOU</td>
<td>0.749</td>
<td>-2.321</td>
</tr>
<tr>
<td>4. BRI</td>
<td>0.772</td>
<td>0.312</td>
</tr>
<tr>
<td>5. ERI</td>
<td>0.694</td>
<td>0.936</td>
</tr>
<tr>
<td>6. FOR</td>
<td>0.935</td>
<td>-1.421</td>
</tr>
<tr>
<td>7. GLY</td>
<td>0.994</td>
<td>-1.352</td>
</tr>
<tr>
<td>8. KNB</td>
<td>0.985</td>
<td>-1.822</td>
</tr>
<tr>
<td>9. LTN</td>
<td>0.936</td>
<td>-6.229</td>
</tr>
<tr>
<td>10. LGM</td>
<td>0.777</td>
<td>-0.798</td>
</tr>
<tr>
<td>11. LVE</td>
<td>0.840</td>
<td>0.147</td>
</tr>
<tr>
<td>12. PTL</td>
<td>0.868</td>
<td>-0.966</td>
</tr>
<tr>
<td>13. ROL</td>
<td>0.734</td>
<td>-3.441</td>
</tr>
</tbody>
</table>

Mean Square Error 0.847 2.291

Temperature in °F
Observation on August 1, 1986 17:15 p.m.
Table 4. Comparison of the processing cost for multiquadric
biharmonic and kriging (average from all job in this
dissertation in ratio and percent to the total time used
to get the solution)\(^a\)

<table>
<thead>
<tr>
<th></th>
<th>Multiquadric</th>
<th></th>
<th>Kriging</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ratio</td>
<td>%</td>
<td>Ratio</td>
<td>%</td>
</tr>
<tr>
<td>1. Preprocessing time</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>75</td>
</tr>
<tr>
<td>2. Preprocessing cost</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>75</td>
</tr>
<tr>
<td>3. Preprocessing degree of difficulty</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>H(^b)</td>
</tr>
<tr>
<td>4. Preprocessing degree of experience dependent</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>H</td>
</tr>
<tr>
<td>5. Preprocessing degree of uncertainty</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>M(^c)</td>
</tr>
<tr>
<td>6. Preprocessing degree of necessity</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>7. Preprocessing degree of conformity among researchers</td>
<td>H</td>
<td>H</td>
<td>L(^d)</td>
<td>L</td>
</tr>
<tr>
<td>8. Overall problem in the processing</td>
<td>0.1</td>
<td>H</td>
<td>1</td>
<td>L</td>
</tr>
<tr>
<td>9. Computation time for the multi-quadric system and the kriging system</td>
<td>0.9</td>
<td>-</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>10. Isarithmic map drawing</td>
<td>1</td>
<td>-</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>11. Total time</td>
<td>0.25</td>
<td>-</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>12. Total cost</td>
<td>0.25</td>
<td>-</td>
<td>1</td>
<td>-</td>
</tr>
</tbody>
</table>

\(^a\)It is impossible to compare reliability in terms of dollars due to the variation in cost based on the program library, time of computation, cost of computation, overhead cost, etc. More details on cost estimation are included in the text. The method that took more time sets equal to 1 in ratio column. The most extreme case is assumed to be 100%. High, medium and low are used to measure items that cannot be mathematically evaluated.

\(^b\)High.

\(^c\)Medium.

\(^d\)Low.
maximum positional error of the multiquadric-biharmonic method is 10.859°F, while it is 13.028°F in the kriging method. This may have happened because of faster increasing and decreasing of the kriging prediction than multiquadric prediction in extrapolation. This happens when dropping the data points located at the edge of the experimental area, while performing the cross validation technique. Again, when applying any interpolation technique, extrapolation is generally avoided. This leads to the general conclusion that both the multiquadric-biharmonic method and the kriging method, under the proper handling process, can be applied to produce closely equal solutions. The most important advantage of the multiquadric-biharmonic method is in the cost and time efficiency rather than its accuracy.

The error of prediction in mapping provided by the stochastic formula on the assumption of error distribution, tends to be an overly optimistic estimation by several orders of magnitude to the cross validation error. This leads investigators to believe that their predictions are better than they actually are (Table 3). From this author's point of view, care must be taken in interpreting the errors produced by the stochastic formula. If the error were used to compare the goodness of two or more theoretical semivariogram models that were used in fitting the experimental semivariogram, this comparison would be theoretically valid because it is based on the same assumption and its theory. However,
to cross compare between the different techniques would be very dangerous.

In conclusion, I would like to state that the multiquadric biharmonic method predicts one realization of the surface representing the temperature surface in Colorado with advantages of cost, time, and man hours in processing. The multiquadric-biharmonic does not depend upon any stochastic assumption. Cross validation error is recommended for use to cross compare the system error between the multiquadric-biharmonic and kriging methods that have different theoretical support. The stochastic error prediction formula should be used only for comparing the methods based on the same theory or for comparing the goodness of the theoretical mathematical semivariogram model in representing the spatial structure of the regionalized variables under investigation.

Wind Velocity in Colorado

Description of data

The wind velocity data in Colorado came from the same source and belongs to the same geographical area as the temperature data. The observation stations were at the same location. Again, the position of the observation stations was given in the latitude and longitude values. A computer program (Appendix B) was used to transform the latitude and longitude into the state plane coordinates. The major
difference that justifies this case study is that it is vectorial data, and cannot be mapped as contours.

**Purpose of study**

The purpose of using these data was basically the same as the temperature data comparison between the multiquadric-biharmonic and the kriging methods to determine problems in both methods. An additional purpose was to find an appropriate new method to use with the vectorial data. I found no evidence that either kriging or the MQ-B method has been applied to interpolation of wind velocity data. A program to plot the wind velocity vector was written.

**Methods of application**

There are two obvious approaches to the vectorial data, which has two components, the magnitude of wind velocity and the direction of wind. The first approach is to apply both the multiquadric biharmonic and the kriging geostatistical methods to the direction and the magnitude separately. The second approach is to decompose the wind velocity vector into two components, Vx and Vy, in east-west and north-south directions, respectively. Both interpolation methods were used to fit each component independently. Then, the multiquadric biharmonic system and the kriging system were used to predict the unsampled points. The Vx and Vy components of the wind velocity vector was then recomposed to give the magnitude of wind velocity and the direction of the wind. The cross validation technique was applied
independently to both components, $V_x$ and $V_y$. The error of each component was then recomposed to give the error in magnitude and direction. The stochastic error prediction formula was also used to confirm the conclusion made in the temperature case.

**Discussion**

In the first approach, both the multiquadric-biharmonic method and the kriging method failed to fit the directional data independently. This is due to the characteristic of the directional data where the arithmetic mean cannot be used to represent the most probable value. Using this approach, the magnitude of the wind velocity at the data points was fitted exactly and the prediction at the unsampled point was found to be correct. But, the prediction of the direction failed.

The second approach was successfully applied to the wind velocity data. Both the multiquadric-biharmonic and the kriging method predicted the magnitude and direction of wind velocity at the data point exactly. The Gaussian model was chosen as the theoretical semivariogram. The cone model of the multiquadric-biharmonic method was used as in the temperature data. This was because there was no physical basis strong enough to believe that the biharmonic source should be located under the reference plane.

In the preprocessing process in the kriging method, the spherical model was initially chosen as the theoretical semivariogram, but it could not be fitted successfully to the experimental semivariogram
Figure 14. Plot of semivariogram of wind velocity data for Vx component
Figure 15. Plot of semivariogram of wind velocity data for Vy component
Figure 16. Vector diagram of wind velocity by multiquadric-biharmonic
Figure 17. Vector diagram of wind velocity from kriging
due to the negative nugget effect that implicitly appeared in the semi-
variogram of point value. Finally, the Gaussian model was used. The
plot of the experimental semivariogram and the theoretical semivariogram
are shown in Figures 14 and 15.

The vector diagram of the wind velocity prediction from the multi-
quadric-biharmonic and the kriging are presented in Figures 16 and
17. The vector diagram of observed data is presented in Figure 18.
The computer program for plotting the vector data is documented in
Appendix E.

It is obvious that the cost of preprocessing the wind velocity
data in the kriging method is twice the cost of preprocessing the temper­
ature data. The multiquadric-biharmonic did not have any cost for
preprocessing. It was not subjected to the problem of choosing the
appropriate theoretical mathematical model. If one chose, in fitting
the mathematical model to the experimental semivariogram, to use the
graphical eye fitting method which is generally used, the problem of
the least squares fitting is avoided. But again, this results in an
uncertainty in the choice of the semivariogram model. Theoretically,
any mathematical model chosen will provide an exact fit to the data
points, but will cause variation in the error of prediction, as mentioned
earlier. This leads to the immediate question: why not omit the pre-
processing step and use the method that doesn't require this step?
Again, there is a theoretical gap between the error of prediction,
based on the variation of the theoretical mathematical model used to
Figure 18. Vector diagram of wind velocity, original data
fit the experimental semivariogram and the real error of prediction. Care must be taken in the interpretation between the kriging error estimation and the real error.

When considering the vector diagrams that are produced by both methods, the multiquadric and the kriging methods gave a prediction of reasonable magnitude and direction at the locations close to the data points. At the southeast and southwest corners, the extrapolation area in the wind incoming direction, both methods predicted almost the same magnitude and the same direction of wind velocity. At the northwest corner, which is the extrapolation area in the wind outgoing direction, the multiquadric-biharmonic predicted less magnitude of wind velocity than the kriging method while the overall predicted directions are virtually the same. At the northeast corner, which is the extrapolation area in the wind outgoing direction, the multiquadric predicted a larger magnitude of the wind velocity than kriging, while the directions were predicted the same. This author does not have enough information to justify the behavior of the predicted value in the extrapolation regions at the northeast and northwest corners. More research on the vector data is suggested.

The results from the cross validation technique in estimating the error of prediction are presented in Table 5. The results from the cross validation technique and the stochastic error prediction formula are presented in Table 6. Again, by the cross validation error, the multiquadric-biharmonic gave a better accuracy on both wind
Table 5. Comparison of multiquadric-biharmonic and kriging method, fitting wind speed and wind direction

<table>
<thead>
<tr>
<th>Point Missing</th>
<th>Speed Error</th>
<th>Azimuth Error</th>
<th>Velocity Vector</th>
<th>Speed Error</th>
<th>Azimuth Error</th>
<th>Velocity Vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ARV</td>
<td>2.389</td>
<td>-0.070</td>
<td>2.510</td>
<td>4.051</td>
<td>-0.053</td>
<td>4.084</td>
</tr>
<tr>
<td>2. AVR</td>
<td>-4.606</td>
<td>-0.157</td>
<td>5.053</td>
<td>-3.261</td>
<td>-0.234</td>
<td>4.494</td>
</tr>
<tr>
<td>3. BGD</td>
<td>4.815</td>
<td>-0.017</td>
<td>4.860</td>
<td>-8.623</td>
<td>-0.060</td>
<td>8.670</td>
</tr>
<tr>
<td>4. BRI</td>
<td>4.564</td>
<td>-0.140</td>
<td>4.925</td>
<td>2.028</td>
<td>-0.234</td>
<td>3.266</td>
</tr>
<tr>
<td>5. BYE</td>
<td>13.654</td>
<td>0.489</td>
<td>14.630</td>
<td>6.391</td>
<td>0.487</td>
<td>7.624</td>
</tr>
<tr>
<td>6. ELB</td>
<td>-3.821</td>
<td>0.419</td>
<td>5.962</td>
<td>-1.105</td>
<td>0.509</td>
<td>6.358</td>
</tr>
<tr>
<td>7. ERI</td>
<td>-3.029</td>
<td>0.157</td>
<td>3.466</td>
<td>-0.197</td>
<td>-0.226</td>
<td>2.695</td>
</tr>
<tr>
<td>8. EPK</td>
<td>-2.035</td>
<td>2.723</td>
<td>15.641</td>
<td>4.171</td>
<td>2.426</td>
<td>20.817</td>
</tr>
<tr>
<td>9. FOR</td>
<td>-7.577</td>
<td>-0.349</td>
<td>9.172</td>
<td>-7.410</td>
<td>-0.437</td>
<td>9.816</td>
</tr>
<tr>
<td>10. FTM</td>
<td>-9.475</td>
<td>-0.367</td>
<td>11.719</td>
<td>-13.043</td>
<td>-0.268</td>
<td>13.743</td>
</tr>
<tr>
<td>11. GLY</td>
<td>-4.009</td>
<td>0.000</td>
<td>4.009</td>
<td>-10.647</td>
<td>-0.060</td>
<td>10.690</td>
</tr>
<tr>
<td>12. ISG</td>
<td>0.051</td>
<td>1.361</td>
<td>8.898</td>
<td>4.985</td>
<td>0.113</td>
<td>5.090</td>
</tr>
<tr>
<td>13. KMB</td>
<td>-6.683</td>
<td>0.209</td>
<td>7.582</td>
<td>-8.712</td>
<td>0.137</td>
<td>8.953</td>
</tr>
<tr>
<td>14. LTN</td>
<td>-0.203</td>
<td>-0.035</td>
<td>0.636</td>
<td>-0.072</td>
<td>-0.052</td>
<td>0.620</td>
</tr>
<tr>
<td>15. LGM</td>
<td>-4.712</td>
<td>-0.750</td>
<td>7.737</td>
<td>1.011</td>
<td>-0.778</td>
<td>8.772</td>
</tr>
<tr>
<td>16. LVE</td>
<td>1.048</td>
<td>0.738</td>
<td>0.087</td>
<td>-1.127</td>
<td>-0.226</td>
<td>3.023</td>
</tr>
<tr>
<td>17. NUN</td>
<td>-2.229</td>
<td>0.140</td>
<td>3.932</td>
<td>-10.812</td>
<td>-0.058</td>
<td>10.850</td>
</tr>
<tr>
<td>18. PTL</td>
<td>4.138</td>
<td>0.140</td>
<td>4.553</td>
<td>0.919</td>
<td>-0.051</td>
<td>1.091</td>
</tr>
<tr>
<td>19. ROL</td>
<td>-5.604</td>
<td>1.030</td>
<td>7.834</td>
<td>2.992</td>
<td>1.876</td>
<td>17.022</td>
</tr>
<tr>
<td>20. WRD</td>
<td>1.805</td>
<td>-0.157</td>
<td>2.038</td>
<td>7.039</td>
<td>1.864</td>
<td>14.309</td>
</tr>
</tbody>
</table>

MSE         5.354       0.784       7.490       6.256       0.852       9.663

Wind speed in nautical miles/hour
Azimuth from N in radian
Observation on August 1, 1986, 17:15 p.m.
speed and direction, as well as vectorial accuracy. The minimum wind speed error is 0.051 nautical miles per hour (nmph), the maximum is 13.654 nmph. The minimum and maximum azimuthal errors are 0.00 and 2.723 degrees respectively, while the minimum and maximum vector errors are 0.087 and 15.641 nmph respectively, in the multiquadric-biharmonic method. The minimum and maximum wind speed errors are 0.072 and 13.674 nmph. The minimum and maximum azimuthal errors are 0.051 and 2.426 degrees respectively, while the minimum and maximum vector errors are 0.620 and 20.817 nmph respectively, in the kriging method. This implies that both methods can yield almost the same prediction accuracy if appropriately used. The multiquadric is more advantageous in that it does not require a preprocessing cost and man hours for this step. The preprocessing step in the kriging technique for vector data costs as much as scalar data.

The error predicted by the stochastic formula is again overly optimistic (Table 6). This author would like to reiterate that this formula should be used only to compare the value among the available theoretical semivariograms to represent the spatial structure of the regionalized variables. It is emphasized that this method cannot be used for cross comparison between the different methods based on different theories!
Table 6. Comparison of estimated wind velocity error by stochastic formula and by cross validation using the kriging technique

<table>
<thead>
<tr>
<th>Point missing</th>
<th>Error by formula</th>
<th>Error by cross validation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ARV</td>
<td>1.4663</td>
<td>4.084</td>
</tr>
<tr>
<td>2. AUR</td>
<td>1.4660</td>
<td>4.494</td>
</tr>
<tr>
<td>3. BRI</td>
<td>1.4660</td>
<td>3.266</td>
</tr>
<tr>
<td>4. ERI</td>
<td>1.4660</td>
<td>2.695</td>
</tr>
<tr>
<td>5. FOR</td>
<td>1.4675</td>
<td>9.816</td>
</tr>
<tr>
<td>6. GLY</td>
<td>1.4677</td>
<td>10.690</td>
</tr>
<tr>
<td>7. KNB</td>
<td>1.4681</td>
<td>8.953</td>
</tr>
<tr>
<td>8. LTN</td>
<td>1.4674</td>
<td>0.620</td>
</tr>
<tr>
<td>9. LGM</td>
<td>1.4665</td>
<td>8.772</td>
</tr>
<tr>
<td>10. LVE</td>
<td>1.4660</td>
<td>3.023</td>
</tr>
<tr>
<td>11. PTL</td>
<td>1.4660</td>
<td>1.091</td>
</tr>
<tr>
<td>12. ROL</td>
<td>1.4658</td>
<td>17.022</td>
</tr>
<tr>
<td><strong>MSE</strong></td>
<td><strong>1.467</strong></td>
<td><strong>7.762</strong></td>
</tr>
</tbody>
</table>

Observation on August 1, 1986, 17:15 p.m.
Iron Ore Grade in the Bahariya Mine, Egypt

Description of data

The data consisting of point values of iron ore percentages was obtained from Dr. M. Z. Rashad, visiting assistant professor at Iowa State University in 1986. A crude manually contoured map of the hydrothermal deposit of iron ore located at the Bahariya Oasis in the western desert is shown in Figure 21. The data points provided in this area are on this map as an overlay. This ore deposit is the only resource for steel production in Egypt. It is contaminated with gangue minerals, namely manganese oxide, chlorine and silica. The average thickness of the ore is 10 meters, which generally is covered by sand and quartzite.

The dimensions of the area in this study are about 2250 x 2250 square meters. The sample points lie in a grid pattern in which some of the grid nodes are missing (Figure 19). The iron grade magnitude was recorded in a range from 44 to 66%. The edges of the experimental area lie in east-west and north-south directions. These data are treated as two-dimensional data.

Purpose of study

This study emphasizes fitting the multiquadric-biharmonic and the kriging methods to the real mineral resources data. Isarithmic maps of both methods were compared, including cross-validation errors. The cone model using the multiquadric constant, \( \delta \), was also investigated.
Figure 19. Crude map of iron ore grade with overlay of data points
(Chapter 2). A new formula for computing the semivariogram point values was tested. The kriging block variance and the multiquadric-biharmonic positional block variation was also investigated.

Methods of application

The multiquadric-biharmonic process was applied to 56 data points. The iron grade data in computer-ready format are in Appendix F. The kriging process was also applied to these data. Both the spherical model and the exponential model were chosen as theoretical semivariograms (Figures 20 and 21). Next, the cross validation method was applied to both the multiquadric-biharmonic and the kriging techniques for both theoretical semivariograms. Cross validations was also compared with the stochastic error formula. The isarithmic map for iron ore grade was plotted by the "Contr2" computer graphics routine.

The multiquadric-biharmonic method with several constants $\delta$, was used to fit the iron grade. Then, the cross validation method was used to find the standard error of prediction for each value of $\delta$.

A new formula, namely Hardy and Sirayanone's formula, for computing experimental semivariograms was applied to the data.

This formula is intended to overcome the difficulty with unequal weight of the semivariogram value when the lag distances are farther apart. The formula computes an experimental semivariogram by always using the same number of distance vectors in each lag interval. The
Figure 20. Semivariogram of iron grade (exponential model)
Figure 21. Semivariogram of iron grade (spherical model)
formula to compute the magnitude of semivariogram value is the same as used by Matheron (1963). This is

\[ \gamma(h) = \frac{1}{2(N-h)} \sum_{i=1}^{N-h} (g_i - g_{i+h})^2 \]  

(76)

The difference in application occurs in assigning the proper magnitude of the distance, h, to the value computed from Eq. (79). The average value of the unequal intervals resulting from the equal number of distance vectors will be used instead of the equal increments of the lag intervals. This leads to a formula for the magnitude of lag interval in each increment,

\[ h_j = \frac{1}{(N-j)} \sum_{i=1}^{N-j} g_i g_{i+j} \]  

(100)

This is called Hardy and Sirayanone's formula. To simplify the concept of this formula, consider the following as an example.

We averaged an equal number of distances and their associated regional variable values in each non-uniform distance interval, such as:

<table>
<thead>
<tr>
<th>Distance</th>
<th>Magnitude of regionalized variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>70</td>
</tr>
<tr>
<td>150</td>
<td>60</td>
</tr>
<tr>
<td>250</td>
<td>90</td>
</tr>
<tr>
<td>475</td>
<td>80</td>
</tr>
</tbody>
</table>
In this problem note the unequal intervals of lag distance from 50 to 150, 150 to 250, and 250 to 475, respectively. The semivariogram can be computed by Eq. (76), say $\gamma(h_1)$. Then, the $h_1$ value can be computed by Eq. (100) with $N = 4$, and $j = 1$.

$$h_1 = \frac{1}{4-1} \sum_{i=1}^{(4-1)} (g_i g_{i+j})$$

$$= \frac{1}{3} (100 + 100 + 225)$$

$$= 141.67 \text{ units}$$

The same method is used to compute $h_2$, $h_3$, etc. The program for using Hardy and Sirayanone's formula is contained in Appendix G.

The kriging block variance was computed by using the F-chart (Figures 6 and 7, Chapter 2) for the arbitrary block size and shape. This was used for comparison with the multiquadric-biharmonic positional block variation. Many arbitrary locations in the iron ore deposit were chosen as the center of the blocks of the same size used in computing the kriging block variance and the multiquadric positional block variation.

For comparative purposes of behavior in extrapolation outside the boundary of the data area, predictions were made by both the multiquadric-biharmonic and kriging spherical models.
Discussion

The shape of the semivariogram of the iron grade showed strong continuous spatial structures (Figures 20 and 21). See Matheron (1963) concerning basic shapes of a semivariogram and its behavior at the origin. However, care must always be taken when one interprets the semivariogram. In fact, one may question whether or not there is any objective way of interpreting a semivariogram by itself for geological structures. For example, if the configuration of the data points' location is changed, the experimental semivariogram in the iron deposit field will show a difference in the spatial structure of the iron grade. Also see Henley (1981), page 34.

The spatial structure analysis of the multiquadric-biharmonic was done on the isarithmic map (Figure 22). Based on the frequency of change in the isograde lines, it was concluded that the iron deposit is the isotropic continuous type. On the other hand, if the frequency of changing the isograde lines is not smooth enough, one could interpret the iron deposit as not continuous or with a very steep gradient. If there were variations in the frequency of changing the isograde line, it will be an anisotropic type deposit.

Comparing the isarithmic maps produced from the multiquadric-biharmonic method and the kriging exponential and spherical models shows that they are almost identical, except in the extrapolation regions (Figures 22 - 24). Considering the cross validation errors of both
Figure 22. Isarithmic map of iron grade by the multiquadric-biharmonic method
Figure 23. Isarithmic map of iron grade by the kriging method (exponential model)
Figure 24. Isarithmic map of iron grade by the kriging method (spherical model)
maps, the kriging exponential model gave slightly less accuracy than
the multiquadric-biharmonic method (Table 7). However, it can be con-
cluded that both methods yield approximately equal accuracy, if one
uses both methods properly. Again, the preprocessing step in the kriging
method causes ambiguity in performing the technique. The processing
step, problems, cost and time were compared in Table 4.

Table 8 compares the cross validation errors between the spherical
and exponential models of the kriging method. The accuracy of prediction
by both theoretical semivariograms are almost equal. This result is
due mostly to the geometrical configuration and the density of the
data points. The exponential model predicted a little better than
the spherical model.

With Hardy and Sirayanone's formula, the shape of the experimental
semivariogram changed as expected. Compare Figure 25 with Figure 21.
Again, the shape of the isograde maps are almost identical, except
for the extrapolation regions (Figure 26). When comparing the cross
validation estimation errors from the classical Matheron formula to
Hardy and Sirayanone's formula, the errors were almost equal (Table
9). Hardy and Sirayanone's formula yielded slightly better results.
This is, in part, due to the geometry and density of the data points.
The new formula was developed to fulfill the theoretical requirements
that the weight of the experimental semivariogram of all lag intervals
should be equal.
Figure 25. Semivariogram of iron grade (spherical model) with Hardy and Sirayanone's formula (using equal number of data points in all lag distances)
Figure 26. Isarithmic map of iron grade (by kriging spherical model with Hardy and Sirayanone's semivariogram formula)
Table 7. A comparison of standard errors of the multiquadric biharmonic method and the kriging method exponential model in the estimation of iron grade

<table>
<thead>
<tr>
<th>Multiquadric</th>
<th>Kriging</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.348%</td>
<td>4.354%</td>
</tr>
</tbody>
</table>

Table 8. A comparison of standard errors of the prediction by the spherical model and the exponential model in the kriging method in the estimation of iron grade

<table>
<thead>
<tr>
<th>Spherical mode</th>
<th>Exponential mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.802%</td>
<td>4.754%</td>
</tr>
</tbody>
</table>
Table 9. A comparison of the standard error using classical semivariogram formula and the Hardy and Sirayanone formula in kriging spherical model

<table>
<thead>
<tr>
<th>Classical formula</th>
<th>Discovered formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.8022%</td>
<td>4.8011%</td>
</tr>
</tbody>
</table>

Table 10. A comparison of standard error when varying the constant term, $\delta$, in the multiquadric biharmonic method in the estimation of the iron grade

<table>
<thead>
<tr>
<th>Magnitude of constant = $0^2$</th>
<th>Standard error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.7818</td>
</tr>
<tr>
<td>0.134 E-12</td>
<td>4.7818</td>
</tr>
<tr>
<td>1.36</td>
<td>4.7882</td>
</tr>
<tr>
<td>1.86</td>
<td>4.7893</td>
</tr>
<tr>
<td>5.47</td>
<td>4.7947</td>
</tr>
<tr>
<td>7.58</td>
<td>4.7970</td>
</tr>
<tr>
<td>14.14</td>
<td>4.8606</td>
</tr>
<tr>
<td>57.46</td>
<td>4.8238</td>
</tr>
<tr>
<td>200.00</td>
<td>5.8674</td>
</tr>
</tbody>
</table>
Table 10 shows that the best result is obtained with $\delta$ very small or $\delta = 0$.

Table 11 compares the multiquadric-biharmonic positional block variation and the kriging block variance. The kriging block variance is restricted by the stationarity assumption and the average property of the experimental semivariogram computation. This leads to the theoretical conclusion that the kriging block variance depends upon the size and shape of the ore block, but not upon the location of that block. Contrary to the kriging idea, the basic theories supporting the multiquadric-biharmonic are not based on any stochastic assumption, but have deterministic properties and a physical meaning as mentioned in Chapter 2. The multiquadric block variation changes with location, size, and shape of the block. Table 11 gives evidence which shows the block variance should be changed with the location of the block.

In this table, two kriging block variances are presented. Both of these were computed using the spherical model. This illustrates that kriging block variances are highly sensitive to the scale of observation and choice of lag distance in the semivariogram. Hence semivariograms are arbitrary, even for the same model and same data, if you use different lag distances. This has been confirmed by authoritative geostatistical text book authors, Henley (1981, 1987) and Journel (1988), who have alluded to variograms as having no a priori genetic significance. It is worth noting that both kriging block variance columns report significantly higher percentages of "variance" than
Table 11. Kriging block variance vs. multiquadric-biharmonic positional block variance of the block size 10 x 10 x 10 meter$^3$, in estimation of iron ore grade

<table>
<thead>
<tr>
<th>Location (x, y, z)</th>
<th>Kriging block$^a$ variance (%)$^2$</th>
<th>Kriging block$^b$ variance (%)$^2$</th>
<th>Multiquadric$^c$ positional block variance (%)$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>155,367,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0043</td>
</tr>
<tr>
<td>225,450,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0076</td>
</tr>
<tr>
<td>420,1600,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0045</td>
</tr>
<tr>
<td>800,380,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0270</td>
</tr>
<tr>
<td>1000,400,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0284</td>
</tr>
<tr>
<td>1200,1800,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0041</td>
</tr>
<tr>
<td>1400,2000,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0087</td>
</tr>
<tr>
<td>1500,1950,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0249</td>
</tr>
<tr>
<td>600,1470,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0585</td>
</tr>
<tr>
<td>650,550,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0147</td>
</tr>
<tr>
<td>1120,2040,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0028</td>
</tr>
<tr>
<td>320,490,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0083</td>
</tr>
<tr>
<td>540,1700,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0264</td>
</tr>
<tr>
<td>975,2020,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0123</td>
</tr>
<tr>
<td>610,395,0</td>
<td>0.6579</td>
<td>0.3922</td>
<td>0.0039</td>
</tr>
</tbody>
</table>

$^a$Where sill = 1, range of influence = 2370 meters.
$^b$Where sill = 0.6, range of influence = 800 meters.
$^c$Refer to Figure 24 (isarithmic map of iron grade by the multiquadric-biharmonic method), the original of the coordinate system at the lower left corner.
the MQ-B "positional variation." It is obvious from MQ-B map interpretation (in a sense, structural analysis) that the kriging variance is not only arbitrary and ambiguous but is not even close to the truth. That the kriging block variance can actually be useful to mining engineers or mill operators appears to be a fallacy to this author.

The MQ-B approach, with positional block variation, provides a flexible and more user-friendly tool for possible use by mining engineers and mill operators.

Table 12 compares the errors of estimation computed by the stochastic formula and the cross validation errors. The stochastic formula yields an overly optimistic result. Again, the estimation error from the stochastic formula is suited to compare the same method when varying the kernel function, rather than the cross comparison among different methods.

The extrapolated isarithmic maps of the predicted iron grade, by both the multiquadric-biharmonic model and the kriging spherical model, are shown in Figures 27 and 28. In general, the interpolation technique should not be used to extrapolate because there will be no theoretical support that the spatial characteristic of the regionalized variable or the phenomenon will be the same. However, to interpret the results of the extrapolation scheme, both maps were produced by estimating unsampled points 1000 meters from all border lines. This is about one-half of the width and length of the sampling area. The maps show the predictions flatten out from the border of the sampling
Table 12. A comparison of standard error by cross validation method and by the stochastic formula in estimation of iron grade

<table>
<thead>
<tr>
<th></th>
<th>Cross validation</th>
<th>Stochastic formula</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4.354%</td>
<td>0.340%</td>
</tr>
</tbody>
</table>

Table 13. A comparison of standard errors of the multiquadric-biharmonic and kriging spherical model in the estimation of percentage manganese

<table>
<thead>
<tr>
<th>Model</th>
<th>Multiquadric</th>
<th>Kriging</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.605%</td>
<td>0.601%</td>
</tr>
</tbody>
</table>

Table 14. A comparison of standard error by cross validation method and by the stochastic formula in estimation of manganese percentage

<table>
<thead>
<tr>
<th></th>
<th>Cross validation</th>
<th>Stochastic formula</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.601%</td>
<td>0.292%</td>
</tr>
</tbody>
</table>
Figure 27. Isarithmic map of iron grade showing the extrapolation region, as estimated by the multiquadric-biharmonic
Figure 28. Isarithmic map of iron grade showing the extrapolation region, as estimated by the kriging spherical model
area in both methods. In this single case the trend at the boundary
generally persists for greater distances with the multiquadric-biharmonic
method than with the kriging spherical model, but this is not conclusive.

In the following cases the author will avoid presenting unnecessary
or similar diagrams. Similar reasons given earlier will not be repeated,
only concise conclusions and opinions will be examined and discussed
on the following case.

Manganese Gangue in the Bahariya Mine, Egypt

Description of data

The data, consisting of point values of manganese gangue percent­
ages, was made available by Dr. M. Z. Rashad for the same geographic
location as the iron ore (Figure 29). This figure shows a crude manual
map upon which is shown as an overlay the percentage of manganese gangue
that was assayed at the same boreholes as the iron grade (Appendix
F). The mining engineer needs to know the amount of gangue mineral
that contaminates the same block of the valuable ore. Knowing the
contamination enables the engineer to regulate and control the mineral
processing procedure. The magnitude of manganese gangue ranged from
0.53 to 1.80%.

Purpose of the study

Due to the small magnitude of the manganese gangue percentages,
a study of the behavior of both methods was justified. Both methods
are seldom used to fit very small data values. The isarithmic maps
Figure 29. Crude map of manganese gangue with overlay of data points
produced by the prediction of the multiquadric-biharmonic and the kriging methods will be plotted. The cross validation error of the prediction by both methods will be investigated. The results from the stochastic error estimation formula will be compared to the cross validation error.

Methods of application

The same model of the multiquadric-biharmonic, as in the iron grade case, was used. The experimental semivariogram was produced by the Hardy and Sirayanone formula (Figure 30). This formula was used because it gave equal weight to the experimental semivariogram at all lag intervals, which had given better results in the iron grade case than unequal weights. The spherical model was used as the theoretical semivariogram. Then both methods were used to estimate the unsampled points. The isarithmic maps were plotted by the "Contr2" computer graphics routine (Figures 31 and 32). The cross validation errors for both methods are shown in Table 13 and the stochastic errors vs. the cross validation errors are shown in Table 14.

Discussion

The multiquadric-biharmonic method and the kriging method are capable of interpreting small magnitude data with no problem. The experimental semivariogram of the manganese gangue showed a sudden change. However, the spherical model was fitted successfully. Based on geostatistical theory, the shape of the experimental semivariogram showed abrupt changes in the spatial structure of the regionalized
Figure 30. Semivariogram of manganese gangue, spherical model, using equal number of distance vector in all lag intervals.
Figure 31. Isarithmic map of manganese gangue by the multiquadric-harmonic method
Figure 32. Isarithmic map of manganese gangue by the kriging spherical model
variable. The least squares line followed an approximately average path among the semivariogram ordinate points. The isarithmic maps (Figures 31 and 32) of the manganese gangue were almost identical, except in the extrapolated region. The predicted values produced by the multiquadric-biharmonic method in the extrapolation area (especially the southeast corner) flattened out faster than the kriging prediction map. The cross validation error of the multiquadric-biharmonic method was slightly larger in this case than the kriging method. This again shows practically equal accuracy of both methods, if the researcher uses the data properly (Table 13). Cost, time, and man hours in the preprocessing step of the kriging method took the same proportion as in the iron grade case. The stochastic formula of the prediction error overly optimized the magnitude of error as happened before in the other cases, when compared to the cross validation error.

Simulated Data

Data description

One set of simulated irregular spaced data were created, there was no conscious effort to give this data any particular identity. They could be anything measured at points in a volume, e.g. the quantity of nuclear radiation in a cubical part of the atmosphere. The data set has 64 points contained in a cubical volume of three units in each coordinate axis, and was purposely intended to realistically simulate the variation of some variable quantity in the volume. This data is
obviously non-stationary, which is the case generally in practice. The computer-ready format of these data is documented in Appendix H.

**Purpose of the study**

The simulated data were used for three main purposes: 1) to observe the behavior of both interpolation techniques on the same simulated data, 2) to use both interpolation methods in a 3-dimensional scheme, and 3) to write a computer graphics program to generate stereopairs of isarithmic maps representing $\mathbb{R}^3$ space. The anaglyph and the photographic methods will be experimented with in producing the stereopair.

**Method of application**

A multiquadric-biharmonic function in a 3-dimensional scheme was used to fit the data. Selected horizontal sections are presented in Figures 33, 35, 37, and 39. The kriging method with the spherical model was also used to study the possibility of handling 3-dimensional data (Figures 34, 36, 38, and 40). A computer subroutine in FORTRAN language was written for the multiquadric-biharmonic method in 3-dimensional space. Under the supervision of Drs. A. A. Read and R. L. Hardy at Iowa State University, the program "meshsurface" was written to utilize the predicted values in a grid pattern from the multiquadric-biharmonic subroutine. Meshsurface prepares the predicted data in a computer-graphics format. Then, the data are read into the AGRAPH computer graphics routine (Read, 1985). The stereopair in 3-dimensional
Figure 33. Multiquadric predicted map of erratic irregular spaced simulated data at level 1.4 units
Figure 34. Kriging predicted isarithmic map of irregularly spaced simulated data at level 1.4 units
Figure 35. Multiquadric predicted isarithmic map of irregularly spaced simulated data at level 1.2 units
Figure 36. Kriging predicted isarithmic map of irregularly spaced simulated data at level 1.2 units.
Figure 37. Multiquadric isarithmic map of irregularly spaced simulated data at level 1.0 units
Figure 38. Kriging isarithmic map of irregularly spaced simulated data at level 1.0 units.
Figure 39. Multiquadric predicted map of irregularly spaced simulated data at level 0.8 units
Figure 40. Kriging predicted isarithmic map of irregularly spaced simulated data at level 0.8 units
space of the isarithmic surfaces was generated (Figures 41 through 43).

Discussion

In the regularly spaced data case, the multiquadric-biharmonic gave almost the same isarithmic map as did the kriging method for 3-dimensional data, especially at the level that has the most data points.

Figures 33 and 34 represent the multiquadric-biharmonic and kriging maps, respectively, of the topmost layer at elevation 1.4 in a four layer construction of a stereopair illustrated in Figures 41 through 43. The next lower layers (below the topmost layer), namely at elevations 1.2, 1.0, and 0.8 are shown in the following pairs: Figures 35 and 36, Figures 37 and 38, and Figures 39 and 40, respectively. As for the topmost layer, the odd numbered figure in each pair is a multiquadric-biharmonic map, and the even numbered figure in each is a kriging map. At each level the multiquadric-biharmonic and kriging is very similar.

Figures 41 through 45 were produced by the application of computer graphics to stereoscopic theory as is commonly used in the field of geodesy and photogrammetry. Only the multiquadric-biharmonic method was applied to the stereo-production. It is more or less obvious that 3-dimensional kriging could produce similar products because of the similarity of the map in each layer as previously mentioned. However the subroutine of kriging cannot be written in the same, relatively
easy method, as accomplished by the multiquadric method, and it is anticipated that there would be many other difficulties. Included was the difficulty associated with several thousand lines of programming steps versus not more than several hundred programming steps with the multiquadric method.

Considering the usual stereoscopic theory as applied in geodesy and photogrammetry, a three-dimensional model was produced by spacing transparencies of the layers in Figures 33, 35, 37, and 39, vertically from top (Figure 33) to bottom (Figure 39) at the same horizontal and vertical scale. Then a stereopair of perspective views of the model were obtained by a 35 mm camera at two exposure stations. This resulted in the expected 3-dimensional image which could be viewed successfully with a stereoscope. The main problem with this approach is the depth of focus of the camera lens. The depth of focus was not deep enough to give sharp isarithmic lines at all layers, especially if the depth was more than six centimeters in the model.

The anaglyph technique, which is also well-known in geodesy and photogrammetry field, was also used. Theoretically if the proper materials are used, this approach can give results equivalent to separate stereopairs as above. An anaglyph involves an overlapping pair in which one views each image separately by color separation and color filters. The major problem was the color of the isarithmic lines which were supposed to be red and blue-green for the left and right eye, respectively, but the proper colors were not available at Iowa State.
They are available at some places. Also, when viewing both left and right images, the proper filters were not available. In order to view stereoscopically, each eye is supposed to receive the impression of only one image. Improvement of both techniques can be developed from improved instruments, cameras, and supplies.

Figures 41 - 43 are selected examples of the plot from the AGRAPH program combined with the meshsurface program and the multiquadric-biharmonic subroutine (Appendix I).

When using a pocket stereoscope to study Figure 41, one should notice that the "slicing" of the three dimensional function in four levels, i.e. 1.4, 1.2, 1.0 and 0.8 has generally produced four isarithmic lines with the same functional value. With proper visualization one will notice that these isarithmic lines in two dimensions have become isarithmic surfaces in the three dimensional map. Notice also that a highly anomalous source occurs at a data point near the center of the stereogram, with a value of 23.9 units at level 1.0 (see Figure 37) which is substantially the equivalent of one of the perspectives in Figure 41. In the stereogram in Figure 41, notice this point is surrounded by a nearly circular intersection of a "spherical bubble surface" of isarithmic surface value of 24 as seen in Figure 37. This is surrounded by a distorted elliptical intersection of a bubble surface of isarithmic value of 25.0 at level 1.2 (Figure 35). The above described bubble-like surfaces are causing a 3-dimensional saddle-like distortion in the 25.0 isarithmic surface. In fact, this saddle-like forma-
Figure 41. Stereopair of isarithmic surfaces of simulated irregularly spaced data at standard perspective pair orientation, multiquadric
Figure 42. Stereopair of isarithmic surfaces of simulated irregular spaced data at an arbitrary orientation, multiquadric
Figure 43. Stereopair of a lattice plot of simulated irregular spaced data at an arbitrary orientation, multiquadric
tion (topographic feature in two dimensions) is, in military terminology, a two dimensional salient, but as an anatomical 3-dimensional interpretation, for example, is like an aneurism (weak area in an arterial membrane). Figures 42 and 43 are the stereopairs of Figure 41 with a different orientation including a look at the bottom of the array instead of the top. The lattice plot is an array of lattice points in an isarithmic surface connected by lines to assist in the visualization of an isarithmic surface, which may be especially useful in some cases. For the case in point, i.e. this simulated data case, it appears that the standard orientation in Figure 41 is actually the best. However, Figures 42 and 43 represent the options that are available in the programming logic to search for advantageous points of view in other cases. Also these options are available for viewing in stereo-color vision on a CRT screen.

The multiquadric-biharmonic turns out to be an ideal kernel function in this type of built-in subroutine to the main program such as the meshsurface. This is due to the fact that a preprocessing step dependent on human experience is not required. Also, the multiquadric-biharmonic has proven to be a versatile, simple, economical and accurate method which adjusts itself to problems in many scientific fields. See the literature review section.
Pollution in an Iowa Reservoir

Description of data

The Environmental and Water Quality Operational Studies (EWQOS) Project at Iowa State University provided the data for the phosphorus pollution content in Red Rock Reservoir on the Des Moines River at Knoxville, Iowa for this research. It is known that the construction of the reservoir and the dam has effects on both the aquatic environment of the river and surrounding terrestrial habitats. The point source pollution entering Red Rock Reservoir is from the City of Des Moines, due to its strategic location. Some means of assessing the environmental implications are imperative.

The phosphorus and other attributes of pollution content were sampled at predicted stations in the reservoir. The location of these stations was laid out in five transects (Figure 47). Each transect has five stations. The sampling at each location was done at one-meter depth intervals. The transects were laid down approximately in a northeast to southwest direction across the reservoir. The phosphorus pollution content data in computer-ready format is provided in Appendix J.

Purpose of the study

The geometric configuration of the data points (designed in 1979) was not appropriate to be used with the interpolation technique, since it was not anticipated that it would be used in any but a tabular form.
Figure 44. Original map of Red Rock Reservoir
Thus it was a severe challenge to existing interpolation schemes (multi-quadric and kriging) for mapping purposes. The length of each transect was about 3,200 feet. The first transect in the west was 23,134 feet away from the last transect in the east. The separation of each transect was about 5,800 feet, while the separation of the data points on each transect was only about 800 feet. Besides that, the separation of the data points in the vertical direction was only one meter. The maximum depth of the reservoir was approximately 20 meters for this data. The depth varies with the locality. The purpose of this study was to see the feasibility of both methods in the three-dimensional interpolating of the phosphorus pollution content and prediction of a 3-dimensional map. Experience with simulated data in the previous section was used as a basis for both the analytical solution and the graphical representation.

Method of application

The multiquadric-biharmonic method was applied to the 3-dimensional scheme with a 300 times vertical exaggeration. The exaggeration was justified by preliminary experimentation with the fitting of a 3-dimensional multiquadric-biharmonic scheme to this data and the discovery that the close spacing of vertical data tends to produce almost identical results at each level. Then it was found that the vertical exaggeration of 300 times, which artificially provided an approximately equal horizontal and vertical data spacing, avoided the difficulty mentioned above,
Figure 45. Multiquadric isarithmic map of phosphorus content at Red Rock Reservoir by 3-dimensional scheme at the 5-meter depth level.
Figure 46. Multiquadric isarithmic map of phosphorus content at Red Rock Reservoir by 3-dimensional scheme at the 10-meter depth level
Figure 47. Multiquadric isarithmic map of phosphorus content at Red Rock Reservoir by 3-dimensional scheme at the 13-meter depth level.
Figure 48. Multiquadric isarithmic map of phosphorus content at Red Rock Reservoir by 3-dimensional scheme at the 15-meter depth level
Figure 49. Multiquadric isarithmic map of phosphorus content at Red Rock Reservoir by 3-dimensional scheme at the 19-meter depth level
i.e. lack of variability in different layers. Then, the multiquadric system was used to predict the phosphorus value at unsampled points. Isarithmic maps at selected depths were plotted. The selected plots are presented in Figures 45 through 49.

Discussion

The 3-dimensional multiquadric-biharmonic method was capable of yielding a realization of the isarithmic map of the phosphorus content in Red Rock Reservoir. The kriging method could not yield a reasonable map because of the weakness of the geometric configuration of the data points which was not different than that available for the multiquadric method. It is easy to visualize that a geometric exaggeration of one dimension (the Z coordinate) would play havoc with the stochastic theory and stationarity assumption support, as needed for kriging. Geometry, mathematics, and physics seem to be a firmer foundation for the interpolation of physical processes.

The multiquadric-biharmonic method was capable of giving an isarithmic map down to a depth that has only two data points. This was done by incorporating the biharmonic sources in $\mathbb{R}^3$ space, instead of $\mathbb{R}^2$ space. Figure 49 is an example of this case.

Iowa Coal

Description of data

Two coal seams in the vicinity of Madrid, Iowa were described by Reese (1975). Isopach maps from Reese (Figures 50 and 51) were
Figure 50. Isopach map of Iowa Coal C from Reese (1975)
Figure 51. Isopach map of Iowa Coal E from Reese (1975)
SURFACE 1000 feet above sea level
850 feet above sea level in
Des Moines River valley

Pennsylvanian--Pleistocene Contact

KEY

1. Shale
2. Dark Carbonaceous Shale, some shale
3. Sandy Shale, Shale
4. Shale, some sandstone
5. Dark Carbonaceous Shale, Coal
6. Sandstone, some shale
7. Limestone
8. Glacial Drift

Coal--3 to $3\frac{1}{2}$ feet average, 700 feet above sea level

Coal--3\frac{1}{2} to 4 feet average, 650 feet above sea level

Mississippian-Pennsylvanian Contact

Figure 52. Composite section of Iowa coal field at Madrid
used to generate the point data that was used to test the multiquadric-biharmonic method in 3-dimensions. This data, in computer-ready format, are documented in Appendix K. In addition, it should be noted that the "no coal" condition between the two layers is equivalent, numerically, to some number of zero thicknesses being placed in the "no coal" region. A composite section of the coal field near Madrid, Iowa is shown in Figure 52. The Pennsylvanian formation consisted of various types of shale, coal, a limited amount of limestone, and sandstone. There were at least nine coal-bearing horizons in the Madrid field. Coal seams C and E were the only two seams that were mined on a large scale.

**Purpose of the study**

The study focuses on applying the 3-dimensional multiquadric-biharmonic to the discontinuous coal data. There are two obvious discontinuities in the coal; horizontally and vertically. The horizontal discontinuity appears at the border of the coal field that changed from coal to "no coal". The vertical discontinuity appears at the top and bottom of each of the seams. It is evident that data collection here is similar to "track" data problems encountered and discussed by Foley (1987) and Kansa (1988). Foley (1987) applied the 2-dimensional multiquadric to Monterey Bay data which was collected in tracks, similar to the transect data described in the section on pollution in Iowa reservoirs. In Foley's Monterey Bay case the data consisted of water temperatures
collected along widely spaced tracks. In a sense Foley's problem was easier than dealing with coal seams. Temperature is certainly continuous and there were no observations of "no temperature" in Monterey Bay, either real or imagined. However, due to the geometric configuration of track data, the surfaces produced by the multiquadric biharmonic, exhibited undesirable oscillations. Kansa (1988) observed that "track data", i.e., data which is closely spaced along one coordinate direction, and widely spaced along the orthogonal direction, caused large errors in the multiquadric process. Kansa solved the problem by mapping the data onto a unit line for one dimensional problems and on to a unit square for two dimensional problems. Kansa conjectured that a number of nearly equal distances occurred in track data and causes an inaccurate multiquadric result. He solved this problem by transforming the coordinate system so that the transformed distances are uniformly scattered.

Data manipulation similar to that of Kansa (1988) will be applied here using 3-dimensional coal data. This is expected to yield a better result than using unmanipulated data, and thus avoid the difficulties encountered by Foley (1987).

Methods of application

Both the real depth of the coal seams and the transformed depth were used in the computation. The transformation was done by exaggerating the vertical dimension. The 3-dimensional multiquadric-biharmonic method was applied to these data. The 2-dimensional multiquadric was
also applied to both coal seams independently. Both the original isopach maps and the maps predicted by the 2-dimensional multiquadric-biharmonic were used as the standard for testing the 3-dimensional multiquadric-biharmonic. This is because the 2-dimensional multiquadric-biharmonic method has provided good results in previous cases. The 3-dimensional kriging method cannot be applied because of the stationarity assumption. Nature, in this case, demonstrates concisely that it is incorrect to assume stationarity as the basis for an interpolation scheme.

Discussion

Figures 53 and 54 are the results of using the 2-dimensional multiquadric-biharmonic scheme. Figures 55 and 56 are the isopach maps from using the 3-dimensional multiquadric-biharmonic without exaggerating the vertical dimension. Figures 57 and 58 are the isopach maps from using the 3-dimensional multiquadric-biharmonic with an exaggerated vertical dimension.

The 2-dimensional multiquadric-biharmonic maps (Figures 53 and 54) are very similar to the maps in Figures 50 and 51. Figures 55 and 56 are identical when using the non-exaggerated depth in the 3-dimensional multiquadric-biharmonic scheme. Also the data point values in both seams were fitted exactly as well as the zeroes located in the region between seams. Thus the identical results in the two seams have been provided by a technically correct interpolation procedure. Nevertheless, the results are unacceptable because we know from geology
Figure 53. Multiquadric isopach map of Coal C, 2-dimensional data
Figure 54. Multiquadric isopach map of Coal E, 2-dimensional data
Figure 55. Multiquadric isopach map of Coal C using no vertical exaggeration in the Z coordinates
Figure 56. Multiquadric isopach map of Coal E using no vertical exaggeration in the Z coordinates
Figure 57. Multiquadric isopach map of Coal C using vertical exaggeration of Z coordinates.
Figure 58. Multiquadric isopach map of Coal E using vertical exaggeration of Z coordinates
that both coal seams should be uncorrelated. Later two tentative explanations are offered, but first I will describe another data arrangement that worked.

Using uniformly scattered data, produced by exaggerating the depth, the isopach maps of Coal C and Coal E are acceptable and are the same shape as those in the 2-dimensional case. Compare Figures 57 and 58 with 50 and 51, respectively. In reaching this result the distance between Coal C and Coal E was gradually exaggerated. Good predictions were obtained when the exaggeration is 20 times or more.

I conjecture that this distance may represent the true separation between Coal C and Coal E at some time in the geological past. Because of various geological processes, e.g. the erosional process, the distance between two coal seams could have been changed tremendously. In the 3-dimensional multiquadric-biharmonic experimentation with exaggerated distance, it should be noted that the isopach values in between the coal seams were set equal to zero. The zero values located close to both seams were helpful in showing steep gradients at or near the boundaries of the coal seams. The zero values farther away from the coal seams were of less significance in the mapping process.

Another conjecture concerning the undesirable identity of figures 55 and 56 is concerned with data management and a coupling with biharmonic potential theory. It has been observed in previous cases that extrapolation with the multiquadric-biharmonic method continues the trend at the boundary for some distance, then diminishes or smooths
out to zero or a mean constant level. The latter effect, i.e. increased smoothing with distance, is an expected result in potential theory because the influence should decrease to zero at infinity. This is accomplished with the multiquadric-biharmonic form by forcing the sum of the coefficients to be zero. This suggests that we should be concerned about extrapolation from one layer to another in this case. The two layers are about 50 feet apart vertically, but the horizontal extent of the layers is on the order of one or two miles. For the data in each single layer a prediction by the other layer is an extrapolation. Hence the map in each layer consists of two components, the two-dimensional component for the data in its own layer and the extrapolated content of the other layer. This is an exactly symmetric relation, i.e. the sum of the two sources is the same in each layer and hence the maps are identical. In the at least 20 times exaggeration of the vertical dimension it seems reasonable that the extrapolation effect of one layer on the other had diminished enough that the mapping in each layer is dependent mainly on the data in that layer.

Further experimentation with these two conjectures, or others, would be an ideal problem for studies in mathematical geology, or in mathematical geodesy, as the case may be.
CHAPTER 4. DISCOVERIES

Many new techniques and characteristics of both methods were dis­covered in this research. This chapter will present some of the discov­ered cases in detail and list others in Table 15.

Unbiasedness of the Multiquadric Method

The multiquadric system, with a condition for physical reasons, that the sum of the multiquadric coefficients is equal to zero, yields also an unbiased prediction automatically, i.e., satisfies the condition that the sum of the statistical weight is one (1). To amplify this statement, write both the kriging system and the multiquadric system in matrix form

\[
Z^*(X_p) = 
\begin{bmatrix}
\gamma_{p1} \cdots \gamma_{pn}, \\
\vdots & \ddots & \vdots \\
\gamma_{n1} \cdots \gamma_{nn} \\
1 \cdots 1
\end{bmatrix}
\begin{bmatrix}
y_{p1} \\
\vdots \\
y_{pn} \\
1
\end{bmatrix}
= 
\begin{bmatrix}
z(x_1) \\
\vdots \\
z(x_n) \\
0
\end{bmatrix}
\tag{100}
\]

and

\[
T^*(X_p) = 
\begin{bmatrix}
Q_{p1} \cdots Q_{pn}, \\
\vdots & \ddots & \vdots \\
Q_{n1} \cdots Q_{nn} \\
1 \cdots 1
\end{bmatrix}
\begin{bmatrix}
\hat{y}_{p1} \\
\vdots \\
\hat{y}_{pn} \\
1
\end{bmatrix}
= 
\begin{bmatrix}
\hat{z}(x_1) \\
\vdots \\
\hat{z}(x_n) \\
0
\end{bmatrix}
\tag{101}
\]

Equation (100) is the kriging system and Eq. (101) is the multiquadric system. Rewrite Eq. (100) and Eq. (101) in the matrix notation as
Table 15. Discoveries in the multiquadric and kriging techniques

<table>
<thead>
<tr>
<th>No.</th>
<th>Discovered or proposed</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Multiquadric</strong></td>
</tr>
<tr>
<td>1</td>
<td>Unbiasedness of the multiquadric method</td>
</tr>
<tr>
<td>2</td>
<td>Positive definiteness of the multiquadric coefficient matrix $(Q^TQ)$</td>
</tr>
<tr>
<td>3</td>
<td>Isotropic versus non-isotropic consideration</td>
</tr>
<tr>
<td>4</td>
<td>Ill-conditioning prevention by scalar multiplication</td>
</tr>
<tr>
<td>5</td>
<td>Underflow and overflow prevention by scalar multiplication</td>
</tr>
<tr>
<td>6</td>
<td>Vector analysis and fitting vector data (see Chapter 3)</td>
</tr>
<tr>
<td>7</td>
<td>Contour surfaces (see simulated data case study, Chapter 3)</td>
</tr>
<tr>
<td></td>
<td><strong>Kriging</strong></td>
</tr>
<tr>
<td>8</td>
<td>Formula for equal weight semivariogram (Hardy and Sirayanone's formula) (see iron grade case study)</td>
</tr>
</tbody>
</table>
The products of the elements in vector \( \gamma_{pj} \) and matrix \( \gamma_{ij}^{-1} \) are the variable kriging coefficients (weights). The products of the elements in matrix \( Q_{ij}^{-1} \) and vector \( T(x_j) \) are the constant multiquadric coefficients. The kriging weights change with the predicted point \( p \) and have no physical meaning. The multiquadric coefficients are unchanged and have the physical meaning as mentioned previously.

The last element in the \( \gamma_{pj} \) vector equals one, which results from the condition that the sum of the variable kriging weights must be one at every predicted point in order to give an unbiased prediction. This is based on geostatistical theory. The last element in the \( z(x_j) \) vector equals zero. This is because the constant resulting from the Lagrangian technique is not used in the prediction.

The last element in the \( Q_{pj} \) vector equals one because the constant \( \alpha_0 \) produced by the Lagrangian technique is used in the prediction based on multiquadric theory. It can be concluded that the one \( (1) \) in multiquadric vector \( Q_{pj} \) also satisfies an unbiased prediction as Eq. (100) does.
Positive Definiteness of the Multiquadric Coefficient Matrix ($Q^TQ$)

The multiquadric system can be solved for the multiquadric coefficients by the system in Eq. (61)

$$[p] = [Q^TQ]^{-1}[QT]$$

(61)

The matrix $[Q^TQ]$ can be proven to be positive definite. The positive definite matrix is defined as meeting the following condition:

If $X^TQX > 0$ then $Q$ is positive definite

if $X = a$ non-zero vector $\in \mathbb{R}^n$ and $Q$ is a non-singular symmetric matrix.

In the multiquadric system, the $Q$ matrix is always non-singular and symmetric (Michelli, 1986).

Proof:

$QX \neq 0$

Thus

$(QX)^T(QX) > 0$

$x^T(Q^TQ)x > 0$ (because $(AB)^T = B^T A^T$)

Hence $(Q^TQ)$ satisfies the positive definite definition. If the multiquadric system is solved by Eq. (61), the matrix $(Q^TQ)$ is always positive definite.
Isotropic Versus Non-Isotropic Considerations

One may ask how the multiquadric-biharmonic method deals with data that gives the appearance of being non-isotropic, since this is a matter of considerable concern in the kriging method. In the kriging method it appears that non-isotropic data is best handled by a non-isotropic semivariogram. The determination of an azimuth dependent semivariogram to handle predictions at non-data points in a non-isotropic data field seems to be an extremely difficult problem.

It may be of interest to learn that the multiquadric-biharmonic method need not be concerned at all about dealing with so-called non-isotropic data structures. From a physical point of view it has been discovered that, for many if not all problems to be handled by multiquadric analysis, anisotropy is an illusion. The reason is strikingly simple. Newton's laws are isotropic. Take gravity anomalies in Iowa as an example, which involves an anomaly ridge called the mid-continent gravity high. Viewed as a random process, the gravity anomaly map of Iowa would be said to be non-isotropic. But on the other hand we know that this so-called anisotropy results from the gravitational attraction generated by irregular magnitudes of sources that fulfill Newton's isotropic laws of attraction. In other words, highly irregular surfaces or maps (i.e. non-isotropics) can be produced by a set of isotropic functions, and the process is reversible. That is, the so-
called non-isotropic gravity anomaly field can be decomposed into a set of linearly combined isotropic kernel functions.

This confirms that mathematical physics is the foundation of the multiquadric-biharmonic method. I suggest this alternative way of viewing anisotropy could have some beneficial effect on the practice of kriging. As a goal it should help reduce the complexity in computing non-isotropic semivariograms.

**Ill-Conditioning Prevention by Scalar Multiplication**

The ill-conditioning in inversing the $Q$ or $Q^TQ$ matrices in Eqs. (51) and (52) results from many causes, e.g., large magnitude of the multiquadric constant ($\delta$), too small elements in the $Q$ or $Q^TQ$ matrix, etc. A successful mathematical technique was used to prevent ill-conditioning during the research period. This was done by increasing the magnitude of all elements in the $Q$ or $Q^TQ$ matrices by multiplying all coordinates by some scalar. The results of prediction by using this technique was found to be exactly the same as if the original matrix was used. This applied to either the $Q$ or $Q^TQ$ matrix.

**Underflow and Overflow Prevention by Scalar Multiplication**

Most computer systems will accept underflow and not accept overflow. The underflow value was treated as zero. The overflow will stop the execution of the computer program. The underflow is generally caused by dividing a small number with a very large number, or multiplying by a very small number. The overflow is generally caused by dividing...
a large number by a very small number, or multiplying by a very large number.

In Eq. (52), underflow and overflow may happen when multiplying the \((Q^TQ)^{-1}\) with \((Q^TT)\) matrices or similarly when multiplying \((Q)^{-1}\) with \((T)\) matrices. This will cause an erratic prediction at some sampled or unsampled points that are affected by the underflow coefficients. By experimentation, this problem may be prevented by multiplying to the data point value or vector \(T\) by some scalar. This results in changing the magnitude of the multiquadric coefficients as a function of the scalar. A prediction by the multiquadric system yields the product of the estimated value with that scalar. In order to get the real estimated value, predicted values are divided by that scalar.
CHAPTER 5. CONCLUSIONS

The multiquadric method (harmonic, biharmonic) is a deterministic method that has a physical meaning. The kriging geostatistical method is a stochastic method based on the theory of regionalized variables. The kernel function of the multiquadric method is a predetermined function, the distance. The kernel function of the kriging method was obtained by fitting the proper mathematical functions to the semivariogram. The multiquadric method interprets the spatial structure of the variable under investigation by the isarithmic map, while the structural analysis in the kriging method was obtained by interpreting the semivariogram. The shape of the experimental semivariogram in the kriging method is dependent upon the choice of lag interval and the scale of observation rather than the real unknown spatial structure of the phenomenon itself.

By experimentation, the accuracy of the multiquadric method is equal to the kriging method. The multiquadric method has no costly preprocessing step. The kriging method has a preprocessing step that is based mainly on the experience and judgment of the researcher. The preprocessing step takes at least three times the total time and cost of processing. This figure varies with the method of handling the preprocessing technique.

A principle characteristic of the kriging method is that it is difficult to understand and has a statistical meaning. Many kriging
users complain about these difficulties. The multiquadric method is more user-friendly.

The kriging or geostatistical method uses terminology similar to classical statistics, but it is not based on the same theories due to the stationarity assumption.

The estimation variance and the block variance in the kriging method are independent of the location of the point or block. The estimation variance in the kriging method is not a valid justification for an assertion of superior interpolation properties over a physically deterministic method such as MQ-B. It is only valid for comparing the kriging methods that use different theoretical semivariograms. This is because they are based on the same assumption.

The MQ-B positional block variation is dependent upon the location of the point or block. This is because the multiquadric method is not subject to any stochastic assumption.

The multiquadric method has been applied to scattered data without any computation problems. An important requirement for the kriging method is the data must be regular grid-spaced in order to obtain the highest efficiency possible for the technique. Both methods yield an unbiased estimation.

The kriging weights are not equivalent to the multiquadric coefficients. The kriging weights change with the location of the predicted points, while the multiquadric coefficients are always the same. The sum of the kriging weights is set equal to one to give an unbiased
estimation, based on statistical theory. The sum of the multiquadric coefficients are set to equal zero to satisfy the minimum energy concept in the physical theory.

The classical multiquadric is comparable to the simple point or punctual kriging. Both the osculating mode of multiquadric and universal kriging utilize additional polynomial terms.

Both methods can be applied to 3-dimensional data, but the kriging method is subjected to additional assumptions.

In the temperature case study, both methods could handle two dimensional scattered data successfully. The multiquadric-biharmonic method required only the data values to be input into the written computer ready program. On the other hand, there were difficulties with the kriging method in constructing an experimental semivariogram because of the inadequacy of the number of data points and the irregularity of the geometric configuration of data point locations. The cross validation error of the multiquadric method showed less error than the kriging method. The kriging error computed by the stochastic formula yielded an overly optimistic absolute value of error. The isarithmic map produced by the kriging method was similar to the isarithmic map from the multiquadric method.

In the wind velocity vector case study, both the multiquadric-biharmonic and the kriging methods failed to fit the direction quantity directly. A meaningful approach was obtained by decomposing the wind velocity vector into two components, north-south and east-west direc-
tions. Both methods were applied to the decomposed vectors. The magnitude and direction of the wind velocity vector were restored by recomposing the two components. This method doubled the cost of the kriging preprocessing step. The cross validation error of the multiquadric biharmonic showed less error than the kriging, cross validation error in the wind velocity case.

In the iron ore grade case study, the short statistical range of the magnitude of the ore grade yielded a continuous semivariogram. The shape of the isarithmic maps of iron grade, produced by both the multiquadric-biharmonic method and the kriging method, are almost identical. The accuracy of the multiquadric-biharmonic method was higher than both of the kriging methods. The kriging exponential model yielded slightly more accurate results than the spherical model in predicting iron grade. Using the Hardy and Sirayanone formula, the kriging method yielded a slightly better accuracy than the classical Matheron formula. The MQ-B positional block variation changes with the size, location, and shape of the block of ore. The stochastic error formula for kriging yielded an overly optimistic magnitude of error as compared to the cross validation error. In a comparison of extrapolation schemes for predicting iron grade, the multiquadric biharmonic trend at the data boundary persisted for greater distances than the kriging spherical model.

In the manganese gangue case, the multiquadric-biharmonic method was almost as accurate as the kriging method. The kriging method yielded
a slightly better cross validation error at the third decimal point. Both methods were capable of being applied to the data that has a small magnitude such as the manganese gangue. An abrupt change in the semi-variogram shape caused difficulty in fitting the theoretical semivariogram to the experimental semivariogram. The isarithmic maps produced by both methods are almost identical. The stochastic error formula yielded an overly optimistic magnitude of error.

In the simulated data case, the multiquadric-biharmonic and kriging methods yielded almost identical isarithmic maps. The kriging method could be applied to the 3-dimensional data with additional assumptions, while the multiquadric-biharmonic method did not need any stochastic assumptions, i.e. assumptions of stationarity in 3-dimensional space. Both methods were applied successfully to irregular data in 3-dimensional space. The preprocessing step for 3-dimensional schemes in the kriging method had more difficulty than the 2-dimensional schemes, especially for the irregular spaced data.

In producing a stereopair, the anaglyph and the photographic techniques yielded the expected solution. The availability of suitable materials hindered the construction of anaglyphs. The photographic technique was limited by the depth of focus of the camera. The results of incorporating the AGRAPH computer graphics routine, the meshsurface program, and the multiquadric-biharmonic subroutine yielded an optimal solution for producing the stereopair image in 3-dimensional space.
An advantage of the stereopair in 3-dimensional space is that the data or variable embedded in an invisible space can be presented. This technique produced a contour surface. The multiquadric-biharmonic was proven to be the most versatile, simple, economical and accurate method while using the meshsurface and AGRAPH computer graphics routines.

In the phosphorus pollution case, the kriging method could not be applied because of the negative effect of the geostatistical assumptions. This was because the data configuration geometry was very erratic, while the ratio of length:width:depth of data space was about 2,000:250:1. The multiquadric-biharmonic method of the 3-dimensional scheme was capable of yielding a realization of the isarithmic map of phosphorus pollution in Red Rock Reservoir. The 3-dimensional scheme of the multiquadric-biharmonic method even fitted the data magnitude exactly at the deepest level which had only two data points.

Many new techniques and characteristics of both methods that were not discovered before are presented in this dissertation and will be concluded here.

Hardy and Nelson's equation, that is, the physical meaning of the multiquadric-biharmonic coefficients as related to the disturbing potential, was named during the research period. The multiquadric theories and processes were organized in Chapter 2. Hardy and Sirayanoone's formula for equal weighted semivariograms was discovered and named. This formula diminishes the complaints that can occur about the weakness of the semivariogram at greater distances.
The multiquadric system was found to yield an unbiased estimation. The multiquadric matrix was proven to be always positive definite.

A method of preventing ill-conditioning when inverting the matrix by the computer was discovered. This was done by multiplying the matrix element to be inversed by an appropriate scalar. The underflow and overflow in computer routines was discovered to be preventable by multiplying the data by some scalar. Then, the correct estimated data could be obtained by dividing the predicted value with the same scalar. This was shown to yield a more accurate prediction, especially in overcoming the erratic estimation problem that is caused by an underflow in computer execution.

A method for handling the vectorial data was discovered. This was done by decomposing the vectorial data into two components perpendicular to each other. The predicted vectorial data can be restored by recomposing the two predicted perpendicular components. This method yielded an accurate prediction for both the magnitude and direction of the variable under investigation.

A computer program for the lattice plot was written. This program is very useful for displaying data that are imbedded in invisible space.

Finally, the isotropic versus the non-isotropic considerations were discussed. This discussion provides a strong perception of the stochastic and deterministic processes.
Many minor discoveries that either have too low a priority or a need to be proven in more depth were left out of this dissertation. Many minor case studies are also omitted.


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I wish to give my highest gratitude to Dr. Rolland L. Hardy, my major professor, for his academic and moral support, guidance, suggestions, understanding, and for his effort in finding me the assistantship that enabled me to fulfill my academic goal. To you, Dr. Hardy, let me put it my way, "I am proud to have been your advisee."

I wish to thank Dr. John Lemish, my co-major professor, who always understood my situation and provided me great support in the Earth Sciences Department.

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I would be ungrateful if I did not mention the names of my previous Thai professors that consistently gave me moral support while I was
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Lastly, I would like to give my highest thanks and gratitude to my father and mother, who trained me in the correct method directly or indirectly. Those things indirectly made me come farther than I could believe.

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C_j \quad \text{multiquadric coefficient}
q \quad \text{multiquadric function}
x_j, y_j, x, y \quad \text{data point cartesian coordinates in } R^2 \text{ space}
z \quad \text{function of } x_j, y_j \text{ in } R^2 \text{ space case}
\delta \quad \text{multiquadric kernel function constant}
r \quad \text{radius to source point}
R \quad \text{radius of the sphere}
\theta_j, \lambda_j, \theta, \lambda \quad \text{data point spheroidal coordinates}
k_i, A_i, B_i \quad \text{coefficients of polynomial function}
N \quad \text{geoid undulation}
C(s) \quad \text{covariance function of lag distance } s
m(x) \quad \text{trend function}
e(x) \quad \text{random function}
\lambda_i \quad \text{stochastic weights}
\gamma(h) \quad \text{semivariogram of lag distance } h
C_0 \quad \text{nugget effect}
C \quad \text{sample variance}
h \quad \text{lag distance}
a \quad \text{range of influence}
\alpha_j, \rho_j \quad \text{multiquadric coefficients}
z_j \quad \text{cartesian coordinates in } R^3 \text{ space case}
T_j \quad \text{data point value in multiquadric method}
Q_{ij} \quad \text{multiquadric matrix element at } i\text{th row and } j\text{th column}
V  discrepancy
S  standard error of sample
W  weight matrix
n  number of data points
\n  Laplacian operator
U, V  function U, V in three variables
$\xi_{pq}$  distance from point p to q
V  volume of sphere
$\delta$  density anomaly
T  disturbing potential
G  Newtonian constant
$\partial n$  partial derivative in normal direction to the sphere surface
$\beta_j$  multiquadric coefficient representing triharmonic source
E  expectation
Var  variance
g_i  regionalized variable at point i
E  nugget effect by Royle (1980)
\Psi  for all
$\mu$  constant produced by Lagrangian technique
Cov  covariance
$\epsilon_i$  error at point i
$\sigma$  population standard deviation
APPENDIX B: PROGRAM CHANGING LATITUDE AND LONGITUDE TO STATE PLANE COORDINATES
This program is used to change latitude and longitude to state plane coordinates. The details and references are in the text.

The order of input is as follows:

Name of station
Latitude
Longitude

Output:

The output will list the value of latitude, longitude, station name, state plane coordinates, scale factor and convergence of Lambert projection.
PROGRAM TO CHANGE LATITUDE,LONGITUDE OF COLORADO NORTH ZONE TO STATE PLANE COORDINATES.

LET P(1)=VALUE OF LATITUDE IN DEGREE AND MINUTE, EXPRESSED IN MINUTES, P(2)=VALUE OF SECOND PORTION OF LATITUDE EXPRESSED IN SECOND.

DLA,MLA,SLA, ARE DEGREE,MINUTE,SECOND, OF LATITUDE.

DLO,MLO,SLO, ARE DEGREE,MINUTE,SECOND, OF LONGITUDE.

RLAT IS A VALUE OF LATITUDE IN RADIANS.

R IS THE MAP RADIUS.

Z IS A CONVERGENCE.

LAM IS A VALUE OF LONGITUDE EXPRESSED IN SECOND.

K IS VALUE OF SCALE FACTOR.

PI=VALUE OF PI=3.141592654

IF YOU WANT TO STOP INPUT DLA=-1, THEN INPUT ANY NUMBER FOR MLA AND SLA.

CHARACTER POINT*30

INTEGER IDLA,IMLA,IDLO,IMLO

REAL DLA,MLA,DL0,ML0

DOUBLE PRECISION L1,L2,L3,L4,L5,L6,L7,L8,L9,L10,L11,PI

DOUBLE PRECISION S,RLAT,R,Z,LAM,X,Y,K

PARAMETER(L1=2000000.00,L2=379800.00,L3=24751897.68)

PARAMETER(L4=25086068.20,L5=0.9999568475)

PARAMETER(L6=0.6461334829,L7=2406.0,L8=24.62308,L9=3.81044)

PARAMETER(L10=3.85610,L11=0.00)

PARAMETER(PI=3.141592654)

READ*,POINT

READ*,DLA,MLA,SLA

IF(DLA.EQ.-1) GO TO 50

READ*,DLO,MLO,SLO

P(1)=60.*DLA+MLA

P(2)=SLA

RLAT=(DLA+MLA/60.+SLA/3600.)*PI/180.

S=101.2794065*(60.*(L7-P(1))+L8-P(2)+((1052.893882-((4.483344-0.023520*(COS(RLAT))*2)*(COS(RLAT))*2)*SIN(RLAT)**2)*SIN(RLAT)**2)*SIN(RLAT)*COS(RLAT))

R=L3+S*L5*(L1+((S/1E8)**2)*(L9-(S/1E8)*L10+((S/1E8)**2)*L11))

Z=L6*(L2-(DLO*3600.+MLO*60.+SLO))

IDLA=INT(DLA/1)

IMLA=INT(MLA/1)

IDLO=INT(DLO/1)

IMLO=INT(MLO/1)

PRINT*,'COLORADO NORTH LAT.=N,LONG.=W'

PRINT*,POINT

PRINT*, 'LATITUDE=',IDLA,' ',IMLA,' ',SLA

PRINT*, 'LONGITUDE=',IDLO,' ',IMLO,' ',SLO

X=L1+R*SIN(Z/3600.*PI/180.)

Y=L4-R+2.*R*(SIN(Z/7200.*PI/180.))*2)
K = L6 * R * (1 - (0.0067686580 * (SIN(RLAT)**2))**0.5) / 20925832.16

* / COS(RLAT)
PRINT*, 'X = ', X, ' FEET'
PRINT*, 'Y = ', Y, ' FEET'
PRINT*, 'SCALE FACTOR = ', K
PRINT*, 'CONVERGENCE = ', Z, ' SECOND'
PRINT*, '
PRINT*, '
GO TO 20
STOP
END

'ARV'
39 48 0.0
105 05 0.0
'AUR'
39 45 0.0
104 52 0.0
'BGD'
40 38 0.0
104 20 0.0
'BRI'
40 00 0.0
104 48 0.0
'BYE'
39 45 0.0
104 08 0.0
'ELB'
39 13 0.0
104 38 0.0
'ERI'
40 02 0.0
105 00 0.0
'EPK'
40 22 0.0
105 34 0.0
'FTH'
40 35 0.0
105 08 0.0
'GLY'
40 26 0.0
104 38 0.0
'ISG'
39 40 0.0
105 30 0.0
COLORADO NORTH LAT.=N, LONG.=W

ARV
LATITUDE=  39  48  0.0000000000000000E+00
LONGITUDE=  105  5  0.0000000000000000E+00
X  =  2117075.620832428  FEET
Y  =  170264.5169944760  FEET
SCALE FACTOR=  0.9999875855470194
CONVERGENCE=  969.200224350000  SECOND

******************************************************************************
COLORADO NORTH LAT.=N, LONG.=W
AUR
LATITUDE= 39 45 0.0000000000000000E+00
LONGITUDE= 104 52 0.0000000000000000E+00
X = 2178084.166081685 FEET
Y = 152412.3120191252 FEET
SCALE FACTOR= 0.9999947879780459
CONVERGENCE= 1473.18431012000 SECOND

COLORADO NORTH LAT.=N, LONG.=W
BGD
LATITUDE= 40 38 0.0000000000000000E+00
LONGITUDE= 104 20 0.0000000000000000E+00
X = 2323809.660922756 FEET
Y = 475687.1486021745 FEET
SCALE FACTOR= 0.9999791265088839
CONVERGENCE= 2713.760628180000 SECOND

COLORADO NORTH LAT.=N, LONG.=W
BRI
LATITUDE= 40 0 0.0000000000000000E+00
LONGITUDE= 104 48 0.0000000000000000E+00
X = 2196110.628414092 FEET
Y = 243616.4193176450 FEET
SCALE FACTOR= 0.9999663718369977
CONVERGENCE= 1628.256376908000 SECOND

COLORADO NORTH LAT.=N, LONG.=W
BYE
LATITUDE= 39 45 0.0000000000000000E+00
LONGITUDE= 104 8 0.0000000000000000E+00
X = 2384274.938430829 FEET
Y = 154737.6562414832 FEET
SCALE FACTOR= 0.9999947879780459
CONVERGENCE= 3178.976735868000 SECOND

******************************************************************************************************************
APPENDIX C: PROGRAM COMPUTING THE MULTIZQUADRIC-BIHARMONIC METHOD FOR TEMPERATURE DATA IN COLORADO
This program is used to predict the temperature at unsampled points by the multiquadric-biharmonic method and gives the cross validation error and plots the isarithmic map. Input data are attached at the end of the program. The order of input is as follows:

name of station, x coordinate, y coordinate, temperature

The output will list the multiquadric coefficient, the prediction at sample points, cross validation error, the predicted temperature in a grid pattern and a plot of an isarithmic map. (Details of related theory are in Chapter 2.)
C MAIN PROGRAM FOR COMPUTING MULTISURFACES USING CONE MODEL.
CHARACTER POINT*30, DUMMY(21)*20, TPOINT*30
DOUBLE PRECISION X(20), C(21), A(21, 21), AA(21, 21), B(21), CI
1, Y(20), YC
DOUBLE PRECISION Z(40, 33), XX(33), YY(40), WKAREA(990),
1 VAL, ZIJ, BC, XC
DOUBLE PRECISION AVER, BCK(20), RANG, ZMAX, ZMIN,
1 TEMPSM, TEST, TOT
DOUBLE PRECISION OX, OY, OB, PB, RES, TOX, TOY, TOB, TB(20)
DIMENSION YZ(40, 33)
READ*, POINT, OX, OY, OB
N=20
DO 15 I=1, N
READ*, DUMMY(I), X(I), Y(I), B(I)
DO 111 IC=1, N+1
CALL PLOTS(0, 0, 31)
IF(IC.GT.1) THEN
TPOINT=POINT
TOX=OX
TOY=OY
TOB=OB
IM=IC-1
POINT=DUMMY(IM)
OX=X(IM)
OY=Y(IM)
OB=B(IM)
DUMMY(IM)=TPOINT
X(IM)=TOX
Y(IM)=TOY
B(IM)=TOB
END IF
PRINT*, 'INPUT DATA (X, Y, TEMPERATURE)'
WRITE(6, 200)(DUMMY(I), X(I), Y(I), B(I), I=1, N)
FORMAT(2X, A20, 3F20.5)
RANG=535079.13636+41298.03801
VAL=RANG/39.
XC=1981423.02292
YC=535079.13636
XX(I)=XC
YY(I)=YC
DO 20 I=2, 40
YY(I)=YC-VAL
20 XC=YY(I)
29. DO 25 I=2,33
30. XX(I)=XC+VAL
31. 25 XC=XX(I)
32. C CENTERING THE DATA
33. 201 TEMPSM=0.0
34. DO 30 I=1,N
35. 30 TEMPSM=TEMPSM+B(I)
36. AVER=TEMPSM/N
37. PRINT*,"****AVERAGE OF TEMPERATURE=',AVER
38. DO 35 I=1,N
39. 35 TEMPSM=0.0
40. DO 40 J=1,N
41. 40 B(I)=B(I)-AVER
42. PRINT*,"SUM OF CENTERED DATA=',TEST,'(SHOULD BE ZERO)'
43. C COMPUTATION
44. DO 33 I=1,N
45. 33 A(N+1,I)=1.0
46. 33 A(I,N+1)=1.0
47. DO 40 I=1,N
48. 40 A(I,J)=DSQRT((X(J)-X(I))**2+(Y(J)-Y(I))**2)
49. CALL LINV2P(A,N+1,N+1,AA,0,WKAREA,IER)
50. DO 45 I=1,N+1
51. 45 CI=0
52. 45 CI=CI+AA(I,J)*B(J)
53. 45 C(I)=CI
54. 5 TOTAL=TOTAL+C(I)
55. DO 43 I=1,N
56. 43 C(I)=C(I)
57. 43 TOTAL=TOTAL+C(I)
58. PRINT*,"SUM OF COEFFICIENTS=',TOTAL
59. 25 PRINT*,"DELTA G ZERO=',C(N+1)
60. WRITE(6,5)(C(1),I=1,N)
61. C SOLUTION OF COEFFICIENTS ',/(D25.14)
62. C COMPUTE THE OBSERVED STATIONS
63. DO 55 I=1,N
64. BC=0.0
65. DO 60 J=1,N
66. 60 BC=BC+C(J)*DSQRT((X(J)-X(I))**2+(Y(J)-Y(I))**2)
67. 55 BCK(I)=BC+AVER+C(N+1)
68. WRITE(6,37)(BCK(I),I=1,N)
69. 37 FORMAT' COMPUTED TEMPERATURE BY USING COEFFICIENTS',/,
70. 1(D25.14)
71. C READ AND COMPUTE THE OMITTED POINT, ALSO THE REQUIRED
72. 05 C POSITION.
73. 11 PRINT*,' COORDINATES OF OMITTED OR REQUIRED POINTS'
DO 271 K=1,N
PRINT*, 'POINT NAME= ', POINT X Y TEMP
WRITE(6,371) OX, OY, OB
FORMAT(3X,3F20.5)
PB=0.0
DO 372 I=1,N
PB=PB+C(I)*DSQRT((X(I)-OX)**2+(Y(I)-OY)**2)
PB=PB+AVER+C(N+1)
RES=PB-OB
PRINT*, 'PREDICTED TEMPERATURE OF ', POINT, ' = ', PB
PRINT*, 'RESIDUAL= ', RES
PRINT*,'++++++++++++++++++++++++++++++++++'
C
COMPUTE TEMPERATURE IN GRID PATTERN
C
DO 65 I=1,40
DO 65 J=1,33
ZIJ=0.0
DO 70 K=1,N
ZIJ=ZIJ+C(K)*DSQRT((X(K)-XX(J))**2+(Y(K)-YY(I))**2)
ZIJ=ZIJ+AVER+C(N+1)
IF(I.EQ.1) THEN
ZMAX=ZIJ
ZMIN=ZIJ
END IF
ZMAX=MAX(ZMAX,ZIJ)
ZMIN=MIN(ZMIN,ZIJ)
Z(I,J)=ZIJ
ICLO=INT(ZMIN)+1
CLO=FLOAT(ICLO)
ICL1=INT(ZMAX)
CL1=FLOAT(ICL1)
PRINT*, 'LOWEST CONTOUR LEVEL= ', CLO
PRINT*, 'HIGHEST CONTOUR LEVEL= ', CL1
PRINT*, '******** DATA USED TO PLOT CONTOUR *******'
DO 72 K=1,30,5
K5=K+4
WRITE(6,18)(XX(LL), LL=K,K5)
FORMAT('1',5F13.3,/) WRITE(6,22)(YY(I),(Z(I,J), J=K,K5), I=1,40)
FORMAT(F15.3,5F13.3,//)
CONTINUE WRITE(6,38)(XX(LL), LL=31,33)
FORMAT('1',3F13.3,/) WRITE(6,42)(YY(I),(Z(I,J), J=31,33), I=1,40)
FORMAT(F15.3,3F13.3,//)
DO 75 I=1,40
DO 75 J=1,33
YZ(I,J)=Z(I,J)
CALL CONTR2(YZ,39,32,CL0,1.0,CL1,1,5.0,1.0,90.0,40)
CALL PLOT(50.0,0.0,999)
DO 352 I=1,N
B(I)=TB(I)
CONTINUE
STOP
END

//GO.SYSIN DD *

'AUR' 2178084.16608 152412.31202 74.3
'BOD' 2323809.66092 475687.14860 73.8
'BRI' 2196110.62841 243616.41932 73.3
'BYE' 2384274.93843 154737.65624 79.4
'ELB' 2245591.03317 -41298.03801 79.9
'ERI' 2140011.26811 255379.36696 74.2
'EPK' 1981423.02292 376417.40240 76.2
'FOR' 2101846.71670 455552.06008 75.9
'PTM' 2470423.50585 308019.39629 73.0
'GLY' 2241259.52325 401875.45550 75.9
'ISG' 2000000.00000 121421.21835 62.0
'KNB' 2279881.15692 268705.09438 76.2
'LTN' 2155066.56635 85475.94331 80.4
'LG8' 2093158.54513 303729.28134 76.3
'LVE' 2129974.34732 388895.40480 75.6
'NUN' 219844.48808 535079.13636 72.3
'PFL' 2176783.33781 334541.94817 75.8
'ROL' 2004676.19562 206416.22548 70.1
'WRD' 1990665.86622 254986.38804 65.5
'ARV' 2117075.62083 170264.51699 74.0
/*
APPENDIX D: PROGRAM COMPUTING THE KRIGING METHOD FOR TEMPERATURE DATA IN COLORADO
This is the program for computing all possible pairs of distance vectors between data points. The details of related procedures are in Chapter 3. An example of input data is attached at the end of the program. The order of input is as follows:

- number of data points
- name of point, x coordinate, y coordinate, regionalized variable
- name of point, x coordinate, y coordinate, regionalized variable

Output:

The output will be a list of sorted distance vectors from minimum to maximum magnitude.
PROGRAM TO COMPUTE ALL POSSIBLE DISTANCES

CHARACTER POINT(56)*10
INTEGER SWITCH,NTEMP,N,NAME(56,56)
DOUBLE PRECISION D(56,56),T(56),X(56),Y(56),TEM(56,56),TEMP
DATA SWITCH/1/
READ*,N
READ*,(POINT(I),X(I),Y(I),T(I),I=1,N)
DO 10 I=1,N-1
DO 10 J=I+1,N
D(I,J)=DSQRT((X(J)-X(I))**2+(Y(J)-Y(I))**2)
TEM(I,J)=T(J)
NTEMP=I*1000000+J
10 NAME(I,J)=NTEMP
20 IF(SWITCH.EQ.1)THEN
   SWITCH=0
   DO 30 1=1,N-1
      DO 40 J=I+1,N
         IF(D(I,J).LT.D(I,J-1))THEN
            TEMP=D(I,J)
            D(I,J)=D(I,J-1)
            D(I,J-1)=TEMP
            TEMP=TEM(I,J)
            TEM(I,J)=TEM(I,J-1)
            TEM(I,J-1)=TEMP
            NTEMP=NAME(I,J)
            NAME(I,J)=NAME(I,J-1)
            NAME(I,J-1)=NTEMP
            SWITCH=1
            END IF
         40 CONTINUE
   IF(I.EQ.N-1) GO TO 30
   IF(D(I+1,I+2).LT.D(I,N))THEN
      TEMP=D(I+1,I+2)
      D(I+1,I+2)=D(I,N)
      D(I,N)=TEMP
      TEM=TEM(I+1,I+2)
      TEM(I+1,I+2)=TEM(I,N)
      TEM(I,N)=TEM
      NTEMP=NAME(I+1,I+2)
      NAME(I+1,I+2)=NAME(I,N)
      NAME(I,N)=NTEMP
      SWITCH=1
   ENDIF
30 CONTINUE
GO TO 20
ENDIF
PRINT*, '*******SORTED DISTANCES*******'
PRINT*, 'LIST OF DISTANCE, ITS VALUES, TEMPERATURE OF ENDED POINT'
DO 50 I=1,N-1
DO 50 J=I+1,N
50 PRINT*, NAME(I,J), D(I,J), TEM(I,J)
STOP
END

56

'6' 150 361.5 58.35
'9' 706.5 367.5 59.10
'10' 916.5 367.5 50.6
'11' 1095 403.5 57.7
'12' 517.5 634.5 62.8
'13' 705 645 55.63
'14' 897 645 60.11
'15' 1291.5 562.5 57.0
'16' 358.5 858 62.08
'17' 720 843 59.40
'18' 1335 810 52.2
'19' 393 1020 59.61
'20' 718.5 1035 60.55
'21' 1522.5 1005 56.76
'22' 568.5 1213.5 58.65
'23' 915 1198.5 61.27
'24' 1125.0 1246.5 62.8
'25' 918 1432.5 58.49
'26' 1111.5 1413.0 58.0
'27' 751.5 1650 44.30
'28' 930 1636.5 63.10
'29' 1156.5 1618.5 55.74
'30' 390 1846.5 49.30
'31' 763.5 1843.5 51.63
'32' 1350 1803 57.44
'33' 1681.5 1858.5 44.28
'34' 1411.5 1980 55.5
'35' 345 345 59.7
'36' 525 357 61.4
'37' 165 639 52.6
'38' 375 630 60.82
'39' 1117.5 630 59.20
'40' 1470.0 532.5 55.70
'41' 1132.5 855 66.45
'42' 178.5 1047 58.60
'43' 532.5 1005 61.51
'44' 928.5 1018.5 60.66
'45' 1168.5 1020 61.0
'46' 187.5 1273.5 60.86
'47' 405 1213.5 60.44
<table>
<thead>
<tr>
<th>Year</th>
<th>Value1</th>
<th>Value2</th>
<th>Value3</th>
</tr>
</thead>
<tbody>
<tr>
<td>'51'</td>
<td>1305</td>
<td>1227.0</td>
<td>62.2</td>
</tr>
<tr>
<td>'52'</td>
<td>1500</td>
<td>1195.5</td>
<td>58.76</td>
</tr>
<tr>
<td>'53'</td>
<td>720</td>
<td>1458</td>
<td>61.90</td>
</tr>
<tr>
<td>'56'</td>
<td>1327.5</td>
<td>1423.5</td>
<td>60.56</td>
</tr>
<tr>
<td>'57'</td>
<td>412.5</td>
<td>1650</td>
<td>58.70</td>
</tr>
<tr>
<td>'58'</td>
<td>600</td>
<td>1650</td>
<td>50.9</td>
</tr>
<tr>
<td>'62'</td>
<td>1335</td>
<td>1633.5</td>
<td>58.74</td>
</tr>
<tr>
<td>'63'</td>
<td>1875</td>
<td>1576.5</td>
<td>55.5</td>
</tr>
<tr>
<td>'65'</td>
<td>555</td>
<td>1845</td>
<td>59.73</td>
</tr>
<tr>
<td>'67'</td>
<td>1140</td>
<td>1800</td>
<td>54.90</td>
</tr>
<tr>
<td>'70'</td>
<td>1935.0</td>
<td>1738.5</td>
<td>52.25</td>
</tr>
<tr>
<td>'71'</td>
<td>712.5</td>
<td>2073.00</td>
<td>53.2</td>
</tr>
<tr>
<td>'72'</td>
<td>1020</td>
<td>2050</td>
<td>59.66</td>
</tr>
<tr>
<td>'73'</td>
<td>1245</td>
<td>1995</td>
<td>56.96</td>
</tr>
<tr>
<td>'75'</td>
<td>1665</td>
<td>1980</td>
<td>47.85</td>
</tr>
</tbody>
</table>
This program is used to average the regionalized variable in the chosen interval of distance vectors. The input data come from the output of the distance vector computation program after choosing the interval of lag distance. The related procedure is explained in Chapter 2. The output will list the lag distance and its regionalized variable magnitude.
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION DIS(20), AV(20)
I=0.0
SAV=0.0
20 READ*, DIST, N
IF(DIST.EQ.-1000) GO TO 30
I=I+1
SUM=0.0
DO 10 J=1, N
READ*, DUMMY1, DUMMY2, TEMP
10 SUM=SUM+TEMP
AVER=SUM/N
DIS(I)=DIST
AV(I)=AVER
SAV=SAV+AVER
GO TO 20
30 TAVER=SAV/I
PRINT*, 'AVERAGE TEMPERATURE=', TAVER
PRINT*, 'DISTANCE, TEMPERATURE, TEMPERATURE-AVERAGE'
DO 40 J=1, I
TEMN=AV(J)-TAVER
40 PRINT*, DIS(J), AV(J), TEMN
STOP
END
This program is used to compute the experimental semivariogram as explained in Chapter 2. The input data come from the previous program (program that gives the lag distance and the magnitude of the regionalized variable). The output will list the lag distance and semivariogram of point value.
$ TYPE POM1.FOR$
C MAIN PROGRAM FOR VARIOGRAMS COMPUTATION.
DIMENSION G(40),Z(20)
READ *,N
M=N-1
DO 10 I=1,N
10 READ *,DUMMY1,G(I)
WRITE(6,35)(G(I),I=1,N)
35 FORMAT(2X,5E12.4)
DO 20 K=1,M
NN=N-K
SUM=0.0
DO 50 I=1,NN
J=I+K.
SUM=SUM+(G(I)-G(J))**2
Z(K)=SUM/(2.0*FLOAT(N-K))
50 CONTINUE
20 CONTINUE
DO 70 I=1,M
WRITE(6,30)I,Z(I)
30 FORMAT(2X,'r',I3,E12.4)
70 CONTINUE
STOP
END
This program is used to compute the unknown variables in the spherical model as fitted to the experimental semivariogram. After the modeler decides that the appropriate model is the spherical model, the output from the experimental semivariogram is input to this program. The related procedure and the details of the spherical model are explained in Chapter 2.

The output will list the required variable values and compute the theoretical semivariogram every 0.25 units of interval.
PROGRAM TO FIT SPHERICAL MODEL
DIMENSION AH(15),R(15)
READ*,N,A
PRINT*, 'SEMIVARIOGRAM OF RAW DATA'
RMAX=-10000
DO 10 I=1,N
READ*,AH(I),R(I)
RMAX=MAX(RMAX,R(I))
10 PRINT*,AH(I),R(I)
PRINT*, 'NORMALIZED SEMIVARIOGRAM'
DO 12 I=1,N
R(I)=R(I)/RMAX
12 PRINT*,AH(I),R(I)
A=A-2.05
B=A+5
SMIN=10000
AMN=10000
20 DOWHILE(A.LE.B)
A=A+.05
SUM=0.0
DO 30 I=1,N
SUM=SUM+((3*AH(I)*A**2-AH(I)**3)/(2*A**3)-R(I))
* ((-3*AH(I)*A**2+3*AH(I)**3)/(2*A**4))
30 CONTINUE
TSMIN=SMIN
TAMN=AMN
SMIN=MIN(SMIN,ABS(SUM))
IF(SMIN.NE.TSMIN)THEN
AMN=A
ENDIF
PRINT*, 'A=',A,'RESULT=',SUM
GO TO 20
END DO
PRINT*, 'VALUE OF A THAT MINIMIZES THE FUNCTION=',AMN
X1=AMN
Y1=1
PRINT*, 'A=',X1,'UNIT'
PRINT*, 'C=',Y1
C PROGRAM TO COMPUTE R(H) FROM SPHERICAL MODEL.
PRINT*, 'R(H) FROM SPHERICAL MODEL'
P=0.0
55 DOWHILE(P.LE.X1)
D=Y1*((3*P/(2*X1))-(P**3/(2*X1**3))}
PRINT*,P,D
P=P+0.25
GO TO 55
END DO
STOP
END
This program is used to find the unknown variables in the exponential model that fit the experimental semivariogram. The exponential model and related details are explained in Chapter 2. The input data come from the experimental semivariogram program after the modeler decides on a shape appropriate to the exponential model.

The output will list the required variables and compute the theoretical semivariogram each 0.25 units of interval.
C PROGRAM TO COMPUTE VALUE OF a AND C (COEFFICIENT OF THE C EXPONENTIAL MODEL TO FIT SEMIVARIOGRAM)

IMPLICIT INTEGER(I-N), REAL*8(A-H, O-Z)
DIMENSION AA(40, 2), Z(40, 1), XX(2, 1), AAT(2, 40), AATA(2, 2)
 1 , AATAI(2, 2), AATZ(2, 1), G(40, 1)
READ *, M
DO 50 I=1, M
50 READ *, Z(I, 1)
PRINT*,'VALUE OF R(H) FROM POINT VALUES'
PRINT*, (Z(I, 1), I=1, M)
DO 52 I=1, M
52 Z(I, 1) = LOG(Z(I, 1))
CSET "A" MATRIX
DO 10 I=1, M
 TA=I
 AA(I, 1) = 1
 AA(I, 2) = -TA
10 CONTINUE
IAA=40
IBB=2
CALL LEASTSQ(AA, XX, AAT, Z, AATA, AATAI, AATZ, IAA, IBB, M, 2)
PRINT*, 'VALUE OF XX(1, 1) =', XX(1, 1)
PRINT*, 'XX(2, 1) =', XX(2, 1)
PRINT*, 'VALUE OF a AND C'
X1=1/(XX(2, 1))
Y1=EXP(XX(1, 1))
PRINT*, 'A =', X1, 'UNIT'
PRINT*, 'C =', Y1
C PROGRAM TO COMPUTE R(H) FROM EXPONENTIAL MODEL.
PRINT*, 'R(H) FROM EXPONENTIAL MODEL'
P=0.0
55 DOWHILE(P.LE.M+3)
 R=Y1*(1-EXP(P/X1))
 PRINT*, P, R
 P=P+0.25
 GO TO 55
 END DO
END

SUBROUTINE LEASTSQ(A, X, AT, DL, ATA, ATAI, ATDL, IA
 1 , IB, MM, NN)
IMPLICIT INTEGER(I-N), REAL*8(A-H, O-Z)
DIMENSION A(IA, IB), DL(IA, 1), X(IB, 1), AT(IB, IA),
 1 ATA(IB, IB), ATAI(IB, IB), ATDL(IB, 1)
CALL ATRANS(A, AT, IA, IB, MM, NN)
CALL AXB(AT, A, ATA, IB, IA, IB, NN, MM, NN)
CALL VERS(ATA, ATAI, 2, 2)
CALL AXB(A, AT, DL, ATDL, IB, IA, 1, NN, MM, 1)
CALL AXB(ATAI,ATDL,X,IB,IB,1,NN,NN,1)
RETURN
END
C
SUBROUTINE ATRANS(TBB,TBBT,NI,NM,K,L)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION TBB(NI,NM),TBBT(NM,NI)
DO 10 J=1,L
  DO 12 I=1,K
    TBBT(J,I)=TBB(I,J)
  12 CONTINUE
10 CONTINUE
RETURN
END
SUBROUTINE AXB(AA,BB,R,N1,N2,N3,LL,MM,NN)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION AA(N1,N2),BB(N2,N3),R(N1,N3)
DO 15 I=1,LL
  DO 16 J=1,NN
    R(I,J)=0.0
    DO 17 K=1,MM
      R(I,J)=R(I,J)+AA(I,K)*BB(K,J)
  17 CONTINUE
16 CONTINUE
15 CONTINUE
RETURN
END
SUBROUTINE VERS(AA,BB,NI,NM)
IMPLICIT REAL*8(A-H,O-Z)
DOUBLE PRECISION AA(NI,NM),BB(NI,NM),P(257)
N=NI-1
MI=NM-1
DO 1 J=1,NI
  DO 1 K=1,NI
    BB(J,K)=AA(J,K)
  1 CONTINUE
DO 2 J=1,MI
  P(J)=BB(1,J+1)/BB(1,1)
2 CONTINUE
P(NM)=1.DO/BB(1,1)
DO 4 L=1,N
  DO 3 J=1,MI
    BB(L,J)=BB(L+1,J+1)-BB(L+1,1)*P(J)
3 CONTINUE
BB(L,NM)=-BB(L+1,1)*P(NM)
4 CONTINUE
BB(NI,J)=P(J)
RETURN
END
This program is used to predict the unsampled points by the kriging method with the exponential model. The input data are attached at the end of the program. The order of input is as follows:

name of point, x coordinate, y coordinate, data value
name of point, x coordinate, y coordinate, data value

The output will list the kriging coefficient (not kriging weight), the predicted values of the sampled points, the predicted unsampled points in a grid pattern, and plot the isarithmic maps of regionalized variables. More related details are in Chapter 2.
C MAIN PROGRAM FOR COMPUTING TEMPERATURE SURFACE USING
KRIGING METHOD.

CHARACTER POINT*30,DUMMY(21)*20,TPOINT*30

DOUBLE PRECISION X(20),C(21),A(21,21),AA(21,21),B(21),CI

1,Y(20),YC,AZ,DD

DOUBLE PRECISION Z(40,33),XX(33),YY(40),

1 WKAREA(990),VAL,ZIJ,BC,XC

DOUBLE PRECISION AVER,BCK(20),RANG,ZMAX,

1 ZMIN,TEMPSM,TEST,TOT

DOUBLE PRECISION OX,OY,OB,PB,RES,TOX,TOY,TOB,TB(20)

DIMENSION YZ(40,33)

N=20

READ*,AZ

PRINT*,"VALUE OF *a* FOR COVARIANCE FUNCTION=",AZ

READ*,POINT,OX,OY,OB

DO 15 I=1,N

READ*,DUMMY(I),X(I),Y(I),B(I)

IF(IC.GT.1) THEN

TPOINT=POINT

TOX=OX

TOY=OY

TOB=OB

IM=IC-1

POINT=DUMMY(IM)

OX=X(IM)

OY=Y(IM)

OB=B(IM)

DUMMY(IM)=TPOINT

X(IM)=TOX

Y(IM)=TOY

B(IM)=TOB

END IF

PRINT*,"INPUT DATA (X,Y,TEMPERATURE)"

WRITE(6,200)(DUMMY(I),X(I),Y(I),B(I),I=1,N)

FORMAT(2X,A20,3F20.5)

RANG=535079.13636+41298.03801

VAL=RANG/39.

XC=1981423.02292

YC=535079.13636

XX(I)=XC

YY(I)=YC

DO 20 I=2,40
29. \( \text{YY}(I) = \text{YC} - \text{VAL} \)
30. \( \text{YC} = \text{YY}(I) \)
31. \( \text{DO} 25 \ I = 2,33 \)
32. \( \text{XX}(I) = \text{XC} + \text{VAL} \)
33. \( \text{XC} = \text{XX}(I) \)
34. \( \text{C} \)
35. \( \text{CENTERING THE DATA} \)
36. \( \text{TEMP}\text{SM} = 0.0 \)
37. \( \text{DO} 30 \ I = 1,N \)
38. \( \text{AVER} = \text{TEMP}\text{SM}/N \)
39. \( \text{PRINT}*, '',**AVG**ERAGE OF TEMPERATURE=',',AVER \)
40. \( \text{DO} 35 \ I = 1,N \)
41. \( \text{TB}(I) = \text{B}(I) \)
42. \( \text{PRINT}*, '',SUM OF CENTERED DATA=',',TEST,'(SHOULD BE ZERO)' \)
43. \( \text{COMPUTATION} \)
44. \( \text{DO} 33 \ I = 1,N \)
45. \( \text{A}(N+1,I) = 1.0 \)
46. \( \text{A}(I,N+1) = 1.0 \)
47. \( \text{A}(N+1,N+1) = 0.0 \)
48. \( \text{DO} 40 \ I = 1,N \)
49. \( \text{DO} 40 \ J = 1,N \)
50. \( \text{DD} = ((X(J)-X(I))^2+(Y(J)-Y(I))^2)^{0.5}/60000 \)
51. \( \text{DD} = \text{DD}^2 \)
52. \( \text{A}(I,J) = \text{EXP}((-AZ)*\text{DD}) \)
53. \( \text{CALL LINV2F(A,N+1,N+1,AA,0,WKAREA,IER)} \)
54. \( \text{DO} 45 \ I = 1,N+1 \)
55. \( \text{CI} = 0 \)
56. \( \text{DO} 50 \ J = 1,N+1 \)
57. \( \text{CI} = \text{CI} + \text{AA}(I,J) * \text{B}(J) \)
58. \( \text{C}(I) = \text{CI} \)
59. \( \text{TOT} = 0.0 \)
60. \( \text{DO} 43 \ I = 1,N \)
61. \( \text{TOT} = \text{TOT} + C(I) \)
62. \( \text{PRINT}*, '',SUM OF COEFFICIENTS=',',TOT \)
63. \( \text{PRINT}*, '',\text{DELTA G ZERO}=',',C(N+1) \)
64. \( \text{WRITE}(6,5)(C(I),I=1,N) \)
65. \( \text{FORMAT}(' \text{SOLUTION OF COEFFICIENTS '},/(D25.14)) \)
66. \( \text{COMPUTE THE OBSERVED STATIONS} \)
67. \( \text{DO} 55 \ I = 1,N \)
68. \( \text{BC} = 0.0 \)
69. \( \text{DO} 60 \ J = 1,N \)
70. \( \text{DD} = ((X(J)-X(I))^2+(Y(J)-Y(I))^2)^{0.5}/60000 \)
71. \( \text{DD} = \text{DD}^2 \)
72. \( \text{BC} = \text{BC} + C(J) * \text{EXP}((-AZ)*\text{DD}) \)
73. \( \text{BCK}(I) = \text{BC} + \text{AVER} + C(N+1) \)
WRITE(6,37)(BCK(I),I=1,N)
37
FORMAT(1(D25.14))
C
READ AND COMPUTE THE OMITTED POINT, ALSO THE REQUIRED
C
POSITION.
C
PRINT*, 'COORDINATES OF OMITTED OR REQUIRED POINTS'
C
DO 271 K=1,1
271
PRINT*, 'POINT NAME=',POINT
PRINT*, X
PRINT*, Y TEMP
WRITE(6,371) OX,OY,OB
371
FORMAT(3X,3F20.5)
PB=0.0
DO 372 I=1,N
372
DD=(((X(I)-OX)**2+(Y(I)-OY)**2)**0.5/60000
PB=PB+C(I)*EXP((-AZ)*DD)
RES=PB-OB
PRINT*, 'PREDICTED TEMPERATURE OF ',POINT,'=',PB
PRINT*, 'RESIDUAL=',RES
PRINT*, '++++++++++++++++++++++++++++++++++++'
C
C
COMPUTE TEMPERATURE IN GRID PATTERN
C
DO 65 I=1,40
65
DO 65 J=1,33
ZIJ=0.0
DO 70 K=1,N
70
DD=(((X(K)-XX(J))**2+(Y(K)-YY(I))**2)**0.5/60000
ZIJ=ZIJ+C(K)*EXP((-AZ)*DD)
ZIJ=ZIJ+AVER+C(N+1)
IF(I.EQ.1)THEN
ZMAX=ZIJ
ZMIN=ZIJ
END IF
ZMAX=MAX(ZMAX,ZIJ)
ZMIN=MIN(ZMIN,ZIJ)
Z(I,J)=ZIJ
ICL0=INT(ZMIN)+1
CLO=FLOAT(ICL0)
ICL1=INT(ZMAX)
CL1=FLOAT(ICL1)
PRINT*, 'LOWEST CONTOUR LEVEL=',CLO
PRINT*, 'HIGHEST CONTOUR LEVEL=',CL1
PRINT*, '******* DATA USED TO PLOT CONTOUR *******'
DO 72 K=1,30,5
118. K5=K+4
119. WRITE(6,18)(XX(LL),LL=K,K5)
120. 18 FORMAT('1',5F13.3,/) 
121. WRITE(6,22)(YY(I),(Z(I,J),J=K,K5),I=1,40)
122. 22 FORMAT(F15.3,5F13.3,/) 
123. 72 CONTINUE 
123.02 WRITE(6,38)(XX(LL),LL=31,33)
123.04 38 FORMAT('1',3P13.3,/) 
123.06 WRITE(6,42)(YY(I),(Z(I,J),J=31,33),I=1,40)
123.08 42 FORMAT(F15.3,3F13.3,/) 
124. DO 75 I=1,40
125. DO 75 J=1,33 
126. 75 YZ(I,J)=Z(I,J) 
127. CALL CONTR2(YZ,39,32,CL0,1.0,CL1,1.5,0.1,0.9,0.0,40) 
128. CALL PLOT(50.0,0.0,999) 
128.05 DO 352 I=1,N 
128.07 352 B(I)=TB(I) 
128.09 111 CONTINUE 
129. STOP 
130. END 
131. //GO.SYSIN DD * 
151.5 0.6852541036529575 
152. 'AUR' 2178084.16608 152412.31202 74.3 
152.05 'BOU' 2070023.02855 249012.20206 75.9 
152.1 'BGD' 2323809.66092 475687.14860 73.8 
152.15 'BRI' 2196110.62841 243616.41932 73.3 
152.2 'BYE' 2384274.93843 154737.65624 79.4 
152.25 'ELB' 2245591.03317 -41298.03801 79.9 
152.3 'ERI' 2140011.26811 255379.36696 74.2 
152.35 'EPK' 1981423.02292 376417.40240 76.2 
152.4 'FOR' 2101846.71670 455552.06008 75.9 
152.45 'GLY' 2241259.52325 401875.45550 75.9 
152.5 'ISG' 2000000.00000 121421.21835 62.0 
152.6 'KNB' 2279881.15692 268705.09438 76.2 
152.65 'LTM' 2155066.56635 85475.94331 80.4 
152.7 'LMG' 2093158.54513 303729.28134 76.3 
152.75 'LVE' 2129974.34732 388895.40480 75.6 
152.8 'NUN' 2198424.48808 535079.13636 72.3 
152.85 'PL1' 2176783.33781 334541.94817 75.8 
152.9 'ROL' 2004676.19562 206416.22548 70.1 
152.95 'WDR' 1990665.86622 254986.38804 65.5 
153. 'ARV' 2117075.62083 170264.51699 74.0 
154. */
APPENDIX E: PROGRAM FOR PLOTTING VECTOR DIAGRAM
This program is used to plot vector diagrams. It is useful to present vectorial data of any kind. It was originally written for plotting wind velocity vector. The input is attached at the end of the program. The order of input is as follows:

\[ x \text{ coordinate}, y \text{ coordinate}, \text{magnitude}, \text{azimuth} \]
\[ x \text{ coordinate}, y \text{ coordinate}, \text{magnitude}, \text{azimuth} \]

The program is written in IGL (Interactive Graphic Library) code. It was successfully run on the Tektronix computer.

The output and vector diagrams will be both onscreen and in hard copy. Examples of output are in Chapter 3.
CHARACTER POINT*10,TEX*7
DIMENSION X(50),Y(50),VEL(50,50),AZ(50,50)
C
LIMITING NAME OF POINT TO 10 CHARACTERS
1000 READ*,POINT
CALL GRSTRT(4014,1)
CALL NEWPAG
CALL MOVE(10.0,80.0)
CALL CMCLOS
PRINT*, 'WHEN THE PROGRAM FINISHING PLOT MAKE THE HARD COPY'
PRINT*, 'FIRST THEN HIT<> TO STOP THE PROGRAM'
PRINT*, 'THIS PROGRAM DESIGNED TO WORK WITH MAX. WIND VELOCITY
1 30 UNITS AND MAX. LENGTH OF AREA 595000 UNITS, IN ORDER TO
1 BE OVER THIS LIMIT, SCALES ARE TO BE CHANGED'
CALL CMOPEN
CALL MOVE(0.0,0.0)
CALL GETUTX(10, 'HIT RETURN', 1, D2, D3)
CALL NEWPAG
CALL MOVE(110.0,90.0)
CALL TEXT(10,POINT)
CALL MOVE(0.0,100.0)
CALL TEXT('N')
CALL MOVE(0.0,100.0)
CALL DRAW(0.0,0.0)
CALL DRAW(100.0,0.0)
CALL DRAW(100.0,100.0)
CALL DRAW(0.0,100.0)
CALL CMCLOS
READ*,M,N
TEX='NO'
SCALE=8500
VECSC=3.0
XMIN=1981423.02292-85000
YMIN=-41298.03801-85000
C
ASSUME MAXIMUM WIND VELOCITY 30 NAUTICAL MILE/HR
C
MAXIMUM LENGTH OF AREA=595000 FEET
READ*,(X(I),I=1,N)
DO 10 I=1,M
READ*,Y(I),(VEL(J,I),AZ(J,I),J=1,N)
10 CONTINUE
137 DO 20 I=1,M
   DO 20 J=1,N
      SX=(X(I)-XMIN)/SCALE
      SY=(Y(J)-YMIN)/SCALE
      PI=3.141592654
      AZI=450-AZ(I,J)
      IF(AZI.GT.360.0) THEN
      AZI=AZI-360.0
     ENDIF
      DX=(VEL(I,J)*SIN(AZ(I,J)*PI/180))/VECSC
DY=(VEL(I,J)*COS(AZ(I,J)*PI/180))/VECSC
OX=SX+DX
OY=SY+DY
ANG1=AZI-150.0
ANG2=AZI+150.0
CALL CMOPEN
CALL MOVE(SX,SY)
CALL DRAW(OX,OY)
CALL PIVOT(OX,OY)
CALL MPOLAR(0.5,ANG1)
CALL DRAW(OX,OY)
CALL PIVOT(OX,OY)
CALL MPOLAR(0.5,ANG2)
CALL DRAW(OX,OY)
CALL CMCLOSES
IF(TEX.EQ.'NO') GO TO 20
TX=SX
TY=SY-2.5
CALL CMOPEN
CALL MOVE(TX,TY)
CALL TEXT(7,TEX)
CALL CMCLOSES
20 CONTINUE
M=1
N=1
C LIMITING NAME OF POINT TO BE PRINTED ON THE SCREEN TO 3
C 3 CHARACTERS
READ*,TEX
READ*,X(1),Y(1),VEL(1,1),AZ(1,1)
IF(TEX.EQ.'NOMORE') GO TO 125
IF(TEX.EQ.'LAST') GO TO 145
GO TO 137
C GET HARD COPY AT THIS STEP
C THEN HIT RETURN TO STOP
125 CALL CMOPEN
CALL MOVE(0.0,0.0)
CALL GETUTX(1,' ',1,D2,D3)
CALL CMCLOSES
GO TO 1000
145 CALL CMOPEN
CALL MOVE(0.0,0.0)
CALL GETUTX(1,' ',1,D2,D3)
CALL GRSTOP
STOP
END
APPENDIX F: DATA FOR IRON GRADE AND MANGANESE PERCENT IN
BAHARIYA OASIS, EGYPT
This is computer ready data for both iron grade and manganese gangue. The order of data input is:

name of point, x coordinate, y coordinate, data value
name of point, x coordinate, y coordinate, data value

The discussion of the experimentation with these data is in Chapter 3.
<table>
<thead>
<tr>
<th>IRON GRADE DATA</th>
</tr>
</thead>
<tbody>
<tr>
<td>'6' 150 361.5 58.35</td>
</tr>
<tr>
<td>'7' 345 345 59.7</td>
</tr>
<tr>
<td>'8' 525 357 61.4</td>
</tr>
<tr>
<td>'9' 706.5 367.5 59.10</td>
</tr>
<tr>
<td>'10' 916.5 367.5 50.8</td>
</tr>
<tr>
<td>'11' 1095 403.5 57.7</td>
</tr>
<tr>
<td>'12' 165 639 52.6</td>
</tr>
<tr>
<td>'13' 375 630 60.82</td>
</tr>
<tr>
<td>'14' 517.5 634.5 62.8</td>
</tr>
<tr>
<td>'15' 706.5 367.5 59.10</td>
</tr>
<tr>
<td>'16' 916.5 367.5 50.8</td>
</tr>
<tr>
<td>'17' 1095 403.5 57.7</td>
</tr>
<tr>
<td>'18' 1156.5 1633.5 55.5</td>
</tr>
<tr>
<td>'19' 1335 1633.5 55.5</td>
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<tr>
<td>'20' 1522.5 1633.5 55.5</td>
</tr>
<tr>
<td>'21' 1648.5 1633.5 55.5</td>
</tr>
<tr>
<td>'22' 1765 1633.5 55.5</td>
</tr>
<tr>
<td>'23' 1882.5 1633.5 55.5</td>
</tr>
<tr>
<td>'24' 2000 1633.5 55.5</td>
</tr>
<tr>
<td>'25' 2117.5 1633.5 55.5</td>
</tr>
<tr>
<td>'26' 2235 1633.5 55.5</td>
</tr>
<tr>
<td>'27' 2352.5 1633.5 55.5</td>
</tr>
<tr>
<td>'28' 2470 1633.5 55.5</td>
</tr>
<tr>
<td>'29' 2587.5 1633.5 55.5</td>
</tr>
<tr>
<td>'30' 2705 1633.5 55.5</td>
</tr>
<tr>
<td>'31' 2822.5 1633.5 55.5</td>
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MANGANESE GANGUE DATA

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APPENDIX G: PROGRAM FOR COMPUTING EXPERIMENTAL SEMIVARIOGRAM BY HARDY AND SIRAYANONE'S FORMULA
This program is used to compute the semivariogram by Hardy and Sirayanone's formula as explained in Chapter 3. The input data come from the program that computes the possible distance vectors between data points as explained in Appendix D. The output will list the semivariogram of point value.
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION DIST(20),G(20),R(20),Z(20)
I=0.0
SAV=0.0
READ*,UNIT
PRINT*, 'UNIT=', UNIT
READ*,N
IF(N.EQ.-1000) GO TO 30

20 I=I+1
SUM=0.0
DISS=0.0
DO 10 J=1,N
READ*, DUMMY1, DIS, TEMP
IF(DUMMY1.EQ.-100000) THEN
   I=I-1
   GO TO 30
END IF
DISS=DISS+DIS
SUM=SUM+TEMP
AVER=SUM/N
DISS=DISS/N
DIST(I)=DISS
G(I)=AVER
SAV=SAV+AVER
GO TO 20

30 TAVER=SAV/I
N=I
PRINT*, 'AVERAGE FE%= ', TAVER
PRINT*, 'DISTANCE, FE%, FE%-AVERAGE'
DO 40 J=1,I
TEMN=G(J)-TAVER
PRINT*, DIST(J), G(J), TEMN
40 CONTINUE

C MAIN PROGRAM FOR VARIOGRAMS COMPUTATION.
M=N-1
DO 80 K=1,M
NN=N-K
SUM=0.0
DISS=0.0
DO 50 I=1,NN
   J=I+K
   SUM=SUM+(G(I)-G(J))**2
   DISS=DISS+(ABS(DIST(J)-DIST(I)))
   Z(K)=SUM/(2.0*FLOAT(N-K))
50 CONTINUE
R(K)=DISS/((N-K)*UNIT)
80 CONTINUE
PRINT*, 'SEMIVARIOGRAM OF RAW DATA, EQUAL NUMBER OF POINTS IN 1 EACH INTERVAL'
DO 70 I=1,M
PRINT*, 'i', ',', R(I), Z(I)
CONTINUE
STOP
END
APPENDIX H: SIMULATED DATA IN COMPUTER READY FORMAT
(IRREGULAR SPACED DATA)
These are data provided by the supervisor of this dissertation as explained in Chapter 3. Data are listed in computer ready format in the following form:

    named point, x, y, z coordinates, data value
    named point, x, y, z coordinates, data value
<table>
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<th>4'</th>
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<td>26.70</td>
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</tbody>
</table>

Note: The table represents data with values rounded to the nearest tenth.
APPENDIX I: MESHSURFACE PROGRAM AND MULTIGRUCIC SUBROUTINE
This program is used to do multiplane plots as explained in Chapter 3. It is to be used with the multiquadric general subroutine which is attached at the end of the program.

The multiquadric subroutine will ask for the data file. The order of variables in the data file is:

- number of data points
- name of point, x, y, z coordinates, data value
- name of point, x, y, z coordinates, data value

The output from the meshsurface program will be in a format ready for the AGRAPH computer graphics routine. References and related details of the AGRAPH routine are in the text.
MESHSURFACE PROGRAM
C MESHSUR.FOR, A ROUTINE TO GENERATE AN AGRAPH SEE THROUGH
C WIRE MESH DIAGRAM OF CONSTANT SURFACES OF THE SCALAR
C DESIGNATED P(X,Y,Z) WHERE P CAN BE POTENTIAL, CONCENTRATION,
C ETC.
REAL V(81,81,21),XX(81),YY(81),ZZ(21),VCONT(16)
CHARACTER * 1 KOLORS(16),KAR,KOL,YES,COLOR(81)
CHARACTER * 50 TITLE
CHARACTER * 80 INCOLR
designation (color(1),incolor)
DATA KOLOR/0/,nx,ny,nz/81,81,21/,NOX,NOY,NOZ/0,0,0/
DATA KOLORS/16*0'/',BIG/1.0E+35/,DX,DY,DZ/0.0,0.0,0.0/
500 FORMAT(A1)
505 FORMAT(A30)
508 FORMAT(A50)
509 FORMAT(A80)
510 FORMAT(1,'..WHAT IS THE NAME OF YOUR INPUT DATA FILE?','$)
512 FORMAT(1,'..WHAT IS THE NAME OF YOUR OUTPUT FILE?','$
514 FORMAT(1,* LIST via the KEYBOARD your COLOR Sequence, e.g.',
1 'R,B,Y,etc','/','$)
518 format(1,' Do you want to predict by Multiquadric biharmonic
1 ?',
1 '//,' Y or (N=RETURN) '?','$)
C
START=TIMER(TF,TS)
612 format(1,'..How many grid point in XYZ direction you want
1 ',//,'..maximum 81 81 21','$
613 format(1,'..Input X width','$
615 format(1,'..Input Y width, same unit as X','$
616 format(1,'..Input depth or height between top and
1 bottom layers, same unit','$
617 format(1,'..Do you want to enlarge or reduce scale
1 ',//,'..in the x or y or z direction, yes or no?','$)
write(1,612)
read*,nx,ny,nz
write(1,613)
read*,xwid
write(1,614)
read*,ywid
write(1,615)
read*,zwid
xfa=1
yfa=1
zfa=1
write(1,616)
read(*,500)yes
if(yes.eq.'y'.or.yes.eq.'Y') then
write(*,617)
format(/,'...Input X enlarging or reducing factor, 1 decimal point accept, 1 if not')
read*,xfa
write(*,618)
format(/,'...Input Y enlarging or reducing factor, 1 decimal point accept, 1 if not enlarge')
read*,yfa
write(*,619)
format(/,'...Input Z enlarging or reducing factor, 1 decimal point accept, 1 if not enlarge')
read*,zfa
end if
write(*,518)
read(*,500) yes
if(yes.eq.'y'.or.yes.eq.'Y') then
call mqb(v,nx,ny,nz,vmin,vmax)
do 100 i=1,nx
   xx(i)=float(i-1)*xfa
   yy(i)=(float(i-1))*(nx/ny)*(ywid/xwid)*yfa
   if(i.le.nz) zz(i)=(float(i-1))*(nx/nz)*(zwid/xwid)*zfa
continue
write(*,620)
format(/,'...Input title of your plot')
read(*,508)title
end if

WRITE(*,510)
READ(*,505) INFILE
OPEN(17,FILE=INFILE,STATUS='OLD')
READ(17,508) TITLE
READ(17,*) NXX,NYY,NZZ

INPUT OR EVALUATE THE X,Y,AND Z GRID ARRAYS

NX=IABS(NXX)
if(NXX.LT.0) then
   READ(17,*)(XX(I),I=1,NX)
   DX=XX(NX)-XX(1)
else
   DD=8000.0/FLOAT(NX-1)
   do 102 I=1,NX
      XX(I)=DD*FLOAT(I-1)
   continue
endif
NY=IABS(NYY)
if(NY.LT.0) then
   READ(17,*)(YY(J),J=1,NY)
DY=YY(NY)-YY(1)

ELSE
  DD=8000.0/FLOAT(NY-1)
  DO 105 J=1,NY
     YY(J)=DD*FLOAT(J-1)
  ENDIF

105 CONTINUE

ENDIF

NZ=IABS(NZZ)

IF(NZ.LT.0) THEN
  READ(17,*) (ZZ(K), K=1,NZ)
  DZ=ZZ(NZ)-ZZ(1)
ELSE
  DD=8000.0/FLOAT(NZ-1)
  DO 110 K=1,NZ
     ZZ(K)=DD*FLOAT(K-1)
  ENDIF

110 CONTINUE

C

DS=ABS(DX)

IF(ABS(DY).GT.DS) DS=ABS(DY)
IF(ABS(DZ).GT.DS) DS=ABS(DZ)

DD=8000.0/DS

IF(NXX.LT.0) THEN
  DO 112 I=1,NX
     XX(I)=DD*(XX(I)-XX(1))
  CONTINUE
END IF

IF(NYY.LT.0) THEN
  DO 114 J=1,NY
     YY(J)=DD*(YY(J)-YY(1))
  CONTINUE
END IF

IF(NZZ.LT.0) THEN
  DO 116 K=1,NZ
     ZZ(K)=DD*(ZZ(K)-ZZ(1))
  CONTINUE
END IF

C

INPUT THE THREE DIMENSIONAL MATRIX WITH THE K=1 XY PLANE FIRST,K=2 XY PLANE NEXT, ETC.

C

WRITE(*,*)'..READING YOUR',NX*NY*NZ,'..INPUT VALUES'
  DO 120 K=1,NZ
     WRITE(*,*)'..READING Z-PLANE NO.', K
     DO 120 J=1,NY
        WRITE(*,*)'..READING YOUR',NX*NY,NZ,'..INPUT VALUES COMPLETE'
VMIN=BIG
VMAX=-VMIN
DO 130 K=1,NZ
  DO 130 J=1,NY
    DO 130 I=1,NX
      VV=V(I,J,K)
      IF(ABS(VV).LT.BIG) THEN
        IF(VV.GT.VMAX) VMAX=VV
        IF(VV.LT.VMIN) VMIN=VV
      ENDIF
  130 CONTINUE
C
140 CONTINUE
WRITE(*,520) VMIN,VMAX
520 FORMAT(/*,4X,'YOUR SCALAR FUNCTION VALUES RANGE FROM', /,
  1PE14.5,' TO',1PE14.5,/,4X,'DO YOU WISH TO CHANGE THESE ',
  ! 'LIMITS? Y or N. ',$)
read(*,500) yes
if(yes.eq.'y'.or.yes.eq.'Y') then
  write(*,521)
521 format(/,' Specify your upper and lower contour limits',$)
  read(*,*) vlow,vhigh
else
  vlow=vmmin
  vhigh=vmax
end if
C
write(*,522)
522 format(/*,4X,'HOW MANY CONTOUR SURFACES ', /,
  1 'DO YOU WISH TO GENERATE?',/,'LIMIT=15. ',$)
READ(*,*) KONTRS
KONTRS=IABS(KONTRS)
IF(KONTRS.GT.15) THEN
  WRITE(*,*) 'EXCESSIVE NUMBER OF CONTOUR TRY AGAIN'
  GO TO 140
ENDIF
if(kontrs.lt.2) then
  kontrs=1
  VCONT(1)=(VLOW+VHIGH)/2.0
else
  DV=(VHIGH-VLOW)/FLOAT(KONTRS-1)
  DO 145 K=1,KONTRS-1
    VCONT(K)=VLOW+DV*FLOAT(K-1)
  145 CONTINUE
  VCONT(KONTRS)=VHIGH
end if
C
WRITE(*,525)
525 FORMAT(/*,5X,'DO YOU WANT YOUR CONTOUR PLOTS IN COLOR? ',$)
READ(*,500) YES
IF(YES.EQ.'y'.OR.YES.EQ.'Y') THEN
  KOLOR=1
  WRITE(*,514)
  READ(*,509) INCOLR
C
K=0
J=0
150 CONTINUE
  K=K+1
  IF(K.GT.80) GO TO 154
  IF(COLOR(K).NE.' ') THEN
    J=J+1
    KOLORS(J)=COLOR(K)
    GO TO 150
  ENDIF
  GO TO 150
154 CONTINUE
  IF(J.LT.KONTRS)THEN
    JJ=2*J
    IF(J.GT.15) GO TO 160
    IF(JJ.GT.15)JJ=15
    DO 158 I=J+1,JJ
      KOLORS(J)=KOLORS(I-J)
    CONTINUE
    J=JJ
    GO TO 154
  ENDIF
C
END IF
160 CONTINUE
C
format(/,3x,'You Can Elect to Plot Contours ONLY on Certain',1
  'Planes.','-3x,'Do You Want to Specify Certain Planes? ','
  2 'Y or N. ',$)
format(/,3x,'Do You Want X-Plane Contours Plotted? Y or N',$)
format(/,3x,'Do You Want Y-Plane Contours Plotted? Y or N',$)
format(/,3x,'Do You Want Z-Plane Contours Plotted? Y or N',$)
C
write(*,540)
read(*,500) YES
if(yes.eq.'y'.or.yes.eq.'Y') then
  nox=1
  noy=1
  noz=1
  write(*,541)
  read(*,500) YES
  if(yes.eq.'y'.or.yes.eq.'Y') nox=0
  write(*,542)
read(*,500) yes
if(yes.eq.'y'.or.yes.eq.'Y') noy=0
write(*,543)
read(*,500) yes
if(yes.eq.'y'.or.yes.eq.'Y') noz=0
end if

WRITE(*,530) timer(tf,ts) - tstart
530 FORMAT(/,3X,'PRELIMINARIES COMPUTED: ALL DATA INPUTTED.',
1 ' ',3X,'CPU TIME SO FAR=',0pFl0.2,' s',3X,'READY ','
1 'TO COMPUTE CONSTANT SURFACE MESHES',3X,'AND OUTPUT TO',
1 ' FILE.',5X,'SPECIFY YOUR OUTPUT FILE NAME. ','$)
READ(*,505) OUTFIL
OPEN(18,FILE=OUTFIL,STATUS='NEW')
write(18,*) '$3D 100000$
write(18,*) '$2D 100000$

WRITE(18,560)TITLE
560 FORMAT('T',A50,/,'-1.0E+35')

Output frame around region of interest
WRITE(*,730)
730 FORMAT(/,'...Do you want frame around region of
1 interest (it may make you figure',/,'...
1 'confusing in some case)? yes or no(',',$)
read(*,500)yes
if(yes.eq.'y'.or.yes.eq.'Y') then
call fill(xx(1),yy(1),zz(1),'0')
call fill(xx(nx),yy(1),zz(1),'!')
call fill(xx(nx),yy(ny),zz(1),'!')
call fill(xx(1),yy(ny),zz(1),'!')
call fill(xx(1),yy(1),zz(1),'!')
call fill(xx(1),yy(1),zz(nz),'!')
call fill(xx(nx),yy(ny),zz(nz),'!')
call fill(xx(nx),yy(1),zz(nz),'!')
call fill(xx(nx),yy(1),zz(1),'0')
call fill(xx(nx),yy(ny),zz(1),'0')
call fill(xx(nx)+2.0,yy(1),zz(1),'0')
call fill(xx(1),yy(1),zz(nz),'!')
call fill(xx(nx),yy(ny),zz(nz),0')
call fill(xx(nx),yy(ny),zz(nz),'!')
call fill(xx(nx),yy(1),zz(nz),'!')
call fill(xx(nx),yy(1),zz(nz),'0')
call fill(xx(nx),yy(ny),zz(1),'0')
call fill(xx(nx),yy(1),zz(1),'0')
call fill(xx(nx),yy(ny),zz(1),'!')
call fill(xx(nx),yy(1),zz(1),'!')
call fill(xx(nx)+2.0,yy(1),zz(1),'!')
call fill(xx(nx),yy(1),zz(1),'0')
call fill(xx(1)+2.0,yy(1),zz(1),'0')
call fill(xx(nx),yy(ny)+2.0,zz(1),'!')
call fill(xx(nx),yy(1),zz(nz),'0')
call fill(xx(nx),yy(1),zz(nz)+2.0,'!')
endif
KOUNT=0
KAR='I'
DO 400 M=1,KONTRS
   KOL=KOLORS(M)
   VC=VCONT(M)
   WRITE(*,*)' MESH M=',M,';MESH LINES=',KOUNT
   IF(VEO.ZE0.0) GO TO 225
   KOUT=5
   DO 220 K=1,NZ
      KOUT=KOUT+1
      IF(KOUT.GT.5) THEN
         WRITE(*,556) M,VC,K,TIMER(TF,TS)-TSTART,KOUNT
         FORMAT(' MESH',I3,' at =',1PE14.5,2X,A1)
      END IF
   END DO 220 K=1,NZ
   Z=Z(Z(K))
   DO 220 J=1,NY-1
      YL=YY(J)
      YH=YY(J+1)
      DO 220 I=1,NX-1
         XL=XX(I)
         XH=XX(I+1)
         XA=CROSS(XL,XH,V(I,J,K),V(I,J+1,K),VC)
         IF(XA.LT.-BIG) THEN
            CALL FILL(XL,YL,Z,KOL)
            CALL FILL(XL,YH,Z,KAR)
            KOUNT=KOUNT+1
            GO TO 220
         END IF
9C  CONTOUR VC IS ALONG THE LOWER EDGE OF XY CELL PLANE
         YA=CROSS(YL,YH,V(I,J,K),V(I,J+1,K),VC)
         IF(YA.LT.-BIG) THEN
            CALL FILL(XL,YL,Z,KOL)
            CALL FILL(XL,YH,Z,KAR)
            KOUNT=KOUNT+1
            GO TO 220
         ENDIF
3C  CONTOUR VC IS ALONG LEFT EDGE OF XY CELL PLANE
         IF(XA.LT.BIG.AND.YA.LT.BIG) THEN
            CALL FILL(XA,YL,Z,KOL)
            CALL FILL(XL,YA,Z,KAR)
            KOUNT=KOUNT+1
            GO TO 220
         END IF
550 FORMAT(1P3E14.5,2X,A1)
ENDIF
XB=CROSS(XL,XH,V(I,J+1,K),V(I+1,J+1,K),VC)
IF(XB.LT.-BIG) GO TO 220
C CONTOUR VC IS ALONG TOP EDGE OF XY CELL PLANE BUT WILL BE DRAWN ON PASS AT J+1
IF(XA.GT.BIG.AND.XB.GT.BIG.AND.YA.GT.BIG) GO TO 220
C CONTOUR VC DOES NOT ENTER XY CELL
IF(XA.LT.BIG.AND.XB.LT.BIG) THEN
  call fill(XA,YL,Z,KOL)
call fill(XB,YH,Z,KAR)
  KOUNT=KOUNT+1
GO TO 220
ENDIF
IF(XB.LT.BIG.AND.YA.LT.BIG) THEN
  call fill(XL,YA,Z,KOL)
call fill(XB,YH,Z,KAR)
  KOUNT=KOUNT+1
END IF
YB=CROSS(YL,YH,V(I+1,J,K),V(I+1,J+1,K),VC)
IF(ABS(YB).GT.BIG) GO TO 220
C CONTOUR VC EITHER DOES NOT ENTER CELL XY OR IS ON RIGHT EDGE AND WILL BE PLOTTED FOR I+1
IF(XA.LT.BIG.AND.YB.LT.BIG) THEN
  call fill(XA,YL,Z,KOL)
call fill(XH,YB,Z,KAR)
  KOUNT=KOUNT+1
GO TO 220
END IF
IF(YA.LT.BIG.AND.YB.LT.BIG) THEN
  call fill(XL,YA,Z,KOL)
call fill(XH,YB,Z,KAR)
  KOUNT=KOUNT+1
GO TO 220
ENDIF
IF(XB.LT.BIG.AND.YB.LT.BIG) THEN
  call fill(XB,YH,Z,KAR)
call fill(XH,YB,Z,KAR)
  KOUNT=KOUNT+1
GO TO 220
END IF
C CONTINUE
220 CONTINUE
WRITE(*,*)' ALL',NZ,' K-PLANES ARE COMPLETE.-CONTINUING'
write(*,*) ' '}
C continue
225 continue
IF(N0Y.NE.0) GO TO 245
KOUTS=5
DO 240 J=1, NY
   kouts=kouts+1
   if(kouts.gt.5) then
     write(*,557) m, vc, j, timer(tf, ts)-tstart, kount
     format(' MESH', i3, ' at =', lpe10.3, ' ON J=', i3, ' PLANE',
     ' CPU TIME=', 0pf10.1, ' s lines =', 16)
     kouts=1
   end if
   Y=YY(J)
   DO 240 K=1, NZ-1
      ZL=ZZ(K)
      ZH=ZZ(K+1)
      DO 240 I=1, NX-1
         XL=XX(I)
         XH=XX(I+1)
         XA=CROSS(XL, XH, V(I, J, K), V(I+1, J, K), VC)
         IF(XA.LT.-BIG) THEN
            C CONTOUR VC ALONG LOWER EDGE XZ CELL
            call fill(XL, Y, ZL, KOL)
            call fill(XH, Y, ZL, kar)
            KOUNT=KOUNT+1
            GO TO 240
         ENDIF
         ZA=CROSS(ZL, ZH, V(I, J, K), V(I, J, K+1), VC)
         IF(ZA.LT.-BIG) THEN
            C CONTOUR VC IS ALONG LEFT EDGE XZ CELL
            call fill(XL, Y, ZL, KOL)
            call fill(XL, Y, ZH, kar)
            KOUNT=KOUNT+1
            GO TO 240
         ENDIF
         IF(XA.LT.BIG.AND.ZA.LT.BIG) THEN
            call fill(XA, Y, ZL, KOL)
            call fill(XB, Y, ZH, kar)
            KOUNT=KOUNT+1
            GO TO 240
         ENDIF
         XB=CROSS(XL, XH, V(I, J, K+1), V(I+1, J, K+1), VC)
         IF(XB.LT.-BIG) GO TO 240
         C CONTOUR VC IS ALONG TOP EDGE OF XZ CELL PLANE BUT WILL
         C BE DRAWN ON PASS AT NEXT K
         IF(XA.GT.BIG.AND.XB.GT.BIG.AND.ZA.GT.BIG) GO TO 240
         C CONTOUR VC DOES NOT ENTER XZ CELL AT J
         IF(XA.LT.BIG.AND.XB.LT.BIG) THEN
            call fill(XA, Y, ZL, KOL)
            call fill(XB, Y, ZH, kar)
            KOUNT=KOUNT+1
            GO TO 240
         ENDIF
if(xb.lt.big.and.za.lt.big) then  
call fill(xl,y,za,kol)  
call fill(xb,y,zh,kar)  
kount=kount+1  
go to 240  
end if  

ZB=CROSS(ZL,ZH,V(I+1,J,K),V(I+1,J,K+1),VC)  
IF(ABS(ZB).GT.BIG) GO TO 240  

CONTINUE  
WRITE**,*)' ALL',NY,' J-PLANES ARE COMPLETED-CONTINUING'  
write(*,*)' '  

CONTINUE  
WRITE(*,*)' ALL',NY,' J-PLANES ARE COMPLETED-CONTINUING'  
write(*,*)' '  

continue  
if(nox.ne.0) go to 265  
kouts=5  
DO 260 I=1,NX  
kouts=kouts+1  
if(kouts.gt.5) then  
write(*,558) m,vc,i,timer(tf,ts)-tstart,kount  
format(' MESH',i3,' at =',1pe10.3,' ON I=',i3,' PLANE',  
': CPU TIME=',0pf10.1,' s: lines =',i6)  
kouts=1  
end if  
X=XX(I)  
DO 260 K=1,NZ-1  
ZL=ZZ(K)  
ZH=ZZ(K+1)
DO 260 J=1, NY-1
  YL=YY(J)
  YH=YY(J+1)
  YA=CROSS(YL, YH, \(V(I, J, K), V(I, J+1, K), VC\))
  IF(YA.LT.-BIG) THEN
    CONTOUR VC IS ALONG LOWER EDGE OF YZ CELL
    call fill(X, YL, ZL, KOL)
    call fill(X, YH, ZL, kar)
    KOUNT=KOUNT+1
    GO TO 260
  ENDIF
  ZA=CROSS(ZL, ZH, \(V(I, J, K), V(I, J, K+1), VC\))
  IF(ZA.LT.-BIG) THEN
    CONTOUR VC IS ALONG LEFT EDGE OF YZ CELL
    call fill(X, YL, ZL, KOL)
    call fill(X, YL, ZH, kar)
    KOUNT=KOUNT+1
    GO TO 260
  ENDIF
  IF(YA.LT.BIG.AND.ZA.LT.BIG) THEN
    call fill(X, YA, ZL, KOL)
    call fill(X, YB, ZH, kar)
    KOUNT=KOUNT+1
    GO TO 260
  ENDIF
  YB=CROSS(YL, YH, \(V(I, J, K), V(I, J, K+1), VC\))
  IF(YB.LT.-BIG) GO TO 260
  CONTOUR VC IS ALONG TOP EDGE OF YZ CELL AND WILL
  BE DRAWN ON PASS AT NEXT K
  IF(YA.GT.BIG.AND.YB.GT.BIG.AND.ZA.GT.BIG) GO TO 260
  CONTOUR VC IS NOT IN YZ CELL
  IF(YA.LT.BIG.AND.YB.LT.BIG) THEN
    call fill(X, YA, ZL, KOL)
    call fill(X, YB, ZH, kar)
    KOUNT=KOUNT+1
    GO TO 260
  ENDIF
  IF(YB.LT.BIG.AND.ZA.LT.BIG) THEN
    call fill(X, YL, ZL, KOL)
    call fill(X, YB, ZH, kar)
    KOUNT=KOUNT+1
    GO TO 260
  ENDIF
  IF(YA.LT.BIG.AND.ZA.LT.BIG) THEN
    call fill(X, YL, ZL, KOL)
    call fill(X, YB, ZH, kar)
    KOUNT=KOUNT+1
    GO TO 260
  ENDIF
  ZB=CROSS(ZL, ZH, \(V(I, J+1, K), V(I, J+1, K+1), VC\))
  IF(ABS(ZB).GT.BIG) GO TO 260
  CONTOUR VC EITHER DOES NOT ENTER CELL YZ OR IS ON
  THE RIGHT EDGE AND WILL BE PLOTTED ON J+1
IF(YA.LT.BLG.AND.ZB.LT.BLG) THEN
  call fill(X,YA,ZL,KOL)
  call fill(X,YH,ZB,kar)
  kount=kount+1
  go to 260
end if
if(yb.lt.big.and.zb.lt.big) then
  call fill( x,yb,zh,kol)
  call fill( x,yh,zb,kar)
  kount=kount+1
  go to 260
end if
if(za.lt.big.and.zb.lt.big) then
  call fill(X,YL,ZA,KOL)
  call fill(X,YH,ZB,kar)
  kount=kount+1
  go to 260
ENDIF

CONTINUE

WRITE(18,*) ' -1.0E+35'
WRITE(*,570) KONTRS,KOUNT,TIMER(TF,TS)-TSTART
FORMAT(I7,' CONSTANT MESH SURFACES HAVE BEEN COMPLETED',/,
1 17,' MESH LINES TO BE DRAWN',/ ,5X,'TIME REQUIRED=',
1 0pFl0.2,' s')
STOP
END

SUBROUTINE FILL

SUBROUTINE FILL(X,Y,Z,K)
  CHARACTER*1 K
  WRITE(18,550) X,Y,Z,K
  RETURN
END

FUNCTION CROSS(S1,S2,V1,V2,VC)
  IF((V1.LE.VC.AND.V2.GE.VC).OR.(V1.GE.VC.AND.V2
1  .LE.VC)) THEN
DV = V2 - V1
IF (ABS(DV).LT.1.0E-20) THEN
    CROSS = -1.0E+36
ELSE
    CROSS = S1 + (S2 - S1) * (VC - V1) / DV
ENDIF
ELSE
    CROSS = +1.0E+36
ENDIF
RETURN
END

FUNCTION VV(V, RSQR)
 This is a test function to give a 3D distribution when called
 subroutine VOLT which in turn is called by MESHSUR

function vv(v, rsqr)
    rr = sqrt(rsqr) / 30.0
    vv = v * (2.0 - (rsqr/900.0) * (2.0 - rr))
return
end

SUBROUTINE VOLT

A test routine to provide three sets of overlapping spherical
surfaces from diffuse sources along a diagonal through a
20 by 20 by 20 cube. The two sources near the corners is
positive and the center negative. The system is neutral.

subroutine volt(v, nx, ny, nz, vmin, vmax)
real v(81, 81, 21), aa(3, 3), b(3)
character*1 yes
character*30 outfil
data x1, y1, z1, b(1)/6., 6., 6., 21.08741/,
1   x2, y2, z2, b(2)/11., 11., 11., 0.0/, kdump/0/,
1   x3, y3, z3, b(3)/16., 16., 16., -21.08741/

500 format(a1)
510 format(a30)
520 format(/,' Test Generation Program VOLT. Do you want ',
1   'these data sent', '/', ' to a file? Y or N? ', $)
530 format(/,' What is the name of your output file? ', $)
540 format(/,' Do You Want 1 or 2 Sources? Ans. 1 or 2. ', $)

nx = 21
ny = 21
nz = 21
write(*, 540)
read(*,*) nsour
if(nsour.lt.1.or.nsour.gt.2) nsour=1

write(*,520)
read(*,500) yes
if(yes.eq.'y'.or.yes.eq.'Y') then
write(*,530)
read(*,510) outfil
open(16,file=outfil,status='new')
write(16,*), 'A SHORT TEST FILE FOR MESHSUR'
write(16,*), nx,ny,nz
kdump=1
end if
r12=(x1-x2)**2+(y1-y2)**2+(z1-z2)**2
r13=(x1-x3)**2+(y1-y3)**2+(z1-z3)**2
r23=(x2-x3)**2+(y2-y3)**2+(z2-z3)**2
if(nsour.eq.l) then
b(1)=0.0
b(2)=5.0
b(3)=0.0
end if

vmin=1.0e+35
vmax=-vmin
do 150 k=1,nz
z=real(k)
z1=(z-z1)**2
z2=(z-z2)**2
z3=(z-z3)**2
do 145 j=1,ny
y=real(j)
yz1=(y-y1)**2+z1
y2=(y-y2)**2+z2
y3=(y-y3)**2+z3
do 140 i=1,nx
x=real(i)
xyz1=(x-x1)**2+y1
xyz2=(x-x2)**2+y2
xyz3=(x-x3)**2+y3
v(i,j,k)=v(v(b(1),xyz1)+v(v(b(2),xyz2)+v(v(b(3),xyz3)
if(vmin.gt.v(i,j,k)) vmin=v(i,j,k)
if(vmax.lt.v(i,j,k)) vmax=v(i,j,k)
140 continue
if(kdump.gt.0) then
write(16,*) (v(i,j,k),i=1,nx)
end if
continue
write(*,*) k, ' z-plane calculations completed. Continuing'

continue
close(16)

return
end
GENERAL MULTIQUADRIC-BIHARMONIC SUBROUTINE

C SUBROUTINE MQB
C THIS IS A SUBROUTINE TO COMPUTE 3 DIMENSIONAL DATA
C \( F(x,y,z) \) FOR PLOTTING IN AGRAPH MESHSUR PROGRAM.

SUBROUTINE MQB(VV,NX,NY,NZ,VMIN,VMAX)
INTEGER NX,NY,NZ
REAL VV(81,81,21),ZZ(21)
CHARACTER PNAME(100)*20,LEV*1,OUTFIL*30,YES*1,CON*1
CHARACTER*30 INFILL
DIMENSION X(100),Y(100),Z(100),A(101,101)
1,B(101),C(101),XX(81),YY(81),BCK(101)
C READ DATA
C
PRINT*, 'NOW YOU ARE IN THE MULTIQUADRIC BIHARMONIC SUBROUTINE'
PRINT*, 'HOW MANY DATA POINT YOU HAVE?'
READ*, ND
PRINT*, 'READING DATA'
BIG=1.0E+35
XMAX=-BIG
XMIN=BIG
YMAX=-BIG
YMIN=BIG
ZMAX=-BIG
ZMIN=BIG
KOUNT=0
WRITE(*,310)
310 FORMAT(WHAT IS THE NAME OF YOUR INPUT FILE*,/,&
! ' FOR MULTIQUADRIC BIHARMONIC?...$,*)
READ(*,311)INFILL
311 FORMAT(A30)
OPEN(14,FILE=INFILL,STATUS='OLD')
PRINT*, 'MULTIQUADRIC DATA ARE:'
DO 15 I=1,ND
READ(14,*)PNAME(I),X(I),Y(I),Z(I),B(I)
KOUNT=KOUNT+1
PRINT*,PNAME(I),X(I),Y(I),Z(I),B(I)
XMAX=MAX(XMAX,X(I))
XMIN=MIN(XMIN,X(I))
YMAX=MAX(YMAX,Y(I))
YMIN=MIN(YMIN,Y(I))
ZMAX=MAX(ZMAX,Z(I))
15 ZMIN=MIN(ZMIN,Z(I))
PRINT*, 'FINISHED READING DATA=',KOUNT,' POINTS'
500 FORMAT(A1)
510 FORMAT(A30)
520 FORMAT(/,'DO YOU WANT THE DATA GENERATED BY MQB',/,&
! ' SENT TO A FILE? YES OR NO?',$)
```
FORMAT(/,' WHAT IS THE NAME OF YOUR OUTPUT FILE?')
WRITE(*,520)
READ(*,500)YES
IF(YES.EQ.'y'.OR.YES.EQ.'Y')THEN
WRITE(*,530)
READ(*,510)OUTFILE
OPEN(16,FILE=OUTFILE,STATUS='NEW')
WRITE(16,'*') ' FILE CREATED BY MULTIQUEADRIC BIHARMONIC'
WRITE(16,'*') NX,NY,NZ
KDUMP=1
ENDIF
RANGX=XMAX-XMIN
RANGY=YMAX-YMIN
RANG=MAX(RANGX,RANGY)
GRID=MAX(NX,NY)
VAL=RANG/(GRID-1)
WRITE(*,612)
FORMAT(/,'...DO YOU WANT THE LEFT EDGE STARTED WITH', //,'...ZERO OR NOT, YES OR NO?')
READ(*,500) YES
IF(YES.EQ.'y'.OR.YES.EQ.'Y') THEN
    XC=0
GO TO 205
ENDIF

XC=XMIN
YC=YMIN
XX(1)=XC
YY(1)=YC
 DO 20 I=2,NX
       XX(I)=XC+VAL
       XC=XX(I)
 20  DO 25 J=2,NY
       YY(J)=YC+VAL
       YC=YY(J)
 25  PRINT*,' DO YOU WANT TO DEFINE THE DEPTH OF EACH LAYER'
 700 FORMAT(A1)
 IF(LEV.EQ.'y'.OR.LEV.EQ.'Y') THEN
    PRINT*,' HOW MANY LEVELS YOU WANT(MAX=20)?'
    READ*,NZ
    PRINT*,' NOW ENTER DEPTH OF THOSE LEVELS FROM TOP'
    PRINT*,' TO BOTTOM'
    DO 251 I=1,NZ
           PRINT*,' ENTER LEVEL',I,'='
           READ*,ZZ(I)
    251  ENDIF
```
COMPUTING \( ZZ(K) \) COORDINATES FOR EACH LAYER

\[
\text{IF(LEV.EQ.'Y'.or.LEV.EQ.'y')GO TO 27}
\]
\[
\text{RANZ}=Z_{\text{MAX}}-Z_{\text{MIN}}
\]
\[
\text{ZVAL}=\frac{\text{RANZ}}{(NZ-1)}
\]
\[
\text{ZC} = Z_{\text{MIN}}
\]
\[
\text{ZZ(1)} = \text{ZC}
\]
\[
\text{DO } 30 \text{ K}=2,\text{NZ}
\]
\[
\text{ZZ(K)} = \text{ZC}-\text{ZVAL}
\]
\[
\text{ZC} = \text{ZZ(K)}
\]

1. PRINT*, ' NOW YOU ARE GOING TO COMPUTE', NX, '*', NY, '*', NZ,

1. ' GRID MESHES'

1. PRINT*, ' THE REAL COORDINATES OF YOUR VOLUME AT THE CORNERS'

1. PRINT*, ' ARE:

1. PRINT*, ' XMIN=', XX(1), 'XMAX=', XX(NX)

1. PRINT*, ' YMIN=', YY(1), 'YMAX=', YY(NY)

1. PRINT*, ' Z COORDINATES OF TOP LAYER=', ZZ(1)

1. PRINT*, ' Z COORDINATES OF BOTTOM LAYER=', ZZ(NZ)

CENTERING THE DATA

\[
\text{TEMPSM}=0.0
\]
\[
\text{DO } 32 \text{ I}=1,\text{ND}
\]
\[
\text{TEMPSM}=\text{TEMPSM}+B(I)
\]
\[
\text{AVER}=\frac{\text{TEMPSM}}{\text{ND}}
\]
\[
\text{DO } 35 \text{ I}=1,\text{ND}
\]
\[
\text{B(I)}=\text{B(I)}-\text{AVER}
\]
\[
\text{B(ND+1)}=0.0
\]

PRINT*, ' COMPUTING THE MQ-B COEFFICIENT MATRIX'

\[
\text{DO } 37 \text{ I}=1,\text{ND}
\]
A(ND+1,1) = 1.0
A(1,ND+1) = 1.0
A(ND+1,ND+1) = 0.0
DO 40 I = 1, ND
   DO 40 J = 1, ND
40   A(I,J) = SQRT((X(J) - X(I))**2 + (Y(J) - Y(I))**2 + 
                    (Z(J) - Z(I))**2)
C
CALL EZSIMEQS(A, B, ND+1, 101, KS)
IF(KS.EQ.0) WRITE(*,'(A')'...SOLUTION COMPLETED, CONTINUED'
IF(KS.NE.0) THEN
   WRITE(*,'(A')'...SINGULAR MATRIX QUIT'
   STOP
ENDIF
C
PRINT*, ' GET MQ-B COEFFICIENTS: NEXT COMPUTING THE'
PRINT*, ' MATRIX VV(I,J,K) CONTAINING THE DATA TO PLOT'
PRINT*, ' CONTOUR BY AGRAPH'
WRITE(16,*)' COMPUTE THE OBSERVED POINTS FOR CHECK'
WRITE(16,*)' RESULT ARE:'
DO 55 I = 1, ND
   BC = 0.0
   DO 60 J = 1, ND
55   WRITE(16,*)' POINT', I, ', PREDICTED=', BCK(I)
   PRINT*, 'BE PATIENT!
C
KOUT = 0
DO 721 L = 1, NZ
   KOUT = KOUT + 1
   DO 65 I = 1, NY
55   DO 65 J = 1, NX
   70   BT = SQRT((X(K) - XX(J))**2 + (Y(K) - YY(I))**2 + (Z(K) - ZZ(L))**2)
   ZIJ = ZIJ + B(K) * BT
   ZIJ = ZIJ + AVER + B(ND+1)
   IF(I.EQ.1.AND.J.EQ.1) THEN
      ZMAX = ZIJ
      ZMIN = ZIJ
   ENDIF
   ZMAX = MAX(ZMAX, ZIJ)
   ZMIN = MIN(ZMIN, ZIJ)
   VV(I, J, L) = ZIJ
C
   IF(L.EQ.1) THEN

VMIN=ZMIN
VMAX=ZMAX
ENDIF
VMIN=MIN(VMIN,ZMIN)
VMAX=MAX(VMAX,ZMAX)

PRINT*, 'FINISHED COMPUTING LAYER', ZZ(L)
PRINT*, 'LOWEST VALUE IN THIS LAYER=', ZMIN
PRINT*, 'HIGHEST VALUE IN THIS LAYER=', ZMAX
IF(L.LT.NZ.AND.L.GT.1)THEN
  PRINT*, 'COMPUTING NEXT LAYER BE PATIENT!
ENDIF

IF(KDUMP.GT.0)THEN
  WRITE(16,*) 'LAYER', ZZ(L)
  WRITE(16,*) 'MIN VALUE=', ZMIN
  WRITE(16,*) 'MAX VALUE=', ZMAX
  WRITE(16,*) 'DATA ARE:
  DO 72 I=1,NY
    DO 72 K=I,NX,5
      KL=K+4
      WRITE(16,18) (VV(I,J,L), J=K, KL)
    CONTINUE
  ENDIF
  CONTINUE
  WRITE(16,*) 'FINISHED COMPUTED', KOUT, 'LAYERS'
  CLOSE(16)
  PRINT*, 'FINISHED COMPUTED ALL OF', KOUT, 'LAYERS'
  PRINT*, 'MAX VALUE OF ALL LAYERS=', VMAX
  PRINT*, 'MIN VALUE OF ALL LAYERS=', VMIN
  PRINT*, 'CONTINUE COMPUTING IN MESHSUR PROGRAM'
  PRINT*, 'IF YOU WANT TO CONTINUE TYPE YES'
  READ(*,28) CON
  FORMAT(A1)
  IF(CON.EQ.'y'.OR.CON.EQ.'Y') GO TO 73
  CONTINUE
  RETURN
END
APPENDIX J: PHOSPHORUS CONTENT DATA IN IOWA RED ROCK RESERVOIR IN COMPUTER READY FORMAT
This is the phosphorus pollution data in Red Rock Reservoir in computer ready format. The order of data input is:

name of point, x, y, z coordinates, data value

These data were used in the case study in Chapter 3.
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| 2.  | '521' | 2121786 | 515052 | 1 0.19 |
| 3.  | '531' | 2121333 | 514274 | 1 0.15 |
| 4.  | '541' | 2120880 | 513497 | 1 0.06 |
| 5.  | '551' | 2120377 | 512632 | 1 0.06 |
| 6.  | '411' | 2129986 | 512722 | 1 0.09 |
| 7.  | '421' | 2128416 | 511712 | 1 0.07 |
| 8.  | '431' | 2127847 | 510659 | 1 0.12 |
| 9.  | '441' | 2127277 | 509603 | 1 0.17 |
|10.  | '451' | 2126802 | 508723 | 1 0.13 |
|11.  | '311' | 2135400 | 509057 | 1 0.05 |
|12.  | '321' | 2134979 | 508269 | 1 0.13 |
|13.  | '331' | 2134069 | 507701 | 1 0.17 |
|14.  | '341' | 2133089 | 506701 | 1 0.07 |
|15.  | '351' | 2132108 | 505702 | 1 0.12 |
|16.  | '211' | 2140590 | 504704 | 1 0.11 |
|17.  | '221' | 2139579 | 503946 | 1 0.19 |
|18.  | '231' | 2139670 | 502897 | 1 0.14 |
|19.  | '241' | 2137744 | 501847 | 1 0.15 |
|20.  | '251' | 2136685 | 500648 | 1 0.16 |
|21.  | '111' | 2143511 | 500204 | 1 0.21 |
|22.  | '121' | 2142878 | 501564 | 1 0.13 |
|23.  | '131' | 2142245 | 500924 | 1 0.16 |
|24.  | '141' | 2141682 | 500356 | 1 0.07 |
|25.  | '151' | 2140908 | 499574 | 1 0.05 |
|26.  | '412' | 2122239 | 515820 | 2 0.17 |
|27.  | '522' | 2121786 | 515052 | 2 0.05 |
|28.  | '532' | 2121333 | 514274 | 2 0.13 |
|29.  | '542' | 2120880 | 513497 | 2 0.09 |
|30.  | '552' | 2120377 | 512632 | 2 0.11 |
|31.  | '412' | 2129986 | 512722 | 2 0.11 |
|32.  | '422' | 2128416 | 511712 | 2 0.11 |
|33.  | '432' | 2127847 | 510659 | 2 0.10 |
|34.  | '442' | 2127277 | 509603 | 2 0.26 |
|35.  | '452' | 2126802 | 508723 | 2 0.13 |
|36.  | '312' | 2135400 | 509057 | 2 0.06 |
|37.  | '322' | 2134979 | 508269 | 2 0.11 |
|38.  | '332' | 2134069 | 507701 | 2 0.12 |
|39.  | '342' | 2133089 | 506701 | 2 0.09 |
|40.  | '352' | 2132108 | 505702 | 2 0.08 |
|41.  | '212' | 2140590 | 504704 | 2 0.15 |
|42.  | '222' | 2139579 | 503946 | 2 0.13 |
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DATA IN COMPUTER READY FORMAT
COAL DATA USING EXAGGERATED DEPTH

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APPENDIX L: COMPUTER OUTPUT OF COAL THICKNESS BETWEEN COAL C
AND COAL E USING ORIGINAL DEPTH AND EXAGGERATED DEPTH
This is the predicted coal thickness at locations between Coal C and Coal E layers. The predicted values are supposed to approach zero, since there is no coal present at these elevations. The discussion of these results is in Chapter 3.
**Predicted Unsampling Points Between Coal E and Coal C Using Exaggerated Depth**

**X, Y, Z, Thickness**

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### PREDICTED UNSAMPLING POINTS BETWEEN COAL E AND COAL C USING ORIGINAL OR GEOMETRICAL DEPTH

**X, Y, Z, THICKNESS**

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