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Paraxial approximations for ultrasonic beam propagation in liquid and solid media with applications to nondestructive evaluation

Byron Peel Newberry

Iowa State University

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Paraxial approximations for ultrasonic beam propagation in liquid and solid media with applications to nondestructive evaluation

Newberry, Byron Peel, Ph.D.

Iowa State University, 1988
Paraxial approximations for ultrasonic beam propagation in liquid and solid media with applications to nondestructive evaluation

by

Byron Peel Newberry

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I. INTRODUCTION

A. Overview

Many of the applications of ultrasonic techniques to the nondestructive evaluation (NDE) of materials involve the generation of a collimated, finite width beam of ultrasound within the material. The ultrasonic energy recorded at a receiver after the beam has either been scattered back from or transmitted through the material is then used to try to deduce some aspect of the material's character. This is generally some material property, some geometrical property, or, in the case of most interest in this work, the nature of possible defects within the material. In order to determine how useful a particular ultrasonic inspection procedure of this type will be in terms of yielding the desired information, a quantitative analysis of the procedure is necessary. This has traditionally been accomplished through extensive experimentation. An alternative approach which is aimed at reducing the monetary and time expense of such experimental programs is to develop theoretical models to predict the results of ultrasonic inspections. A discussion of the importance of ultrasonic modeling for this purpose, along with several examples of cases where this approach has been applied, is given by Thompson and Gray (1).

The processes involved in an ultrasonic measurement which must be considered in the development of a model are the generation of an ultrasonic beam by a transducer, the propagation of the beam from the transducer to the point of interest, which encompasses the effects of diffraction, refraction, focusing, and attenuation, the scattering of the ultrasound by a defect, the propagation of the scattered ultrasound to and its detection by a receiving transducer, and the noise, both ultrasonic and electronic, which enter into the
received signal. As has been shown (1-3), the use of reciprocity relations allow for a straightforward formulation of such a model, in principle. In practice, the application of this type of formalism requires making appropriate approximations such that the computations are both sufficiently tractable and accurate. This has led to a modular approach to the modeling. That is, the various processes, such as beam propagation and ultrasound-flaw interaction, are formally treated separately and are then interfaced in the appropriate manner to achieve the final result. This is possible if the system of physical processes can be approximated as a linear invariant one, which is generally the case for ultrasonic NDE. In this manner, theoretical advances which have been made in recent years in the solution of canonical wave scattering and beam propagation problems can be directly applied. A substantial list of references for advances in wave scattering theory is given by Thompson and Thompson (4) and Thompson (5). Beam propagation modeling is the topic of the present paper and will be discussed in detail below. The usefulness of this overall modeling approach for predicting the results of ultrasonic measurements has been demonstrated in practical applications (1,3,6-8).

As stated, the subject of the present work is the development of approximate models for the ultrasonic fields radiated by transducers such as those used in NDE. The objective has been to develop models to be used as components in the broader modeling scheme described above. In that capacity, the models are required to have enough generality to accurately treat a wide range of problems without a large computational time expenditure. In general, the transducer in an ultrasonic inspection is positioned outside of the material to be inspected and hence the ultrasound must pass from the transducer to the material of interest through some coupling medium, either liquid or solid. Consequently, a model must encompass not only beam propagation and diffraction in a single medium, but transmission effects at interfaces as well. Also, since most materials to be inspected are
solids, the case of waves in an elastic medium must be considered along with the acoustic wave case. To further complicate matters, the solid medium may possibly exhibit anisotropy of the elastic properties. This case has become important in the NDE of advanced composite materials and certain metallic materials with preferentially aligned grain structure. The philosophy of the present beam modeling effort has been to trade strict mathematical rigor in the treatment of these problems for computational and conceptual simplicity wherever possible without severely compromising the accuracy of the calculations. This is crucial since the models are intended to be integrated into engineering tools which require more expediency than detailed research models.

The ultrasonic transducers of interest here have a piezoelectric disk which oscillates when a voltage pulse is applied across it, thus producing an ultrasonic wave emanating from the disk face. Detailed discussions of piezoelectric transducer theory are given by Kino (9) and Sachse and Hsu (10). This type of transducer can generally be modeled as a disk which oscillates with uniform velocity over its face, while the remainder of the plane containing the face is assumed to be perfectly rigid. Although not rigorously correct, this is generally valid as long as the dimension of the disk is much larger than the wavelength, as is often the case for NDE applications. This type of acoustic radiator is referred to as a rigid piston and is most often taken to be flat and circular. Also of interest is the case in which a lens is applied to the face of the transducer to produce a focused beam for the purpose of concentrating the ultrasonic energy in a small area to obtain higher resolution. Still other cases might call for pistons with noncircular shapes such as ellipses or rectangles. Another type of transducer of interest is one which produces a Gaussian vibration amplitude across its face. The advantage of this type of probe is that it produces a beam of simple shape which does not have undesirable interference patterns characteristic of piston probes. However, Gaussian piezoelectric transducers are not
trivial to manufacture (9,11,12).

B. Prior Work

The radiation of a circular piston into a fluid has been exhaustively studied in both the frequency and time domains (9,10,13-18). This great interest is due to the technological importance of circular piston transducers in many different engineering fields, including NDE. Considerable attention has also been given to the focusing of this radiation by axially symmetric lenses (9,19-21). The radiation of rectangular (9,18,22) and elliptical (23) pistons has been studied as well, along with the theory of Gaussian beam propagation (9,11,24). Harris (14) gives a review of the most widely used mathematical techniques for analyzing piston radiation. These techniques can generally be applied to non-piston sources as well. A technique which has proven quite useful for acoustic, elastic and electromagnetic beam propagation problems and which is employed extensively in this work is that of representing a beam as an angular spectrum of plane waves (9,23-27). Methods such as this one, in which a complex solution is decomposed into elementary solutions, while not guaranteeing computational simplicity, have the advantage of interpretive simplicity since the behavior of elementary solutions such as plane waves is generally well understood. This is a distinct advantage when trying to seek simplifying approximations to obtain an analytical result.

For cases where the wavelength is much smaller than the radius of the piston, \( ka \ll 1 \) where \( k \) is the wavenumber and \( a \) is the radius, an approximation is often made to yield a simplified solution to the vibrating piston problem. This is known as the Fresnel (also parabolic, paraxial) approximation (9,15,23-25). A generalization of this approximation to beams in solid media has been developed in this work and will be reported in the following sections.
As pointed out by Rose and Meyer (28), the problem of transducer beam radiation in isotropic solids has received very little direct attention. Miller and Pursey (29) derive asymptotic expressions for the far field radiation produced by strip and disk radiators of finite dimension placed on the surface of a semi-infinite solid. Rose and Meyer (28) and Singh and Rose (30) employ a point source Green's function method for computing the fields due to an arbitrary shaped radiator on the surface of a solid half-space. Harris (31,32) derives asymptotic expressions for the fields of two-dimensional and axisymmetric Gaussian and piston radiators on the surface of a solid halfspace while Weight (33) approximates the impulse response for a circular radiator vibrating normally to a solid halfspace. The treatments mentioned above are for beams which originate on the surface of the solid rather than in a medium adjacent to the solid. For the latter problem, Coffey and Chapman (34) used the scalar theory for beams in a fluid to approximate the fields in a solid and treated passage through an interface by defining a virtual source within the solid. More recently, the propagation of Gaussian beams through liquid-solid interfaces has been studied (24,35,36) and asymptotic analysis has been applied to the propagation of Gaussian type wave packets in isotropic, anisotropic, and inhomogeneous solids and through interfaces (37-40). Also, numerical integrations using the angular spectrum of plane waves approach have been used to compute the exact fields produced by a transducer beam passing through a liquid-isotropic solid interface (41,42). A big jump seems to have been made past the isotropic solid problem directly to the anisotropic solid problem, as there is currently quite a lot of research being done on beam propagation in anisotropic solids, most of which is valid for isotropic materials as well.

As stated above, there is currently a great interest in wave propagation in anisotropic
materials. This is due to their increasing importance as engineering materials. However, investigators studying seismic wave propagation have been concerned with the anisotropic nature of geologic structures for quite a number of years. The current interest is stimulated mainly by the inspection of metals with anisotropy due to grain alignment and of advanced composite materials which are anisotropic due to preferentially oriented fibers.

The study of the basic principles of wave theory in anisotropic materials can be found in several texts (26,43-45). In some early work in finite beam propagation, Papadakis (46,47) and Cohen (48) used a generalization of the Rayleigh integral to approximate the fields due to a piston transducer applied to the surface of an anisotropic crystal and directed along certain symmetry axes. The diffraction of two-dimensional surface wave beams in an anisotropic medium was studied extensively in the analysis of surface acoustic wave devices (49-54). In that body of work, the angular spectrum of plane waves approach and the Fresnel approximation were used extensively. Kharusi and Farnell (55) computed beam profiles in crystals using an angular spectrum of plane waves formulation. In fact, Musgrave (56) suggests that the angular spectrum of plane waves approach to modeling beam propagation in anisotropic media is the most feasible one, and Vezzetti (57) gives a formal development of the method. More recently, there has been a wealth of literature on anisotropic beam propagation. For example, Ogilvy (58,59) has employed ray tracing techniques to determine the effects of anisotropy and inhomogeneity on ultrasonic beams. You et al. (60) have used finite element techniques to numerically predict two-dimensional anisotropic beam propagation phenomena. Rose et al. (61) and Tverdokhlebov and Rose (62) have developed a point source Green's function model for weakly anisotropic materials. A high frequency asymptotic theory for Gaussian pulse propagation in anisotropic solids has been developed by Norris (40,63).
Roberts and Kupperman (64) and Roberts (27) have developed an angular spectrum of plane waves model for the transmission of an ultrasonic beam from an isotropic halfspace into a transversely isotropic halfspace. Finally, in conjunction with the present work, Thompson and Newberry (65) and Newberry et al. (66) have developed a Fresnel approximation to the angular spectrum of plane waves representation of an arbitrarily shaped beam.

C. Present Work

In Section II, approximate solutions are derived for the radiation of elliptical and bicylindrically focused piston transducers after passing through bicylindrically curved liquid-isotropic solid interfaces. These solutions are generalized from the work of Thompson et al. (23) and Thompson and Gray (67). First, solutions are presented for the on-axis fields. These consist of simple analytic formulae for certain cases, while for the most general cases a simple numerical evaluation of a definite integral is required. The development of the model for on-axis fields was motivated by the detection of flaws small compared to the beam width, for which the incident field can be approximated as a plane wave with an amplitude equal to the on-axis beam amplitude at the flaw location. In general, however, the off-axis field profile is important. Thus, it is shown that the full field solution can be expressed as a superposition of Gauss-Hermite beam modes, which are bound beam solutions to the wave equation under the Fresnel approximation. Gauss-Hermite beam modes have been used in the optics literature for quite a number of years (68-70). Mason (49-51) applied Gauss-Hermite beam modes to the analysis of surface wave beam diffraction on anisotropic substrates. More recently, Cavanagh and Cook (20,71), Cook and Arnoult (72), and Thompson et al. (23) have applied the Gauss-Hermite beam method, and the related Gauss-Laguerre method, to the analysis of piston
radiation fields in a fluid. Extensions of this method to beams radiated through liquid-isotropic solid interfaces have been given by Thompson and Lopes (73) and Newberry et al. (74).

Gauss-Hermite beam modes have the property of retaining their transverse functional form as they propagate, so that their diffraction properties are simply characterized. Also, they lend themselves well to the analysis of paraxial optical device systems (70). This is important since the acoustic/elastic analogs of the optical devices are such things as acoustic lenses and curved interfaces, which are of interest here. Consequently, much of the theory built up in the optics literature can be applied in the ultrasonics case. The Gaussian beam is found to be the lowest order of the Gauss-Hermite modes. Methods related to the Gauss-Hermite method, based on the Gaussian beam, have been developed recently. Wen and Breazeale (75) have shown that the radiation into a fluid of an axisymmetric source can be expressed as a superposition of Gaussian beams with different widths located at different axial positions. Also, Felsen et al. (76) have developed a Gaussian beam superposition method for the radiation of a two-dimensional source into an isotropic solid.

In Section III, starting from the angular spectrum of plane waves approach, the Gauss-Hermite beam model is derived for the case of beams in anisotropic materials. This form of the model is found to be basically the same as before with the exception of new parameters which govern the beam skew and excess beam divergence phenomena which arise in anisotropic materials due to nonspherical slowness surfaces.

In order to validate the models which have been developed, comparisons have been made to experimentally mapped transducer beam profiles in a range of materials. These results are presented in Section IV. In the isotropic case, beam maps were acquired in both fused quartz and equi-axed cast stainless steel specimens. For comparison with the
anisotropic model, experiments were performed with both graphite/epoxy composite samples and a columnar grained cast stainless steel sample. In the fused quartz case, the beams were mapped by pulse-echo measurements from small reflectors imbedded in the material. In the remaining materials, beams were mapped using a small point receiver or "microprobe" which was scanned past the field transmitted through the sample. Both of these techniques are commonly used (10,41,77). The present experimental work expands on earlier work to validate the Gaussian and Gauss-Hermite beam models (78-80).

Section V describes some early applications of the beam models. The fact that the Gauss-Hermite model can be readily applied to any localized source distribution, so long as that distribution is within the bounds of the Fresnel approximation, allows one to use the model as an aide in transducer design by predicting the radiation patterns of novel types of transducers. This has been done for the case of axicon, or conically focused, transducers (74,81) which have been suggested for use due to their apparent large depth of focus (82,83). Reported here is the use of the model to predict the fields of several types of focusing radiators including axicons, pyramidal radiators, and radiators whose surfaces are various conic sections (84).

Another application of the models which will be reported, is the prediction of beam distortions due to nonsmooth interfaces. An extension of the model will be described which will treat the passage of a beam through an interface which has a step discontinuity (85). This follows the earlier work of Thompson and Lopes (73) in which beam aberrations at a smooth interface were treated using a hybrid diffraction/ray tracing approach. The problem arises when an ultrasonic inspection in the field, such as the inspection of a weld overlay, encounters a surface which may have steps, bumps, or waves due to the welding and/or machining processes. Such a surface will decrease the
effectiveness of the inspection because of the blockage and/or redirection of the beam due to the surface discontinuities. The purpose of the modeling is to quantify these effects.

Finally, the results are summarized and conclusions are given in Section VI.
II. THEORY FOR FIELDS IN ISOTROPIC MEDIA

In this section, the theory for fields radiated into isotropic media is derived. The results are formulated in terms of pressure in a fluid, and displacement potentials in a solid. Both on-axis and full field solutions are derived for time harmonic waves.

A. Axial Fields

1. Piston radiation in a fluid

Consider an elliptical piston radiator placed in a motionless baffle, as shown in Fig. 1. The piston has semi-axis lengths of a and b and may be bicylindrically focused with focal lengths \( f_1 \) and \( f_2 \) in the x-z and y-z planes, respectively. It vibrates with peak sinusoidal velocity \( V_0 \) (the time dependence \( e^{j\omega t} \) is assumed throughout this Section).

Thompson et al. (23) showed that, within the Fresnel approximation, a general formalism for the radiated sound pressure, \( p \), can be written as

\[
p(x,y,z) = p_0 e^{-jkz} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \hat{A}(u,v) e^{j\pi \lambda z (u^2 + v^2)} e^{-j2\pi(ux + vy)}
\]

where

\[
\hat{A}(u,v) = \int_0^{2\pi} d\theta \int_0^{\min(\sqrt{b^2\cos^2\theta + a^2\sin^2\theta},\infty)} dr \ e^{j2\pi(r\cos\theta + v\sin\theta)} \times \ e^{j\pi^2(\cos^2\theta/f_1 + \sin^2\theta/f_2)}
\]

(1)
Figure 1. Geometry of elliptical piston source
and \( p_0 = \rho_0 c_0 V_0 \), where \( \rho_0 \) is the density, \( c_0 \) is the wavespeed, \( k \) is the wave number and \( \lambda \) is the wavelength. Equation 1 is an angular spectrum of plane waves representation of a beam and Eqn. 2 gives the complex plane wave amplitude spectrum associated with the source.

2. Refraction through a planar interface

   a. General formalism The representation of the radiation field of a piston as an angular spectrum of plane waves lends itself well to approaching the problem of transmission through a planar liquid-solid interface since one may analyze individual plane wave components and superimpose the results.

   Consider the problem depicted in Fig. 2 in which a piston is aimed with incident angle \( \theta_0 \) at a planar interface. The axial distance from the piston to the interface is \( d \). The radiation field of the piston is defined by Eqns. 1 and 2 in the \((x_0, y_0, z_0)\) coordinate system shown in Fig. 2. In order to conveniently describe the fields incident on the interface, a rotation and translation of the coordinate system from the transducer coordinate system, \((x_0, y_0, z_0)\), to the interface coordinate system, \((x_i, y_i, z_i)\), is employed. Under this transformation, Eqn. 1 becomes

   \[
   p(x_i, y_i, z_i) = \rho_0 e^{-jk_0(c_0 \zeta_i + \sin \theta_0 x_i + d)} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \hat{A}(u, v) \times \]

   \[
   e^{i\pi\lambda_0(u^2 + v^2)(\cos \theta_0 z_i + \sin \theta_0 x_i + d)} e^{-j2\pi u(-\sin \theta_0 z_i + \cos \theta_0 x_i)} e^{-j2\pi v y_i} \tag{3}
   \]

   where the subscript 0 denotes the properties of the liquid medium. Each incident plane wave in the angular spectrum is traveling with wave vector components
Figure 2. Geometry of piston aimed at liquid-solid interface
\[ k_{0x} = k_0[\sin \theta_0 - \lambda_0^2(u^2+v^2)\sin \theta_0/2 + \lambda_0 u \cos \theta_0] \] (4.a)
\[ k_{0y} = k_0 [\lambda_0 v] \] (4.b)
\[ k_{0z} = k_0[\cos \theta_0 - \lambda_0^2(u^2+v^2)\cos \theta_0/2 - \lambda_0 u \sin \theta_0] \] (4.c)

A plane pressure wave with unit amplitude traveling with this wave vector and incident on the liquid solid interface will excite longitudinal and shear disturbances in the solid of the form

\[ \phi = \frac{c_L}{c_0 \rho_0 \omega^2} T_L(u,v)e^{-jk_L(u,v) \cdot z_t} \] (5.a)
\[ \psi = \frac{c_T}{c_0 \rho_0 \omega^2} T_T(u,v)\hat{d}(u,v)e^{-jk_T(u,v) \cdot z_t} \] (5.b)

where \( \phi \) and \( \psi \) are the scalar and vector displacement potentials, and the underscore denotes a vector. The transmission coefficients \( T_L \) and \( T_T \) are the usual displacement amplitude transmission coefficients for longitudinal and shear waves respectively. The wavespeeds \( c_L \) and \( c_T \) are those for longitudinal in solid and shear in solid, respectively. The longitudinal and shear waves travel with wave vectors \( k_L \) and \( k_T \), respectively, and \( \hat{d} \) is the polarization of the vector potential. The term polarization is used loosely here to describe the direction associated with the vector potential. This is orthogonal to the polarization of the actual motion. The scalar potential as well as each component of the vector potential satisfy a scalar wave equation. The displacement is given by the relation

\[ \mathbf{u} = \nabla \phi + \nabla \times \psi. \]

The wave vectors \( k_L \) and \( k_T \) are determined by recognizing that the components parallel to the interface must be conserved so that \( k_{Lx} = k_{Tx} = k_{0x} \) and \( k_{Ly} = k_{Ty} = k_{0y} \). The components perpendicular to the interface are determined from the relations
\[ k_{Lz} = \sqrt{k_L^2 - k_{Lx}^2 - k_{Ly}^2} \]  
\[ k_{Tz} = \sqrt{k_T^2 - k_{Tx}^2 - k_{Ty}^2} \]  

where \( k_L \) and \( k_T \) are the longitudinal and shear wave numbers, respectively.

Substituting Eqns. 4 into Eqns. 6 and making use of Snell's law, \( k_\beta \sin \theta_\beta = k_0 \sin \theta_0 \), where \( \beta = L \) or \( T \), and \( \theta_\beta \) is the angle the transmitted wave vector makes with the interface normal, gives

\[
k_{\beta z} = k_0 \cos \theta_\beta \left[ 1 - \frac{\lambda_\beta^2 \lambda_\beta^2 (u^2 + v^2)^2 \sin^2 \theta_0}{4 \cos^2 \theta_\beta} - \frac{\lambda_\beta^2 u^2 \cos^2 \theta_\beta}{\cos^2 \theta_\beta} \\
+ \frac{\lambda_\beta^2 (u^2 + v^2) \sin^2 \theta_0}{\cos^2 \theta_\beta} - \frac{2 \lambda_\beta^2 u \cos \theta_0 \sin \theta_0}{\lambda_0 \cos^2 \theta_\beta} \\
+ \frac{\lambda_\beta^2 \lambda_0 (u^2 + v^2) u \cos \theta_0 \sin \theta_0}{\cos^2 \theta_\beta} - \frac{\lambda_\beta^2 v^2}{\cos^2 \theta_\beta} \right]^{1/2}
\]

Here, \( \lambda_\beta \) is the wavelength in the solid. In keeping with the paraxial approximation, one considers the beam energy to be concentrated at small values of \( u \) and \( v \) (plane waves traveling close to the forward direction). Expanding the square root in Eqn. 7 in a power series and neglecting terms of order \( O(u^3, v^3) \) yields
With the wave vector thus determined, the potentials in the solid are constructed by combining Eqns. 4, 5, and 8. Doing this, along with a rotation of the coordinate system to one aligned with the refracted central ray of the beam, \((x_1, y_1, z_1)\) (see Fig. 2), yields

\[
\Phi(x_1, y_1, z_1) = \Phi_0 e^{-j(k_0 d + k_1 z_1)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \hat{A}(u,v) T(u,v) \tilde{d}(u,v) e^{j\lambda_1 z_1(y^2 u^2 + v^2)} \times e^{-j\pi x_1 [2\gamma - \lambda_1 u^2 \alpha_1 - \lambda_1 v^2 \alpha_2]} e^{-j2\pi x_1 e^{j\pi k_0 d(u^2 + v^2)}}
\]

where

\[
\gamma = \cos\theta_0/\cos\theta_1 \tag{10}
\]

\[
\alpha_1 = \sin(\theta_0 + \theta_1) \sin(\theta_0 - \theta_1)/\sin\theta_1 \cos^3\theta_1 \tag{11}
\]

\[
\alpha_2 = \sin(\theta_0 + \theta_1) \sin(\theta_0 - \theta_1)/\sin\theta_1 \cos\theta_1 \tag{12}
\]

\[
\Phi_0 = \frac{p_0 c_1}{c_0 \rho_0 \omega^2} = \frac{V_0 c_1}{\omega^2} \tag{13}
\]

Here, \(\Phi\) can represent either the scalar potential, \(\phi\), or the vector potential \(\psi\). The
subscript β(=L,T) has been dropped since the mode will be obvious from which potential is being considered. The subscript 1 distinguishes a property of the solid. The quantity \( \hat{d}(u,v) \) is defined to be unity when \( \Phi = \phi \) and is the vector \( \hat{d} \) when \( \Phi \) is taken to be \( \psi \).

Equations 9-13 represent a general formalism for the displacement potentials within the solid due to a piston radiating in the fluid. The equations are rigorous within the Fresnel approximation. The discussions that follow explore the use of this formalism for computing the axial radiation in the solid. In what follows, longitudinal and shear disturbances will be considered separately since a complete field may be constructed by superposition.

b. Axial field of longitudinal mode The integrals in Eqn. 9 for the scalar potential are generally intractable analytically due to the complicated dependence of the transmission coefficient, \( T \), on the spatial frequencies \( u \) and \( v \). However, it has been assumed that the energy is concentrated within a small angular range. If one assumes that the transmission coefficient is sufficiently slowly varying, then over that small range of angles it may be considered a constant equal to the value at the central ray, \( T(\theta_0) \). This assumption would be invalid near critical angles, or other points at which the transmission coefficient varies rapidly with angle. With this assumption and considering the axial field, \( x_1 = y_1 = 0 \), Eqn. 9 reduces to

\[
\phi(0,0,z_1) = \phi_0 T(\theta_0) e^{-j(k_0d+k_1z_1)} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \hat{A}(u,v) \times e^{j\pi\lambda_1z_1(y^2+u^2+v^2)} e^{j\pi\lambda_0d(u^2+v^2)}
\]

Substituting Eqn. 2 into Eqn. 14 and using standard integral tables (86), Eqn. 14 may be reduced to
\[ \phi(0,0,z_1) = \phi_0 T(\theta_0) e^{-i(k_0 d + k_1 z_1)} C_0(S,\Delta S) \]

where

\[ C_0(S,\Delta S) = \left( \frac{1}{2\pi} \right) \sqrt{S^2 - \Delta S^2} \int_0^{2\pi} d\theta \left[ S + \Delta S \cos 2\theta - \frac{ab(S^2 - \Delta S^2)(f - \delta \cos 2\theta)}{\lambda_0 f_1 f_2} \right]^{-1} \times \]

\[ \left\{ 1 - \exp \left[ -j\pi \left( S + \Delta S \cos 2\theta - \frac{ab(S^2 - \Delta S^2)(f - \delta \cos 2\theta)}{\lambda_0 f_1 f_2} \right) \right] \right\} \]

\[ (15.b) \]

and

\[ S = \lambda_0 d/ab + \lambda_1 z_1 (1 + \gamma^2)/2ab \]
\[ \Delta S = \lambda_1 z_1 (1 - \gamma^2)/2ab \]
\[ f = (f_1 + f_2)/2 \]
\[ \delta f = (f_1 - f_2)/2 \]
\[ \sigma = (a^2 + b^2)/2ab \]
\[ \delta \sigma = (a^2 - b^2)/2ab \]

\[ (15.c) \]
\[ (15.d) \]
\[ (15.e) \]
\[ (15.f) \]
\[ (15.g) \]
\[ (15.h) \]

The function \( C_0 \) can be rapidly evaluated with a simple numerical integration. It is possible that there will be values of \( \theta \) for which the first factor in the integrand becomes infinite. This is compensated for, however, by the behavior of the complex exponential
for small arguments. Also $\sigma > \delta \sigma$ and $S > \Delta S$ so that there is no singularity in the exponential.

For an unfocused piston radiating in a fluid only, $f_1 = f_2 = \infty$ and $z_1 = 0$, Eqns. 15 can be put into an identical form as given in Ref. (23) for that case. Note, however, that the present definitions of $S$ and $\Delta S$ differ from those in Ref. (23). For the case of a focused piston in a fluid, the present $C_0$ function does not have the same form as in Ref. (23) although both forms yield the same result. The difference is due to a different sequence for evaluating the multiple integrals. The present form is preferable since the previous form contained a singularity in the exponential for some cases and thus required a complex integration contour.

If the piston is circular, $a = b$, spherically focused, $f_1 = f_2 = f$, and directed normal to the interface, $\gamma = 1$, then $\Delta S = 0$ and the integral may be evaluated analytically to yield

$$C_0 = \frac{1 - e^{-j\pi S} e^{ja^2/\lambda_0f}}{1 - a^2S/\lambda_0f} \quad (16)$$

Once again, the singularity at $1 - a^2S/\lambda_0f = 0$ is compensated for by the behavior of the exponential.

For the circular, unfocused case Eqn 15.b may again be evaluated analytically with the result taking the form of the infinite series

$$C_0 = 1 - e^{-j\pi S/(s^2-\Delta S^2)} \sum_{n=0}^{\infty} a_n (-j)^n \left. \frac{\pi \Delta S}{s^2-\Delta S^2} \right] \quad (17.a)$$
where \( a_0 = 1 \) and

\[
a_n = 2 \left( \frac{\sqrt{1 - (\Delta S/S)^2} - 1}{(\Delta S/S)} \right)^n, \quad n > 0
\]  

(17.b)

It can be seen that \( a_n \approx [(\Delta S/S)^n + \text{higher order terms}] \). Thus, for cases where \( (\Delta S/S) \ll 1 \) the series in Eqn. 17.a may be truncated after only a couple of terms to yield a good approximation. These cases will occur near normal incidence \( (\gamma = 1) \) and also when the liquid path, \( d \), is large compared to the solid path, \( z_1 \).

c. Axial field of shear mode

Obtaining a useful equation for the axial variation of the vector displacement potential is more difficult, in principle, than for the scalar potential. As can be seen in Eqn. 9, the integrals are complicated not only by the dependence of the transmission coefficient on \( u \) and \( v \), but also by the dependence of the polarization vector, \( \hat{d}(u,v) \). It was shown in the case of the scalar potential that assuming \( T(u,v) \) to be constant was sufficient to obtain the result. One should consider, therefore, whether the same assumption can be made regarding the components of \( \hat{d} \). In fact, this may be a reasonable assumption based on the following argument.

For a plane shear wave traveling in the direction of the central ray of the refracted field (see Fig. 2), the polarization of the motion is in the \( x_1 \) direction and the field has only a \( z_1 \) dependence. Thus, the vector displacement potential is polarized in the \( y_1 \) direction. Since most of the beam energy is assumed to be concentrated in plane waves traveling at small angles to the central ray, an appropriate assumption would be that the dominant component of the vector potential for all the waves is the \( y_1 \) component. Thus, setting the \( y_1 \) component equal to unity and neglecting the others yields a result identical to Eqns. 15 where \( \phi \) is replaced by \( \psi y_1 \) and the material parameters now
represent values for shear rather than longitudinal waves. Also $\psi_{x_1} = \psi_{z_1} = 0$. As with the assumption on the transmission coefficient, this is invalid at points where the polarization varies rapidly with angle.

These approximations are expected to provide a satisfactory prediction of the radiation field, particularly for well collimated beams such as those often used in ultrasonic nondestructive evaluation. Assuming that $T(u,v)$ and $d(u,v)$ are constant amounts to neglecting terms of order $O(u,v)$ and above as multiplying factors in the integrand. Terms of order $O(u^2,v^2)$ were retained in the exponential phase factor as is necessary due to its rapid fluctuation. This relative level of approximation for terms appearing within exponentials, as compared to premultiplying factors, is consistent with the Fresnel approximation commonly made in the Rayleigh diffraction integral where the radial position $r$ is replaced by the first two terms of a binomial expansion in the exponential and by only the first term in the denominator.

3. Refraction through a bicylindrical interface

a. General formalism  Consider the problem depicted in Fig. 3 in which a piston is directed toward a bicylindrically curved interface. The angle of incidence, $\theta_0$, which the central ray of the piston makes with the normal to the interface is defined to be in the $x$-$z$ plane of Fig. 3. At the point of intersection between the central ray of the beam and the interface normal, the interface is assumed to have radii of curvature $B_x$ and $B_y$ in the $x$-$z$ and $y$-$z$ planes respectively. The problem of the curved interface is much more difficult from the angular spectrum of plane waves approach than is the planar interface case. The interface no longer simply transforms planar incident waves into planar transmitted waves. However, the approach developed here may still be applied to obtain an
Figure 3. Geometry of piston aimed at bicylindrically curved liquid-solid interface
approximation for the fields refracted through a curved interface provided that the physical effects of the interface on the beam are properly treated, as discussed below.

The curved interface has two main effects on the beam. The first is refraction, just as in the planar interface case. The second is focusing due to the curvature of the surface. It is therefore proposed to treat the problem as the sequential application of two separate physical processes. First (see Fig. 4.a), the transmitted fields are determined for the problem of oblique incidence on a planar surface. Then, those fields are transformed as if a bicylindrical lens were placed perpendicular to the direction of the refracted central ray, as shown in Fig. 4.b. The relationships between the focal lengths, $F_1$ and $F_2$, of the hypothetical lens and the radii of curvature of a bicylindrical interface are determined through the laws of geometrical optics and are given by

\[
F_1 = \frac{B_x \cos^2 \theta_1}{\frac{c_1}{c_0} \cos \theta_0 - \cos \theta_1} \quad (18.a)
\]

\[
F_2 = \frac{B_y}{\frac{c_1}{c_0} - 1} \quad (18.b)
\]

Here, $B_x$ and $B_y$ are defined to be positive for a concave solid surface.

In this manner the gross effects of both focusing and refraction are accounted for. Aberrations induced in the beam at the interface are neglected. It is expected that this method will yield suitable results for cases where the interface radii are much larger than the beam dimensions and when the angle of incidence is not too severe.

The first step of the solution, refraction through a planar interface, has already been
Figure 4. Conceptualization of curved interface transmission problem: (a) refraction through planar interface, (b) passage through bicylindrical lens.
analyzed and is given by Eqns. 9-13, where once again it is assumed that both the transmission coefficient and polarization vector are constant. This assumption will not be quite as good for the curved interface since the curvature will give rise to a wider range of incident angles. Assuming that the beam passes through a lens immediately after passing through the interface, the field emerging from the lens, call it \( \Phi(x_1,y_1,0^+) \), is given by

\[
\Phi(x_1,y_1,0^+) = \Phi(x_1,y_1,0) e^{i\pi(x^2/\lambda_1 F_1 + y^2/\lambda_2 F_2)}
\]  

(19)

where \( \Phi(x_1,y_1,0) \) is given by Eqn. 9 with \( z_1 = 0 \). Equation 19 is the thin lens transformation law. Then radiation field at some distance \( z_1 \) in the solid is given by Eqn. 1, where \( (x,y,z) \rightarrow (x_1,y_1,z_1), p \rightarrow \Phi, p_0 \rightarrow 1, k \rightarrow k_1, \lambda \rightarrow \lambda_1 \), and \( \hat{A}(u,v) \rightarrow \hat{A}'(u,v) \) and is now given by

\[
\hat{A}'(u,v) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ \Phi(x,y,0^+) e^{2\pi(ux+vy)}
\]  

(20)

b. Axial field  

The substitution of Eqn. 9 into Eqn. 19, Eqn. 19 into Eqn. 20, and then Eqn. 20 into Eqn. 1 results in a rather formidable expression. Considering only the axial field, \( x_1 = y_1 = 0 \), that expression may be reduced with the use of integral tables to

\[
\Phi(0,0,z_1) = \Phi_0 e^{-j(k_0 d + k_1 z_1)} T_0 \left( \frac{F_1 F_2}{(F_1 - z_1)(F_2 - z_1)} \right)^{1/2} C_0(S, \Delta S)
\]  

(21.a)

Where \( C_0 \) is the same function defined in Eqn. 15.b. The parameters \( S \) and \( \Delta S \) have
been altered and are given by

\[ S = \frac{\lambda_0 d}{ab} + \frac{\lambda_1 z_1}{2ab} \left( \frac{F_2}{F_2 - z_1} + \frac{\gamma^2 F_1}{F_1 - z_1} \right) \quad (21.b) \]

\[ \Delta S = \frac{\lambda_1 z_1}{2ab} \left( \frac{F_2}{F_2 - z_1} - \frac{\gamma^2 F_1}{F_1 - z_1} \right) \quad (21.c) \]

When \( F_1 = F_2 = \infty \) (planar interface), Eqns. 21 reduce to Eqns. 15. The apparent singularities when \( F_1 - z_1 = 0 \) and \( F_2 - z_1 = 0 \) can be removed by some algebraic manipulation. Unlike the planar interface case, however, there now exist cases for which \( S^2 - \Delta S^2 = 0 \), resulting in a singularity in Eqn. 15.b. These cases occur when \( \Delta S = S \), \( \Delta S = -S \), and \( \Delta S = S = 0 \). These singularities are removable, though, and the values at these points are given by

\[ C_0(S,S) = \sqrt{\frac{2j\lambda_0 f_2/ab}{2S - \lambda_0 f_2/ab}} \left\{ C \left( b \sqrt{\frac{2S - \lambda_0 f_2/ab}{S\lambda_0 f_2}} \right) + jS \left( b \sqrt{\frac{2S - \lambda_0 f_2/ab}{S\lambda_0 f_2}} \right) \right\} \quad (22.a) \]

\[ C_0(S,-S) = \sqrt{\frac{2j\lambda_0 f_1/ab}{2S - \lambda_0 f_1/ab}} \left\{ C \left( a \sqrt{\frac{2S - \lambda_0 f_1/ab}{S\lambda_0 f_1}} \right) + jS \left( a \sqrt{\frac{2S - \lambda_0 f_1/ab}{S\lambda_0 f_1}} \right) \right\} \quad (22.b) \]
where \( C \) and \( S \) are the Fresnel integrals. In practice, one must be careful when numerically integrating Eqn. 15.b very near these points since the complex exponential will be oscillating rapidly.

B. Full Fields

In attempting to apply the previously derived formalisms to the evaluation of off-axis fields, the problem becomes considerably more difficult. The general results are given in Section II.A. However, integrals which were evaluated analytically for the axial fields are now intractable for the off-axis case. Numerical evaluation of the multiple integrals is required. This will, in general, be quite tedious, particularly if one wishes to examine beam profiles by computing the field at many points.

An alternative approach which has proven quite useful is the expansion of the radiated fields in a set of complete, orthogonal solutions to the wave equation within the Fresnel approximation. The application of this method to piston radiation in a fluid was described in Ref. (23). Here it is shown that this method can be approximately extended to the liquid-solid interface problem.

1. Fields in a fluid

In Ref. (23) it was shown that the scalar wave equation for time harmonic disturbances has, in the Fresnel approximation, solutions of the form
\[ \Phi_{mn}(x,y,z) = \sqrt{\frac{w_x(0)w_y(0)}{w_x(z)w_y(z)}} e^{-j{kz} \cdot j[(2m+1)\psi_x(z)+(2n+1)\psi_y(z)]} \times \]
\[ e^{-\left(\frac{jk}{2}\right)[x^2/q_x(z)+y^2/q_y(z)]} H_m \left[ \frac{\sqrt{\psi_x(z)}}{w_x(z)} \right] H_n \left[ \frac{\sqrt{\psi_y(z)}}{w_y(z)} \right] \] (23.a)

where \( H_m \) is a Hermite polynomial of order \( m \) and

\[ q_{x,y}(z) = q_{x,y}(0) + z \] (23.b)

\[ w_{x,y}(z) = \left[ -\frac{1}{2} k \Im \left( \frac{1}{q_{x,y}(z)} \right) \right]^\frac{1}{2} \] (23.c)

\[ \psi_{x,y}(z) = \frac{1}{2} [\angle q_{x,y}(0) - \angle q_{x,y}(z)] \] (23.d)

\[ R_{x,y}(z) = \left[ \Re \left( \frac{1}{q_{x,y}(z)} \right) \right]^{-1} \] (23.e)

The parameters associated with the \( x \) and \( y \) directions are found by taking the subscripts \( x \) and \( y \) in turn. Here, \( w_{x,y} \) and \( R_{x,y} \) are the beam widths and radii of phase curvature, respectively, although the latter do not appear in the solution explicitly. The excess phase beyond that associated with a plane wave is given by \( \psi_{x,y} \). The \( z \)-dependence of the solution is contained in the complex parameters \( q_{x,y} \). They are sometimes referred to as the complex radius of phase curvatures. They contain the beam width and radius of phase curvature information. In the transverse directions, \( x \) and \( y \), the solution is seen to
have the form of Hermite polynomials modulated by Gaussian exponentials, and hence the name Gauss-Hermite solutions. The definition of the solution is completed with the choice of the initial parameters $q_{x,y}(0)$ or, equivalently, the choice of initial beam widths and radii of phase curvatures, $w_{x,y}(0)$ and $R_{x,y}(0)$, through

$$\frac{1}{q_{x,y}(0)} = \frac{1}{R_{x,y}(0)} - \frac{j2}{k w_{x,y}^2(0)}$$ (24)

The $\Phi_{mn}$ of Eqn. 23.a represent a complete set of eigensolutions for time harmonic bound beams. Therefore, a general beam may be represented as a sum over the $\Phi_{mn}$.

The representation for pressure in a fluid is

$$p(x,y,z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} \Phi_{mn}(x,y,z)$$ (25)

where the $C_{mn}$ are complex constant coefficients. The coefficients are determined by the initial beam profile $p(x,y,0) = p_0(x,y)$. Using this initial data along with the orthogonality property of the Gauss-Hermite functions, the coefficients are found to be

$$C_{mn} = \frac{1}{w_x(0)w_y(0)2^{m+n-1}m!n!} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ p_0(x,y)\Phi_{mn}^*(x,y,0)$$ (26)

In general, for a source function $p_0(x,y)$, the two-dimensional integration will be intractable analytically. This is the case for a bicylindrically focused, elliptical piston.
\[ p_0(x,y) = \rho_0 c_0 V_0 e^{i(k/2)(x^2/r_1 + y^2/r_2)} \]  

(27)

and the limits of integration extend only over the piston face. For a localized source such as this, a numerical integration is straightforward and time efficient. Once the coefficients are computed in this manner, the field at some observation point is computed from Eqn. 25. The number of terms necessary for successful convergence of this series will depend on the choice of the now arbitrary convergence parameters \( q_{x,y}(0) \). A discussion about the appropriate choice of \( q_{x,y}(0) \) can be found in Ref. (23).

2. Fields transmitted through an interface

As was discussed in Section A, longitudinal and shear beams in a solid can be represented by scalar and vector displacement potentials, respectively. The scalar potential and the components of the vector potential all satisfy scalar wave equations and can thus be represented by the solution form of Eqn. 25. Following the reasoning of Section B, the shear wave case will again be approximated by considering only one dominant component of the vector potential, \( \psi_y \). Therefore, for the case depicted in Fig. 3, there will be an incident field, \( p \), and transmitted fields, \( \phi \) and \( \psi_y \).

Each beam is represented in the form of Eqn. 25, where of course each beam has its appropriate wave number and the z-axis for each beam is aligned with its central ray. For the incident field, initial parameters \([q_{x,y}(0)]_i\) are chosen and the coefficients \([C_{mn}]_i\) are computed from Eqn. 26, where the subscript "i" denotes incident field quantities. The key to defining the transmitted fields is the determination of the \([C_{mn}]_t\) and initial
[\textbf{[q_{x,y}(0)]}_t], which must be related to the incident field and to the properties of the interface. The subscript "t" denotes transmitted field quantities, which will of course be different for the longitudinal and shear modes.

The coefficients, \( \mathbf{C}_{mn} \), represent the initial complex amplitudes of the individual Gauss-Hermite beams of which the incident field is composed. The amplitudes of the transmitted beams will consequently be equal to the amplitudes of the incident beams, at the interface, multiplied by the appropriate transmission coefficient. This is written as

\[
[C_{mn}]_t = \Phi_0[C_{mn}]_iT(\theta_0) \left[ \frac{w_x(0)w_y(0)}{w_x(d)w_y(d)} \right]^{1/2} e^{-jk_0d} \times e^{j[(2m+1)\psi_x(d)]} \times e^{j[(2n+1)\psi_y(d)]} \quad (28)
\]

where \( d \) is the liquid path distance as in Fig. 3, and \( \Phi_0 = (c_1/c_0p_0\omega_0^2) \).

Next, the \([q_{x,y}(0)]_t\) must be determined. Recall that the \( q_{x,y} \) carry beam width and radii of phase curvature information. These are precisely the quantities which are affected by the interface. Transformations were obtained for \( q_{x,y} \) following the laws of paraxial ray optics, e.g. Ref. (70), with the result, for the geometry of Fig. 3,

\[
\left[ \frac{1}{q_x(0)} \right]_t = \left[ \frac{1}{q_x(d)} \right]_i \left( \frac{k_0\cos^2\theta_0}{k_1\cos^2\theta_1} \right) - \left( \frac{k_0\cos\theta_0/k_1 - \cos\theta_1}{B_x\cos^2\theta_1} \right) \quad (29.a)
\]

\[
\left[ \frac{1}{q_y(0)} \right]_t = \left[ \frac{1}{q_y(d)} \right]_i \left( \frac{k_0}{k_1} \right) - \left( \frac{k_0/k_1 - 1}{B_y} \right) \quad (29.b)
\]
The small angle expansions used to derive these transformations neglect aberrations which lead to asymmetries in the transmitted beam. This error becomes greatest for large angles of incidence and for small interface radii. An analysis of this error is given in Ref. (24).

The solution procedure is now complete. For a piston given by Eqn. 27 and a choice of initial parameters \([q_{x,y}(0)]_i\), the coefficients of the incident field are computed by numerically evaluating Eqn. 26. The coefficients and initial parameters of the transmitted beam are then given by Eqns. 28 and 29, respectively. The solution at any point in the solid is computed by summing Eqn. 25 where \(p\) is replaced by a displacement potential, \(\phi\) or \(w_y\). Note that the first step, computation of the incident field coefficients, is independent of the interface, depending only on the piston. Thus, for a given piston, this only needs to be done once and can then be applied to as many interface shapes or angles of incidence as desired. This fact allows the solution at many field points, for many geometries, to be computed very quickly. Note also that the source is not limited to being a piston. This method works equally well for any source function \(p_0(x,y)\), provided that it is sufficiently localized to make the numerical integration practical.

C. Examples: Comparison of Axial and Full Field Solutions

In Ref. (23), the axial fields predicted by the axial and Gauss-Hermite theories were compared for piston radiation into a fluid, and good results were obtained. Here, analogous comparisons are made for the fluid-solid interface case. Computer codes have been written to perform the computations required by the methods of Sections A and B. In both cases, the required integrals were evaluated using a simple trapezoidal algorithm.

For the Gauss-Hermite method, good convergence of the series has been obtained by choosing the initial parameters \(R_x(0) = f_1, R_y(0) = f_2, w_x(0) = a/\sqrt{5},\) and \(w_y(0) = b/\sqrt{5},\)
where the latter values have been determined empirically (23). For the results presented here, 30 x 30 terms were taken in the Gauss-Hermite series. These programs were run on a Micro-Vax II computer. The evaluation of the 900 Gauss-Hermite coefficients required approximately 15 seconds, real time. The field at an observation point was then computed at a rate of approximately 5 points per second. Consequently, the computational effort required by the Gauss-Hermite method is quite small. The axial field theory requires substantially less time.

The first comparison of the two methods is for a circular piston directed at a planar interface. The liquid and solid (longitudinal) wavespeeds are taken to be 0.15 cm/μs and 0.6 cm/μs, respectively. The piston radius is 0.635 cm and the liquid path, d, is 5 cm. The axial fields in the solid at a frequency of 10 MHz are shown in Figs. 5.a and 5.b for refracted longitudinal waves of 0 and 45 degrees, respectively. The horizontal axis in the figures is the position along the refracted beam axis, with the interface being the origin. The vertical axis is the field amplitude (transmission coefficient and constant amplitude factor not included). It is seen that the agreement between the two methods is excellent. For the normal incidence case (Fig. 5.a) the beam retains its circular shape and a typical axial profile is obtained (the nulls are not perfect in the figure due to the resolution of the computational points). In the oblique incidence case (Fig. 5.b) the beam becomes elliptical due to the refraction and the on-axis peaks and nulls are damped due to imperfect cancellation. Also shown in Fig. 5.b is the prediction of Eqn. 17.a using only one term of the series. This approximation yields good results and, in fact, when the second term is included as well, the profile becomes indistinguishable from that obtained with the numerical integration. Consequently, for plane, circular pistons and flat interfaces, Eqn. 17.a is very useful.

Figures 6.a and 6.b show the axial profiles for the same geometry, but for an
elliptical piston with $a = 0.635$ cm and $b = 0.3175$ cm. Once again the agreement between the 2 methods is quite good. Due to the strong ellipticity, the axial oscillations are strongly damped.

The final case, shown in Figs. 7.a and 7.b, is the same as the previous one with the exception that the interface is now cylindrically curved, with $B_x = 7.62$ cm. The interface focuses the beam in accordance with Eqn. 24.a. The focal length is 2.54 cm at normal incidence and 1.18 cm at a 45 deg. refracted angle. The agreement is again excellent.
Figure 5. Axial field predictions for the case of a plane circular piston directed at a plane interface [Eqn. 15 ———; Eqn. 25 ———]. (a) Normal incidence, (b) 45° refracted L-wave [Eqn. 17 —·——·—·]
Figure 6. Axial field predictions for the case of a plane, elliptical (2:1) piston directed at a plane interface [Eqn. 15 ———- ; Eqn. 25 ———- ]. (a) Normal incidence, (b) 45° refracted L-wave
Figure 7. Axial field predictions for the case of a plane, elliptical (2:1) piston directed at a cylindrical interface [Eqn. 15 ——— ; Eqn. 25 ———]. (a) Normal incidence, (b) 45° refracted L-wave
III. THEORY FOR FIELDS IN ANISOTROPIC MEDIA

In Section II the full fields in the isotropic solid were formulated in terms of displacement potentials. This allowed for the use of the scalar wave equation, for which the Gauss-Hermite beam modes are solutions, in the Fresnel approximation. For the anisotropic medium the problem is not as straightforward since displacement potentials cannot be defined. Consequently, the problem must be formulated in terms of the displacements, and it is not obvious that the Gauss-Hermite solutions apply. However, it is shown in this section that, beginning with the angular spectrum of plane waves approach, the Gauss-Hermite beam method may still be used. Also, whereas in Section II harmonic time dependence was always assumed, general time dependence will be formally assumed in this section and pulsed solutions will be discussed. This result can easily be applied to the theory for isotropic media.

A. Theory for Wave Propagation

1. Plane wave propagation in anisotropic media

The theory for plane wave propagation in anisotropic solids is well known (26,43-45). When the equations of motion and constitutive relations of anisotropic elasticity are combined, a wave equation results which has three solutions for the displacement field \( \mathbf{u} \) of waves propagating in a unit direction \( \hat{\mathbf{p}} \). The solutions have the form

\[
\mathbf{u} = \hat{\mathbf{d}} \exp[j\omega(t - \hat{\mathbf{p}} \cdot \mathbf{r}/\mathbf{v}_p)]
\]

(30)

where \( \omega \) is the angular frequency and \( t \) and \( \mathbf{r} \) are the temporal and spatial coordinates.
The phase velocity $v_p$ and the polarization $\hat{d}$ are determined, respectively, by the eigenvectors and eigenvalues of the Christoffel equation

$$[C_{ijkl}P_i^0P_j^0 - \rho v_p^2\delta_{jk}]d_k^0 = 0 \quad (31)$$

Where $C_{ijkl}$ is the elastic constant tensor and $\rho$ is the material density. The superscript 0 is used to denote tensor components evaluated in a coordinate system aligned with the axes of material symmetry. This will be dropped in a subsequent section when the axes are rotated to align with the direction of wave propagation. When $\hat{d}$ lies along certain symmetry directions, there is one longitudinally polarized solution and two transversely polarized solutions. For other propagation directions, the polarization is generally neither parallel nor perpendicular to $\hat{d}$. The solutions are sometimes called quasi-longitudinal and quasi-transverse modes. The general form of these solutions is determined by the symmetry of the material, as reflected by the form of the elastic constant tensor. Detailed discussions of the various crystal classes may be found in the literature (26,43-45).

Figure 8 represents an example of a map of inverse phase velocity (slowness) as a function of propagation direction in the x-z plane of a material exhibiting hexagonal symmetry, transversely isotropic austenite. For any propagation direction $\hat{d}$, the phase velocity is equal to the inverse of the slowness. A second property of interest is the group velocity, which determines the direction and rate of energy flow. It is well known that the direction of the group velocity is the normal to the slowness surface, and that the magnitude of the group velocity is given by

$$v_g = v_p / \cos \psi \quad (32)$$

where $\psi$ is the angle separating the two.
Figure 8. Slowness in the (100) plane of a hexagonal material — transversely isotropic austenite [after Ogilvy (58)]
2. Beam propagation in anisotropic media

a. Angular spectrum of plane waves representation

Consider next the case of a beam propagating in the direction \( \hat{p} = (p^*_1, p^*_2, p^*_3) \). It is often useful to represent the beam as an angular spectrum of plane waves propagating in a direction centered about \( \hat{p} \), and it is most convenient to rotate the coordinate system such that the z-axis coincides with the central axis of the beam. This type of representation for anisotropic problems is well known (26,56,57) and has been used extensively in the analysis of acoustic surface wave devices (50,54). Recognizing that there are three plane wave solutions for each propagating direction, the general angular spectrum of plane waves solution has the form

\[
\psi(x,y,z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \sum_{i=1}^{3} \phi^i(k_x,k_y,\omega) \hat{d}^i(k_x/\omega,k_y/\omega) \times \\
\exp[i(\omega t - k_x x - k_y y - k_z z)]
\]

(33)

where \( \hat{d}^i(k_x/\omega,k_y/\omega) \) is the aforementioned polarization of a plane wave solution and \( k_z \) is defined by the corresponding slowness surface. The index "i" denotes the three branches of the slowness surface. To differentiate the two coordinate systems, alphabetic indices are used for the system aligned with the propagation direction while numeric indices are used for the system aligned with the material symmetry directions.

If the displacement field is known in an initial plane \( z=0 \), then taking the inverse Fourier transform of Eqn. 33 yields the result

\[
\sum_{i=1}^{3} \phi^i(k_x,k_y,\omega) \hat{d}^i(k_x/\omega,k_y/\omega) = H(k_x,k_y,\omega)
\]

(34)
where

\[
H(k_x,k_y,\omega) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ u(x,y,0,t) \exp[j(\omega t + k_x x + k_y y)]
\]  

(35)

For a given propagation direction, the \( \hat{d}^i(\phi) \) are orthogonal. However, specification of \( k_x \) and \( k_y \) generally leads to different propagation directions for different branches, and the \( \hat{d}^i(k_x/\omega, k_y/\omega) \) cannot be assumed to be orthogonal. This is illustrated in Fig. 9. It is therefore necessary to project the right hand side of Eqn. 34 onto these three unit vectors to obtain a unique determination of the \( \phi^i(k_x,k_y,\omega) \). The result is

\[
\phi^1 = \{(H \cdot \hat{d}^1)[1-(\hat{d}^2 \cdot \hat{d}^2)^2] + (H \cdot \hat{d}^2)[(\hat{d}^1 \cdot \hat{d}^3)(\hat{d}^2 \cdot \hat{d}^3) - (\hat{d}^1 \cdot \hat{d}^2)] \\
+ (H \cdot \hat{d}^3)[(\hat{d}^1 \cdot \hat{d}^2)(\hat{d}^3 \cdot \hat{d}^2) - (\hat{d}^1 \cdot \hat{d}^3)]\} / (1 + 2(\hat{d}^1 \cdot \hat{d}^2)(\hat{d}^2 \cdot \hat{d}^3)(\hat{d}^3 \cdot \hat{d}^1) \\
- [\hat{d}^1 \cdot \hat{d}^2)^2 + (\hat{d}^1 \cdot \hat{d}^3)^2 + (\hat{d}^2 \cdot \hat{d}^3)^2] 
\]  

(36)

with similar forms for \( \phi^2 \) and \( \phi^3 \) obtained by permutation of indices.

b. Fresnel approximation The above represents a general approach to the computation of the evolution of propagating fields. Given an initial field \( u(x,y,0,t) \), the \( H \) are computed form Eqn. 35, the \( \phi^1 \) from Eqn. 36, and the full propagating fields from Eqn. 33. However, this procedure may be computationally tedious. Here, simplifying approximations are presented for the case in which the beam profile varies sufficiently slowly that only a small range of \( k_x \) and \( k_y \) values make significant contributions. Under these conditions the Fresnel approximation can be employed.

The essential feature of this approximation is a Taylor series expansion of the
Figure 9. Graphical illustration that specification of \( \frac{k_z}{\omega} \) leads to different propagation directions for different wave modes.
slowness surface in the vicinity of the propagation direction. After rotation of the coordinate system to the axes aligned with the beam, the slowness \((k/\omega)\) will have a form such as that illustrated in Fig. 10. For anisotropic media, in the vicinity of the \(z\)-axis, \((k/\omega)\) must have the form

\[
(k/\omega) \equiv S_0 + A(k_x/\omega) + B(k_y/\omega) + C(k_x^2/\omega^2) + D(k_x k_y/\omega^2) + E(k_y/\omega)^2 \tag{37}
\]

where the Taylor series has been truncated after the second order terms. Here \(S_0\) is the slowness of plane waves propagating in the \(z\)-direction. \(A\) and \(B\) determine the rate of change of slowness with propagation direction, and hence determine the group velocity.

From Eqn. 22, it follows that the phase and group velocities of the central ray are given by

\[
\mathbf{v}_p = (1/S_0)\mathbf{\hat{a}}_z \tag{38.a}
\]

\[
\mathbf{v}_g = (-A\mathbf{\hat{a}}_x - B\mathbf{\hat{a}}_y + \mathbf{\hat{a}}_z)/S_0 \tag{38.b}
\]

where \(\mathbf{\hat{a}}_{x,y,z}\) are unit vectors in the indicated directions. Equation 38.b predicts that the group velocity exceeds the phase velocity by the factor \((1+A^2+B^2)^{1/2}\). The parameters \(C, D,\) and \(E\) define the curvature of a plot of slowness versus \((k_x,k_y)\) in rectangular coordinates. As will be shown below, these will determine the rate of divergence or convergence of the beam due to diffraction.

This description of the slowness surface can be combined with the relationship
Figure 10. Schematic slowness surfaces with respect to coordinate system aligned with beam
\[(k/\omega)^2 = (k_x/\omega)^2 + (k_y/\omega)^2 + (k_z/\omega)^2 \quad (39)\]

to solve for \(k_z\) as required in the evaluation of beam profiles as predicted by Eqn. 33.

The result is

\[
(k_y/\omega) \equiv S_0 + A(k_x/\omega) + B(k_y/\omega) + [C-(1/2S_0)](k_x/\omega)^2 + D(k_xk_y/\omega^2) + [E-(1/2S_0)](k_y/\omega)^2 \quad (40)
\]

For an isotropic medium, the terms linear in \(k_x\) and \(k_y\) vanish, but quadratic terms remain as required by Eqn. 39.

This Fresnel (also paraxial, parabolic) approximation to the slowness surface in the vicinity of the beam axis is a widely used assumption for anisotropic wave propagation (46-56). It is well known that the coefficients of the first and second order terms in the expansion of the slowness surface govern the beam skew and divergence, respectively.

In addition to this near-axis expansion of the slowness surface, it will be assumed that

\[\hat{d}_i'(k_x/\omega,k_y/\omega) = \hat{d}_i'(0,0) \quad (41)\]

i.e., the variation of the polarization with propagation direction will be neglected. It might initially be assumed that this is a much weaker approximation than the expansion for \((k_z/\omega)\), which retained terms to second order in \((k_x/\omega)\) and \((k_y/\omega)\). Note, however, that much greater accuracy is required for \(k_z\) since it appears in the rapidly varying factor.
exp(-jk\textsuperscript{2}z), as discussed in detail in the previous chapter.

The approximation will be completed by assuming that the incident field contains only one polarization mode. Of course, more complex cases can be treated by superposition. If one writes the initial displacement field as

\[ u(x,y,0,t) = f(x,y,t)\hat{d}(0,0) \quad (42) \]

then the general procedures presented in Eqns. 33-36 reduce to

\[
\phi^i(k_x,k_y,\omega) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ f(x,y,t)\exp[j(-\omega t + k_x x + k_y y)] \quad (43)
\]

and

\[
u(x,y,z,t) = (1/2\pi)^3 \hat{d}(0,0) \int_{-\infty}^{\infty} d\omega \ \exp[j\omega(t-S_0z)] \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \ \
\phi^i(k_x,k_y,\omega)\exp\{-j[k_x(x+Az) + k_y(y+Bz)]\} \ \
\exp\{-jz[(k_x^2/\omega)(C-1/2S_0) + k_x k_y/\omega + (k_y^2/\omega)(E-1/2S_0)]\} \quad (44)\]

In the isotropic limit (A=B=C=D=E=0), the integral has the same functional form as for the anisotropic case. This suggests that those isotropic cases which have been treated analytically with the Fresnel approximation can be treated in the anisotropic limit as well. The cases of Gaussian and Gauss-Hermite beams will be discussed below.
3. Gaussian beams

The case in which the initial beam profile, \( f \), has a Gaussian shape can be treated quite simply. Let

\[
f(x,y,t) = u_0 \exp\{j\omega_0 t\} \exp\left\{-\left[\left(\frac{x}{w_{x0}}\right)^2 + \left(\frac{y}{w_{y0}}\right)^2\right]\right\} \times \\
\exp\left\{-\frac{j\pi}{\lambda_z} \left[\frac{x^2}{R_{x0}} + \frac{y^2}{R_{y0}}\right]\right\}
\]

(45)

where \( \omega_0 \) is the angular frequency, \( w_{x0} \) and \( w_{y0} \) are the initial beam widths, \( R_{x0} \) and \( R_{y0} \) are the initial radii of phase curvature, and \( \lambda_z \) is the wavelength associated with a plane wave traveling in the z-direction.

Substitution of Eqn. 45 into Eqn. 43 and evaluation of the integrals (86) gives

\[
\phi^i(k_x,k_y,\omega) = -2\pi \delta(\omega-\omega_0) u_0 k_z \sqrt{\frac{\pi w_{x0} w_{y0}}{q_{x0} q_{y0}}} \exp\left\{\frac{j\lambda_z}{4\pi} \left[q_{x0} k_x^2 + q_{y0} k_y^2\right]\right\}
\]

(46)

where, by definition

\[
\frac{1}{q_{x0,y0}} = \frac{1}{R_{x0,y0}} - \frac{j\lambda_z}{\pi w_{x0,y0}^2}
\]

(47)

**a. Propagation in a symmetry plane**  When Eqns 44 and 46 are combined, difficulties in the analytical evaluation of the integrals ensue associated with the cross-term \( \exp(-jk_x k_y D/\omega) \) in the integrand. Assuming that \( D \) vanishes implies that the beam axis is restricted to lying in a plane of material symmetry. Since this corresponds to
many practical situations, this assumption will be made first. The more general case will be discussed afterwards. Since it is not necessary to assume that either A or B vanish, although one or the other generally would if $D=0$, both will be retained in the analysis.

When $D=0$, the multiple use of tabulated integral relationships (86) leads to the form

$$u(x,y,z,t) = d'(0,0)u_0 \exp[j\omega_0(t-S_0z)] \sqrt{\frac{q_{x0}q_{y0}}{(q_{x0}+\Lambda_xz/\Lambda_z)(q_{y0}+\Lambda_yz/\Lambda_z)}} \times \exp\left\{ -\frac{j\pi}{\Lambda_z} \left( \frac{(x+Az)^2}{q_{x0}+\Lambda_xz/\Lambda_z} + \frac{(y+Bz)^2}{q_{y0}+\Lambda_yz/\Lambda_z} \right) \right\} \tag{48}$$

where

$$\Lambda_x = \frac{2\pi}{\omega_0 S_0} (1-2CS_0) \tag{49.a}$$

$$\Lambda_y = \frac{2\pi}{\omega_0 S_0} (1-2ES_0) \tag{49.b}$$

This result can be put into a more compact form which is analogous to the form for the isotropic case given in Ref. (24). The result is

$$u(x,y,z,t) = d'(0,0)u_0 \sqrt{\frac{w_xw_y}{w_x(z)w_y(z)}} \exp[j\omega_0(t-S_0z)] \times \exp\{j[\psi_x(z) + \psi_y(z)]\} \exp\left\{ -\frac{j\pi}{\Lambda_z} \left[ \frac{(x+Az)^2}{q_x(z)} + \frac{(y+Bz)^2}{q_y(z)} \right] \right\} \tag{50}$$

where
Here, $w_{x,y}$ and $R_{x,y}$ are the beam widths and wavefront radii of curvature, respectively, although the $R_{x,y}$ do not appear explicitly in the solution. The excess phase, beyond that associated with a plane wave, is given by $\psi_{x,y}$. As in the isotropic case, the $z$-dependence of the solution is contained in the complex parameter $q_{x,y}(z)$. In the isotropic limit, $A=B=C=E=0$, the above equations reduce to the expression in Ref (24).

The following observations can be made regarding the relationship between the isotropic and anisotropic cases. a) The beam has the phase variation $\exp(-j\omega_0 S_0 z)$ rather than $\exp(-jkz)$. This is consistent with the phase velocity predicted by Eqn. 38.a. b) The transverse beam profile is centered on the line $x=-Az$, $y=-Bz$. This is consistent with the direction of group velocity predicted by Eqn. 38.b. The effect is often referred to as beam skew. c) For the isotropic case $\Lambda_{x,y} = \Lambda = \lambda$, and therefore $q_{x,y}$ in Eqn. 51 varies directly with $z$. In the anisotropic case, the $z$ dimension is scaled in Eqn. 51 by the factor $(\Lambda_{x,y}/\Lambda_z)$. This parameter, which contains terms related to the second derivative of slowness with angle, can be thought of as an "anisotropy factor", determining the
degree to which the material differs from the isotropic case. The quantity \((\Lambda_x, \sqrt{\Lambda_z})z\) can be thought of as the equivalent distance needed to be traveled in an isotropic medium to achieve the same diffraction effects that occur when traveling a distance \(z\) in the anisotropic medium. The curvature of the slowness surface therefore controls the rate of beam spread. The fact that the diffraction in an anisotropic material is related to that in an isotropic material by a distance scaling factor has been noted by other authors (49,50,52). In fact, the anisotropy parameter "\(\gamma\)" of Szabo and Slobodnik (52) is related to the present parameters by \(\gamma = (\Lambda/\lambda) - 1\).

b. Propagation out of a symmetry plane

If the beam axis is not restricted to lying in a symmetry plane, i.e., \(D\neq 0\), the \(k_x\) and \(k_y\) integrals are no longer separable and one would suspect that the \(x\) and \(y\) dependencies of the solution are no longer separable as well. This complicates the integrals; however, with some effort, the integrals may still be evaluated analytically for \(D=0\). Substitution of Eqn. 46 into Eqn. 44 and using integral tables (86) yields

\[
\psi(x,y,z,t) = \hat{d}^{(0,0)}_0 u_0 \exp[\imath \omega_0 (t - S_0 z)] \sqrt{\frac{q_x q_y}{q_x(z) q_y(z) - (D S_0 z)^2}} \times 
\exp \left\{ \frac{-j \pi}{\lambda z} \left[ \frac{q_x(z)(y+Bz)^2 + q_y(z)(x+Az)^2 + (2D S_0 z)(x+Az)(y+Bz)}{q_x(z) q_y(z) - (D S_0 z)^2} \right] \right\} 
\]

(55)

where \(q_{x,y}(z)\) is defined by Eqn. 51. The formalism of Norris (40) for Gaussian pulse propagation can be put, in the long pulse (harmonic) limit, in a form equivalent to Eqn. 55. Norris' envelope tensor \(M\) and spreading matrix \(N\) can be identified as equivalents to the present \(q\)'s and \(\Lambda/\lambda\)'s, respectively.

The exponential now has a term which is a cross product in \(x\) and \(y\), apparently
disrupting the Gaussian profile. When $D=0$, Eqn. 55 reduces to Eqn. 48. It is 
interesting to examine the behavior of this solution with regard to whether the beam 
remains Gaussian or not. The Gaussian exponential is composed of two terms -- a 
Gaussian phase variation and a Gaussian amplitude distribution. Considering these two 
parts separately, the xy cross product term in each may be eliminated by a rotation of the 
coordinate system about the z-axis by some angle $\theta$. If the beam is initially circular, 
$q_{x0}=q_{y0}$, then this angle $\theta$ is constant, is the same for both the phase and amplitude 
components and is given by $\tan(2\theta) = D/(C-E)$. The beam is Gaussian with respect to 
this coordinate system, which is the principal coordinate system for the second order 
slowness surface. If the beam is initially elliptical, $q_{x0} \neq q_{y0}$, then the angle $\theta$ with 
respect to which the phase and amplitude are Gaussian varies with distance $z$ and is 
different for the phase and amplitude components. At $z=0$, $\theta=0$ and as $z$ increases the 
angle rotates until in the farfield it approaches the limits $\tan(2\theta) = D/(C-E)$ for the phase 
distribution and for the amplitude component

$$
\tan(2\theta) = \frac{2DS_{0}[(1-2CS_{0})\text{Im}(q_{y0})+(1-2ES_{0})\text{Im}(q_{x0})]}{(DS_{0})^{2}\text{Im}(q_{y0}+q_{x0})+(1-2ES_{0})^{2}\text{Im}(q_{x0})-(1-2CS_{0})^{2}\text{Im}(q_{y0})}
$$

The amplitude and phase components are therefore always Gaussian for this case, but 
with respect to different coordinate axes. This can be thought of as a generalized 
definition of a Gaussian beam. The physical reasons for this result are not fully 
understood.
4. Gauss-Hermite beams

The Gaussian function is the lowest order member of a complete set of orthogonal functions, known as the Gauss-Hermite functions, which were described in Section II. The steps of Section III.A.3 for obtaining the solution for a Gaussian beam will be repeated here with the initial beam profile now given by a general Gauss-Hermite function of order \( m,n \). Let

\[
f_{mn}(x,y,t) = u_0 \exp\{j\omega_0 t\} \exp\left\{-\frac{j\pi}{\lambda_z} \left[ \frac{x^2}{q_x(z)} + \frac{y^2}{q_y(z)} \right]\right\} H_m \left[ \frac{\sqrt{2}}{w_x(z)} \right] H_n \left[ \frac{\sqrt{2}}{w_y(z)} \right]
\]

where \( H_m \) is a Hermite polynomial of order \( m \). When \( m=n=0 \), Eqn. 56 is identical to Eqn. 45 since \( H_0(x)=1 \).

The field produced by this initial profile is found by substituting Eqn. 56 into Eqn. 43 and then Eqn. 43 into Eqn. 44 as done previously for the Gaussian case. Once again the assumption that \( D=0 \), i.e., propagation in a symmetry plane, is made. The ensuing integrals may be evaluated analytically by substituting the explicit polynomial expressions for the Hermite polynomials and then making use of tabulated integral relationships (86). The result can be greatly simplified with the notation introduced earlier and the field is given by

\[
u_{mn}(x,y,z,t) = \hat{d}(0,0)u_0 \sqrt{\frac{w_x(z)w_y(z)}{w_x(z)w_y(z)}} \exp\{j\omega_0(t-S_0t)\} \times \exp\{j[(2m+1)\psi_x(z) + (2n+1)\psi_y(z)]\} \exp\left\{-\frac{j\pi}{\lambda_z} \left[ \frac{(x+Az)^2}{q_x(z)} + \frac{(y+Bz)^2}{q_y(z)} \right]\right\} \times H_m \left[ \frac{\sqrt{2}}{w_x(z)}(x+Az) \right] H_n \left[ \frac{\sqrt{2}}{w_y(z)}(y+Bz) \right]
\]

(57)
where \( q_{x,y}, w_{x,y}, \) and \( \psi_{x,y} \) are given by Eqns. 51-53. When \( m=n=0 \), Eqn. 57 reduces to Eqn. 50 for the Gaussian case.

It is clear from Eqn. 57 that the Hermite polynomials as well as the Gaussian exponential are centered about the line \( x=-Az, y=-Bz \), resulting in the appropriate beam skew. Once again the \( z \)-dependence enters through the parameter \( q_{x,y} \) and the slowness surface curvature controls the rate of beam spread through the terms \( A_{x,y} \). No solution has been found as yet for the case when \( D\neq0 \). The analysis of Gauss-Hermite beam modes for the 2-dimensional surface wave beam case in an anisotropic medium has been considered by Mason (49-51).

5. Construction of more complicated beams

As found in Section II, the \( \psi_{mn}(x,y,z) \), where the time dependence \( \exp(j\omega_0 t) \) is implied but has been dropped from the notation for convenience, represent a complete set of eigenfunctions for time harmonic bound beams in an anisotropic material within the Fresnel approximation. Therefore, a general beam may be represented as a sum over the \( \psi_{mn} \) provided that the Fresnel approximation is valid for the specific case. That is, provided that the angular spectrum of the beam is contained within a sufficiently narrow angular range. The representation is

\[
\psi(x,y,z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} \psi_{mn}(x,y,z) \tag{58}
\]

where the \( C_{mn} \) are complex constant coefficients. As in Section II.B, these coefficients are determined by the initial beam profile \( \psi(x,y,0) = f(x,y)\delta(0,0) \). Equation 26 is used, replacing \( p_0(x,y) \) by \( f(x,y) \) and \( \Phi_{mn}(x,y,0) \) by \( \psi_{mn}(x,y,0) \).
6. Application to pulsed beams

In the derivation of the Gaussian and Gauss-Hermite solutions, for both isotropic and anisotropic materials, harmonic time dependence was assumed. However, practical applications generally deal with beams of finite pulse length. If one seeks a pulsed solution, one straightforward computational approach is to calculate the harmonic solution at many frequencies and then numerically Fourier transform that data into the time domain. This approach has been used successfully by Margetan et al. (8). Alternatively, it may be possible to find analytical solutions for some cases.

As an example, consider an initial field of the form

\[
\mathbf{f}(x,0,t) = u_0 \exp\left\{-\left(\frac{x}{w_{x0}}\right)^2\right\} \exp\{j\omega_0 t\} \exp\left\{-\left(\frac{t}{\Delta}\right)^2\right\} \tag{59}
\]

For simplicity, the initial field is assumed to be infinite in extent in the y-direction, although this is not necessary. This initial field is a planar Gaussian with a center frequency of \(\omega_0\) and modulated by a Gaussian envelope of duration \(\Delta\). Substituting Eqn. 59 into Eqn. 43, evaluating the integrals, and then inserting that result into Eqn. 44 yields

\[
\mathbf{l}(x,z,t) = u_0 \frac{w_{x0}^z}{4\pi} \hat{d}(0,0) \int_{-\infty}^{\infty} d\omega \exp\{j\omega(t-S_0 z)\} \exp\left\{-\left(\frac{\Delta^2}{4}\right)(\omega-\omega_0)^2\right\} \times \int_{-\infty}^{\infty} dk_x \exp\{-jk_x(x+Az)\} \exp\left\{-\left(\frac{w_{x0}^z}{4}\right)k_x^2\right\} \exp\left\{\left(\frac{j\Lambda_x z}{2\Lambda_z S_0 \omega}\right)k_x^2\right\} \tag{60}
\]

When \(\Delta \gg 1/\omega_0\), the argument of the last exponential in Eqn. 60 will vary slowly over the values of \(\omega\) contributing significantly to the integral. The term \(1/\omega\) in that argument
can be expanded in a Taylor series about \( \omega_0 \)

\[
\frac{1}{\omega} = \frac{1}{\omega_0} - \frac{(\omega - \omega_0)}{\omega_0^2} + \ldots
\]  

(61)

Keeping only the first two terms and integrating yields

\[
y(x,z,t) = d'(0,0) \exp\left\{j\omega_0(t - S_0z)\right\} \exp\left\{-\frac{(t - S_0z)^2}{\Delta^2}\right\} \times \sqrt{\frac{q_{x0}}{q_x(z) - j(\Lambda_x/\Lambda_z)z(t - S_0z)/(\Delta^2\omega_0)}} \times \\
\exp\left\{-\frac{-j(\pi/\Lambda_z)(x + A_z)^2}{q_x(z) - j(\Lambda_x/\Lambda_z)z(t - S_0z)/(\Delta^2\omega_0)}\right\}
\]  

(62)

Equation 62 is the solution for a Gaussian modulated pulse. When the pulse duration \( \Delta \) becomes infinite, this reduces to the harmonic solution of Eqn. 50. The term \( j(\Lambda_x/\Lambda_z)z(t - S_0z)/(\Delta^2\omega_0) \) appears in the solution in two places. If this term is neglected in both places, Norris' solution for a Gaussian pulse is obtained (40). This term can be neglected when \( (t - S_0z)/\Delta << \Delta\omega_0 \). It has already been assumed that \( \Delta\omega_0 >> 1 \), so that this constraint will generally be satisfied. In this case, the solution is seen to be identical to the harmonic solution with the exception that it is multiplied by the Gaussian modulation envelope which remains constant in shape as it propagates.

This result is easily generalized to include the third dimension since, for \( D=0 \), the solution is separable in the transverse coordinates. Also, when the term \( j(\Lambda_x/\Lambda_z)z(t - S_0z)/(\Delta^2\omega_0) \) is neglected, i.e., when only the zeroth order term is retained in the Taylor
series in Eqn. 61, the Gauss-Hermite solution may be obtained. This also applies to the isotropic media solutions of Section II.

B. Transmission of a Beam Through an Interface

The transmission of elastic waves through an interface between two materials of different elastic properties adds two elements to the problem: refraction due to the difference in wavespeeds plus focusing if the interface is curved. This problem is an important one since often the source of the beam is outside of the material which is being examined. This is true in nondestructive evaluation, where a transducer is generally coupled to a part through a solid wedge or through a water bath. A rigorous solution to the problem is, in general, extremely difficult. Here, the transformation formulae for the Gauss-Hermite modes presented in Section II.B.2 are generalized for the case of beam transmission through an interface between two anisotropic materials.

Consider a beam of a particular wave mode traveling in medium "a" as shown in Fig. 11. This beam is incident upon the curved interface between media "a" and "b" with angle $\theta_a$ between the central wave vector and the normal to the interface. This angle will be taken to lie in the x-z plane. The surface is assumed to have radii of curvature $B_x$ and $B_y$ in the x-z and y-z planes respectively. The radii of curvature are taken to be positive for a concave surface. Since propagation from a liquid into a solid with higher ultrasonic velocity is of primary interest in this paper, this convention leads to positive $B_{x,y}$ for focusing situations. Attention will be restricted to the case when the x-z plane is a plane of material symmetry for both medium "a" and medium "b". This implies that $D=B=0$ for both media.

Within the framework of this assumption and the approximations made for beam
Figure 11. Schematic of transmission of a beam through a curved interface
propagation, there will be, at most, two beam modes excited in medium "b" by the incident beam. If the incident beam has either longitudinal or vertical shear polarization, then beams of each of these polarizations will generally be transmitted. If the incident beam has a horizontal polarization, then only this type will be transmitted since this is a pure mode in a symmetry plane. The amplitudes of the transmitted beams relative to the incident beam will be approximated by using the appropriate plane wave transmission coefficients. These are computed by assuming that a plane wave of the appropriate mode is incident with angle $\theta_a$ on a plane boundary between media "a" and "b". Details on how to compute these coefficients may be found in the literature (26,43,87). This approximation will be valid if the transmission coefficient varies slowly over a small angular range about the incident wave vector direction. Near critical angles this may not be the case.

The formalism presented for Gauss-Hermite beams provides a powerful framework for treating the effects of interface transmission. Recall that all of the beam properties depend on the complex parameters $q_{x,y}$, which change linearly with distance as the beam propagates. Since $q_{x,y}$ define the radii of curvature of the wavefronts and the widths of the beam, the interfacial problem reduces to the determination of these parameters after transmission through an interface. If the values of $q_{x,y}$ are determined after transmission, then these can be used as the initial values for the transmitted beam.

To obtain the transformations for $q_{x,y}$, the laws of geometrical optics are derived for rays whose angles deviate slightly from that of the central ray. This derivation follows the procedure outlined in Ref (70). For the anisotropic case, the result is
\[
\frac{1}{q_x} = \left( \frac{S_a \cos^2 \psi_b \cos^2 \phi_b}{S_b \cos^2 \psi_a \cos^2 \phi_a} \right) \frac{1}{q_x} I - \frac{\cos^2 \psi_b (S_a \cos \theta_a / S_b - \cos \theta_b)}{B_a \cos^2 \phi_b} \quad (63.a)
\]

\[
\frac{1}{q_y} = \left( \frac{S_a}{S_b} \right) \frac{1}{q_y} I - \frac{(S_a / S_b - 1)}{B_y} \quad (63.b)
\]

where the "T" indicates the transmitted quantity and the "I" indicates the incident quantity. This result must be applied to each transmitted mode of interest. \(S_a\) and \(S_b\) are the slownesses associated with the incident and transmitted wave vector directions of the beam axis, respectively. The angles are defined in Fig. 11. The transmitted slowness and angle are determined from Snell's law and the slowness surface of medium "b".

Snell's law for an anisotropic material is

\[
S_a(\theta_a) \sin \theta_a = S_b(\theta_b) \sin \theta_b \quad (64)
\]

Equations 63 are equivalent to those derived by Norris (40) for the transmission of Gaussian pulses through an interface between anisotropic media.

The procedure for using this interface transmission technique is the same as that given for the isotropic medium in Section II.B.2. A source in medium "a" a distance "d" from the interface excites a Gauss-Hermite beam of order \(m,n\). As illustrated in Fig. 11, the distance "d" is measured along a normal to the source plane to the point at which the beam center hits the interface. If medium "a" is anisotropic, beam skew may make the actual distance from the source to the interface greater than "d". The complex amplitude of the incident beam will be the complex amplitude of the incident beam multiplied by the transmission coefficient, \(T(\theta_b)\). The amplitude of the transmitted beam is given by
\[(C_{mn})_T = (C_{mn})_I \sqrt{\left(\frac{w_x w_y}{w_x(d)w_y(d)}\right)} \exp\left[-j\omega_0 S_x d\right] \times \exp\left[j(2m+1)\psi_x(d) + (2n+1)\psi_y(d)\right]_I\]  

(65)

where \((C_{mn})_I\) is the initial amplitude of the incident beam, and the subscript "I" denotes properties of the incident beam. The initial q values of the transmitted beam are given by Eqns. 63 with \((1/q_{x,y})_I\) given by \([1/q_{x,y}(d)]_I\). The incident beam is defined in the \((x_a,y_a,z_a)\) coordinate system and the transmitted beam is defined in the \((x_b,y_b,z_b)\) system.

This procedure can be used to treat a more general source. If a source is constructed as a sum of Gauss-Hermite functions through Eqn. 58, each term is the series is governed by the same parameters \(q_{x,y}\). Therefore, each term in the series may be transformed in the manner of described above.

C. Examples

1. Calculation of anisotropy parameters

The application of the theory which has been derived requires the knowledge of the anisotropy properties, which depend on the elastic constants, \(C_{ijkl}\). These properties are introduced through the parameters \(A,B,C,D,\) and \(E\), defined in Section III.A.2 as the coefficients of a Taylor series expansion of the slowness surface about the propagation direction. For an arbitrary propagation direction in an anisotropic material, the slowness is defined by the roots of the characteristic equation, Christoffel's equation. These roots are most conveniently found with the aid of a computer. Consequently, the parameters
A, B, C, D, and E can be computed by first computing points on the slowness surface in the region of the propagation direction and then making a fit of the function given by Eqn. 37 to those points by a method such as least squares.

Under certain symmetry conditions, however, the characteristic equation will factor and it may be possible to derive analytic expressions for the anisotropy parameters. For example, if a material exhibits cubic symmetry, the slowness for propagation in a cube face can be written analytically (43,44). If one considers a beam propagating along a material symmetry axis, then the slowness is symmetric about the propagation direction in both the x-z and y-z planes. This implies that A = B = D = 0. The parameters C and E may be derived by taking the expressions for the slowness in the x-z and y-z planes (cube faces) and expanding them in Taylor series about the z-axis. For a cubic material, C and E will be the same and are given, for the quasi-longitudinal mode, by

\[
C = E = \frac{1}{2\sqrt{\rho c_{11}}} \frac{(c_{12}+c_{11})(c_{12}c_{11}+2c_{44})}{(c_{11}-c_{44})}
\]

(66)

where the \( c_{ij} \)'s are the elastic constants given in abbreviated subscript notation (43,44). This quantity is essentially identical to the anisotropy factor, "b", which Papadakis (46,47) derives in a parabolic approximation to the slowness about the [100] direction in a cubic crystal. The relationship is \( C = b/S_0 \). The parameter which controls the rate of beam spread was found to be \( \Lambda/\lambda_z \). Combining Eqns. 66 and 49 gives

\[
\frac{\Lambda}{\lambda_z} = \frac{(c_{12}^2+2c_{12}c_{44}+c_{11}c_{44})}{(c_{11}^2-c_{11}c_{44})}
\]

(67)
For an isotropic material $c_{12} = c_{11} - 2c_{44}$ and $\Lambda/\lambda_z$ reduces to unity as it should.

The $\Lambda/\lambda_z$ may be written for other crystal types as well. For instance, it may be written for propagation along a material axis in an orthorhombic material. For propagation in the [001] direction of an orthorhombic material $C$ and $E$ are, in general, different and the result is, for the quasilongitudinal mode

$$\frac{\Lambda_x}{\lambda_z} = \frac{c_{55}}{c_{33}} + \frac{(c_{13}+c_{55})^2}{c_{33}(c_{33}-c_{55})}$$

$$\frac{\Lambda_y}{\lambda_z} = \frac{c_{44}}{c_{33}} + \frac{(c_{23}+c_{44})^2}{c_{33}(c_{33}-c_{44})}$$  \hspace{1cm} (68.a)

Equations 68 are equivalent to those derived by Norris (40) for the subcase of a transversely isotropic (hexagonal) material. For propagation along the [001] axis of a hexagonal material, the anisotropy factor for the horizontal shear (pure) mode is

$$\frac{\Lambda}{\lambda_z} = \frac{3}{2} - \frac{c_{66}}{2c_{44}}$$  \hspace{1cm} (69)

These are just a few examples of cases for which $\Lambda/\lambda_z$ may be derived. Propagation along principal axes for other crystal classes may be tractable as well. Also, it may be possible in some cases to obtain an analytical solution for propagation in directions other than along a principle axis; along a cube face diagonal in a cubic material, for example. Papadakis derives his anisotropy parameter for quite a few cases (47). Since there is a simple relationship between his parameter and the present ones, one may refer to Papadakis for other cases.
2. Discussion of anisotropy factor $\Lambda/\Lambda_z$

It has been stated that the factor $\Lambda/\Lambda_z$ determines how the rate of beam spread will differ from that of an isotropic material. This arises from the fact that $\Lambda/\Lambda_z$ is related to the second derivative of the slowness with angle through the parameter C or E. In order to physically illustrate this point, the anisotropy factor is compared with the divergence parameter of Ogilvy (58).

In order to obtain a measure of the beam spread in an anisotropic material, Ogilvy defines a parameter called the divergence

$$D^* = \left| \frac{\Delta \theta_g}{\Delta \theta_p} \right|$$

(90)

where $\theta_g$ is the angle of the group velocity and $\theta_p$ is the angle of the phase velocity. The * is used to distinguish this quantity from the D which appears in the Taylor series expansion of the slowness surface. The divergence, $D^*$, is therefore the rate of change of group velocity direction with respect to phase velocity direction. For an isotropic material $D^*=1$. When the slowness surface is highly curved $D^*>1$ and the beam is diverging rapidly. When the slowness surface is slightly curved $D^*<1$ and the beam is not diverging very fast.

Ogilvy gives the specific example of transversely isotropic austenite which has the same macroscopic elastic symmetry as a hexagonal crystal. The slowness in the (010) plane is shown in Fig. 8. A computer program was written to compute $\Lambda_y/\Lambda_z$ at points along the slowness surface in the plane of Fig. 8. This was done by first computing B and E at each point by making a least squares fit of a second order polynomial to a local region of the slowness curve. The resulting plot of $|\Lambda_y/\Lambda_z|$ versus propagation angle is
shown in Fig. 12. Comparing Fig. 12 with an equivalent plot of $D^*$ by Ogilvy (58) reveals that the anisotropy factor $\Lambda/\lambda_x$ is virtually identical to the divergence parameter of Ogilvy. There are some subtle quantitative differences between the two, however. The relationship between the two parameters is found to be $\Lambda/\lambda_x = (1+B^2)D^*$, where $B$ is the slope of the slowness surface at a point with respect to the wave vector direction, as defined in the Taylor series approximation. While $D^*$ is a good heuristic measure of beam divergence, the more appropriate measure is $\Lambda/\lambda_x$ since it accounts for the slope of the slowness surface through the factor $(1+B^2) = (\cos\psi)^2$ where $\psi$ is the skew angle defined in Eqn. 32. Although the curvature is generally the dominant influence in beam spread, the slope also plays a role. The physical explanation for this is that when $B=0$, i.e., the beam skews, the aperture of energy flow is decreased and hence the diffraction rate is increased.

3. Example of interface transmission and beam propagation

Consider the problem depicted in Fig. 13 in which a Gaussian transducer is immersed in water and is directed normally to a plane interface between the water and an anisotropic halfspace. The techniques developed in this paper can be used to determine the transducer beam profile as it propagates through the water, across the interface, and into the solid. For this example, the solid will be taken to be the transversely isotropic austenite discussed in the previous section. Let the normal to the surface (directed into the solid) be at an 80° angle to the [001] axis. Since the beam in the fluid is normally incident, the transmitted wave vector will be normal to the surface as well, and consequently the beam will skew.

In the water, $A_a=B_a=C_a=D_a=E_a=0$ and the slowness is $S_a=6.757\times10^{-4}\text{s/m}$. 
Figure 12. Anisotropy parameter as a function of propagation angle for the three wave modes in transversely isotropic austenite.
Figure 13. Geometry of example calculation: Gaussian transducer directed normally at water/transversely isotropic austenite interface.
Considering the transmitted quasi-longitudinal wave, the appropriate parameters for the solid are computed to be $S_b=1.697 \times 10^{-4} \text{s/m}$, $A_b=0.207$, $C_b=-1476.3 \text{m/s}$, $B_b=D_b=0$, and $E_b=0$. The Gaussian source is taken to be circular with an initial width of $w_{x_0}=w_{y_0}=0.2 \text{cm}$ and initial phase curvature of $R_{x_0}=R_{y_0}=\infty$. The standoff in the water is $d=15 \text{cm}$.

The magnitude of the beam profile in the x-z plane is shown in Fig. 14. The back plane in the figure is the source plane. The initial region of large mesh is the path in the water, while the fine mesh is the path in the solid. The profile is computed up to 25 cm in the solid. As a comparison, Fig. 15 depicts an analogous situation in which the solid is isotropic with a slowness equivalent to $S_b$ for the anisotropic case. For both cases, the beam profile in the water is the same, with the beam approaching the interface normally. In the isotropic case, the beam continues straight into the solid with an increase in the rate of beam spread over that in the water due to the greater velocity in the solid. In the anisotropic case, however, the beam is seen to skew to the left of the inward normal direction. The skew angle is $\psi_b=\arctan(A_b)=11.7^\circ$. It is also apparent that the beam spreads faster in the anisotropic case than in the isotropic case. This can be seen by comparing the widths of the beams at the farthest distance in the solid. This is consistent with the anisotropy factor for this case, $\Lambda_x/\lambda_z=1.5$. Further points to note about these figures are: a) the transmission coefficients have not been included since this would not add any insights and would make the figures less intelligible by scaling down the profiles in the solid, and b) the transverse and axial distances in the figures are not on the same scale since the distance traveled is much greater than the beam width.
Figure 14. Beam profile in x-z plane as beam propagates through water/anisotropic steel interface
Figure 15. Beam profile in x–z plane as beam propagates through water/isotropic steel interface
IV. EXPERIMENTAL VALIDATION OF BEAM MODELS

Two methods exist for validating the use of the approximate beam models derived in the previous sections. The first is to compare the predictions of the models with those of a more rigorous theory. This will indicate how well the models treat the mathematical idealization of the physical problem. The second approach is to compare the model predictions with experimentally observed data. This is more satisfying from a practical viewpoint since, ultimately, the validity of the models depends on their ability to predict observations. For this reason, the predictions of the ultrasonic beam models presented in Sections II and III are compared to experimentally measured beam profiles for a range of materials and geometries. Data from both isotropic and anisotropic materials are presented. The "model" or "theory" referred to in this section in all cases denotes the Gauss-Hermite model for the full fields.

A. Beam Maps in Isotropic Materials

1. Fused quartz

Experimental beam profile maps were acquired for transmission through a planar water/fused quartz interface for both longitudinal and shear waves transmitted over a wide range of angles and for several frequencies. The fused quartz material has a very homogeneous, isotropic character with low attenuation and consequently is easy to work with experimentally.

The experiments were carried out in an immersion tank using a 0.635cm diameter circular, unfocused broadband transducer with a nominal center frequency of 10MHz. The test specimen was a 2.54cm thick rectangular block of fused quartz with the
properties: longitudinal wavespeed = \( v_L = 0.597 \text{cm/\mu s} \), shear wavespeed = \( v_T = 0.376 \text{cm/\mu s} \), and density = \( \rho = 2.2 \text{g/cm}^3 \) (these values are found in Ref. (88) and agree with measured values). The test specimen contained several approximately spherical air bubbles at various depths below the surface which were on the order of 200 microns in diameter. Beam profiles were measured by scanning the transducer such that the beam transmitted into the solid passed across a bubble and the signals produced by backscatter from the bubble were recorded. Since the bubbles are small compared with the beam dimensions, they act as point reflectors and the amplitude of the backscattered signal is therefore related in simple way to the local amplitude of the beam incident on the bubble. Figure 16 illustrates the experimental geometry.

Referring to Fig. 16, it can be noted that, for obliquely incident beams, the beam profile mapped in the solid is parallel to the part surface and is not perpendicular to the beam axis. Consequently, as the refracted angle increases from 0 to near 90 degrees, the profile changes from transverse to near axial. This is undesirable in that it makes it more difficult to compare beam profiles taken at different angles. However, this does correctly measure the beam profile incident upon a flaw in a scanned inspection.

Since the beam models have been formulated for time harmonic beams, and the experiments employ pulsed beams, the experimental waveforms acquired at each scan point are fast-Fourier-transformed and the amplitude at a particular frequency is plotted versus scan position. In this manner a comparison can be made between theory and experiment. It can be shown that the amplitude of a frequency component in a signal backscattered from the bubble is proportional to the square of the incident beam amplitude (3). Consequently, the model predictions are squared for the comparisons. The results will be shown for frequencies of 5, 10, and 15MHz, which spans the bandwidth of the 10MHz transducer.
Figure 16. Experimental geometry
The beam profile comparisons are for shape only, and not for absolute amplitude. Comparisons of absolute amplitudes are difficult since such factors as transducer efficiency, attenuation, and bubble scattering amplitude must be known.

Beam profiles were mapped at refracted beam angles of 30°, 45°, 60°, and 70° for both longitudinal and shear waves. A longitudinal profile was measured at normal incidence as well. Table 1 gives the geometrical data for each experiment. Refer to Fig. 16 for parameter definitions.

Table 1. Experimental geometrical data

<table>
<thead>
<tr>
<th>Wave Mode</th>
<th>Refracted Angle (deg)</th>
<th>Incident Angle (deg)</th>
<th>d (cm)</th>
<th>z (cm)</th>
<th>H (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>0</td>
<td>0</td>
<td>2.23</td>
<td>1.14</td>
<td>1.14</td>
</tr>
<tr>
<td>L</td>
<td>30</td>
<td>7.1</td>
<td>1.47</td>
<td>1.52</td>
<td>1.14</td>
</tr>
<tr>
<td>L</td>
<td>45</td>
<td>10.1</td>
<td>3.75</td>
<td>0.76</td>
<td>0.55</td>
</tr>
<tr>
<td>L</td>
<td>60</td>
<td>12.4</td>
<td>2.46</td>
<td>1.08</td>
<td>0.55</td>
</tr>
<tr>
<td>L</td>
<td>75</td>
<td>13.9</td>
<td>3.00</td>
<td>2.13</td>
<td>0.55</td>
</tr>
<tr>
<td>T</td>
<td>30</td>
<td>11.4</td>
<td>5.20</td>
<td>0.64</td>
<td>0.55</td>
</tr>
<tr>
<td>T</td>
<td>45</td>
<td>16.2</td>
<td>2.73</td>
<td>1.61</td>
<td>1.14</td>
</tr>
<tr>
<td>T</td>
<td>60</td>
<td>19.9</td>
<td>1.04</td>
<td>2.27</td>
<td>1.14</td>
</tr>
<tr>
<td>T</td>
<td>75</td>
<td>22.4</td>
<td>1.41</td>
<td>2.13</td>
<td>0.55</td>
</tr>
</tbody>
</table>

For each case in Table 1 and each frequency, the parameter "S" may be calculated from Eqn. 15.c. This parameter is a generalization of the non-dimensional axial "distance" commonly used in piston analyses. The region of the beam for which S<1 is considered the near field and the region for which S>1 is considered the far field, with
the region around S=1 being a transition between the two. Since S is proportional to the wavelength, evaluating a beam profile at several frequencies for a fixed geometry will yield information at various cross-sections in the evolution of the beam. For example, in the 0° L-wave case at 10MHz, S has a value of 1.0. At 5 and 15MHz, S has values of 2.0 and 0.67, respectively. Thus information is obtained about the beam profile in the near field, transition, and far field regions. In the results which follow, the value of S for each case appears on the figure and will be considered in the discussion.

In the following figures, all of the amplitudes have been normalized so that the comparisons are of beam shape and not of absolute amplitude, as discussed earlier. The horizontal axis in the figures indicates transducer position along the scan path with zero being the starting transducer position. The ranges of the experimental scans were chosen to lie between the points at which the reflected signal dropped to a few percent of the maximum signal amplitude obtained during the scan. The theoretical calculations were made for the same scan ranges with the center of the range on the beam axis. Since experimentally it was difficult to accurately determine the absolute position of the transducer relative to the bubble, the two ranges, experimental and theoretical, may be slightly shifted relative to one another, particularly for the higher refracted angle cases. No attempt has been made to realign the curves since any such shifting would be entirely subjective.

Figure 17 shows the beam profiles for the normal incidence L-wave case. As mentioned previously, the three frequencies, 5, 10, and 15MHz, correspond to values of S of 2.0, 1.0, and 0.67, respectively. The model compares well in the far field and transition regions, although the experimental profile has spread slightly more at S=2 than is predicted. In the near field, the overall beam width is predicted well; however, the structure in the main lobe of the experimental profile is more erratic than the theory
Figure 17. Comparison of theory and experiment for 0° L refracted beam (a) 5MHz, (b) 10MHz, (c) 15MHz; (solid line - theory, triangles - experiment)
predicts. This is most likely due to two factors. The validity of the Fresnel approximation will be suspect for rapidly varying fine structure in the beam profile, both in the near field and possibly in the sidelobe structure in the far field. Also, a real transducer only approximates an ideal piston. Again, this will be most apparent in fine beam structure such as is found in the near field and sidelobes, which depends critically on the interference of signals emanating from various points on the face of the probe.

The results of the longitudinal beam profile comparisons at 30°, 45°, 60°, and 75° are similarly shown in Figs. 18-21. For the moderate angles, the same general comments as above apply. That is, there is good agreement for the far field and transition regions, while in the near field cases there is discrepancy in the fine structure. For the 75° case, as the frequency becomes higher, the experimental profiles become increasingly narrower with respect to the model predictions. At 75°, the aberrations induced in the beam when refracted at the interface become significant. The model does not take this into account. Also, the model is a scalar approximation to vector displacement fields. For a "well-behaved" beam, the displacement field is generally dominated by a particular polarization and thus the scalar approximation is valid. The large aberrations in this case may invalidate that approximation, however.

The comparisons of the shear wave beam profiles are shown in Figs. 22-25 for 30°, 45°, 60°, and 75° beams. The comparisons are, in general, very similar to those for the longitudinal wave beam profiles. One difference, however, is that the comparison between the theory and experiment starts to break down at 60° for the shear wave case rather than at 75°, as in the longitudinal wave case. This may be due to the fact that the scalar model is more severe of an approximation for the shear wave case, since it is governed by a vector rather than scalar displacement potential.
Figure 18. Comparison of theory and experiment for 30° L refracted beam (a) 5MHz, (b) 10MHz, (c) 15MHz; (solid line - theory, triangles - experiment)
Figure 19. Comparison of theory and experiment for 45° L refracted beam (a) 5MHz, (b) 10MHz, (c) 15MHz; (solid line - theory, triangles - experiment)
Figure 20. Comparison of theory and experiment for 60° L refracted beam (a) 5MHz, (b) 10MHz, (c) 15MHz; (solid line - theory, triangles - experiment)
Figure 21. Comparison of theory and experiment for 75° L refracted beam (a) 5MHz, (b) 10MHz, (c) 15MHz; (solid line - theory, triangles - experiment)
Figure 22. Comparison of theory and experiment for 30° T refracted beam (a) 5MHz, (b) 10MHz, (c) 15MHz; (solid line - theory, triangles - experiment)
Figure 23. Comparison of theory and experiment for 45° T refracted beam (a) 5MHz, (b) 10MHz, (c) 15MHz; (solid line - theory, triangles - experiment)
Figure 24. Comparison of theory and experiment for 60° T refracted beam (a) 5MHz, 
(b) 10MHz, (c) 15MHz; (solid line - theory, triangles - experiment)
Figure 25. Comparison of theory and experiment for 75° T refracted beam (a) 5MHz, (b) 10MHz, (c) 15MHz; (solid line - theory, triangles - experiment)
The data presented here for the fused quartz material are for transmission through a planar interface. Reference (79) contains the results of a comparison between the Gauss-Hermite model and experimental data for transmission through a cylindrical interface in the fused quartz. The results in that work were similar to those presented here. That is, the model worked well in predicting beam profiles in the far field and near focal points, but was unable to reproduce near field structure accurately.

2. Equi-axed cast stainless steel

The beam maps in the fused quartz are quite useful for basic evaluation of the model since the material is very homogeneous, isotropic and non-attenuative. Most practical engineering materials will not be so ideal, however. One example is cast stainless steel. This material, like most metals, is a composition of many grains, or crystallites, each anisotropic and with its own orientation. If the grains are equi-axed (approximately the same dimension in each direction) and randomly oriented, then, when averaged over a sufficient volume, the elastic properties of the material appear to be isotropic. In terms of ultrasonic beams, the material can generally be assumed to be isotropic when the wavelength is much larger than the grain size. An equi-axed cast stainless steel sample was obtained from the Electric Power Research Institute, which is interested in the inspection of these materials due to their use in the circulation systems of electric power generation plants. This sample had an average grain size of approximately 100 microns which small compared to the wavelength at frequencies from 0.5 to 3MHz, which is the typical range for inspecting this type of material. The wavespeeds for longitudinal and shear waves in this sample, which was 5.7 cm thick, were measured to be 0.57cm/μs and 0.32cm/μs, respectively.
Unlike the fused quartz samples, the cast steel samples did not contain convenient voids or inclusions which could be used as reflectors for mapping beam profiles. The alternative was to measure the beam profiles in through transmission as shown in Fig. 26. In order to do this, it was necessary to have a receiver which had an effective receiving area that was small compared to the beam width. For this purpose, an ultrasonic microprobe (obtained from G. Posakony of Battelle, Pacific Northwest Laboratories) was used. This transducer had an active element diameter of 0.5mm and could therefore be used to approximate a point receiver. The microprobe was placed in contact with the bottom surface of the sample and the transmitter was scanned parallel to the upper surface, in both the x and y directions of Fig. 26.

Figure 27.a shows a map of a 1MHz longitudinal wave beam transmitted through the sample at normal incidence. The transmitting transducer was a 1MHz, 2.54cm diameter unfocused probe. The standoff in the water was 4.3cm. The corresponding plot of the model predictions is shown in Fig. 27.b. The experiment and theory plots for the same geometry at 3MHz are shown if Figs. 28.a and 28.b. The comparisons for these cases are quite good.

The cast stainless steel materials are used for pipes in electric power generation plants. They are inspected for defects, usually cracks growing radially from the internal diameter near weld joints, by the use of either 45° longitudinal or shear waves since these will reflect well from such cracks. Figures 29 and 30 show comparisons of experiment and theory for 45° longitudinal wave beams at 1 and 3MHz, respectively. The transducer was the same as above and the water path was 5.4 cm. The incident angle in the water was 10.7°. The agreement is excellent for these cases. The 45° shear wave beam maps are shown in Figs. 31 and 32 for 1 and 3MHz, respectively. The same transmitting probe was used again and the water path was 3.7cm. The incident angle in the water for
Figure 26. Through transmission beam mapping using ultrasonic microprobe
Figure 27. Normal incidence longitudinal wave beam map through cast steel sample at 1MHz: (a) experiment, (b) theory
Figure 28. Normal incidence longitudinal wave beam map through cast steel sample at 3MHz: (a) experiment, (b) theory
Figure 29. 45° longitudinal wave beam map through cast steel sample at 1MHz:
(a) experiment, (b) theory
Figure 30. 45° longitudinal wave beam map through cast steel sample at 3MHz: (a) experiment, (b) theory
Figure 31. 45° shear wave beam map through cast steel sample at 1MHz:
(a) experiment, (b) theory
Figure 32. 45° shear wave beam map through cast steel sample at 3MHz:
(a) experiment, (b) theory
producing the 45° refracted shear waves was 19.3°. The agreement between the experiment and theory at 1 MHz is good in terms of overall beam size and shape, although the side lobe structure predicted by the model is not clearly defined in the experimental data. At 3 MHz, the experimental beam profile is slightly narrower than that predicted by the model. Once again the side lobe structure is not clearly defined in the experimental data.

B. Beam Maps in Anisotropic Materials

1. Graphite/epoxy composites

One class of materials which is becoming increasingly important in engineering applications, particularly in the aerospace industries, is advanced composite materials such as graphite/epoxy laminates. This material exhibits a high degree of anisotropy due to preferentially aligned high-strength fibers set in a resin matrix. Generally, a composite is fabricated by stacking layers of unidirectionally oriented fibers upon one another with varying orientations between layers in order to achieve some desired macroscopic stiffness properties. Consequently, the ultrasonic inspection of these materials is affected by not only the anisotropy of the material, but by the inhomogeneity of the fiber/matrix structure and by the periodicity of the layered structure as well. However, when the wavelength of the inspecting ultrasound is large compared to the fiber diameter and the layer thickness, the material may be approximately treated as homogeneous with some macroscopic elastic constants governing wave propagation. In this case, the present models may be used to predict beam patterns in these materials.

The beam models have been employed in an effort to model the through transmission inspection of graphite/epoxy plates containing delaminations (8). In order to validate this usage, beam profiles transmitted through a uniaxial composite plate have been measured.
and compared with the theory (89). The plate was 0.75cm thick and was constructed of 64 layers of "pre-preg" tape with the same fiber direction for all layers; hence the name uniaxial. This fiber distribution gives the composite the same macroscopic elastic constant symmetry as a hexagonal crystal. This is commonly referred to as transverse isotropy, and in this case the transversely isotropic axis is aligned with the fiber direction. Measured speeds of longitudinal and shear waves propagating along symmetry directions of coupons cut from the specimen were used to determine the stiffness constants $c_{11}$, $c_{22}$, $c_{44}$, and $c_{66}$. The fifth independent constant, $c_{12}$, was then chosen to reproduce the off-diagonal Poisson's ratio, $\nu_{12}$, provided by the composite manufacturer, LTV Aerospace and Defense Company. The resulting constants are $c_{11}=139$GPa, $c_{12}=7.2$GPa, $c_{22}=15.6$GPa, $c_{44}=4.0$GPa, $c_{66}=7.4$GPa and the density is $\rho=1.61$gm/cm$^3$.

The beams were generated using a 1.27cm diameter, 7.62cm focal length transducer with a center frequency of 10MHz. The focused probe was used in order to accentuate the effects of the anisotropy on the beam. A focused beam contains a broader angular spectrum of plane waves and consequently a larger region of the material's slowness surface will affect the beam. The beams emerging from the back surface of the plate were mapped with the microprobe described earlier. In order to simulate the through-transmission inspection geometry, the transmitting transducer was aligned so that the beam was normally incident on the plate. This experimental geometry is illustrated in Fig. 33. The microprobe was scanned in both the x and y directions to obtain a two-dimensional beam map.

In order to predict the beam profiles in the composite, the appropriate anisotropy parameters necessary as inputs to the model must be determined. The relevant section of the slowness surface is that about the z, or through-thickness, direction. Figure 34
Figure 33. Through transmission beam mapping through graphite/epoxy composite plate.
Figure 34. Detail of the slowness curve in the x-z plane and two parabolic approximations to it
shows a plot of the exact slowness curve about that direction, in the x-z plane, as computed from the above elastic constants. The y-z plane is the transverse isotropy plane and consequently the slowness curve in that plane is a circle. Also shown is the Taylor series approximation to that slowness curve computed using the anisotropy parameter given by Eqn. 68.a for an orthorhombic material, of which the hexagonal symmetry is a subcase. This leads to a value of $C = -0.18 \text{cm/\mu s}$. However, as stated previously, a focused probe was used so that the beam would contain a broader angular range to enhance the effects of the anisotropy. Consequently, the anisotropy parameter given by Eqn. 68.a may not be suitable. It was estimated that the transducer emits significant energy into the water up to about $6^\circ$ from the forward direction. When refracted into the solid, this translates to about $15^\circ$ from forward. The dashed curve in Fig. 34 is the parabola obtained by minimizing the average error between the parabola and the exact curve over the $15^\circ$ range. This gives a value of $C = -0.4 \text{cm/\mu s}$. A similar analysis in the y-z plane led to a value of $E = 0$, as one would expect.

The experimental and three theoretical two-dimensional beam profiles are shown in Fig. 35 for a frequency of 6MHz. The anisotropic effects are clearly visible in the experimental profile since the beam is no longer axially symmetric, as it would be if the material were isotropic. The model prediction for an isotropic material is shown. The two anisotropic model predictions corresponding to the "Taylor" and "average" parabolas seem to predict the main features of the experimentally observed profile very well, although it is not obvious from these figures which is better. In order to make quantitative comparisons, slices through the beam profile along the x and y axes have been plotted for 2, 4, and 6MHz in Figs. 36-38. It is apparent from these figures that the "average" parabola is the best result, although it is still not perfect since no parabola can closely reproduce the exact slowness surface over such a broad angular range. The
Figure 35. Measured 6-MHz field pattern, and predicted field patterns using three sets of elastic parameters
Figure 36. Measured and predicted beam profiles at 2MHz in the x and y directions
Figure 37. Measured and predicted beam profiles at 4 MHz in the x and y directions.
Figure 38. Measured and predicted beam profiles at 6MHz in the x and y directions
overall result is quite good, however, since all of the main beam features due to the anisotropy have been predicted.

2. Columnar cast stainless steel

In Section IV.A.2, beam propagation in equi-axed cast stainless steel was discussed. Sometimes, depending upon the composition of the alloy and the details of the casting process, the direction of grain growth is preferentially aligned with the radial direction of the pipe, which is the direction of the thermal gradient during cooling. Since the base metal has a cubic structure, each grain is a cubic crystal with a [100] axis aligned with the radial direction. The orientation of the grains in the plane transverse to the radial direction is generally thought to be random. When averaged over many grains, the elastic properties of this type of material take a form analogous to that of a crystal with hexagonal symmetry. A slowness curve for this type of material is shown in Fig. 8. The elastic constants used to generate this curve were approximated by applying an averaging scheme to the elastic constants for the cubic base metal. This type of microstructure is generally referred to as being columnar grained.

A columnar grained pipe sample was obtained from Battelle, Pacific Northwest Laboratories. The sample was 6.1cm thick in the radial direction and had an outer radius of 43.6cm. Wavespeed measurements in the through thickness direction yielded values of 0.54cm/μs and 0.37cm/μs for longitudinal and shear waves, respectively. These values are suggestive of the hexagonal symmetry (sometimes called transverse isotropy) assumed for this microstructure, however they are somewhat different from those predicted as in Fig. 8. Since knowledge of the precise shape of the slowness surface is necessary to determine the parameters for input to the model, more information about the slowness surface of the sample was needed. One way to acquire this information would
be to cut coupons from the sample so that wavespeeds could be measured in various directions. This was not possible since the sample was part of a round robin test and had to remain intact.

As an alternative, measurements were made of the group velocity angle, $\theta_g$ in Fig. 39, as a function of the incident angle in the water, $\theta_i$. This is the angle at which the energy actually propagates, which in general will be different from the phase velocity angle $\theta_p$. The difference between the two angles is the beam skew angle. The group velocity angle was measured by scanning the microprobe along the back surface of the sample to find the point about which a through transmitted beam was centered. The transmitting probe was a 2.25MHz, .635cm radius transducer. The results are shown, for both the quasilongitudinal and quasishear modes, in Fig. 40. These data can be used to estimate the slowness surface by the following procedure. It can be shown that

$$\frac{dS_z}{dS_x} = -\tan \theta_g$$

(91)

where $S_z$ and $S_x$ are the $z$ and $x$ components of the slowness vector, respectively, and $z$ is the radial direction. Integrating Eqn. 91 yields

$$S_z = \int_{0}^{S_x} -\tan \theta_g \, dS_x + (1/v_0)$$

(92)

where $v_0$ is the wavespeed in the radial direction and $S_x = \sin \theta_i/v_w$, where $v_w$ is the wavespeed in water. Equation 92 may be numerically evaluated using the data in Fig. 40 to obtain $S_z$ as a function of $S_x$, which is precisely the slowness surface. The result is
Figure 39. Ultrasonic beam propagation in columnar grained steel
Figure 40. Group velocity angle vs. incident beam angle in water for columnar grained steel
shown in Fig. 41 for both the quasilongitudinal and quasishear modes. Comparing Fig.
41 to Fig. 8, the quasilongitudinal slowness curve looks quite similar to what one would
expect. The quasishear slowness curve initially looks good as well, although it continues
straight rather than turning toward the $S_x$ axis at higher angles of propagation relative to
the radial direction. The data at these higher angles are probably suspect due to the fact
that the beam divergence and skew angle of the quasishear mode are severe past a
refracted angle of 45°. This fact is why in practice 45° longitudinal waves are used to
inspect columnar grained materials rather than the 45° shear waves used for isotropic
materials.

One-dimensional longitudinal wave beam maps at normal incidence and at 45°
refracted angle were made by scanning along the direction of the pipe axis with the
2.25MHz, .635cm diameter unfocused transducer and using the microprobe as receiver.
The water paths for the two cases were 0.8cm at normal incidence and 1.1cm for the 45°
scan. Using the experimental longitudinal slowness curve in Fig. 41, parabolic fits,
using the "average" method, to the slowness curve about the 0° and 45° directions yield
values of the parameter C of -0.28cm/$\mu$s and 0.2cm/$\mu$s, respectively. Comparisons of
the experimental data and theoretical predictions at 2MHz are shown in Figs. 42 and 43
for 0° and 45°, respectively. Also shown in the figures are the model predictions if the
material were assumed to be isotropic with a wavespeed equal to that in the forward
direction of the anisotropic material. At normal incidence, C is negative and consequently
the beam spreads faster than would be expected in an isotropic material. The anisotropic
model accounts for this. At 45°, C is positive and the beam spreads more slowly than it
would in an isotropic material. Once again, the model successfully predicts this
phenomenon.
Figure 41. Slowness curves determined from group velocity angle data (dashed lines indicate expected behavior)
Figure 42. Quasilongitudinal wave beam profiles at normal incidence in columnar grained steel.
Figure 43. Quasilongitudinal wave beam profiles at 45° in columnar grained steel
V. APPLICATIONS OF BEAM MODELS

The previous sections have described the derivation and experimental validation of theoretical ultrasonic beam models. In this section, the initial results of two engineering applications of the beam models are presented. The first is the design of novel ultrasonic transducers. The second is the use of the models to predict the distortion induced in ultrasonic beams when passing through a non-smooth interface.

A. Design of Ultrasonic Transducers

The pressure field in a fluid produced by a planar source is given by Equations 25 and 26, the formalism for the Gauss-Hermite beam model. These equations were derived with a planar or bicylindrically focused piston transducer in mind. However, they are more generally applicable to any localized source which can be described by a field distribution in a plane, \( p(x,y,z=0) \). Consequently, one application of the model is in the design of ultrasonic transducers, where it may be used to analyze the field patterns of prospective source types. This has been done previously for the analysis of axicon, or conically focused, transducers (80,81). The interest in axicons is stimulated by the desire for a transducer with a narrow, extended depth of field (82,83). The present work extends that interest to the analysis of other types of focused radiators (84). The results presented here demonstrate the application of the model to this type of problem, although the research is not mature enough to provide any firm conclusions about the utility of various types of probes.

The most common type of focused transducer is one which is spherically or cylindrically focused by placing a lens on the face of a planar transducer. In this case, the
modeling of the focusing is accomplished with the lens transformation law and the initial planar source distribution is given as in Eqn. 27. For the case of an axicon, however, the probe is generally constructed, not with a lens, but rather by having a radiating surface which is conical in shape, as illustrated in Fig. 44.a. This presents a problem for the Gauss-Hermite model, which requires a planar source distribution to utilize the orthogonality property of the functions through Eqn. 26. This is circumvented by making an approximation equivalent to the thin lens transformation of Eqn. 27. The thin lens law neglects diffraction in the lens. In a similar manner, the axicon case is treated by constructing a virtual planar source which has the appropriate phase variation, linear in this case, such that when rays are traced from the virtual source plane to the actual source surface, the correct field is obtained. This is illustrated in Fig. 44.b. This method may be used for other non-planar radiating surfaces as well, provided that the distance from any point on the radiating surface to the virtual source plane is small enough to make the thin lens law valid.

The types of focused radiators that have been modeled, along with the axicon and spherically focused piston, are shown in Fig. 45. They include a pyramidal radiator, and radiators which are elliptical, parabolic, and hyperbolic in shape. All of the above sources are axially symmetric with the exception of the pyramidal probe, which has a square cross-section. The radiation fields of these probes have been computed and compared with one another with respect to beam width and depth of focus. The parameters of all of the probes were chosen to be roughly equivalent to that of a reference probe, a 0.635cm radius spherically focused piston with a focal length of 7.62cm. Accordingly, each probe was given a radius of 0.635cm, or, in the case of the pyramidal probe, this value represented the probe half-width. The tilt angles of the axicon and pyramidal probes were both selected to be 2.4°. For this angle, the rays emanating from
Figure 44. Axicon transducer: (a) Geometry of an axicon probe, (b) modeling an axicon using a virtual planar source
Figure 45. Geometries of focused radiators used in comparison studies
the points halfway between the center and the edge of the probe will intersect the beam axis at 7.62cm. The focal lengths (position of foci) of all of the conic section probes were selected to be 7.62cm so that each probe would be focused in approximately the same region. The eccentricities of the elliptical and hyperbolic probes were chosen to be 0.5 and 1.5, respectively.

The radiation patterns of the spherical, axicon, pyramidal, hyperbolic, parabolic, and elliptical probes are shown in Figs. 46 through 51, respectively, for a frequency of 5MHz. These plots are of beam amplitude vs position in the x-z plane (the plane containing the beam axis and the x transverse axis). Therefore, these plots show how the transverse beam profiles evolve with propagation distance along the axis. For all of the probes except the pyramidal, this information completely describes the beam since they are axially symmetric. For the pyramidal probe, the plot depicts the beam profile in the principle planes. All of these calculations were performed by taking 30 x 30 terms in the Gauss Hermite expansion. Although no independent data are presented for comparison with these computed beam profiles, one can expect, based on previous experience with the model (as presented in Section IV), that the profiles are likely to be accurate except for points very near to the source plane.

In order to obtain more information about the depth of field for these probes, axial beam amplitude and beam width are plotted in Figs. 52 and 53, respectively, for each of the probes. The beam width in Fig. 53 is the width along a transverse line from the beam axis to the point at which the amplitude of the profile drops to one half the maximum for that line. The steps in Fig. 53 are due to the discrete nature of the calculated field points. From these figures, it is seen that all of the probes produce focusing at about the same point. The spherically focused probe has the highest amplitude at the focal point, but the amplitude drops off rapidly past the focus and the beam width grows rapidly. The rest of
Figure 46. Beam pattern for spherically focused probe (dimensions in cm)
Figure 47. Beam pattern for axicon probe (dimensions in cm)
Figure 48. Beam pattern for pyramidal probe (dimensions in cm)
Figure 49. Beam pattern for hyperbolically focused probe (dimensions in cm)
Figure 50. Beam pattern for parabolically focused probe (dimensions in cm)
Figure 51. Beam pattern for elliptically focused probe (dimensions in cm)
Figure 52. Axial beam amplitude for focused probes
0-spherical, 1-axicon, 2-pyramidal, 3-elliptical, 4-parabolic, 5-hyperbolic
Figure 53. Beam width at half maximum amplitude vs. axial distance for focused probes

0-spherical, 1-axicon, 2-pyramidal, 3-elliptical, 4-parabolic, 5-hyperbolic
the probes behave quite similar to one another, all having a more extended depth of field than the spherically focused probe. The pyramidal probe is perhaps the best since it retains a strong amplitude while having the most gradual beam spread. The fact that it is not axisymmetric may be undesirable in certain situations, however. One drawback, however, to the nonspherically focused probes is that they appear to produce higher side lobe levels than the spherically focused probe, which is a generally undesirable property.

The above example demonstrates how the model can be applied to the analysis of various probe types to determine which types might best produce desired beam properties. Source types other than those shown here may be modeled as well. Also the beam patterns may be analyzed after passing through some interface, if that is a factor which will influence the choice of the probe.

B. Prediction of Surface Induced Beam Distortions

One factor which influences the performance of ultrasonic examinations is the condition of the surface of a component through which the ultrasound must pass to enter the material. For example, often in the steel power plant components discussed in Section IV, factors such as weld overlays, claddings, and diametrical shrink can give part surfaces a wavy, corrugated, or abruptly stepped topography. Having to pass an ultrasonic probe over such a surface during an inspection can result in a redirection of beam energy, beam partitioning, or possibly a partial truncation of the beam. These factors could leave regions of the part uninspected or give rise to mislocation of defects or geometrical reflectors.

Based on a review of the literature and ASME Codes, Good (90) has provided estimates of what surface conditions are likely to exist in the field. One such condition, illustrated in Fig. 54, is an abrupt discontinuity due to an unground or partially ground
Figure 54. A 1.5mm abrupt surface discontinuity causing ultrasonic beam distortion
[after Good (90)]
circumferential weld. Good estimates that steps of this nature as large as 1.5mm may be encountered in the field. The object of this work was to develop a model to quantify the effects of irregular surface conditions such as this on ultrasonic beams. This initial effort has been directed at modeling beam transmission through a surface with a step discontinuity (85).

1. Hybrid Gauss-Hermite/ray tracing model

The models developed thus far for beam transmission through interfaces have assumed that the the interfaces are smooth, whether flat or curved. The interface transformation formulas will not suffice for a non-smooth interface such as in Fig. 54. The non-smooth interface has been treated by an extension of the model following the work of Thompson & Lopes (73) and Newberry et al. (74). In this method, the Gauss-Hermite beam model is joined with a ray tracing scheme in order to propagate the beam through an interface. The technique is illustrated schematically in Fig. 55. The beam model is used to compute the field, produced by a transducer, which is incident upon an interface. The incident field consists of beam amplitude and phase on a grid of points at the interface. Each point on the grid is assigned a ray pointing in the direction of beam propagation. These rays are allowed to refract through the interface according to Snell's law. The refracted rays are then traced to their intersection with a hypothetical plane called the transmitted plane. The change in the phase of each ray is computed based on its path length and the amplitude is modified by the transmission coefficient appropriate for its refracted angle as well as by considering the change in cross-sectional area of a "flux-tube" surrounding the ray as it refracts through the interface. The details of this procedure are given in Refs. (73) and (74).

The result is that a grid of points now exists on the transmitted plane, each point
Figure 55. Schematic illustration of ray tracing through abruptly stepped surface
having an amplitude and phase assigned to it. This data is treated as a source plane for an
expansion into Gauss-Hermite functions. A set of coefficients, $C_{mn}$, is determined by
numerically integrating over this plane and the beam pattern at points subsequent to this
plane may be computed.

2. Experimental comparison

As an initial test of the model, a simple experiment was conducted using a 1.85cm
thick stainless steel plate with a 0.63mm step machined into it. A 0.635cm radius
broadband, planar transducer with a 5MHz center frequency was used to insonify the
plate at normal incidence in an immersion tank. This is illustrated in Fig. 56. The
ultrasonic beam transmitted through the plate was mapped using the microprobe
described earlier. The microprobe was positioned directly underneath the step since that
is the spot at which the sound field is most likely to be adversely affected by the presence
of the step. The transmitting probe was scanned parallel to the surface with the signal
received at the microprobe recorded at small increments in the scan. As a reference, the
same procedure was carried out with both transmitter and receiver positioned in an area of
the plate well away from the step. Figures 57 through 60 show the comparisons of
theory and experiment for 2, 3, 4, and 5MHz, respectfully. Both the step and reference
(no step) scans are shown. The agreement between the theory and experiment is good
for all cases.

There is significant beam distortion at 2 and 5 MHz, but very little at 3 and 4 MHz.
This is understood by examining Fig. 61, which is a plot of through transmitted
amplitude versus frequency when the transmitter is positioned directly over the step and
receiver. These data are normalized to the reference case. As can be seen, there are nulls
in amplitude which are periodic with frequency (note that frequencies less than 1 and
Figure 56. Experimental scan through stepped interface
Figure 57. Beam profile at 2MHz: (a) with step, (b) reference,
[solid line - theory, triangles - experiment]
Figure 58. Beam profile at 3MHz: (a) with step, (b) reference, [solid line - theory, triangles - experiment]
Figure 59. Beam profile at 4 MHz: (a) with step, (b) reference,
[solid line - theory, triangles - experiment]
Figure 60. Beam profile at 5MHz: (a) with step, (b) reference,
[solid line - theory, triangles - experiment]
Figure 61. Through transmitted amplitude vs. frequency in presence of step with transmitter and receiver aligned, [solid line - experiment, dashed line - theory]
greater than 8 MHz are outside the transducer bandwidth. Inspection will show that these nulls occur at frequencies given by

\[ f = \frac{n v_s v_w}{2d(v_s - v_w)}, \quad n=1, 3, 5, \ldots \] (93)

where \( v_s \) is the solid velocity, \( v_w \) is the water velocity, and \( d \) is the step height. These frequencies correspond to the two halves of the beam, on either side of the step, being 180° out of phase. This is the worst case. When \( n=2, 4, 6, \ldots \), the two halves of the beam are in phase and there is no distortion. Referring again to Figs. 57 through 60, the 2 and 5 MHz cases are near the out of phase frequencies while the 3 and 4 MHz cases are nearer the in phase frequencies.

For the simple case examined, the model did an excellent job of predicting the beam profiles due to a step discontinuity on the surface of the sample. If one chooses the maximum beam amplitude obtained at a point in a material as a simplistic measure of the inspectibility of that point, then it is obvious that the inspectibility is degraded severely by the presence of the step for the case shown. It is clear that surface condition plays an important role in the quality of an inspection. The present model shows promise as a tool for quantifying that role so that inspection procedures can be carried out more reliably.
VI. CONCLUSIONS

The objective of this work was to develop ultrasonic beam propagation models which can be used in the quantitative analysis of ultrasonic nondestructive evaluation procedures. These beam models can be used as input to models for wave scattering from defects, which require information about incident wave fields. Using these tools together, the results of ultrasonic inspections may be predicted. This is a valuable asset when trying to validate existing inspection procedures or when trying to develop new ones.

Two models have been derived in this work. The first predicts the radiation field only on the axis of a focused elliptical piston transducer radiating into an isotropic material and possibly through a curved interface. The solution takes the form of either a simple analytical formula for some cases or a simple numerical integration over a finite interval for other cases. The second type of model predicts the full off-axis radiation field produced in either an isotropic or anisotropic material and possibly transmitted through an interface. The solution takes the form of a series summation over a set of Gauss-Hermite eigenfunctions. The coefficients associated with this series are found by utilizing the orthogonality property of the eigenfunctions and numerically integrating over the source field, which can now be more general than a uniform piston. Convergence of the series summation is a function of how rapidly the beam varies in its transverse profile. Consequently, more terms are generally needed in the near field, where the beam is varying rapidly, than are needed in the far field. Most of the results presented here were computed using $30 \times 30 = 900$ terms. This number of terms yields good convergence.
from the far field in to about one half of the near field length.

Both the axial and full field solutions have been derived by making the Fresnel approximation. In essence, this approximation restricts the solution to cases where the ultrasonic beam is well collimated. For propagation in an isotropic material, well collimated generally means that the dimension of the beam is much greater than a wavelength and less than the radius of phase curvature. In general, this holds for propagation in an anisotropic material as well, although the validity of the approximation will vary somewhat from case to case depending on the exact form of the material's slowness surface. When beams are transmitted through interfaces, the criteria for validity are, in general, that the refracted angle not be too severe (approximately <70°) and that the radius of curvature of the interface be much greater than the beam dimension. Otherwise, aberrations caused by the interface will not be accounted for.

The models derived herein have several attributes which make them desirable for the purpose of predicting ultrasonic inspection results for the validation of inspection procedures. First, the models are applicable to a wide range of cases involving various types of probes, materials, and geometries. This provides the user with the flexibility to examine many configurations. Also, if many configurations are to be examined by varying parameters, then the models must be time efficient so that results can be obtained on what could be called a "real time" basis. The present models satisfy this condition, requiring only minimal computational time and effort. Finally, conceptual simplicity is desirable so that the physics of the problem can be visualized in the solution. The Gauss-Hermite model is a representation of an ultrasonic beam in terms of elementary beam modes, each of which has simple rules for propagation and simple parameters describing beam width, phase, and phase curvature. Consequently, much insight can be found by examining the solution itself, in addition to plotting the computed results.
There are more rigorous methods to treat the problem of ultrasonic beam propagation; however, these improvements in accuracy are generally achieved at the expense of one or more of the above attributes. Green's function methods generally require intensive computations to calculate the contribution from a distribution of point sources and anisotropy adds significant complexity. A direct numerical evaluation of the angular spectrum of plane waves representation for a beam is also computationally intensive and, while this method can treat transmission through planar interfaces very readily, curved interfaces present a difficulty. Numerical procedures such as the finite element and finite difference methods require large scale computations as well and sometimes mask the physics of the problem.

Whether or not the use of the present method, as opposed to one of these more rigorous methods, can be justified depends on how well the present method performs. This can be evaluated by either comparison to one of the more rigorous methods, or by comparison to experimental data. Here, the latter approach has been used. Ultrasonic beam profile maps were experimentally acquired in both isotropic and anisotropic materials and compared to the full field Gauss-Hermite solutions. The performance of the model was generally consistent with expectations based on presumed limits of validity of the approximations in the model. In the far field, and at moderate refracted angles, the agreement between model and experiment was excellent. In the near field, the model did not reproduce the fine structure of ultrasonic beam profiles well, although overall beam width was accurate. At large refracted angles in an isotropic material, the model disagreed with the experimental results, presumably due to aberration effects. For anisotropic materials, the agreement between model and experiment was very good. However, all of the experiments were for cases where the material's slowness surface was "well behaved". There are, undoubtably, cases for which the model will not fare as
well. Overall, the results of the comparisons between model and experiment justify the use of the model for predicting ultrasonic beam fields.

The Gauss-Hermite model has already been put to use in the quantitative analysis of ultrasonic inspection phenomena. Two applications, in the early stages of development, have been presented here. One is the use of the model to predict the radiation patterns of new types of transducers for the purpose of transducer design. The other is the use of the model to predict the distortion induced in ultrasonic beams due to transmission through nonsmooth interfaces. The model has proven useful in other applications as well. It has been used, in conjunction with the Kirchhoff approximation for scattering, to predict the signals transmitted through graphite/epoxy composite plates containing delaminations (8). It has been used to provide input fields for boundary element method codes for the scattering of ultrasound from arbitrarily shaped voids (91). Also, it is being used as a component in a code for predicting probability of detection (POD) (92-93). The POD model determines the probability that a prescribed inspection system will locate a certain type of defect in a given component.

The continued use of the model in these and other applications will contribute to its refinement as well as to the refined knowledge of its limits of validity. Hopefully, further work on the model will expand these limits. Subjects which have not been addressed as yet, such as attenuation and inhomogeneity, should be incorporated. Also, the axial field theory should be extended to treat anisotropic materials. Overall, it is expected that these models will play an important role in quantitative nondestructive evaluation analysis.
VII. REFERENCES


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